



# Boundary homogenization in domains with randomly oscillating boundary<sup>☆</sup>

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Received 23 February 2010; received in revised form 25 August 2010; accepted 26 August 2010

Available online 9 September 2010

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## Abstract

We consider a model homogenization problem for the Poisson equation in a domain with a rapidly oscillating boundary which is a small random perturbation of a fixed hypersurface. A Fourier boundary condition with random coefficients is imposed on the oscillating boundary. We derive the effective boundary condition, prove a convergence result, and establish error estimates.

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*Keywords:* Boundary homogenization; Random oscillating boundary; Effective boundary condition

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## 1. Introduction

Many problems in modern material sciences and engineering require the study of the macroscopic behavior of bodies with rough inhomogeneous surfaces. The problem of electromagnetic scattering by an obstacle coated with an absorbing inhomogeneous paint, the dynamics of two-fluid flow in porous media and past rough walls, and the hydrodynamic lubrication of rough surfaces are only a few examples. A fundamental issue is understanding the link between microscopic and macroscopic behavior.

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<sup>☆</sup> The work of the third author was supported in part by the Leading Scientific Schools program (project HIII–7387.2010.1) and was also supported in part by RFBR grant (10-01-00594).

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Recently, many mathematical works have been devoted to the asymptotic analysis of problems in domains with random microstructure. The first rigorous homogenization results for divergence form elliptic operators with random coefficients have been obtained in the pioneer works [26,27,33]. Then the estimates for the rate of convergence were proved in [38].

Homogenization problems in randomly perforated domains were studied in [39,40]. Notice that in [40] a rather general random geometry was considered. In particular, this geometry did not assume the existence of an extension operator.

In [12] the authors introduced the stochastic two-scale convergence in the mean and investigated its main properties. Later on the realizationwise two-scale convergence was defined in [41]; this technique also applies to homogenization of random thin structures and singular measures.

Effective equations of a flow in media with stochastic microstructure were derived in [11].

Homogenization problems for random operators with large lower order terms were considered in [13,21].

Further information on random homogenization and detailed bibliography can be found in the monographs [25,20].

The boundary homogenization for elliptic boundary value problems with randomly alternating kinds of boundary conditions was studied in [8]; the effective boundary condition in a domain randomly perforated along the boundary was obtained in [16]. The paper [17] dealt with the homogenization of a thick junction through a thin random transmission zone.

Another field, that of equations in domains with rapidly oscillating boundary (periodic and locally periodic as well as almost periodic), is also quite well-developed. See, for instance, [1–7,9,10,15,14,18,19,24,29,31,32,35,34,36].

The combination of the two effects, oscillation of the exterior boundary and the randomness of its geometry, appears naturally in applications but leads to additional mathematical difficulties.

As a typical example we mention here the morphology of contacting surfaces that plays an important role in the frictional behavior of deformable bodies. The roughness of the contact surface and the material properties near this surface are the microscopic characteristics which essentially influence the large scale behavior. The most realistic case includes a random statistically homogeneous profile of the oscillating part of the boundary and the Fourier boundary condition.

In biology, when studying the metabolism of infusoria, the cell membrane has a random microstructure. The description of life activity in the cell requires boundary homogenization at the cell membrane.

The most realistic case, when there is a small dissipation at the boundary, is of special interest. The corresponding mathematical description of this effect involves the Fourier boundary condition. The aim of this paper is to investigate a model problem in such a context. We study the Poisson equation in a domain with rapidly oscillating random boundary, in the presence of a small random dissipation at the boundary.

We assume throughout this paper that all the random functions describing both the domain geometry and the coefficients of the boundary operator are statistically homogeneous. We derive the homogenized problem, prove the convergence result, and, under additional mixing conditions, establish error estimates.

The paper is structured as follows. In the next section we introduce necessary notation, describe the family of random domains depending on a small positive parameter  $\varepsilon$  and pose the problem to be studied. In Section 3 we specify the probabilistic framework of our study and make explicit assumptions on the random fields under consideration. Section 4 contains the

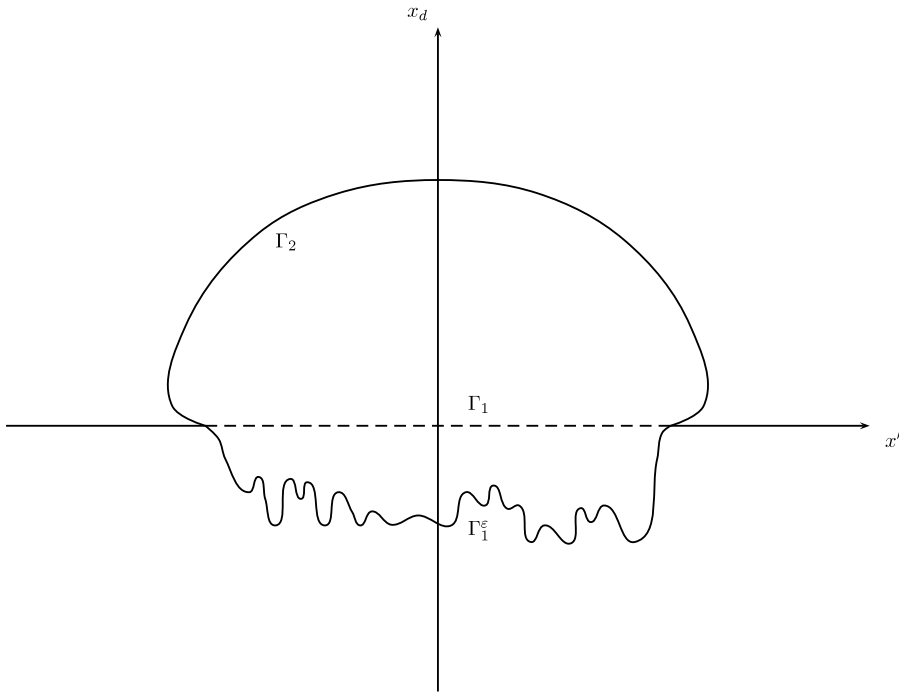


Fig. 1. Domain with random rapidly oscillating boundary.

statements of our main results. Section 5 deals with various technical assertions that are used in our analysis. Sections 6 and 7 are devoted to the proof of the convergence result and obtaining error estimates, respectively.

### 2. Preliminaries and statement of the problem

Let  $D \subset \mathbb{R}^d \cap \{x \mid x_d > 0\}$ ,  $d \geq 2$ , be a smooth bounded domain whose boundary has a nontrivial flat part  $\Gamma_1 = \partial D \cap \{x \mid x_d = 0\}$  with a nonempty  $(d - 1)$ -dimensional interior  $\overset{\circ}{\Gamma}_1$ .

We perturb the flat part of the boundary in such a way that the perturbed domain has an oscillating boundary (see Fig. 1). To this end, we define a smooth nonnegative function  $g(\hat{x})$ ,  $\hat{x} = (x_1, \dots, x_{d-1})$  such that  $\text{supp } g(\hat{x}) \subset \Gamma_0 \Subset \overset{\circ}{\Gamma}_1$ , and, given a statistically homogeneous nonpositive random function  $F(\hat{\xi}, \omega)$ ,  $\hat{\xi} = (\xi_1, \dots, \xi_{d-1})$ , which has smooth realizations and is defined on a standard probability space  $(\Omega, \mathcal{A}, \mu)$ , we set, for  $\varepsilon > 0$ ,

$$\Pi_\varepsilon = \left\{ x \in \mathbb{R}^d : \hat{x} \in \Gamma_1, \varepsilon g(\hat{x}) F\left(\frac{\hat{x}}{\varepsilon}, \omega\right) < x_d \leq 0 \right\}$$

and, finally, introduce the desired domain with random boundary as follows:

$$D^\varepsilon = D \cup \Pi_\varepsilon.$$

For more detailed definitions of randomness we refer the reader to the next section. According to the above construction, the boundary  $\partial D^\varepsilon$  consists of the parts  $\Gamma_2$  and  $\Gamma_1^\varepsilon = \left\{ x \in \partial D^\varepsilon : (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon g(\hat{x}) F\left(\frac{\hat{x}}{\varepsilon}, \omega\right) \right\}$  forming together the domain boundary.

We consider the boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f(x) & \text{in } D^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial v_\varepsilon} + g(\widehat{x})p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)u_\varepsilon = g(\widehat{x})q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) & \text{on } \Gamma_1^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial v} = 0 & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where  $v_\varepsilon$  is an outer normal to  $\Gamma_1^\varepsilon$  and  $v$  is an outward normal to  $\Gamma_2$ ;  $p(\widehat{\xi}, \omega)$  and  $q(\widehat{\xi}, \omega)$  are random statistically homogeneous positive functions.

**Definition 1.** A function  $u_\varepsilon \in H^1(D^\varepsilon)$  is a solution to problem (1) if it satisfies the integral identity

$$\begin{aligned} \int_{D^\varepsilon} \nabla u_\varepsilon(x) \nabla v(x) dx + \int_{\Gamma_1^\varepsilon} g(\widehat{x})p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)u_\varepsilon(x)v(x) ds \\ = \int_{D^\varepsilon} f(x)v(x) dx + \int_{\Gamma_1^\varepsilon} g(\widehat{x})q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)v(x) ds, \end{aligned} \tag{2}$$

for any function  $v \in H^1(D^\varepsilon)$ .

Our aim is to investigate the asymptotic behavior, as  $\varepsilon \rightarrow 0$ , of the solution  $u_\varepsilon$  to problem (1).

### 3. The probabilistic framework and main assumptions

In this section we introduce the probabilistic framework of our problem. We refer the reader to [25] and the references therein for a more detailed description.

Throughout the paper, we assume that all the random fields and random variables are defined on a probability space  $(\Omega, \mathcal{A}, \mu)$ . The random fields considered in the paper are statistically homogeneous.

**Definition 2.** A family of measurable maps

$$T_x : \Omega \rightarrow \Omega, \quad x \in \mathbb{R}^{d-1},$$

is called a  $(\mathbf{d} - 1)$ -dynamical system if the following properties hold true:

- Group property:

$$T_{x+y} = T_x T_y \quad \forall x, y \in \mathbb{R}^{d-1}, \quad T_0 = \text{Id} \quad (\text{Id is the identical mapping});$$

- Isometry property:

$$T_x \mathcal{U} \in \mathcal{A}, \quad \mu(T_x \mathcal{U}) = \mu(\mathcal{U}), \quad \forall x \in \mathbb{R}^{d-1}, \quad \forall \mathcal{U} \in \mathcal{A};$$

- Measurability: for any measurable functions  $\phi(\omega)$  on  $\Omega$ , the function  $\phi(T_x \omega)$  is measurable on  $\Omega \times \mathbb{R}^{d-1}$ , where the space  $\mathbb{R}^{d-1}$  is equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}$ .

**Definition 3.** Let  $\phi(\omega)$  be a measurable function (i.e. a random variable) on  $\Omega$ . The function  $\phi(T_x \omega)$  of  $x \in \mathbb{R}^{(d-1)}$  and  $\omega \in \Omega$  is called a *statistically homogeneous random field*, and, for fixed  $\omega \in \Omega$ , the function  $x \mapsto \phi(T_x \omega)$  is called a *realization* of the random field  $\phi$ .

Let  $L_q(\Omega)$  ( $q \geq 1$ ) be the space of measurable functions and integrable in the power  $q$  with respect to the measure  $\mu$ . The following assertion holds; see [20,25] for the proof.

**Proposition 3.1.** Assume that  $\phi \in L_q(\Omega)$ . Then almost all realizations  $\phi(T_x\omega)$  belong to  $L_q^{\text{loc}}(\mathbb{R}^{(d-1)})$ .

If the sequence  $\{\phi_k\} \subset L_q(\Omega)$  converges in  $L_q(\Omega)$  to the function  $\phi$ , then there exists a subsequence  $\{\phi_{k'}\}$  such that almost all realizations  $\phi_{k'}(T_x\omega)$  converge in  $L_q^{\text{loc}}(\mathbb{R}^{(d-1)})$  to the realization  $\phi(T_x\omega)$ .

**Definition 4.** A measurable function  $\phi(\omega)$  on  $\Omega$  is called *invariant* if, for any  $x \in \mathbb{R}^{d-1}$ ,  $\phi(T_x\omega) = \phi(\omega)$  almost surely.

**Definition 5.** A dynamical system  $T_x$  is said to be *ergodic* if all its invariant functions are almost surely constant.

**Definition 6.** Let  $\theta \in L_1^{\text{loc}}(\mathbb{R}^{d-1})$ . We say that the function  $\theta$  has a spatial average if the limit

$$M(\theta) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|B|} \int_B \theta\left(\frac{x}{\varepsilon}\right) dx$$

exists for any bounded Borel set  $B \in \mathcal{B}$  with  $|B| > 0$ , and moreover this limit does not depend on the choice of  $B$ . The quantity  $M(\theta)$  is called the *spatial average* of the function  $\theta$ .

The following result is proved in [20].

**Proposition 3.2.** Let a function  $\theta$  have a spatial average in  $\mathbb{R}^{d-1}$ , and suppose that the family  $\{\theta(\frac{x}{\varepsilon}), 0 < \varepsilon \leq 1\}$  is bounded in  $L_q(\mathcal{K})$ , for some  $q \geq 1$ , where  $\mathcal{K}$  is a compact in  $\mathbb{R}^{d-1}$  whose interior is not empty and contains the origin. Then,

$$\theta\left(\frac{x}{\varepsilon}\right) \rightharpoonup M(\theta) \text{ weakly in } L_q^{\text{loc}}(\mathbb{R}^{d-1}), \text{ as } \varepsilon \rightarrow 0.$$

Throughout the article, we make use of the Birkhoff ergodic theorem in the following particular form (see, for instance, [20,25] for more details).

**Theorem 3.1 (Birkhoff Ergodic Theorem).** Let  $T_x$  be an ergodic  $(d - 1)$ -dynamical system and let  $\phi \in L_q(\Omega)$ ,  $q \geq 1$ . Then, almost surely (i.e. for almost all  $\omega \in \Omega$ ), the realization  $\phi(T_x\omega)$  admits a spatial average  $M(\phi(T_x\omega))$ . Moreover,

$$\mathbb{E}(\phi) = M(\phi(T_x\omega))$$

where  $\mathbb{E}(\phi)$  is the mathematical expectation of  $\phi$ .

**Definition 7.** A random field  $\zeta(x, \omega)(x \in \mathbb{R}^{d-1}, \omega \in \Omega)$  is called *statistically homogeneous* if the following representation holds:

$$\zeta(x, \omega) = \tilde{\zeta}(T_x\omega),$$

where  $\tilde{\zeta}$  is a random variable on  $(\Omega, \mathcal{A}, \mu)$  and  $T_x$  is a  $(d - 1)$ -dynamical system on  $\Omega$ .

We are now ready to make assumptions on the random fields  $F(\widehat{\xi}, \omega)$ ,  $p(\widehat{\xi}, \omega)$  and  $q(\widehat{\xi}, \omega)$ . First we assume that these random fields are statistically homogeneous, that is

$$F(\widehat{\xi}, \omega) = \tilde{F}(T_{\widehat{\xi}}\omega), \quad p(\widehat{\xi}, \omega) = \tilde{p}(T_{\widehat{\xi}}\omega), \quad q(\widehat{\xi}, \omega) = \tilde{q}(T_{\widehat{\xi}}\omega),$$

for all  $\widehat{\xi} \in \mathbb{R}^{d-1}$ , where  $\widetilde{F}$ ,  $\widetilde{p}$  and  $\widetilde{q}$  are random variables on  $(\Omega, \mathcal{A}, \mu)$ , and  $T_{\widehat{x}}$  is an ergodic  $(d - 1)$ -dynamical system on  $\Omega$ .

Moreover, we assume that  $\widetilde{F}$  has, almost surely, continuously differentiable or locally Lipschitz realizations. We define

$$\partial_{\omega}^i \widetilde{F}(\omega) = \partial_{\xi_i} \widetilde{F}(T_{\widehat{\xi}}\omega)|_{\widehat{\xi}=0}, \quad \partial_{\omega} \widetilde{F}(\omega) = \nabla_{\widehat{\xi}} \widetilde{F}(T_{\widehat{\xi}}\omega)|_{\widehat{\xi}=0}.$$

We have  $\nabla_{\widehat{\xi}} F(\widehat{\xi}, \omega) = \partial_{\omega} \widetilde{F}(T_{\widehat{\xi}}\omega)$  (see, for instance, [25]).

Finally, we make the following assumptions on the functions  $\widetilde{F}$ ,  $\widetilde{p}$  and  $\widetilde{q}$ :

- (h1)  $\widetilde{F} \in L_{\infty}(\Omega)$ ,  $\widetilde{F}(\omega) \leq 0$  a.s.;
- (h2)  $\partial_{\omega} \widetilde{F} \in (L_2(\Omega))^{d-1}$ ;
- (h3)  $\widetilde{p} \in L_{\infty}(\Omega)$ ,  $\widetilde{p}(\omega) \geq 0$  a.s.,  $\mu\{\omega : \widetilde{p}(\omega) > 0\} > 0$ ;
- (h4)  $\widetilde{q} \in L_2(\Omega)$ ,  $\widetilde{q}\partial_{\omega} \widetilde{F} \in (L_2(\Omega))^{d-1}$ .

Several assertions formulated in this work are valid under a positiveness condition on  $\widehat{p}$  stronger than (h3). This stronger condition reads as follows:

$$(h3') \quad p^- \leq \widetilde{p}(\omega) \leq p^+ \text{ a.s. for deterministic } p^- \text{ and } p^+ \text{ with } p^- > 0.$$

Also in a number of statements we assume that

$$(h2') \quad \partial_{\omega} \widetilde{F} \in (L_2(\Omega))^{d-1} \text{ if } d < 5; \partial_{\omega} \widetilde{F} \in (L_{d/2}(\Omega))^{d-1} \text{ if } d \geq 5.$$

Parts of the results on the rate of convergence are obtained under the following condition:

$$(h2'') \quad \partial_{\omega} \widetilde{F} \in (L_{\infty}(\Omega))^{d-1}.$$

#### 4. Main results

In this section we describe the homogenized problem for (1) and formulate the convergence results. Applying the formal asymptotic technique, one can obtain the effective boundary conditions for the limit problem (see [4–7,19], for the periodic case). The homogenized problem reads

$$\begin{cases} -\Delta u_0 = f(x) & \text{in } D, \\ -\frac{\partial u_0}{\partial x_d} + g(\widehat{x})P(\widehat{x})u_0 = g(\widehat{x})Q(\widehat{x}) & \text{on } \Gamma_1, \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{3}$$

where

$$\begin{aligned} P(\widehat{x}) &= \mathbb{E} \left( \widetilde{p}(\omega) \sqrt{1 + (g(\widehat{x})\partial_{\omega} \widetilde{F}(\omega))^2} \right), \\ Q(\widehat{x}) &= \mathbb{E} \left( \widetilde{q}(\omega) \sqrt{1 + (g(\widehat{x})\partial_{\omega} \widetilde{F}(\omega))^2} \right). \end{aligned} \tag{4}$$

The variational formulation associated with problem (3) reads

$$\begin{aligned} & \int_D \nabla u_0(x) \nabla v(x) dx + \int_{\Gamma_1} g(\widehat{x})P(\widehat{x})u_0(x)v(x) d\widehat{x} \\ &= \int_D f(x)v(x) dx + \int_{\Gamma_1} g(\widehat{x})Q(\widehat{x})v(x) d\widehat{x} \end{aligned} \tag{5}$$

for any function  $v \in H^1(D)$ .

By the standard regularity results for elliptic equations and thanks to the smoothness of  $\partial D$ , the solution  $u_0$  of problem (3) belongs to the space  $H^2(D)$ .

**Remark 4.1.** By construction the function  $u_0$  is not defined in the whole domain  $D^\varepsilon$ . Applying the technique of symmetric extension (see e.g. [28]) allows us to extend  $u_0$  into a larger domain, say  $D^+$ , which comprises the domains  $D^\varepsilon$ , for all  $\varepsilon \in (0, 1]$ ; we keep the same notation  $u_0$  for the extended function. In particular, for all  $\varepsilon \in (0, 1]$  we have  $\|u_0\|_{H^2(D^\varepsilon)} \leq C \|u_0\|_{H^2(D)}$ , where  $C$  does not depend on  $\varepsilon$ .

The limit behavior of the solution  $u_\varepsilon$  of problem (1) is described by the following statement.

**Theorem 4.1.** Assume that  $f \in L_2^{\text{loc}}(\mathbb{R}^d)$ , assumptions (h1)–(h4) are fulfilled, and  $F(x, \omega)$  has, almost surely, continuously differentiable realizations. Then, almost surely for any sufficiently small  $\varepsilon > 0$ , problem (1) has a unique solution, and we have almost surely

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u_0\|_{L_2(D^\varepsilon)} = 0, \tag{6}$$

where  $u_0$  is the solution of problem (3). If in addition assumption (h3') is satisfied, then we have

$$\mathbb{E} (\|u_\varepsilon - u_0\|_{L_2(D^\varepsilon)}) \longrightarrow 0, \tag{7}$$

as  $\varepsilon \rightarrow 0$ .

Under assumptions (h1), (h2'), (h3) and (h4), we have

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u_0\|_{H^1(D^\varepsilon)} = 0, \tag{8}$$

almost surely. Finally, if (h1), (h2'), (h3') and (h4) are fulfilled, then

$$\mathbb{E} (\|u_\varepsilon - u_0\|_{H^1(D^\varepsilon)}) \longrightarrow 0, \tag{9}$$

as  $\varepsilon \rightarrow 0$ .

**Remark 4.2.** Notice that in lower dimensions  $d < 5$  conditions (h2) and (h2') coincide.

**Remark 4.3.** In fact, if  $\partial_\omega \tilde{F} \in L_\infty(\Omega)$  then the condition  $p(\hat{\xi}, \omega) \geq 0$  almost surely, in the statement of Theorem 4.1, can be replaced with the weaker condition  $P(\hat{x}) > 0$ .

The rate of convergence of  $u_\varepsilon$  towards  $u_0$  can be estimated under an additional mixing assumption on the random fields  $F(\hat{\xi})$ ,  $p(\hat{\xi})$  and  $q(\hat{\xi})$ . In order to introduce this assumption we first define the so-called uniform mixing coefficient and maximum correlation coefficient.

For a bounded set  $A$  in  $\mathbb{R}^{d-1}$ , denote by  $\sigma_A$  the  $\sigma$ -algebra  $\sigma\{F(\hat{\xi}, \cdot), p(\hat{\xi}, \cdot), q(\hat{\xi}, \cdot) : \hat{\xi} \in A\}$ , i.e. the  $\sigma$ -algebra generated in  $\Omega$  by  $F(\hat{\xi}, \cdot)$ ,  $p(\hat{\xi}, \cdot)$ ,  $q(\hat{\xi}, \cdot)$ , for  $\hat{\xi} \in A$ .

**Definition 8.** The function  $\alpha(s)$ ,  $s > 0$ , defined by

$$\alpha(s) = \sup_{\substack{A_1, A_2 \subset \mathbb{R}^{d-1}, \\ \text{dist}(A_1, A_2) \geq s}} \sup_{\substack{\mathcal{U}_1 \in \sigma_{A_1}, \\ \mathcal{U}_2 \in \sigma_{A_2}}} |\mu(\mathcal{U}_1 \cap \mathcal{U}_2) / \mu(\mathcal{U}_2) - \mu(\mathcal{U}_1)|,$$

is called the *uniform mixing coefficient* of the random field  $(F, p, q)$ .

The *maximum correlation coefficient*  $\rho(s)$ ,  $s > 0$ , of the random field  $(F, p, q)$  is defined by

$$\rho(s) = \sup_{\substack{A_1, A_2 \subset \mathbb{R}^{d-1}, \\ \text{dist}(A_1, A_2) \geq s}} \sup_{\substack{\eta_1 \in L_2(\Omega, \sigma_{A_1}), \\ \eta_2 \in L_2(\Omega, \sigma_{A_2})}} |\mathbb{E}(\eta_1 \eta_2)|, \tag{10}$$

where the second supremum is taken over all  $\sigma_{A_1}$ -measurable  $\eta_1$  and  $\sigma_{A_2}$ -measurable  $\eta_2$  such that  $\mathbb{E}\eta_j = 0$  and  $\mathbb{E}\{(\eta_j)^2\} = 1, j = 1, 2$ .

Since the domain  $D^\varepsilon$  depends on  $\varepsilon$ , it is convenient to introduce a domain, say  $D^+$ , which contains all the domains  $D^\varepsilon, \varepsilon \leq 1$ .

**Theorem 4.2.** (i) Assume that (h1)–(h4) and (h3') are fulfilled, and that  $F(\widehat{\xi}, \omega)$  has, almost surely, continuously differentiable realizations. Assume also that  $f \in L_2(D^+)$  with  $\text{dist}(\text{supp}(f), \Gamma_1) > 0$ . If, in addition,

$$\int_0^\infty \sqrt{\alpha(s)} ds < \infty \quad \text{or} \quad \int_0^\infty \rho(s) ds < \infty, \tag{11}$$

then the following estimate holds true:

$$\mathbb{E}(\|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)}) \leq K \varepsilon^{1/4}, \tag{12}$$

where  $u_0$  solves problem (3) and the constant  $K$  does not depend on  $\varepsilon$ .

(ii) If conditions (h1), (h2''), (h3') and (h4) are satisfied, and at least one of the conditions in (11) is fulfilled, then for any  $f \in L^2(D^+)$  the estimate (12) holds.

**Remark 4.4.** Both conditions in (11) are fulfilled if the random field  $(F, p, q)$  has finite range of dependence. Also, if the random field  $(F, p, q)$  is Gaussian, then (11) follows from fast enough decay of the correlation function of this field.

In practice, for a generic statistically homogeneous random field  $(F, p, q)$  it might be difficult to check rigorously whether condition (11) holds true. In the engineering applications the supremum in (10) is often replaced with the expression

$$\max_{j,k} \mathbb{E} \left( \varkappa_j(F(0, \cdot), p(0, \cdot), q(0, \cdot)) \varkappa_k(F(\widehat{\xi}, \cdot), p(\widehat{\xi}, \cdot), q(\widehat{\xi}, \cdot)) \right),$$

where  $|\widehat{\xi}| = s$ , and  $\varkappa_1, \varkappa_2, \dots, \varkappa_N$  is a (sufficiently rich) finite collection of Borel functions such that

$$\mathbb{E} \left( \varkappa_j(F(0, \cdot), p(0, \cdot), q(0, \cdot)) \right) = 0, \quad \mathbb{E} \left[ \left( \varkappa_j(F(0, \cdot), p(0, \cdot), q(0, \cdot)) \right)^2 \right] = 1.$$

If this new quantity shows sufficiently fast decay as  $s \rightarrow \infty$ , then it is supposed that condition (11) is fulfilled.

### 5. Preliminary lemmas

This section is devoted to various technical assertions which are used in the further analysis. Some of these assertions have been proved in [19] (see also [5]); for them we do not provide detailed proofs but only stress the difference from the periodic case.

**Lemma 5.1.** *Almost surely, the inequalities*

$$\left\| v \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) - v(\widehat{x}, 0) \right\|_{L_2(\Gamma_1)} \leq C_1 \sqrt{\varepsilon} \|v\|_{H^1(D^\varepsilon)}, \tag{13}$$

$$\|v\|_{L_2(\Pi_\varepsilon)} \leq C_2 \sqrt{\varepsilon} \|v\|_{H^1(D^\varepsilon)}, \tag{14}$$

hold for any function  $v \in H^1(D^\varepsilon)$ , with deterministic positive constants  $C_1$  and  $C_2$ .



If  $u \in H^2(D^+)$  we have, for  $d > 2$ ,

$$\left\| u \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) - u(\widehat{x}, 0) \right\|_{L^{(2d)/(d-2)}(\Gamma_1)} \leq C_3 \varepsilon^{\frac{d+2}{2d}} \|u\|_{H^2(D^+)}, \tag{15}$$

with a deterministic constant  $C_3$ .

**Proof.** The proof of estimates (13) and (14) is completely identical to that of Lemma 1 in [19]. The constants  $C_1$  and  $C_2$  are deterministic due to assumption (h1).

To prove estimate (15) it suffices to justify it for smooth functions; the validity of this estimate for a function of  $H^2(D^+)$  will follow by a density argument. For  $u \in C^\infty(\mathbb{R}^d)$ , by Hölder’s inequality we have

$$\begin{aligned} & \int_{\Gamma_1} \left| u \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) - u(\widehat{x}, 0) \right|^{\frac{2d}{d-2}} d\widehat{x} \\ &= \int_{\Gamma_1} \left| \int_0^{\varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right)} \frac{\partial}{\partial x_d} u(\widehat{x}, x_d) dx_d \right|^{\frac{2d}{d-2}} d\widehat{x} \\ &\leq C \varepsilon^{\frac{d+2}{d-2}} \int_{\Gamma_1} \left| \int_0^{\varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right)} \left| \frac{\partial}{\partial x_d} u(\widehat{x}, x_d) \right|^{\frac{2d}{d-2}} dx_d \right| d\widehat{x} \leq C \varepsilon^{\frac{d+2}{d-2}} \|\nabla u\|_{L^{\left(\frac{2d}{d-2}\right)}(D^+)}. \end{aligned}$$

By the Sobolev embedding theorem (see, for instance, [37]),  $\|\nabla u\|_{L^{\left(\frac{2d}{d-2}\right)}(D^+)} \leq C \|u\|_{H^2(D^+)}$  with a constant  $C$  which does not depend on  $u$ . This yields (15).  $\square$

As a consequence of the previous lemma and the trace theorem we have

$$\left\| v \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \right\|_{L_2(\Gamma_1)} \leq C \|v\|_{H^1(D^\varepsilon)} \tag{16}$$

and

$$\left\| u \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \right\|_{L^{\left(\frac{2d}{d-2}\right)}(\Gamma_1)} \leq C \|u\|_{H^2(D^+)} \tag{17}$$

with a deterministic constant  $C$  which does not depend on  $\varepsilon$ .

When computing boundary integrals over  $\Gamma_1^\varepsilon$ , it is convenient to choose the coordinates  $\widehat{x} = (x_1, \dots, x_{d-1})$  on  $\Gamma_1^\varepsilon$ . Then we need a convenient expression for the element of the  $(d - 1)$ -dimensional volume of  $\Gamma_1^\varepsilon$  in this coordinate system, which is the purpose of the next lemma.

**Lemma 5.2.** *Let  $(ds)$  be an element of the  $(d - 1)$ -dimensional volume of  $\Gamma_1^\varepsilon$ . Then, almost surely,*

$$ds = \sqrt{1 + \left| g(\widehat{x}) \partial_\omega \widetilde{F} \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \right|^2} d\widehat{x} (1 + O(\varepsilon)), \tag{18}$$

where  $|O(\varepsilon)| \leq C\varepsilon$  with a deterministic constant  $C$ .

**Proof.** According to our assumptions, the boundary  $\Gamma_1^\varepsilon$  is defined by the equation

$$x_d - \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) = 0.$$

Hence, omitting the variable  $\omega$  (which is the usual convention), we have

$$ds = \sqrt{(\varepsilon F \partial_{x_1} g + g \partial_{\xi_1} F)^2 + \dots + (\varepsilon F \partial_{x_{n-1}} g + g \partial_{\xi_{n-1}} F)^2 + 1} \Big|_{\widehat{\xi} = \frac{\widehat{x}}{\varepsilon}} d\widehat{x}.$$

Defining

$$S = \sqrt{\varepsilon^2 |\nabla_{\widehat{x}} g|^2 F^2 + 2\varepsilon F g (\nabla_{\widehat{x}} g, \nabla_{\widehat{\xi}} F) + g^2 |\nabla_{\widehat{\xi}} F|^2 + 1},$$

by direct calculations we get, almost surely,

$$\begin{aligned} S \Big|_{\widehat{\xi} = \frac{\widehat{x}}{\varepsilon}} - \sqrt{1 + g^2 |\nabla_{\widehat{\xi}} F|^2} \Big|_{\widehat{\xi} = \frac{\widehat{x}}{\varepsilon}} &= \frac{\varepsilon^2 |\nabla_{\widehat{x}} g|^2 F^2 + 2\varepsilon F g (\nabla_{\widehat{x}} g, \nabla_{\widehat{\xi}} F)}{S + \sqrt{1 + g^2 |\nabla_{\widehat{\xi}} F|^2}} \Big|_{\widehat{\xi} = \frac{\widehat{x}}{\varepsilon}} \\ &\leq C_3 \varepsilon, \end{aligned}$$

where the constant  $C_3$  is deterministic and does not depend on  $\varepsilon$ . This inequality implies (18).  $\square$

The next proposition is a direct consequence of the Sobolev embedding theorem (see e.g. [37]).

**Proposition 5.1.** *The inequality*

$$\left| \int_{\Gamma_1} uv d\widehat{x} \right| \leq C_3 \|u\|_{H^{\frac{1}{2}}(\Gamma_1)} \|v\|_{H^{\frac{1}{2}}(\Gamma_1)}$$

holds uniformly in  $u, v \in H^{\frac{1}{2}}(\Gamma_1)$ .

The uniform coerciveness, with respect to  $\varepsilon$ , of the bilinear form in (2) is the subject of the next statement. It implies, in particular, that problem (1) is well-posed. For the proof of the lemma see, for instance, [30].

**Lemma 5.3.** *Under the assumptions of Theorem 4.1, almost surely for sufficiently small  $\varepsilon > 0$ , the inequality*

$$\int_{D^\varepsilon} |\nabla v|^2 dx + \int_{\Gamma_1^\varepsilon} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^2 ds \geq C_4 \|v\|_{H^1(D^\varepsilon)}^2 \tag{19}$$

holds for any  $v \in H^1(D^\varepsilon)$  with a deterministic constant  $C_4$  that does not depend on  $\varepsilon$ .

**Proof.** Consider, for a given  $p_0 > 0$ , the random variable on  $\Omega$  defined by

$$m_\varepsilon(\omega) = \mathcal{H}^{d-1} \left\{ x \in \tilde{\Gamma}_1 : p \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \geq p_0 \right\},$$

where  $\mathcal{H}^{d-1}$  stands for the  $(d - 1)$ -dimensional Lebesgue measure on  $\Gamma_1$ , and  $\tilde{\Gamma}_1$  is an open Borel subset of  $\Gamma_1$  such that  $g(\widehat{x}) \geq g_0 > 0$  for  $x \in \tilde{\Gamma}_1$ . We have almost surely, due to the Birkhoff theorem,

$$m_\varepsilon = \int_{\tilde{\Gamma}_1} \mathbf{1}_{\left\{ \tilde{p} \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \geq p_0 \right\}} d\widehat{x} \longrightarrow \left| \tilde{\Gamma}_1 \right| \cdot \mathbb{E} \left( \mathbf{1}_{\{\tilde{p}(\omega) \geq p_0\}} \right), \quad \text{as } \varepsilon \rightarrow 0.$$

Then, assumption (h3) ensures the existence of  $p_0 > 0$  and  $m_0 > 0$  such that, for almost all  $\omega \in \Omega$ , there exists  $\varepsilon_0 = \varepsilon_0(\omega)$  such that  $m_\varepsilon(\omega) > m_0$  for  $\varepsilon \leq \varepsilon_0$ . This implies the desired inequality (19). Indeed, if (19) fails to hold, then there is a sequence  $\{w_{\varepsilon_k}\}_{k=1}^\infty$  such that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ , and

$$\lim_{k \rightarrow \infty} \left[ \left( \|w_{\varepsilon_k}\|_{H^1(D^{\varepsilon_k})}^2 \right)^{-1} \left( \int_{D^{\varepsilon_k}} |\nabla w_{\varepsilon_k}|^2 dx + \int_{\Gamma_1^{\varepsilon_k}} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) w_{\varepsilon_k}^2 ds \right) \right] = 0.$$

Without loss of generality we assume that  $\|w_{\varepsilon_k}\|_{L_2(D^{\varepsilon_k})} = 1$ . Then

$$\lim_{k \rightarrow \infty} \|\nabla w_{\varepsilon_k}\|_{L_2(D^{\varepsilon_k})} = 0 \tag{20}$$

and

$$\lim_{k \rightarrow \infty} \int_{\Gamma_1^{\varepsilon_k}} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) w_{\varepsilon_k}^2 ds = 0. \tag{21}$$

It follows from (20) and our normalization condition that, along a subsequence (still denoted by  $\varepsilon_k$ ),  $w_{\varepsilon_k}$  converges in  $H^1(D)$  norm to a constant  $C_D$  equal to either  $|D|^{-1/2}$  or  $-|D|^{-1/2}$ . This implies by the trace theorem that  $w_{\varepsilon_k}$  converges in  $L_2(\Gamma_1)$  to  $C_D$ . Finally, by Lemma 5.1, 5.2 and by the definitions of  $m_0$ ,  $p_0$  and  $\tilde{\Gamma}_1$ , we have

$$\begin{aligned} \int_{\Gamma_1^{\varepsilon_k}} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) w_{\varepsilon_k}^2 ds &\geq \int_{\Gamma_1} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) w_{\varepsilon_k}^2 \left( \widehat{x}, \varepsilon_k g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) \right) d\widehat{x} \\ &\geq \int_{\tilde{\Gamma}_1} g_0 p \left( \frac{\widehat{x}}{\varepsilon_k}, \omega \right) w_{\varepsilon_k}^2(\widehat{x}, 0) d\widehat{x} + O(\sqrt{\varepsilon_k}) \geq \frac{g_0}{|D|} p_0 m_0 + o(1), \end{aligned}$$

where, almost surely,  $o(1)$  tends to zero as  $\varepsilon_k \rightarrow 0$ . The last inequality contradicts (21). This completes the proof.  $\square$

The following result is also a direct consequence of the Birkhoff theorem.

**Lemma 5.4.** *Let  $h(\widehat{\xi}, \omega)$  be a random statistically homogeneous function such that  $\|\tilde{h}\|_{L_\infty(\Omega)} < \infty$  and assume that*

$$\mathbb{E}(h(\widehat{\xi}, \omega)) \equiv 0.$$

*Then, almost surely,*

$$\int_{\Gamma_1} h \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) u^\varepsilon(\widehat{x}) v^\varepsilon(\widehat{x}) d\widehat{x} \rightarrow 0, \tag{22}$$

*as  $\varepsilon \rightarrow 0$ , for any families  $u^\varepsilon, v^\varepsilon \in H^{\frac{1}{2}}(\Gamma_1)$  such that  $\|u_\varepsilon\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C$  and  $\|v_\varepsilon\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C$ .*

*If  $\tilde{h}_0 : \Omega \mapsto \mathbb{R}^k, k \geq 1$ , is a random vector such that  $\tilde{h}_0 \in (L_2(\Omega))^k$ , and a function  $\mathcal{R}(\widehat{x}, z) : \Gamma_1 \times \mathbb{R}^k \mapsto \mathbb{R}$  has the following properties:*

$$\begin{aligned} \mathcal{R} &\in C(\Gamma_1 \times \mathbb{R}^k), \\ |\mathcal{R}(\widehat{x}, \zeta)| &\leq C(1 + |\zeta|) \end{aligned} \tag{23}$$

*for all  $\widehat{x} \in \Gamma_1$  and  $\zeta \in \mathbb{R}^k$ , and*

$$\mathbb{E}\mathcal{R}(\widehat{x}, \tilde{h}_0(\cdot)) = 0 \quad \text{for each } x \in \tilde{\Gamma}_1, \tag{24}$$

then a.s.

$$\int_{\Gamma_1} \mathcal{R} \left( \widehat{x}, \widetilde{h}_0 \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \right) v^\varepsilon(\widehat{x}) d\widehat{x} \xrightarrow{\varepsilon \rightarrow 0} 0 \tag{25}$$

for any family  $v^\varepsilon \in H^1(D^\varepsilon)$  with  $\|v^\varepsilon\|_{H^1(D^\varepsilon)} \leq C$ .

**Proof.** From Proposition 3.2 and the Birkhoff ergodic theorem it follows that

$$h \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \rightarrow 0 \quad \text{weakly in } L_p(\Gamma_1) \quad \forall p \geq 1,$$

as  $\varepsilon \rightarrow 0$ . This limit relation combined with Sobolev embedding theorems implies (22). Then, again by means of the Birkhoff theorem, one can easily deduce from (23), (24) and the bound  $\|\widetilde{h}^0\|_{L_2(\Omega)} \leq C$  that, almost surely,

$$\mathcal{R} \left( \widehat{x}, \widetilde{h}_0 \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{weakly in } L_2(\Gamma_1).$$

According to the trace and Sobolev embedding theorems, the inequality  $\|v^\varepsilon\|_{H^1(D^\varepsilon)} \leq C$  implies that a.s. the family  $v^\varepsilon$  is compact in  $L_2(\Gamma_1)$ . This yields (25) and completes the proof of the lemma.  $\square$

**Lemma 5.5.** *Almost surely, for any  $v^\varepsilon \in H^1(D^\varepsilon)$  such that  $\|v^\varepsilon\|_{H^1(D^\varepsilon)} \leq C$  and  $u \in C^\infty(\mathbb{R}^d)$ , as  $\varepsilon \rightarrow 0$ , the following limit relations hold:*

$$\left| \int_{\Gamma_1^\varepsilon} g(\widehat{x}) q \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^\varepsilon(x) ds - \int_{\Gamma_1} g(\widehat{x}) Q(\widehat{x}) v^\varepsilon(\widehat{x}, 0) d\widehat{x} \right| \rightarrow 0, \tag{26}$$

$$\left| \int_{\Gamma_1^\varepsilon} g(\widehat{x}) p \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^\varepsilon(x) u(x) ds - \int_{\Gamma_1} g(\widehat{x}) P(\widehat{x}) v^\varepsilon(\widehat{x}, 0) u(\widehat{x}, 0) d\widehat{x} \right| \rightarrow 0 \tag{27}$$

with  $P(\widehat{x})$  and  $Q(\widehat{x})$  defined in (4).

**Proof.** Letting

$$I = \left| \int_{\Gamma_1^\varepsilon} g(\widehat{x}) q \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^\varepsilon(x) ds - \int_{\Gamma_1} g(\widehat{x}) Q(\widehat{x}) v^\varepsilon(\widehat{x}, 0) d\widehat{x} \right|,$$

we have, according to Lemma 5.2,

$$\begin{aligned} I &\leq \left| \int_{\Gamma_1} g(\widehat{x}) \left[ q \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^\varepsilon \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \sqrt{1 + \left| g(\widehat{x}) \partial_\omega \widetilde{F} \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \right|^2} \right. \right. \\ &\quad \left. \left. - Q(\widehat{x}) v^\varepsilon(\widehat{x}, 0) \right] d\widehat{x} \right| \\ &\quad + C\varepsilon \int_{\Gamma_1} \left| g(\widehat{x}) q \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) v^\varepsilon \left( \widehat{x}, \varepsilon g(\widehat{x}) F \left( \frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \sqrt{1 + \left| g(\widehat{x}) \partial_\omega \widetilde{F} \left( T_{\frac{\widehat{x}}{\varepsilon}} \omega \right) \right|^2} \right| d\widehat{x} \\ &= I_1 + I_2. \end{aligned}$$

Inequality (16) implies that, almost surely,

$$I_2 \leq C\varepsilon \|v^\varepsilon\|_{H^1(D^\varepsilon)} \left\| \tilde{q} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \sqrt{1 + \left| \partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} \right\|_{L_2(\Gamma_1)}. \tag{28}$$

We also have, by Proposition 5.1 and Lemma 5.1,

$$\begin{aligned} I_1 &\leq \left| \int_{\Gamma_1} \left( g(\hat{x})q \left( \frac{\hat{x}}{\varepsilon}, \omega \right) \right) \left[ v^\varepsilon \left( \hat{x}, \varepsilon g(\hat{x})F \left( \frac{\hat{x}}{\varepsilon}, \omega \right) \right) - v^\varepsilon(\hat{x}, 0) \right] \right. \\ &\quad \times \left. \sqrt{1 + \left| g(\hat{x})\partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} d\hat{x} \right| \\ &\quad + \left| \int_{\Gamma_1} g(\hat{x})v^\varepsilon(\hat{x}, 0) \left[ q \left( \frac{\hat{x}}{\varepsilon}, \omega \right) \sqrt{1 + \left| g(\hat{x})\partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} - Q(\hat{x}) \right] d\hat{x} \right| \\ &\leq C\sqrt{\varepsilon} \|v^\varepsilon\|_{H^1(D^\varepsilon)} \left\| \tilde{q} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \sqrt{1 + \left| \partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} \right\|_{L_2(\Gamma_1)} \\ &\quad + \left| \int_{\Gamma_1} g(\hat{x})v^\varepsilon(\hat{x}, 0) \left[ q \left( \frac{\hat{x}}{\varepsilon}, \omega \right) \sqrt{1 + \left| g(\hat{x})\partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} - Q(\hat{x}) \right] d\hat{x} \right|. \end{aligned}$$

Combining this inequality with (28) we obtain that almost surely, for  $\varepsilon$  small enough,

$$\begin{aligned} I &\leq C\sqrt{\varepsilon} \|v^\varepsilon\|_{H^1(D^\varepsilon)} \left\| \tilde{q} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \sqrt{1 + \left| \partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} \right\|_{L_2(\Gamma_1)} \\ &\quad + \left| \int_{\Gamma_1} g(\hat{x})v^\varepsilon(\hat{x}, 0) \left[ q \left( \frac{\hat{x}}{\varepsilon}, \omega \right) \sqrt{1 + \left| g(\hat{x})\partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} - Q(\hat{x}) \right] d\hat{x} \right|. \tag{29} \end{aligned}$$

Now, let

$$\mathcal{R}(\hat{x}, (\tilde{q}, \partial_\omega \tilde{F}, \tilde{q}\partial_\omega \tilde{F})) = g(\hat{x})\tilde{q}(\omega)\sqrt{1 + \left| g(\hat{x})\partial_\omega \tilde{F}(\omega) \right|^2} - g(\hat{x})Q(\hat{x}).$$

Taking into account the definition of  $Q(\hat{x})$ , we get

$$\mathbb{E} \{ \mathcal{R}(\hat{x}, (\tilde{q}, \partial_\omega \tilde{F}, \tilde{q}\partial_\omega \tilde{F})) \} = 0 \quad \text{for all } x \in \Gamma_1.$$

Hence, by Lemma 5.4 the second term on the right hand side of (29) almost surely tends to zero, as  $\varepsilon \rightarrow 0$ . By the Birkhoff theorem and condition (h4), almost surely and for sufficiently small  $\varepsilon > 0$ , the following inequality holds:

$$\left\| \tilde{q} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \sqrt{1 + \left| \partial_\omega \tilde{F} \left( T_{\frac{\hat{x}}{\varepsilon}} \omega \right) \right|^2} \right\|_{L_2(\Gamma_1)} \leq C \left\| \tilde{q} \sqrt{1 + \left| \partial_\omega \tilde{F} \right|^2} \right\|_{L_2(\Omega)} \leq C_1.$$

This gives (26).

Convergence (27) can be justified in the same way. This completes the proof of Lemma 5.5.  $\square$

### 6. The basic convergence

For the sake of clarity, the argument  $\omega$  will be omitted in the rest of the paper. Moreover, we use the notation  $p^\varepsilon(\widehat{x}) = p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)$  and  $q^\varepsilon(\widehat{x}) = q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)$ .

This section is devoted to the proof of **Theorem 4.1**. The proof relies on the following result.

**Proposition 6.1.** *Under assumptions (h1)–(h4), there exists  $C > 0$  such that, almost surely for all sufficiently small  $\varepsilon > 0$ , the following estimate holds:*

$$\|u_\varepsilon\|_{H^1(D^\varepsilon)} \leq C.$$

If in addition assumption (h3') is fulfilled, then

$$\mathbb{E}(\|u_\varepsilon\|_{H^1(D^\varepsilon)}) \leq C. \tag{30}$$

**Proof.** Choosing  $v = u_\varepsilon$  in the variational formulation (2) yields

$$\begin{aligned} & \|\nabla u_\varepsilon\|_{L_2(D^\varepsilon)}^2 + \int_{\Gamma_1^\varepsilon} g(\widehat{x}) p^\varepsilon(\widehat{x}) u_\varepsilon^2(x) ds \\ &= \int_{D^\varepsilon} f(x) u_\varepsilon(x) dx + \int_{\Gamma_1^\varepsilon} g(\widehat{x}) q^\varepsilon(\widehat{x}) u_\varepsilon(x) ds. \end{aligned} \tag{31}$$

By **Lemma 5.3**, for almost every  $\omega$  there is  $\varepsilon_0(\omega) > 0$  such that, for all  $\varepsilon < \varepsilon_0$ , the bound

$$\|\nabla u_\varepsilon\|_{L_2(D^\varepsilon)}^2 + \int_{\Gamma_1^\varepsilon} g(\widehat{x}) p^\varepsilon(\widehat{x}) u_\varepsilon^2(x) ds \geq C \|u_\varepsilon\|_{H^1(D^\varepsilon)}^2 \tag{32}$$

holds true. By the Birkhoff theorem, almost surely for sufficiently small  $\varepsilon > 0$ , we have

$$\left| \int_{\Gamma_1^\varepsilon} g(\widehat{x}) q^\varepsilon(\widehat{x}) u_\varepsilon(x) ds \right| \leq C \|u_\varepsilon\|_{H^1(D^\varepsilon)} \left\| \tilde{q} \sqrt{1 + |\partial_\omega \tilde{F}|^2} \right\|_{L_2(\Omega)}. \tag{33}$$

Combining this with the Cauchy–Schwarz inequality we obtain

$$\|u_\varepsilon\|_{H^1(D^\varepsilon)} \leq C \left( \|f\|_{L_2(D^+)} + \left\| \tilde{q} \sqrt{1 + |\partial_\omega \tilde{F}|^2} \right\|_{L_2(\Omega)} \right).$$

This yields the first estimate of the proposition. We have also shown that, almost surely and for  $\varepsilon > 0$  small enough,

$$\int_{\Gamma_1^\varepsilon} g(\widehat{x}) p^\varepsilon(\widehat{x}) u_\varepsilon^2(x) ds \leq C \tag{34}$$

with a deterministic constant  $C$ .

To prove (30) we observe that, under assumption (h3'), the bound (32) holds uniformly in  $\varepsilon > 0$  and  $\omega \in \Omega$ . Then, taking the expectation on both sides of (31) and using (32) and the Cauchy–Schwarz inequality, we obtain the desired estimate. This completes the proof of **Proposition 6.1**.  $\square$

**Proof of Theorem 4.1.** The existence and uniqueness of  $u_\varepsilon$  follow from (33), Lemma 5.3 and the Lax–Milgram lemma (see [20] for details). We then deduce from (2) and (5) that, for any  $v \in H^1(D^\varepsilon)$ ,

$$\begin{aligned} & \int_{D^\varepsilon} \nabla(u_0 - u_\varepsilon) \nabla v dx + \int_{\Gamma_1^\varepsilon} g p^\varepsilon (u_0 - u_\varepsilon) v ds \\ &= \int_{D^\varepsilon} \nabla u_0 \nabla v dx - \int_{D^\varepsilon} f v dx - \int_{\Gamma_1^\varepsilon} g q^\varepsilon v ds + \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 v ds \\ &= \int_D \nabla u_0 \nabla v dx - \int_{D^\varepsilon} f v dx - \int_{\Gamma_1^\varepsilon} g q^\varepsilon v ds + \int_{\Pi_\varepsilon} \nabla u_0 \nabla v dx + \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 v ds \\ &= \left( \int_{\Pi_\varepsilon} \nabla u_0 \nabla v dx - \int_{\Pi_\varepsilon} f v dx \right) + \left( \int_{\Gamma_1} g Q v d\hat{x} - \int_{\Gamma_1^\varepsilon} g q^\varepsilon v ds \right) \\ & \quad + \left( \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 v ds - \int_{\Gamma_1} g P u_0 v d\hat{x} \right). \end{aligned} \tag{35}$$

Let us estimate the terms in the right hand side of the last relation. According to the Cauchy–Schwarz inequality, (14) and the regularity of  $u_0$ , we have

$$\left| \int_{\Pi_\varepsilon} \nabla u_0 \nabla v dx \right| \leq \| \nabla u_0 \|_{L_2(\Pi_\varepsilon)} \| v \|_{H^1(D^\varepsilon)} \leq C_2 \sqrt{\varepsilon} \| u_0 \|_{H^2(D^\varepsilon)} \| v \|_{H^1(D^\varepsilon)} \tag{36}$$

and

$$\left| \int_{\Pi_\varepsilon} f v dx \right| \leq \| f \|_{L_2(\Pi_\varepsilon)} \| v \|_{L_2(\Pi_\varepsilon)} \leq C_2 \sqrt{\varepsilon} \| f \|_{L_2(D^\varepsilon)} \| v \|_{H^1(D^\varepsilon)}.$$

Then, according to Lemma 5.5, as  $\varepsilon \rightarrow 0$ , almost surely we have

$$\left| \int_{\Gamma_1^\varepsilon} g q^\varepsilon v ds - \int_{\Gamma_1} g Q v d\hat{x} \right| \rightarrow 0$$

and

$$\left| \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 v ds - \int_{\Gamma_1} g P u_0 v d\hat{x} \right| \rightarrow 0 \tag{37}$$

for any  $v \in C^\infty(\mathbb{R}^d)$ .

It follows from Proposition 6.1 that, for a subsequence  $\varepsilon_k \rightarrow 0$ , we have  $u_{\varepsilon_k} \rightarrow \widehat{u}$  weakly in  $H^1(D)$ , as  $k \rightarrow \infty$ . This implies that, for any  $v \in C^\infty(\mathbb{R}^d)$ ,

$$\lim_{k \rightarrow \infty} \int_{D^{\varepsilon_k}} (\nabla u_0 - \nabla u_{\varepsilon_k}) \nabla v dx = \int_D \nabla(u_0 - \widehat{u}) \nabla v dx.$$

Passing to the limit, as  $k \rightarrow \infty$ , on both sides of (35) and exploiting (36)–(37), we conclude that, for any  $v \in C^\infty(\mathbb{R}^d)$ ,

$$\int_D (\nabla u_0 - \nabla \widehat{u}) \nabla v dx + \int_{\Gamma_1} P(\widehat{x})(u_0 - \widehat{u}) v d\hat{x} = 0.$$

By density arguments the last relation also holds true for any  $v \in H^1(D)$ . This implies that  $u_0 = \widehat{u}$ . Therefore, a.s. the whole family  $u_\varepsilon$  converges to  $u_0$  weakly in  $H^1(D)$ , and (6) follows

from the compactness of the embedding of  $H^1(D)$  in  $L^2(D)$  (see the Rellich–Kondrachov theorem in [22]).

In order to justify (7) we notice that, under assumption (h3'), the estimate (30) holds. The Lebesgue theorem then applies and (7) is a consequence of (6).

We now turn to proving the  $H^1$  convergence (8). We choose  $v = (u_0 - u_\varepsilon)$  as a test function in (35). We then have

$$\begin{aligned} & \int_{D^\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2 dx + \int_{\Gamma_1^\varepsilon} g p^\varepsilon (u_0 - u_\varepsilon)^2 ds \\ &= \int_{\Pi_\varepsilon} \nabla u_0 \nabla(u_0 - u_\varepsilon) dx - \int_{\Pi_\varepsilon} f(u_0 - u_\varepsilon) dx + \left( \int_{\Gamma_1} g Q(u_0 - u_\varepsilon) d\hat{x} \right. \\ & \quad \left. - \int_{\Gamma_1^\varepsilon} g q^\varepsilon (u_0 - u_\varepsilon) ds \right) + \left( \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g P u_0 (u_0 - u_\varepsilon) d\hat{x} \right). \end{aligned} \tag{38}$$

We are going to estimate the four terms in the right hand side of (38). First, in view of (14) we have

$$\begin{aligned} \left| \int_{\Pi_\varepsilon} \nabla u_0 \nabla(u_0 - u_\varepsilon) dx \right| &\leq \|\nabla u_0\|_{L_2(\Pi_\varepsilon)} \|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)} \\ &\leq C_2 \sqrt{\varepsilon} \|u_0\|_{H^2(D^\varepsilon)} \|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)} \leq C \sqrt{\varepsilon} \end{aligned} \tag{39}$$

almost surely for sufficiently small  $\varepsilon$ . Similarly,

$$\left| \int_{\Pi_\varepsilon} f(u_0 - u_\varepsilon) dx \right| \leq \|f\|_{L_2(\Pi_\varepsilon)} \|u_0 - u_\varepsilon\|_{L_2(\Pi_\varepsilon)} \leq C_2 \sqrt{\varepsilon}.$$

Then, using Lemma 5.5 and Proposition 6.1, we deduce that

$$\left| \int_{\Gamma_1^\varepsilon} g q^\varepsilon (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g Q(u_0 - u_\varepsilon) d\hat{x} \right| \rightarrow 0$$

almost surely, as  $\varepsilon \rightarrow 0$ . The most technical part of this proof now consists in estimating

$$J_\varepsilon = \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g P u_0 (u_0 - u_\varepsilon) d\hat{x}.$$

We are going to show that almost surely

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon = 0. \tag{40}$$

In order to prove this, we first introduce the following notation:

$$\begin{aligned} U_0(\hat{x}) &= u_0(\hat{x}, \varepsilon g(\hat{x}) F^\varepsilon(\hat{x})), & U_\varepsilon(\hat{x}) &= u_\varepsilon(\hat{x}, \varepsilon g(\hat{x}) F^\varepsilon(\hat{x})), \\ S^\varepsilon(\hat{x}) &= \left( 1 + g^2(\hat{x}) |\partial_\omega F^\varepsilon(\hat{x})|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{41}$$

Notice that although it is not indicated explicitly, the function  $U_0$  does depend on  $\varepsilon$ . Then we write  $J_\varepsilon$  as the sum of four terms:

$$J_\varepsilon = \left( \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g p^\varepsilon U_0 (U_0 - U_\varepsilon) S^\varepsilon d\hat{x} \right)$$



$$\begin{aligned}
 &+ \left( \int_{\Gamma_1} gp^\varepsilon U_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} - \int_{\Gamma_1} gp^\varepsilon u_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} \right) \\
 &+ \left( \int_{\Gamma_1} gp^\varepsilon u_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} - \int_{\Gamma_1} gp^\varepsilon u_0(u_0 - u_\varepsilon)S^\varepsilon d\widehat{x} \right) \\
 &+ \left( \int_{\Gamma_1} gp^\varepsilon u_0(u_0 - u_\varepsilon)S^\varepsilon d\widehat{x} - \int_{\Gamma_1} gPu_0(u_0 - u_\varepsilon)d\widehat{x} \right).
 \end{aligned}$$

We are going to prove that each of these terms tends to 0 as  $\varepsilon \rightarrow 0$ .

First we note that, under assumption (h2'), by (17), (18) and the Hölder inequality, almost surely for sufficiently small  $\varepsilon$  we have

$$\begin{aligned}
 \int_{\Gamma_1^\varepsilon} p^\varepsilon u_0^2 ds &\leq 2 \int_{\Gamma_1} p^\varepsilon(\widehat{x}) \left( 1 + g^2(\widehat{x}) |\partial_\omega F^\varepsilon(\widehat{x})|^2 \right)^{1/2} u_0^2(\widehat{x}, \varepsilon g(\widehat{x}) F^\varepsilon(\widehat{x})) d\widehat{x} \\
 &\leq C \|\tilde{p}\|_{L^\infty(\Omega)} \left( 1 + \|\partial_\omega F\|_{L^{\frac{d}{2} \vee 2}(\Omega)} \right) \|u_0\|_{H^2(D^+)}^2 \leq C_1 \|u_0\|_{H^2(D^+)}^2 \tag{42}
 \end{aligned}$$

with deterministic constants  $C$  and  $C_1$ ; the notation  $\frac{d}{2} \vee 2$  is used for  $\max(\frac{d}{2}, 2)$ . Combining this bound with (34) we conclude that almost surely for sufficiently small  $\varepsilon$  the following estimate holds:

$$\int_{\Gamma_1^\varepsilon} gp^\varepsilon (u_0 - u_\varepsilon)^2 ds \leq C \tag{43}$$

with a deterministic constant  $C$ . Then, it follows from (42) and (43) that

$$\left| \int_{\Gamma_1^\varepsilon} gp^\varepsilon u_0(u_0 - u_\varepsilon) ds - \int_{\Gamma_1} gp^\varepsilon U_0(U_0 - U_\varepsilon)S^\varepsilon(\widehat{x})d\widehat{x} \right| \leq C\varepsilon.$$

Then, considering (42), (43) and (34), applying Lemma 5.1 and the Hölder inequality we obtain

$$\begin{aligned}
 &\left| \int_{\Gamma_1} gp^\varepsilon U_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} - \int_{\Gamma_1} gp^\varepsilon u_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} \right| \\
 &\leq \int_{\Gamma_1} |U_0 - u_0| \left( \sqrt{gp^\varepsilon S^\varepsilon} |U_0 - U_\varepsilon| \right) \sqrt{gp^\varepsilon S^\varepsilon} d\widehat{x} \\
 &\leq C \|U_0 - u_0\|_{L^{\frac{2d}{d-2}}(\Gamma_1)} \leq C\varepsilon^{\frac{d+2}{2d}} \leq C\sqrt{\varepsilon}.
 \end{aligned}$$

Notice that Lemma 5.1 applies here since  $U_0$  admits an extension  $\tilde{U}_0 \in H^2(D^\varepsilon)$  such that  $\|\tilde{U}_0\|_{H^2(D^\varepsilon)} \leq C\|U_0\|_{L^2(\Gamma_1)}$  with deterministic  $C > 0$  independent of  $\varepsilon$ . This technique fails to work in dimension 2. In order to justify the last inequality in 2D case we use the Hölder continuity of  $u_0$ .

Next we want to show that almost surely

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\Gamma_1} gp^\varepsilon u_0(U_0 - U_\varepsilon)S^\varepsilon d\widehat{x} - \int_{\Gamma_1} gp^\varepsilon u_0(u_0 - u_\varepsilon)S^\varepsilon d\widehat{x} \right| = 0. \tag{44}$$

To this end we observe that, by the Sobolev embedding and trace theorems,  $u_0 \in L_{2(d-1)/(d-4)}(\Gamma_1)$ . Combining this with (15) and assumption (h2') and using the Hölder

inequality, we deduce that almost surely

$$\|gp^\varepsilon \mathcal{S}^\varepsilon u_0(U_0 - u_0)\|_{L_1(\Gamma_1)} \leq C\sqrt{\varepsilon}.$$

Notice now that almost surely the function  $(U_\varepsilon - u_\varepsilon)$  tends to zero in  $L_2(\Gamma_1)$ , as  $\varepsilon \rightarrow 0$ , and by assumption (h2') the family  $gp^\varepsilon \mathcal{S}^\varepsilon$  is bounded in  $L_2(\Gamma_1)$ . Then  $gp^\varepsilon \mathcal{S}^\varepsilon(U_\varepsilon - u_\varepsilon)$  converges to zero in measure on  $\Gamma_1$ , and so does  $gp^\varepsilon \mathcal{S}^\varepsilon u_0(U_\varepsilon - u_\varepsilon)$ . It remains to check that this family is uniformly integrable. The relation (44) then follows from the convergence in measure by the Lebesgue theorem.

In order to prove the uniform integrability of  $gp^\varepsilon \mathcal{S}^\varepsilon u_0 U_\varepsilon$  we represent it as

$$gp^\varepsilon \mathcal{S}^\varepsilon u_0 U_\varepsilon = \left(\sqrt{gp^\varepsilon \mathcal{S}^\varepsilon} U_\varepsilon\right) u_0 \sqrt{gp^\varepsilon \mathcal{S}^\varepsilon}.$$

Since almost surely  $(\sqrt{gp^\varepsilon \mathcal{S}^\varepsilon} U_\varepsilon)$  is bounded in  $L_2(\Gamma_1)$  (see (34)),  $\sqrt{gp^\varepsilon \mathcal{S}^\varepsilon}$  is bounded in  $L_d(\Gamma_1)$  and  $u_0 \in L_{2(d-1)/(d-4)}(\Gamma_1)$ , then the product is bounded in  $L_{(2d^2-2d)/(2d^2-3d-2)}(\Gamma_1)$ , which ensures the uniform integrability of the function  $gp^\varepsilon \mathcal{S}^\varepsilon u_0 U_\varepsilon$ .

Similarly, since almost surely  $u_\varepsilon$  is bounded in  $H^{1/2}(\Gamma_1)$  then, by the Sobolev embedding theorem,  $u_\varepsilon$  is bounded in  $L_{2(d-1)/(d-2)}(\Gamma_1)$ . By the Hölder inequality  $gp^\varepsilon \mathcal{S}^\varepsilon u_0 u_\varepsilon$  is bounded in  $L_{(2d^2-2d)/(2d^2-2d-4)}(\Gamma_1)$  and thus uniformly integrable. This implies (44). Here we assumed that  $d > 4$ ; for lower dimensions the validity of (44) can be justified in a similar way with obvious simplifications.

Now, in order to prove (40) it remains to show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_1} (gp^\varepsilon \mathcal{S}^\varepsilon - P) u_0(u_0 - u_\varepsilon) d\hat{x} = 0. \tag{45}$$

This convergence follows from the Birkhoff ergodic theorem. Indeed, by the definition of  $P$  we have

$$\mathbb{E} \left\{ g(\hat{x}) \tilde{p} \sqrt{1 + g^2(\hat{x}) |\partial_\omega \tilde{F}|^2} - P(\hat{x}) \right\} = 0,$$

for any  $\hat{x} \in \Gamma_1$ . Then, using the Birkhoff theorem, one can easily prove that under assumption (h2'), the function  $(gp^\varepsilon \mathcal{S}^\varepsilon - P)$  converges almost surely to zero weakly in  $L_{2\sqrt{\frac{d}{2}}}(\Gamma_1)$ . Since  $u_0 \in H^2(D^+)$  and  $(u_0 - u_\varepsilon)$  is bounded in  $H^{1/2}(\Gamma_1)$ , the family  $u_0(u_0 - u_\varepsilon)$  is compact in  $L_{2 \wedge \frac{d}{d-2}}(\Gamma_1)$ ; here the notation  $2 \wedge \frac{d}{d-2}$  is used for  $\min\left(2, \frac{d}{d-2}\right)$ . This implies (45).

Combining now (39)–(40) we arrive at the conclusion that all the terms on the right hand side of (38) almost surely tend to zero, as  $\varepsilon \rightarrow 0$ . This yields

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{D^\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2 dx + \int_{\Gamma_1^\varepsilon} p^\varepsilon (u_0 - u_\varepsilon)^2 ds \right| = 0,$$

and by Lemma 5.3 we derive that a.s.

$$\lim_{\varepsilon \rightarrow 0} \|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)} = 0.$$

Then, under assumption (h3'), estimate (30) holds and we obtain (9) by the Lebesgue theorem. The proof of Theorem 4.1 is completed.  $\square$

### 7. The rate of convergence

This section is devoted to the proof of [Theorem 4.2](#) which relies on the following result.

**Lemma 7.1.** *Let  $h(\widehat{\xi}, \omega), \widehat{\xi} \in \mathbb{R}^{n-1}$ , be a statistically homogeneous random field with values in  $\mathbb{R}^k$ , and suppose that at least one of the conditions in [\(11\)](#) is fulfilled. Then, given a smooth function  $\mathcal{R}(x, z), x \in \Gamma_1, z \in \mathbb{R}^k$ , such that*

$$\|\mathcal{R}(x, \widetilde{h}(\cdot))\|_{L_2(\Omega)} \leq C, \quad \mathbb{E}\mathcal{R}(x, h(\widehat{\xi}, \cdot)) = 0, \quad \text{for all } x \in \Gamma_1,$$

we have

$$\mathbb{E} \left( \sup_{\|\theta\|_{H^{1/2}(\Gamma_1)}=1} \left| \int_{\Gamma_1} \mathcal{R} \left( x, h \left( \frac{x}{\varepsilon} \right) \right) \theta(x) dx \right|^2 \right) \leq C \sqrt{\varepsilon}, \tag{46}$$

with a constant  $C$  that does not depend on  $\varepsilon$ .

**Proof.** First, we are going to prove that

$$\mathbb{E} \left( \sup_{\|\theta\|_{H_0^1(\Gamma_1)}=1} \left| \int_{\Gamma_1} \mathcal{R} \left( \widehat{x}, h \left( \frac{\widehat{x}}{\varepsilon} \right) \right) \theta(\widehat{x}) d\widehat{x} \right|^2 \right) \leq C\varepsilon \tag{47}$$

with a constant  $C$  that does not depend on  $\varepsilon$ .

Define

$$H^\varepsilon(x) = \int_0^{x_1} \mathcal{R} \left( r, x', h \left( \frac{r}{\varepsilon}, \frac{x'}{\varepsilon} \right) \right) dr, \quad x' = x_2, \dots, x_{d-1},$$

and

$$R(x^1, x^2, \widehat{\eta}) = \mathbb{E} \left( \mathcal{R}(x^1, h(\widehat{\xi})) \mathcal{R}(x^2, h(\widehat{\xi} + \widehat{\eta})) \right)$$

so that  $R$  is the correlation function of  $\mathcal{R}(x, h(\widehat{\xi}))$ . According to Lemma 3.102, p. 456 in [\[23\]](#), the function  $R$  admits the estimates

$$|R(x^1, x^2, \widehat{\eta})| \leq C(\alpha(|\widehat{\eta}|))^{1/2}, \quad |R(x^1, x^2, \widehat{\eta})| \leq C\rho(|\widehat{\eta}|),$$

uniformly with respect to  $x^1$  and  $x^2$ , where  $\alpha$  and  $\rho$  are the mixing coefficients introduced in [Definition 8](#). If we define

$$\bar{R}(s) = \sup\{|R(x^1, x^2, \widehat{\eta})| : |\widehat{\eta}| = s, x^1 \in \Gamma_1, x^2 \in \Gamma_1\},$$

then, due to [\(11\)](#), the integral  $\int_0^\infty \bar{R}(s) ds$  converges. Clearly,

$$\sup_{\|\theta\|_{H_0^1(\Gamma_1)}=1} \left| \int_{\Gamma_1} \mathcal{R} \left( \widehat{x}, h \left( \frac{\widehat{x}}{\varepsilon} \right) \right) \theta(\widehat{x}) d\widehat{x} \right|^2 \leq \|H^\varepsilon\|_{L^2(\Gamma_1)}^2,$$

and thus to prove [\(47\)](#) it suffices to show that

$$\mathbb{E}(\|H^\varepsilon\|_{L^2(\Gamma_1)}^2) \leq C\varepsilon.$$

We have

$$\begin{aligned} & \mathbb{E} \left( \int_{\Gamma_1} \left( \int_0^{x_1} \mathcal{R} \left( r, x', h \left( \frac{r}{\varepsilon}, \frac{x'}{\varepsilon} \right) dr \right)^2 d\widehat{x} \right) \right) \\ &= \varepsilon^2 \mathbb{E} \left( \int_{\Gamma_1} \left( \int_0^{x_1/\varepsilon} \mathcal{R} \left( \varepsilon r, x', h \left( r, \frac{x'}{\varepsilon} \right) \right) dr \right)^2 d\widehat{x} \right) \\ &= \varepsilon^2 \int_{\Gamma_1} \left( \int_0^{x_1/\varepsilon} \int_0^{x_1/\varepsilon} \mathbb{E} \left( \mathcal{R} \left( \varepsilon r, x', h \left( r, \frac{x'}{\varepsilon} \right) \right) \mathcal{R} \left( \varepsilon s, x', h \left( s, \frac{x'}{\varepsilon} \right) \right) \right) dr ds \right) d\widehat{x} \\ &\leq \varepsilon^2 \int_{\Gamma_1} \int_0^{x_1/\varepsilon} \int_0^{x_1/\varepsilon} \bar{R}(|s - r|) dr ds d\widehat{x} \leq C\varepsilon, \end{aligned}$$

and (47) follows. Finally, applying the interpolation inequality and using (47) we deduce (46), which ends the proof of the lemma.  $\square$

**Proof of Theorem 4.2.** *Case (i).* Since  $f$  vanishes in the vicinity of  $\Gamma_1$ , the solution  $u_0$  is smooth in a sufficiently small neighbourhood of  $\Gamma_1$  and thus has a smooth extension in  $D^+$ ; as above, we keep the same notation  $u_0$  for the extended function.

Consider the identity (38). Using again the notation (41) introduced in the proof of Theorem 4.1, it reads

$$\begin{aligned} & \int_{D^\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2 dx + \int_{\Gamma_1^\varepsilon} g p^\varepsilon (u_0 - u_\varepsilon)^2 ds \\ &= \int_{\Pi_\varepsilon} \nabla u_0 \nabla(u_0 - u_\varepsilon) dx - \int_{\Pi_\varepsilon} f(u_0 - u_\varepsilon) dx \\ &+ \left( \int_{\Gamma_1} g q^\varepsilon \mathcal{S}^\varepsilon (u_0 - u_\varepsilon) d\widehat{x} - \int_{\Gamma_1^\varepsilon} g q^\varepsilon (u_0 - u_\varepsilon) ds \right) \\ &+ \left( \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g p^\varepsilon \mathcal{S}^\varepsilon u_0 (u_0 - u_\varepsilon) d\widehat{x} \right) \\ &+ \int_{\Gamma_1} g(Q - q^\varepsilon \mathcal{S}^\varepsilon)(u_0 - u_\varepsilon) d\widehat{x} + \int_{\Gamma_1} g(p^\varepsilon \mathcal{S}^\varepsilon - P)u_0 (u_0 - u_\varepsilon) d\widehat{x}. \end{aligned} \tag{48}$$

By our standing assumptions, the second term on the right hand side is equal to zero. Then, applying (36) with  $v = u_0 - u_\varepsilon$ , we have

$$\left| \int_{\Pi_\varepsilon} \nabla u_0 \nabla(u_0 - u_\varepsilon) \right| \leq C\sqrt{\varepsilon} \|\nabla(u_0 - u_\varepsilon)\|_{L_2(D^\varepsilon)}.$$

It follows from (13) and Lemma 5.2 that

$$\left| \int_{\Gamma_1^\varepsilon} g q^\varepsilon (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g q^\varepsilon \mathcal{S}^\varepsilon (u_0 - u_\varepsilon) d\widehat{x} \right| \leq C\sqrt{\varepsilon} \|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)}.$$

Taking the smoothness of  $u_0$  in the vicinity of  $\Gamma_1$  into account, one also has

$$\left| \int_{\Gamma_1^\varepsilon} g p^\varepsilon u_0 (u_0 - u_\varepsilon) ds - \int_{\Gamma_1} g p^\varepsilon \mathcal{S}^\varepsilon u_0 (u_0 - u_\varepsilon) d\widehat{x} \right| \leq C\sqrt{\varepsilon} \|u_0 - u_\varepsilon\|_{H^1(D^\varepsilon)}.$$

Now, for  $z_1 \in \mathbb{R}^{d-1}$  and  $z_2 \in \mathbb{R}$ , let us define

$$\mathcal{R}(\widehat{x}, z_1, z_2) = g(\widehat{x})z_2\sqrt{1 + (g(\widehat{x}))^2z_1^2} - g(\widehat{x})Q(\widehat{x}).$$

One easily checks that the function  $\mathcal{R}(\widehat{x}, \partial_\omega \tilde{F}, \tilde{p})$  satisfies the assumptions of Lemma 7.1. Therefore, applying Lemma 7.1 and considering the boundedness of  $(u_0 - u_\varepsilon)$  in  $H^{\frac{1}{2}}(\Gamma_1)$ , we have

$$\mathbb{E} \left\{ \left| \int_{\Gamma_1} g(Q - q^\varepsilon \mathcal{S}^\varepsilon)(u_0 - u_\varepsilon) d\widehat{x} \right| \right\} \leq C\sqrt{\varepsilon}.$$

Similarly, in view of the smoothness of  $u_0$  in the neighborhood of  $\Gamma_1$ , we obtain

$$\mathbb{E} \left\{ \left| \int_{\Gamma_1} g(p^\varepsilon \mathcal{S}^\varepsilon - P)u_0(u_0 - u_\varepsilon) d\widehat{x} \right| \right\} \leq C\sqrt{\varepsilon}. \tag{49}$$

Combining (48)–(49) yields

$$\begin{aligned} & \mathbb{E} \int_{D^\varepsilon} |\nabla u_\varepsilon(x) - \nabla u_0(x)|^2 dx + \mathbb{E} \int_{\Gamma_1^\varepsilon} p\left(\frac{\widehat{x}}{\varepsilon}\right) (u_\varepsilon(x) - u_0(x))^2 ds \\ & \leq C\sqrt{\varepsilon} \mathbb{E} \|u_\varepsilon - u_0\|_{H^1(D^\varepsilon)} + C\sqrt{\varepsilon}. \end{aligned}$$

Thanks to Lemma 5.3 this implies the bound (12).

Case (ii). The proof is rather similar to that of the previous case. All the estimates obtained in the case (i) remain valid. The function  $\mathcal{R}$  in this case is to be defined as follows:

$$\mathcal{R}(\widehat{x}, z_1, z_2) = u_0(\widehat{x}, 0)g(\widehat{x})z_2\sqrt{1 + (g(\widehat{x}))^2z_1^2} - g(\widehat{x})Q(\widehat{x}).$$

We leave the details to the reader. This completes the proof of Theorem 4.2.  $\square$

### Acknowledgements

The research was partially conducted in May 2008 and completed in May 2009, while G.A. Chechkin and A.L. Piatnitski were visiting the Université Blaise Pascal (Clermont-Ferrand, France), whose support is acknowledged.

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