## THE HERMITE-GENOCCHI FORMULA

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ABSTRACT. We present an elementary proof of the Hermite-Genocchi formula for divided differences.

First recall the definition of divided differences.

**Definition.** Let  $I \subset \mathbf{R}$  be an interval,  $f: I \to \mathbf{R}$  and let  $x_0, x_1, \ldots, x_n$  be distinct points in I. The *divided difference of order*  $n, f[x_0, x_1, \ldots, x_n]$ , is the coefficient of  $x^n$  in the polynomial that interpolates f at the points  $x_0, x_1, \ldots, x_n$ .

It follows immediately from the definition that divided differences are invariant under permutations of the nodes.

**Theorem 1.** Let  $\pi$  be any permutation of  $0, 1, \ldots, n$ . Then

$$f[x_{\pi(0)}, x_{\pi(1)}, \dots, x_{\pi(n)}] = f[x_0, x_1, \dots, x_n].$$

In addition, divided differences can be computed recursively.

**Theorem 2** (Recursive computation of divided differences). *Divided* differences satisfy the recursion

$$f[x_0] = f(x_0),$$
  
$$f[x_0, x_1, \dots, x_{n+1}] = \frac{f[x_1, \dots, x_{n+1}] - f[x_0, x_1, \dots, x_n]}{x_{n+1} - x_0}$$

These facts will be used to prove our main result, which says that a divided difference is an integral average of the derivative of the same order. Consult the references [1], [2] and [3] for other proofs and formulations.

**Main Theorem** (Hermite-Genocchi formula). Let  $x_0, x_1, \ldots, x_n$  be distinct points, and let f be n times continuously differentiable on the convex hull of  $x_0, x_1, \ldots, x_n$ . Then

(1) 
$$f[x_0, x_1, \dots, x_n] = \int \cdots \int f^{(n)}(t_0 x_0 + \dots + t_n x_n) dt_1 \cdots dt_n.$$

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In this formula  $t_0 = 1 - \sum_{i=1}^n t_i$  and  $\Sigma_n$  is the n-simplex

$$\Sigma_n = \{(t_1, \dots, t_n) \mid t_i \ge 0 \text{ for } i = 1, \dots, n \text{ and } \sum_{i=1}^n t_i \le 1\}$$

*Proof.* We prove the theorem by induction on n. For n = 1 we have, by the fundamental theorem of calculus,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f'(\xi) \, d\xi$$

and changing variables by  $\xi = x_0 + t_1(x_1 - x_0)$  gives

$$f[x_0, x_1] = \frac{1}{x_1 - x_0} \int_0^1 f' \left( x_0 + t_1 (x_1 - x_0) \right) \left( x_1 - x_0 \right) dt_1$$
$$= \int_0^1 f' \left( (1 - t_1) x_0 + t_1 x_1 \right) dt_1$$

completing the proof in the case n = 1.

For the inductive step we assume that the formula (1) is valid for some n and prove that it is valid for n+1. To this end let  $x_0, x_1, \ldots, x_{n+1}$ be distinct points. Then by the recursive formula for divided differences, Theorem 2, we have

(2) 
$$f[x_0, x_1, \dots, x_{n+1}] = \frac{f[x_1, \dots, x_{n+1}] - f[x_0, x_1, \dots, x_n]}{x_{n+1} - x_0}$$

We apply the symmetry property, Theorem 1, to the first term in the numerator on the right side to rewrite Equation (2) as

(3) 
$$f[x_0, x_1, \dots, x_{n+1}] = \frac{f[x_{n+1}, x_1, \dots, x_n] - f[x_0, x_1, \dots, x_n]}{x_{n+1} - x_0}$$

Next, use the induction hypothesis to write the n-th order divided differences of Equation (3) in integral form:

$$f[x_0, \dots, x_{n+1}] = \frac{1}{x_{n+1} - x_0} \int_{\Sigma_n} \cdots \int \left( f^{(n)}(t_0 x_{n+1} + t_1 x_1 + \dots + t_n x_n) - f^{(n)}(t_0 x_0 + t_1 x_1 + \dots + t_n x_n) \right) dt_1 \cdots dt_n$$

[Recall that  $t_0 = 1 - \sum_{i=1}^n t_i$ .] The integrand is a difference of  $f^{(n)}$ ; use the fundamental theorem of calculus to write it as an integral of  $f^{(n+1)}$ :

$$f[x_0, \dots, x_{n+1}] = \frac{1}{x_{n+1} - x_0} \int \cdots \int \int_{t_0}^{t_0 x_{n+1} + t_1 x_1 + \dots + t_n x_n} f^{(n+1)}(\xi) \, d\xi \, dt_1 \cdots dt_n.$$

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Make the change of variable

 $\xi = (1 - t_1 - \dots - t_n)x_0 + t_1x_1 + \dots + t_nx_n + t_{n+1}(x_{n+1} - x_0)$ 

converting  $\int \cdots d\xi$  to  $(x_{n+1} - x_0) \int_0^{t_0} \cdots dt_{n+1}$ . Then we have

$$f[x_0, \dots, x_{n+1}] = \int_{\sum_{n+1}} \cdots \int f^{(n+1)}(\hat{t}_0 x_0 + t_1 x_1 \dots + t_{n+1} x_{n+1}) dt_1 \dots dt_{n+1}$$

with  $\hat{t}_0 = 1 - \sum_{i=1}^{n+1} t_i$ , and the proof is complete.

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## References

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