# THE HERMITE-GENOCCHI FORMULA 

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#### Abstract

We present an elementary proof of the Hermite-Genocchi formula for divided differences.


First recall the definition of divided differences.
Definition. Let $I \subset \mathbf{R}$ be an interval, $f: I \rightarrow \mathbf{R}$ and let $x_{0}, x_{1}, \ldots, x_{n}$ be distinct points in $I$. The divided difference of order $n, f\left[x_{0}, x_{1}, \ldots, x_{n}\right]$, is the coefficient of $x^{n}$ in the polynomial that interpolates $f$ at the points $x_{0}, x_{1}, \ldots, x_{n}$.

It follows immediately from the definition that divided differences are invariant under permutations of the nodes.
Theorem 1. Let $\pi$ be any permutation of $0,1, \ldots, n$. Then

$$
f\left[x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(n)}\right]=f\left[x_{0}, x_{1}, \ldots, x_{n}\right] .
$$

In addition, divided differences can be computed recursively.
Theorem 2 (Recursive computation of divided differences). Divided differences satisfy the recursion

$$
\begin{aligned}
f\left[x_{0}\right] & =f\left(x_{0}\right), \\
f\left[x_{0}, x_{1}, \ldots, x_{n+1}\right] & =\frac{f\left[x_{1}, \ldots, x_{n+1}\right]-f\left[x_{0}, x_{1}, \ldots, x_{n}\right]}{x_{n+1}-x_{0}}
\end{aligned}
$$

These facts will be used to prove our main result, which says that a divided difference is an integral average of the derivative of the same order. Consult the references [1], [2] and [3] for other proofs and formulations.
Main Theorem (Hermite-Genocchi formula). Let $x_{0}, x_{1}, \ldots, x_{n}$ be distinct points, and let $f$ be $n$ times continuously differentiable on the convex hull of $x_{0}, x_{1}, \ldots, x_{n}$. Then

$$
\begin{equation*}
f\left[x_{0}, x_{1}, \ldots, x_{n}\right]=\int \underset{\Sigma_{n}}{\cdots \int f^{(n)}\left(t_{0} x_{0}+\cdots+t_{n} x_{n}\right) d t_{1} \cdots d t_{n} . . . . ~ . ~} \tag{1}
\end{equation*}
$$

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In this formula $t_{0}=1-\sum_{i=1}^{n} t_{i}$ and $\Sigma_{n}$ is the $n$-simplex

$$
\Sigma_{n}=\left\{\left(t_{1}, \ldots, t_{n}\right) \mid t_{i} \geq 0 \text { for } i=1, \ldots, n \text { and } \sum_{i=1}^{n} t_{i} \leq 1\right\}
$$

Proof. We prove the theorem by induction on $n$. For $n=1$ we have, by the fundamental theorem of calculus,

$$
f\left[x_{0}, x_{1}\right]=\frac{f\left(x_{1}\right)-f\left(x_{0}\right)}{x_{1}-x_{0}}=\frac{1}{x_{1}-x_{0}} \int_{x_{0}}^{x_{1}} f^{\prime}(\xi) d \xi
$$

and changing variables by $\xi=x_{0}+t_{1}\left(x_{1}-x_{0}\right)$ gives

$$
\begin{aligned}
f\left[x_{0}, x_{1}\right] & =\frac{1}{x_{1}-x_{0}} \int_{0}^{1} f^{\prime}\left(x_{0}+t_{1}\left(x_{1}-x_{0}\right)\right)\left(x_{1}-x_{0}\right) d t_{1} \\
& =\int_{0}^{1} f^{\prime}\left(\left(1-t_{1}\right) x_{0}+t_{1} x_{1}\right) d t_{1}
\end{aligned}
$$

completing the proof in the case $n=1$.
For the inductive step we assume that the formula (1) is valid for some $n$ and prove that it is valid for $n+1$. To this end let $x_{0}, x_{1}, \ldots, x_{n+1}$ be distinct points. Then by the recursive formula for divided differences, Theorem 2, we have

$$
\begin{equation*}
f\left[x_{0}, x_{1}, \ldots, x_{n+1}\right]=\frac{f\left[x_{1}, \ldots, x_{n+1}\right]-f\left[x_{0}, x_{1}, \ldots, x_{n}\right]}{x_{n+1}-x_{0}} \tag{2}
\end{equation*}
$$

We apply the symmetry property, Theorem 1 , to the first term in the numerator on the right side to rewrite Equation (2) as

$$
\begin{equation*}
f\left[x_{0}, x_{1}, \ldots, x_{n+1}\right]=\frac{f\left[x_{n+1}, x_{1}, \ldots, x_{n}\right]-f\left[x_{0}, x_{1}, \ldots, x_{n}\right]}{x_{n+1}-x_{0}} \tag{3}
\end{equation*}
$$

Next, use the induction hypothesis to write the $n$-th order divided differences of Equation (3) in integral form:

$$
\begin{aligned}
& f\left[x_{0}, \ldots, x_{n+1}\right]= \\
& \quad \frac{1}{x_{n+1}-x_{0}} \int{ }_{\Sigma_{n}} \cdots \int\left(f^{(n)}\left(t_{0} x_{n+1}+t_{1} x_{1}+\cdots+t_{n} x_{n}\right)-\right. \\
& \\
& \left.\quad f^{(n)}\left(t_{0} x_{0}+t_{1} x_{1}+\cdots+t_{n} x_{n}\right)\right) d t_{1} \cdots d t_{n}
\end{aligned}
$$

[Recall that $t_{0}=1-\sum_{i=1}^{n} t_{i}$.] The integrand is a difference of $f^{(n)}$; use the fundamental theorem of calculus to write it as an integral of $f^{(n+1)}$ :

$$
\begin{aligned}
f\left[x_{0}, \ldots, x_{n+1}\right] & = \\
& \frac{1}{x_{n+1}-x_{0}} \int \cdots \iint_{\Sigma_{0}} \int_{t_{0} x_{0}+t_{1} x_{1}+\cdots+t_{n} x_{n}}^{t_{n+1}+t_{1} x_{1}+\cdots+t_{n} x_{n}} f^{(n+1)}(\xi) d \xi d t_{1} \cdots d t_{n} .
\end{aligned}
$$

Make the change of variable

$$
\xi=\left(1-t_{1}-\cdots-t_{n}\right) x_{0}+t_{1} x_{1}+\cdots+t_{n} x_{n}+t_{n+1}\left(x_{n+1}-x_{0}\right)
$$

converting $\int \cdots d \xi$ to $\left(x_{n+1}-x_{0}\right) \int_{0}^{t_{0}} \cdots d t_{n+1}$. Then we have

$$
\begin{aligned}
f\left[x_{0}, \ldots, x_{n+1}\right] & = \\
& \int_{\Sigma_{n+1}} \cdots \int f^{(n+1)}\left(\hat{t}_{0} x_{0}+t_{1} x_{1} \cdots+t_{n+1} x_{n+1}\right) d t_{1} \cdots d t_{n+1}
\end{aligned}
$$

with $\hat{t}_{0}=1-\sum_{i=1}^{n+1} t_{i}$, and the proof is complete.

## References

[1] K. E. Atkinson, An Introduction to Numerical Analysis, 2nd ed., §3.2, Wiley, New York, 1989.
[2] M. Schatzmann, Numerical Analysis, §4.2.2, Oxford, Clarendon Press, 2002.
[3] F. Stummel and K. Hainer, Introduction to Numerical Analysis, §3.2.1, Scottish Academic Press, Edinburgh, 1980.

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