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ASYMPTOTIC BEHAVIOR OF A SOLUTION TO A BOUNDARY VALUE PROBLEM IN A PERFORATED DOMAIN WITH OSCILLATING BOUNDARY†)‡)

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Introduction

Many contemporary problems of materials technology require studying the macroscopic behavior of micro-inhomogeneous perforated media and bodies with rough surfaces.

The aim of the present article is to consider one model problem for the Poisson equation in a perforated domain with a very rapidly oscillating outer boundary in the presence of small dissipation on the boundaries of holes.

In the recent years many mathematical articles were devoted to asymptotic analysis of problems in perforated domains. Various results on averaging were obtained for periodic, almost periodic, and random structures. We mention the articles [1–4] wherein the reader can find a detailed bibliography. Of particular interest is the most practically realistic case in which we have small dissipation on the boundaries of the holes. The corresponding mathematical statement involves the third boundary condition (the Fourier condition) with a small parameter; the periodic case was elaborated in [5–8].

Another direction of research, dealing with equations in domains with very rapidly oscillating boundary, is well developed too (see, for instance, [2, 3, 9–17]).

The combination of these two phenomena, perforation and oscillation of the outer boundary, is natural but leads to additional mathematical difficulties. In the present article we study a particular case of such medium in which perforation as well as oscillation of the boundary are locally periodic and their structures are assumed to be coordinated. Studying locally periodic perforation, we face another difficulty: the geometry of cavities is not fixed. Using the method of compensated compactness [18] or the method of two-scale convergence [19], we can construct a limit problem but the methods provide no estimates for the error. In the present article we use the technique of asymptotic expansion [20, 21] which requires the data to be regular but enables us to estimate the convergence rate.

In § 1 we introduce the necessary notations, construct a family of domains which depends on a small positive parameter ε, and pose the problem to be studied. § 2 is devoted to constructing the first terms of the formal interior asymptotic expansion for a solution. The technical results of § 3 make it possible to justify this asymptotic expansion and estimate the error. Namely, by Theorem 1 of § 4, two terms of the interior asymptotic expansion guarantee an estimate of order $\sqrt{\varepsilon}$ in the H^1-norm. In § 5, we construct a boundary layer corrector and so improve the estimate for the residual, making the former of order ε. This is the content of Theorem 2.

§ 1. Statement of the Problem

We start with the definition of a perforated domain with oscillating boundary. Let $\Omega \subset \mathbf{R}^d \cap \{x \mid x_d > 0\}$, $d \geq 2$, be a smooth bounded domain whose boundary has a nontrivial flat part $\Gamma_1 = \partial\Omega \cap \{x \mid x_d = 0\}$. We suppose that the $(d-1)$-dimensional interior $\overset{\circ}{\Gamma}_1$ of Γ_1 is nonempty.

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We wish to determine a locally periodic interior perforation in such a way that it vanishes in a neighborhood of $\Gamma_2 = \partial\Omega\backslash\Gamma_1$ and is purely periodic in a neighborhood of Γ_0, where Γ_0 is a compact subset of $\overset{\circ}{\Gamma}_1$. To this end, we introduce a $C_0^\infty(\mathbf{R}^d)$-truncator $\Phi(x)$ such that $0 \le \Phi(x) \le 1$, $\Phi(x) = 0$ in a neighborhood of Γ_2, and $\Phi(x) = 1$ in a neighborhood of Γ_0 and fix an open set Q with smooth boundary such that $\overline{Q} \subset \square = \{\xi \mid -1/2 < \xi_j < 1/2, j = 1, \dots, d\}$. Afterwards, denoting the 1-periodic extension of the characteristic function of Q by $\chi_Q(\xi)$ and denoting the function $\chi_Q(\xi/\Phi(x))$ by $\chi(x, \xi)$, we define the domain $\Omega_1^\varepsilon = \Omega\backslash\{x \mid x_d > \varepsilon/2,\ \chi(x, x/\varepsilon) = 1\}$.

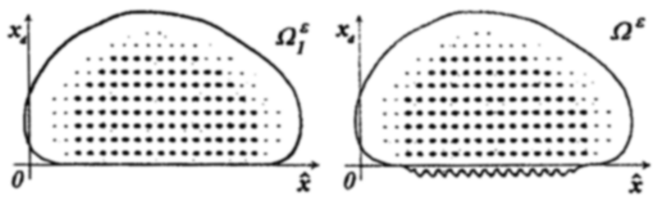

Fig. 1

We now furnish the so-constructed domain Ω_1^ε with an oscillating boundary (see Figs. 1, 2). To this end, given a smooth nonpositive function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, \dots, x_{d-1})$, $\hat{\xi} = (\xi_1, \dots, \xi_{d-1})$, 1-periodic in $\hat{\xi}$ and such that $\operatorname{supp}_x F(\hat{x}, \hat{\xi}) \equiv \overline{\{x | F(\hat{x}, \hat{\xi}) < 0\}} \subset \Gamma_0$ for all $\hat{\xi}$, we put

$$\Pi_\varepsilon = \{x \in \mathbf{R}^d : \hat{x} \in \Gamma_1, \varepsilon F(\hat{x}, \hat{x}/\varepsilon) < x_d \le 0\}$$

and finally define our perforated domain as follows: $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Pi_\varepsilon$. We also use the notations

$$Q_\varepsilon = \{x \mid \chi(x/\varepsilon) = 1\},\ \Omega' = \{x \in \Omega \mid \Phi(x) > 0\},\ \Omega'' = \Omega\backslash\overline{\{x \in \Omega \mid \Phi(x) < 1\}},$$

$$\Omega'_\varepsilon = \Omega' \cap \Omega_1^\varepsilon, \quad \Omega''_\varepsilon = \Omega'' \cap \Omega_1^\varepsilon, \quad \omega = \mathbf{R}^d\backslash\overline{\{\xi \mid \chi_Q(x, \xi) = 1\}}.$$

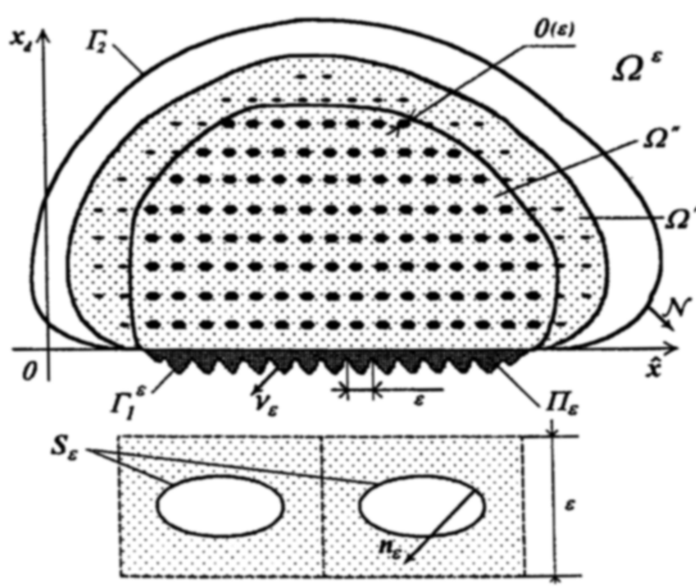

Fig. 2

According to the above construction, the boundary $\partial\Omega^\varepsilon$ consists of Γ_2 and $\Gamma_1^\varepsilon = \{x \in \Gamma_1, x_d = \varepsilon F(\hat{x}, \hat{x}/\varepsilon)\}$, forming the outer boundary, and of the boundary $S_\varepsilon \subset \Omega$ of the cavities, $S_\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$.

We study the asymptotic behavior of a solution $u_\varepsilon(x)$ to the following boundary value problem in the domain Ω^ε as $\varepsilon \to 0$:

$$\begin{gathered} -\Delta u_\varepsilon = f(x) \text{ in } \Omega^\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + p(\hat{x}, \hat{x}/\varepsilon)u_\varepsilon = g(\hat{x}, \hat{x}/\varepsilon) \text{ on } \Gamma_1^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \mathcal{N}} = 0 \quad \text{on} \quad \Gamma_2, \quad \frac{\partial u_\varepsilon}{\partial n_\varepsilon} + \varepsilon^\alpha q(x, x/\varepsilon)u_\varepsilon = 0 \text{ on } S_\varepsilon, \end{gathered} \tag{1}$$

where ν_ε is the outward normal of Γ_1^ε, n_ε is the inward normal to the boundary of the "holes." $\mathcal{N}$ is the outward normal of Γ_2, $p(\hat{x}, \hat{\xi})$ and $g(\hat{x}, \hat{\xi})$ are positive functions 1-periodic in $\hat{\xi}$, and $q(x, \xi)$ is a function 1-periodic in ξ. Moreover, we suppose that the functions p, g, and q are sufficiently smooth. Also, we suppose that $\operatorname{supp}_x(p(\hat{x}, \hat{\xi}))$ and $\operatorname{supp}_x(g(\hat{x}, \hat{\xi}))$ lie in Γ_0 for all $\hat{\xi}$.

DEFINITION. A function $u_\varepsilon \in H^1(\Omega^\varepsilon)$ is a *solution* to (1) if the integral identity

$$\int_{\Omega^\varepsilon} \nabla u_\varepsilon(x) \nabla v(x)\, dx + \varepsilon^\alpha \int_{S_\varepsilon} q(x, x/\varepsilon) u_\varepsilon(x) v(x)\, ds + \int_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) u_\varepsilon(x) v(x)\, ds$$
$$= \int_{\Omega^\varepsilon} f(x) v(x)\, dx + \int_{\Gamma_1^\varepsilon} g(\hat{x}, \hat{x}/\varepsilon) v(x)\, ds \tag{2}$$

holds for every function $v \in H^1(\Omega^\varepsilon)$.

REMARK. In the present article, we study the critical case of $\alpha = 1$. The other cases are the topic of independent research.

§ 2. The Formal Averaging Procedure

In this section we construct the first "locally periodic" terms of the formal asymptotic expansion and then write down the limit problem. Towards this end, we represent a solution $u_\varepsilon(x)$ to (1) as the asymptotic series

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \varepsilon^3 u_3(x, x/\varepsilon) + \dots. \tag{3}$$

Inserting (3) in (1) and using the obvious relation

$$\frac{\partial \zeta}{\partial x}(x, x/\varepsilon) = \left(\frac{\partial \zeta}{\partial x}(x, \xi) + \frac{1}{\varepsilon}\frac{\partial \zeta}{\partial \xi}(x, \xi) \right)\Big|_{\xi = x/\varepsilon}$$

after simple transformations we obtain the following formal equality:

$$\begin{gathered} -f(x) = \Delta_x u_\varepsilon(x) \cong \Delta_x u_0(x) + \varepsilon(\Delta_x u_1(x, \xi))|_{\xi=x/\varepsilon} + 2(\nabla_x, \nabla_\xi u_1(x, \xi))|_{\xi=x/\varepsilon} \\ + \frac{1}{\varepsilon}(\Delta_\xi u_1(x, \xi))|_{\xi=x/\varepsilon} + \varepsilon^2(\Delta_x u_2(x, \xi))|_{\xi=x/\varepsilon} + 2\varepsilon(\nabla_x, \nabla_\xi u_2(x, \xi))|_{\xi=x/\varepsilon} \\ + (\Delta_\xi u_2(x, \xi))|_{\xi=x/\varepsilon} + \varepsilon^3(\Delta_x u_3(x, \xi))|_{\xi=x/\varepsilon} \\ + 2\varepsilon^2(\nabla_x, \nabla_\xi u_3(x, \xi))|_{\xi=x/\varepsilon} + \varepsilon(\Delta_\xi u_3(x, \xi))|_{\xi=x/\varepsilon} + \dots. \end{gathered} \tag{4}$$

Similarly, inserting (3) in the boundary conditions of (1), we obtain

$$\begin{gathered} 0 = \frac{\partial u_\varepsilon}{\partial n_\varepsilon} + \varepsilon q(x, x/\varepsilon) u_\varepsilon \cong (\nabla_x u_0, n_\varepsilon) + \varepsilon q(x, x/\varepsilon) u_0 + \varepsilon(\nabla_x u_1, n_\varepsilon) \\ + (\nabla_\xi u_1|_{\xi=x/\varepsilon}, n_\varepsilon) + \varepsilon^2 q(x, x/\varepsilon) u_1 + \varepsilon^2(\nabla_x u_2, n_\varepsilon) + \varepsilon(\nabla_\xi u_2|_{\xi=x/\varepsilon}, n_\varepsilon) \\ + \varepsilon^3 q(x, x/\varepsilon) u_2 + \varepsilon^3(\nabla_x u_3, n_\varepsilon) + \varepsilon^2(\nabla_\xi u_3|_{\xi=x/\varepsilon}, n_\varepsilon) + \varepsilon^4 q(x, x/\varepsilon) u_3 + \dots \end{gathered} \tag{5}$$

on S_ε.

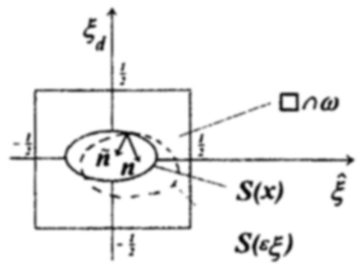

Fig. 3

Observe that the normal vector n_ε depends on x and x/ε in $\Omega'\backslash\Omega''$ and only on x/ε in Ω''. Considering x and $\xi = x/\varepsilon$ to be independent variables as usual, we write n_ε in $\Omega'\backslash\Omega''$ as

$$n_\varepsilon(x, x/\varepsilon) = \tilde{n}(x,\xi)|_{\xi=x/\varepsilon} + \varepsilon n'_\varepsilon(x,\xi)|_{\xi=x/\varepsilon},$$

where $\tilde{n}$ is the normal of $S(x) = \partial\{\xi \mid \xi/\Phi(x) \in Q\}$ and $n'_\varepsilon = n' + O(\varepsilon)$. Simple calculations show that n' is 1-periodic in ξ and

$$n' = \left(\tilde{n}, \frac{\xi}{|\xi|}\right)(\nabla_x\Phi(x) - (\nabla_x\Phi(x), \tilde{n})\tilde{n}), \quad \xi \in \square.$$

Equating the terms of order ε^{-1} in (4) and of order ε^0 in (5), we obtain the following auxiliary problem (see Fig. 3):

$$\Delta_\xi u_1(x,\xi) = 0 \quad \text{in} \quad \square\cap\omega,$$
$$\frac{\partial u_1(x,\xi)}{\partial\tilde{n}} = -(\nabla_x(u_0(x)), \tilde{n}) \quad \text{on} \quad S(x). \tag{6}$$

This problem must be solved in the space of functions 1-periodic in ξ with x a parameter. It represents a standard problem on a "cell" which results from averaging in a perforated domain in the case of the Neumann conditions on the boundary of the holes. It is clear that the solvability condition

$$\int_S (\nabla_x u_0(x), \tilde{n}(\xi))\, d\sigma = 0$$

of the problem (6) is satisfied and the corresponding solution gives the first "interior" corrector in (3).

Our next step consists in equating all terms of order ε^0 in (4) and of order ε^1 in (5). This leads to the problem

$$\Delta_\xi u_2(x,\xi) = -f(x) - \Delta_x u_0(x) - 2(\nabla_\xi, \nabla_x u_1(x,\xi)) \text{ in } \square\cap\omega, \tag{7}$$

$$\frac{\partial u_2(x,\xi)}{\partial\tilde{n}} = -(\nabla_x u_1(x,\xi), \tilde{n}) - (\nabla_\xi u_1(x,\xi), n')$$
$$-(\nabla_x u_0(x), n') - q(x,\xi)u_0(x) \text{ on } S(x).$$

A 1-periodic (in ξ) solution to the last problem is the second term of the interior asymptotic expansion for $u_\varepsilon(x)$.

It is natural to represent a solution $u_1(x,\xi)$ to (6) as

$$u_1(x,\xi) = (\nabla_x u_0(x), M(x,\xi)), \tag{8}$$

where the 1-periodic vector-function $M(x,\xi) = (M_1(x,\xi), \ldots, M_d(x,\xi))$ satisfies the problem

$$\Delta_\xi M_i(x,\xi) = 0 \text{ in } \square\cap\omega, \quad \frac{\partial M_i(x,\xi)}{\partial\tilde{n}} = -\tilde{n}_i \text{ on } S(x). \tag{9}$$

Now, we can rewrite (7) as follows:

$$\Delta_\xi u_2(x,\xi) = -f(x) - \Delta_x u_0(x) - 2\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\frac{\partial M_i(x,\xi)}{\partial\xi_j}$$
$$-2\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial^2 M_i(x;\xi)}{\partial\xi_j\partial x_j} \text{ in } \square\cap\omega, \tag{10}$$
$$\frac{\partial u_2(x,\xi)}{\partial\tilde{n}} = -\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}M_i(x,\xi)\tilde{n}_j - \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial x_j}\tilde{n}_j$$
$$-q(x,\xi)u_0(x) - \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial\xi_j}n'_j - \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}n'_i \text{ on } S(x).$$

Writing down the solvability condition for (10) in the space of 1-periodic functions, we obtain the equation

$$\int\limits_{\Box\cap\omega}\left(f(x)+\Delta_x u_0(x)+2\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right.$$
$$\left.+2\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\partial x_j}\right)d\xi=\int\limits_{S}\left(\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}M_i(x,\xi)\tilde{n}_j\right.$$
$$+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial x_j}\tilde{n}_j+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial \xi_j}n_j'$$
$$\left.+\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}n_i'+q(x,\xi)u_0(x)\right)d\sigma. \tag{11}$$

By the Stokes formula, (11) implies the equality

$$|\Box\cap\omega|\Delta_x u_0(x)+\sum_{i,j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}$$
$$+\sum_{i,j=1}^{d}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}+|\Box\cap\omega|f(x)=Q(x)u_0(x)+\sum_{i=1}^{d}U_i(x)\frac{\partial u_0(x)}{\partial x_i}, \tag{12}$$

which is the limit equation in Ω. Here $\langle\cdot\rangle$ stands for integration over the set $\Box\cap\omega$, $Q(x)=\int_S q(x,\xi)\,d\sigma$, and

$$U_i(x)=\int\limits_S\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}n_j'+n_i'\right)d\sigma.$$

Inspect the function $U_i(x)$ in more detail. Fortunately, there is no need to calculate $U_i(x)$. Instead, recalling that the operators of the original problem are selfadjoint and using convergence of the corresponding bilinear forms, we infer that the G-limit operator is selfadjoint by necessity. Thus, the limit equation (12) takes the form

$$\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)+|\Box\cap\omega|f(x)=Q(x)u_0(x); \tag{13}$$

consequently,

$$U_i(x)=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle-\sum_{j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\partial \xi_j}\right\rangle. \tag{14}$$

It is clear that $\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle$ is a smooth matrix equal to a constant in Ω'' and to the identity in $\Omega\backslash\Omega'$. Arguing as in [3], we can moreover verify that this matrix is positive definite.

We have thus found the limit equation inside the domain. Applying a similar technique, we can derive boundary conditions for the limit problem (see also [13, 14, 16, 17]). Consequently, we arrive at the following averaged problem:

$$\begin{gathered}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)-Q(x)u_0(x)=-|\Box\cap\omega|f(x)\text{ in }\Omega,\\ \sum_{i=1}^{d}\left\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\right\rangle\frac{\partial u_0(x)}{\partial x_i}+P(\hat{x})u_0(x)=G(\hat{x})\text{ on }\Gamma_1,\\ \frac{\partial u_0(x)}{\partial\mathcal{N}}=0\text{ on }\Gamma_2,\end{gathered}\tag{15}$$

where

$$P(\hat{x})=\int\limits_{T}p(\hat{x},\hat{\xi})\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}\,d\hat{\xi},\quad G(\hat{x})=\int\limits_{T}g(\hat{x},\hat{\xi})\,\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}\,d\hat{\xi},$$

and $T=\{\xi:0<\xi_j<1,j=1,\dots,d-1\}$.

The integral identity for (13) has the shape

$$\begin{aligned}&\int\limits_{\Omega}\left(\sum_{i,j=1}^{d}\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}+Q(x)u_0(x)v(x)\right)dx\\ &+\int\limits_{\Gamma_1}P(\hat{x})u_0(x)v(x)\,d\hat{x}=\int\limits_{\Omega}|\Box\cap\omega|f(x)v(x)\,dx+\int\limits_{\Gamma_1}G(\hat{x})v(x)\,d\hat{x}\end{aligned}\tag{16}$$

for all functions $v\in H^1(\Omega)$.

REMARK 2. According to the above construction, the functions $u_0(x)$ and $M(x,x/\varepsilon)$ are defined not in the whole of Ω^ε and $M(x,x/\varepsilon)$ is purely periodic in a neighborhood of Γ_0. Application of the symmetric extension technique [22] enables us to extend $u_0(x)$ to some neighborhood Π_ε and $M(x,\xi)$ to the interior of the "holes" so as to preserve regularity of the functions. We use the same notations for the extended functions.

In particular, in Theorem 1 below, $u_0(x)$ belongs to $C^3(\Omega)$ due to the standard elliptic estimates. Thus, $u_0(x)$ can be extended to Ω^ε as a C^3-function.

The limit behavior of a solution to the problem (1) is described by the following assertion:

Theorem 1. *Assume $f(x)\in C^1(\mathbf{R}^d)$ and suppose that $g(\hat{x},\hat{\xi})$, $p(\hat{x},\hat{\xi})$, and $q(x,\xi)$ are sufficiently smooth functions. Also, suppose that $p(\hat{x},\hat{\xi})$ and $q(x,\xi)$ are nonnegative and at least one of them is strictly positive in at least one point. Then the problem* (1) *has a unique solution for all sufficiently small ε and the following estimate holds:*

$$\|u_0+\varepsilon u_1-u_\varepsilon\|_{H^1(\Omega^\varepsilon)}\leq K_1\sqrt{\varepsilon},\tag{17}$$

where u_0 and u_1 are solutions to the problems (13) *and* (6) *and K_1 is independent of ε.*

REMARK 3. In the statement of Theorem 1, the conditions $q(x,\xi)\geq 0$ and $p(\hat{x},\hat{\xi})\geq 0$ can be replaced with the weaker conditions $Q(x)\geq 0$ and $P(\hat{x})\geq 0$.

§ 3. Preliminary Lemmas

This section is devoted to various technical assertions to be used below. Some of them are proven in [17] (see also [14]). We omit their proofs.

Lemma 1. *The inequalities*

$$\|v(\hat{x}, \varepsilon F(\hat{x}, \hat{x}/\varepsilon)) - v(\hat{x}, 0)\|_{L_2(\Gamma_1)} \le C_1\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}, \tag{18}$$

$$\|v\|_{L_2(\Pi_\varepsilon)} \le C_2\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)} \tag{19}$$

are valid for every $v \in H^1(\Omega^\varepsilon)$.

It is convenient to choose the coordinates $\hat{x} = (x_1, \dots, x_{d-1})$ on Γ_1^ε. The $(d-1)$-dimensional volume element on Γ_1^ε is calculated in the following lemma:

Lemma 2. *Let ds stand for the* $(d-1)$*-dimensional volume element of* Γ_1^ε. *Then*

$$ds = \sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2|_{\hat{\xi}=\hat{x}/\varepsilon}}\, d\hat{x}(1 + O(\varepsilon)).$$

The following assertion is a straightforward consequence of the Sobolev embedding theorem (see, for instance, [23]).

Assertion 3. *The inequality*

$$\left|\int_{\Gamma_1} uv\, d\hat{x}\right| \le C_3\|u\|_{H^{1/2}(\Gamma_1)}\|v\|_{H^{1/2}(\Gamma_1)}$$

holds uniformly in $u, v \in H^{1/2}(\Gamma_1)$.

The uniform (in ε) coerciveness of the bilinear form in (2) is the topic of Lemma 4 which in particular implies that the problem (1) is well-posed.

Lemma 4. *If the conditions of Theorem 1 are satisfied then the inequality*

$$\int_{\Omega^\varepsilon} |\nabla v|^2\, dx + \varepsilon \int_{S_\varepsilon} q(x, x/\varepsilon)v^2 ds + \int_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon)v^2\, ds \ge C_4\|v\|^2_{H^1(\Omega^\varepsilon)}$$

holds for all $v \in H^1(\Omega^\varepsilon)$, *with* C_4 *a constant independent of* ε.

The following assertion will be used below systematically:

Lemma 5. *Suppose that* $h(\hat{x}, \hat{\xi})$ *is a Lipschitz function* 1*-periodic in* ξ *and such that*

$$\int_0^1\int_0^1 \cdots \int_0^1 h(\hat{x}, \hat{\xi})\, d\hat{\xi} \equiv 0. \tag{20}$$

Then the estimate

$$\left|\int_{\Gamma_1} h(\hat{x}, \hat{x}/\varepsilon)u(\hat{x})v(\hat{x})\, d\hat{x}\right| \le C_5\sqrt{\varepsilon}\|u\|_{H^{1/2}(\Gamma_1)}\|v\|_{H^{1/2}(\Gamma_1)} \tag{21}$$

is valid for arbitrary functions $u, v \in H^{1/2}(\Gamma_1)$.

REMARK 4. For smooth functions $h(\hat{x}, \hat{\xi})$ in the two-dimensional case, the assertion was proven in [14] and independently in [24] (see also [25]). A proof for a Lipschitz function is given in [17].

Lemma 6. *The following estimates are valid:*

$$\left|\int_{\Gamma_1^\varepsilon} g(\hat{x}, \hat{x}/\varepsilon)v(x)\,ds - \int_{\Gamma_1} G(\hat{x})v(\hat{x}, 0)\,d\hat{x}\right| \le C_6\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}, \tag{22}$$

$$\left|\int_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon)v(x)u(x)\,ds - \int_{\Gamma_1} P(\hat{x})v(\hat{x}, 0)u(\hat{x}, 0)\,d\hat{x}\right| \le C_7\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}\|u\|_{H^1(\Omega^\varepsilon)}, \tag{23}$$

where

$$P(\hat{x}) = \int_T p(\hat{x}, \hat{\xi})\sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2}\,d\hat{\xi}, \quad G(\hat{x}) = \int_T g(\hat{x}, \hat{\xi})\sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2}\,d\hat{\xi}.$$

The following assertion is essentially a modified version of Lemma 5.

Lemma 7. *If*

$$\frac{1}{|\square \cap \omega|}\int_{\square\cap\omega} Q(x)\,d\xi - \int_S q(x, \xi)\,d\sigma \equiv 0 \tag{24}$$

then

$$\left|\frac{1}{|\square \cap \omega|}\int_{\Omega'_\varepsilon} Q(x)v(x)\,dx - \varepsilon\int_{S_\varepsilon} q(x, x/\varepsilon)v(x)\,ds\right| \le C_8\varepsilon\|v\|_{H^1(\Omega^\varepsilon)} \tag{25}$$

for every $v(x) \in H^1(\Omega^\varepsilon)$, with C_8 a constant independent of ε.

PROOF. It follows from (24) that the problem

$$\Delta_\xi\Psi(x, \xi) = \frac{1}{|\square \cap \omega|}Q(x) \text{ in } \square \cap \omega, \quad \frac{\partial\Psi}{\partial n} = q(x, \xi) \text{ on } S \tag{26}$$

has a solution 1-periodic in ξ. Moreover, this solution is unique up to an additive constant.

Multiply (26) by a function $v(x) \in H^1(\Omega^\varepsilon)$ and integrate over the domain Ω'_ε. Integrating the left-hand side of the so-obtained formula by parts, we obtain

$$\left|\frac{1}{|\square \cap \omega|}\int_{\Omega'_\varepsilon} Q(x)v(x)\,dx - \varepsilon\int_{S_\varepsilon} q(x, x/\varepsilon)v(x)\,ds\right|$$

$$= \left|\int_{\Omega'_\varepsilon} \Delta_\xi\Psi(x, \xi)|_{\xi=x/\varepsilon}v(x)\,dx - \varepsilon\int_{S_\varepsilon} q(x, x/\varepsilon)v(x)\,ds\right|$$

$$= \Bigg|\varepsilon\int_{\Omega'_\varepsilon}(\nabla_x[\nabla_\xi\Psi(x, \xi)|_{\xi=x/\varepsilon}] - ((\nabla_x, \nabla_\xi)\Psi(x, \xi))|_{\xi=x/\varepsilon})v(x)\,dx$$

$$-\varepsilon\int_{S_\varepsilon} q(x, x/\varepsilon)v(x)\,ds\Bigg| \le \varepsilon\left|\int_{\Omega'_\varepsilon}((\nabla_\xi\Psi(x, \xi))|_{\xi=x/\varepsilon}, \nabla_x v(x))\,dx\right|$$

$$+\varepsilon\left|\int_{\Omega'_\varepsilon}((\nabla_x, \nabla_\xi)\Psi(x, \xi))|_{\xi=x/\varepsilon}v(x)\,dx\right|$$

$$+\varepsilon^2\left|\int_{S_\varepsilon}((\nabla_\xi\Psi(x, \xi))|_{\xi=x/\varepsilon}, n'_\varepsilon)v(x)\,ds\right| \le C_8\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}. \tag{27}$$

The lemma is proven.

The following lemma enables us to neglect the right-hand side of (1) in the thin layer Π_ε without worsening the estimate.

Lemma 8. *Let y_ε be a solution to the problem*

$$-\Delta y_\varepsilon = h^\varepsilon(x) \text{ in } \Omega^\varepsilon, \quad \frac{\partial y_\varepsilon}{\partial \nu_\varepsilon} + p(\hat{x}, \hat{x}/\varepsilon) y_\varepsilon = 0 \text{ on } \Gamma_1^\varepsilon,$$
$$\frac{\partial y_\varepsilon}{\partial n_\varepsilon} + \varepsilon q(x, x/\varepsilon) y_\varepsilon = 0 \text{ on } S_\varepsilon, \quad \frac{\partial y_\varepsilon}{\partial \mathcal{N}} = 0 \text{ on } \Gamma_2, \tag{28}$$

where $h^\varepsilon(x) = f(x)$ for $x \in \Pi_\varepsilon$ and 0 otherwise. Then

$$\|y_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_9 \varepsilon. \tag{29}$$

PROOF. Put

$$H^\varepsilon(x) = \int\limits_{\varepsilon F(\hat{x}, \hat{x}/\varepsilon)}^{x_d} h^\varepsilon(x)\, dx_d.$$

It is easy to see that $H^\varepsilon = 0$ on Γ_1^ε and $|H^\varepsilon(x)| \le C_{10}\varepsilon$ in Π_ε. The integral identity for the problem (28) has the shape

$$\int\limits_{\Omega^\varepsilon} (\nabla y_\varepsilon, \nabla \varphi)\, dx + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) y_\varepsilon \varphi\, ds + \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) y_\varepsilon \varphi\, ds = \int\limits_{\Omega^\varepsilon} h^\varepsilon(x) \varphi(x)\, dx \tag{30}$$

for all $\varphi \in H^1(\Omega^\varepsilon)$. Extend y_ε to the "holes" so as to have

$$\|\tilde{y}_\varepsilon\|_{H^1(\Omega \cup \Pi_\varepsilon)} \le C_{11} \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}; \tag{31}$$

where $\tilde{y}_\varepsilon$ denotes the extended function. Putting $\varphi = \tilde{y}_\varepsilon$ in (30), we obtain

$$\int\limits_{\Omega^\varepsilon} |\nabla \tilde{y}_\varepsilon|^2\, dx + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) \tilde{y}_\varepsilon^2\, ds + \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) \tilde{y}_\varepsilon^2\, ds = \int\limits_{\Omega^\varepsilon} \frac{\partial H^\varepsilon}{\partial x_d} \tilde{y}_\varepsilon\, dx. \tag{32}$$

Integrating the right-hand side of (32) by parts and recalling the boundary condition in (28), from (31) and Lemma 4 we infer that

$$\|y_\varepsilon\|^2_{H^1(\Omega^\varepsilon)} \le C_4 \Bigg(\int\limits_{\Omega^\varepsilon} |\nabla \tilde{y}_\varepsilon|^2\, dx + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) \tilde{y}_\varepsilon^2\, ds + \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) \tilde{y}_\varepsilon^2\, ds \Bigg)$$
$$\le C_4 \Bigg(\Bigg| \int\limits_{\Omega^\varepsilon} H^\varepsilon(x) \frac{\partial \tilde{y}_\varepsilon}{\partial x_d}\, dx \Bigg| + \Bigg| \int\limits_{\Gamma_2} H^\varepsilon(x) \tilde{y}_\varepsilon \mathcal{N}_d\, ds \Bigg| \Bigg)$$
$$\le C_{12} \max_{\Omega^\varepsilon} |H^\varepsilon(x)|\, \|\tilde{y}_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_{13}\varepsilon \|y_\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

Now, we obtain (29) immediately. The lemma is proven.

The following assertion can be proven by analogy with Lemma 7:

Lemma 9. *Suppose that a 1-periodic (in ξ) function $w(\xi)$ belongs to $L_2(Q)$ and*

$$\int\limits_Q w(\xi)\, d\xi = 0. \tag{33}$$

Then

$$\Bigg| \int\limits_{Q_\varepsilon \cap \Pi_\varepsilon} w(\xi)|_{\xi = x/\varepsilon} \nabla_x u_0(x) v(x)\, dx \Bigg| \le C_{14} \varepsilon^{3/2} \|w\|_{L_2(Q)} \|v\|_{H^1(\Omega^\varepsilon)} \tag{34}$$

for every $v(x) \in H^1(\Omega^\varepsilon)$, with C_{14} a constant independent of ε.

§ 4. The Main Estimate

PROOF OF THEOREM 1. We have to estimate the H^1-norm of the residual $\|u_0+\varepsilon u_1-u_\varepsilon\|_{H^1(\Omega^\varepsilon)}$. To this end, we extend the functions $u_0(x)$ and $M_i(x,\xi)$ to the layer Π_ε (see Remark 2 above) and insert the expression $z_\varepsilon(x,x/\varepsilon)=u_0(x)+\varepsilon u_1(x,x/\varepsilon)-u_\varepsilon(x)$ in (1) to obtain the equality

$$\Delta_x(z_\varepsilon(x,x/\varepsilon))=\Delta_x u_0(x)+\varepsilon\Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon}$$
$$+2(\nabla_x,\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon})+\frac{1}{\varepsilon}\Delta_\xi u_1(x,\xi)|_{\xi=x/\varepsilon}-\Delta_x u_\varepsilon(x). \tag{35}$$

Using the relations

$$\Delta_\xi u_1(x,\xi)=0\ \forall x\in\Omega_1^\varepsilon,\quad \Delta_x u_\varepsilon(x)=-f(x)\text{ in }\Omega^\varepsilon,$$
$$2(\nabla_x,\nabla_\xi u_1(x,\xi))=2\sum_{i,j=1}^d\frac{\partial M_i(x,\xi)}{\partial\xi_j}\frac{\partial^2u_0(x)}{\partial x_i\partial x_j}+2\sum_{i,j=1}^d\frac{\partial^2M_i(x,\xi)}{\partial x_j\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}. \tag{36}$$

$$\sum_{i,j=1}^d\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)-Q(x)u_0(x)-|\square\cap\omega|f(x)\text{ in }\Omega, \tag{37}$$

we can rewrite (35) in Ω_1^ε as follows:

$$\Delta_x(z_\varepsilon(x,x/\varepsilon))=\varepsilon\Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon}+2\sum_{i,j=1}^d\frac{\partial M_i(x,\xi)}{\partial\xi_j}\frac{\partial^2u_0(x)}{\partial x_i\partial x_j}\bigg|_{\xi=x/\varepsilon}$$
$$+2\sum_{i,j=1}^d\frac{\partial^2M_i(x,\xi)}{\partial x_j\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon}+\Delta_x u_0(x)$$
$$-\frac{1}{|\square\cap\omega|}\sum_{i,j=1}^d\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)+\frac{1}{|\square\cap\omega|}Q(x)u_0(x). \tag{38}$$

Similarly, on S_ε we have

$$\frac{\partial z_\varepsilon(x,x/\varepsilon)}{\partial n_\varepsilon}=-(\nabla_x u_\varepsilon(x),n_\varepsilon)+(\nabla_x u_0(x),n_\varepsilon)$$
$$+\varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon},n_\varepsilon)+(\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon},n_\varepsilon)$$
$$=\varepsilon q(x,x/\varepsilon)u_\varepsilon(x)+(\nabla_x u_0(x),n_\varepsilon)+\varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon},n_\varepsilon)$$
$$+\sum_{i,j=1}^d\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial\xi_j}\tilde n^j(x,\xi)\right)\bigg|_{\xi=x/\varepsilon}+\varepsilon(\nabla_\xi u_1(x,\xi),n_\varepsilon'(x,\xi))|_{\xi=x/\varepsilon},$$

whereas on Γ_1^ε we have

$$\frac{\partial z_\varepsilon(x,x/\varepsilon)}{\partial\nu_\varepsilon}=-(\nabla_x u_\varepsilon(x),\nu_\varepsilon)+(\nabla_x u_0(x),\nu_\varepsilon)$$
$$+\varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon},\nu_\varepsilon)+(\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon},\nu_\varepsilon)$$
$$=p(\hat x,\hat x/\varepsilon)u_\varepsilon(x)-g(\hat x,\hat x/\varepsilon)+(\nabla_x u_0(x),\nu_\varepsilon)$$
$$+\varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon},\nu_\varepsilon)+\sum_{i,j=1}^d\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial\xi_j}\nu_\varepsilon^j\right)\bigg|_{\xi=x/\varepsilon}$$

Finally, on Γ_2 we have the trivial boundary condition

$$\frac{\partial z_\varepsilon(x,x/\varepsilon)}{\partial \mathcal{N}} = \left\{ -\frac{\partial u_\varepsilon(x)}{\partial \mathcal{N}} + \frac{\partial u_0(x)}{\partial \mathcal{N}} + \sum_{i,j=1}^{d} \varepsilon \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} M_i(x,\xi)\mathcal{N}_j \right.$$
$$\left. + \sum_{i,j=1}^{d} \varepsilon \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial x_j}\mathcal{N}_j + \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial \xi_j}\mathcal{N}_j \right\}\Bigg|_{\xi=x/\varepsilon} \equiv 0.$$

Multiplying (38) by $v(x)$ and integrating over Ω^ε, we now obtain

$$\int_{\Omega^\varepsilon} \Delta_x(z_\varepsilon(x,x/\varepsilon))v(x)\,dx = \varepsilon \int_{\Omega^\varepsilon} \Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+2\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\bigg|_{\xi=x/\varepsilon} v(x)\,dx + 2\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+ \int_{Q_\varepsilon \cap \Pi_\varepsilon} \frac{1}{\varepsilon}\Delta_\xi u_1(x,\xi)|_{\xi=x/\varepsilon} v\,dx + \int_{\Omega^\varepsilon} \Delta_x u_0(x)v(x)\,dx$$
$$-\frac{1}{|\square \cap \omega|}\int_{\Omega_1^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) v(x)\,dx$$
$$+\frac{1}{|\square \cap \omega|}\int_{\Omega_1^\varepsilon} Q(x)u_0(x)v(x)\,dx + \int_{\Pi_\varepsilon} f(x)v(x)\,dx. \tag{39}$$

On the other hand, we can transform the left-hand side of (39) by using the Green's formula as follows:

$$\int_{\Omega^\varepsilon} \Delta_x(z_\varepsilon(x,x/\varepsilon))v(x)\,dx = \int_{S_\varepsilon} \frac{\partial z_\varepsilon}{\partial n_\varepsilon} v(x)\,ds + \int_{\Gamma_1^\varepsilon} \frac{\partial z_\varepsilon}{\partial \nu_\varepsilon} v(x)\,ds + \int_{\Gamma_2} \frac{\partial z_\varepsilon}{\partial \mathcal{N}} v(x)\,ds$$
$$-\int_{\Omega^\varepsilon} \nabla z_\varepsilon \nabla v(x)\,dx = \varepsilon \int_{S_\varepsilon} q(x,x/\varepsilon)u_\varepsilon(x)v(x)\,ds + \int_{S_\varepsilon} \frac{\partial u_0(x)}{\partial n_\varepsilon} v(x)\,ds$$
$$+\varepsilon \int_{S_\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$
$$+\int_{S_\varepsilon} \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\tilde{n}^j(x,\xi)\right)\bigg|_{\xi=x/\varepsilon} v(x)\,ds + \int_{\Gamma_1^\varepsilon} p(\hat{x},\hat{x}/\varepsilon)u_\varepsilon(x)v(x)\,ds - \int_{\Gamma_1^\varepsilon} g(\hat{x},\hat{x}/\varepsilon)v(x)\,ds$$
$$+\int_{\Gamma_1^\varepsilon} \frac{\partial u_0}{\partial \nu_\varepsilon} v(x)\,ds + \varepsilon \int_{\Gamma_1^\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)v(x)\,ds + \int_{\Gamma_1^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\bigg|_{\xi=x/\varepsilon} \nu_\varepsilon^j\right) v(x)\,ds$$
$$+\int_{\Gamma_2} \frac{\partial z_\varepsilon}{\partial \mathcal{N}} v(x)\,ds - \int_{\Omega^\varepsilon} \nabla z_\varepsilon(x,x/\varepsilon)\nabla v(x)\,dx. \tag{40}$$

From (39) and (40) we derive

$$\int_{\Omega^\varepsilon} \nabla z_\varepsilon(x, x/\varepsilon)\nabla v(x)\,dx = \varepsilon \int_{S_\varepsilon} q(x, x/\varepsilon)u_\varepsilon(x)v(x)\,ds + \int_{S_\varepsilon} \frac{\partial u_0(x)}{\partial n_\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{S_\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon} \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right)\Big|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon)u_\varepsilon(x)v(x)\,ds - \int_{\Gamma_1^\varepsilon} g(\hat{x}, \hat{x}/\varepsilon)v(x)\,ds + \int_{\Gamma_1^\varepsilon} \frac{\partial u_0}{\partial \nu_\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{\Gamma_1^\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)v(x)\,ds - \int_{\Pi_\varepsilon} f(x)v(x)\,dx$$

$$+\int_{\Gamma_1^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\Big|_{\xi=x/\varepsilon} \nu_\varepsilon^j \right) v(x)\,ds + \int_{\Gamma_2} \frac{\partial z_\varepsilon}{\partial \mathcal{N}} v(x)\,ds$$

$$-\varepsilon \int_{\Omega^\varepsilon} \Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon} v(x)\,dx - 2\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial M_i(x,\xi)}{\partial \xi_j} \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\Big|_{\xi=x/\varepsilon} v(x)\,dx$$

$$-2\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j} \frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon} v(x)\,dx - \int_{\Omega^\varepsilon} \Delta_x u_0(x)v(x)\,dx$$

$$-\int_{Q_\varepsilon \cap \Pi_\varepsilon} \frac{1}{\varepsilon}\Delta_\xi u_1(x,\xi)|_{\xi=x/\varepsilon} v\,dx$$

$$+\frac{1}{|\square \cap \omega|} \int_{\Omega_1^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} \right) v(x)\,dx$$

$$-\frac{1}{|\square \cap \omega|} \int_{\Omega_1^\varepsilon} Q(x)u_0(x)v(x)\,dx. \tag{41}$$

In view of the obvious relation

$$\operatorname{div}_\xi \left(\frac{\partial}{\partial x_j} \left(M_i(x,\xi) \frac{\partial u_0(x)}{\partial x_i} \right) \right)\Big|_{\xi=x/\varepsilon} = \varepsilon \operatorname{div}_x \left(\frac{\partial}{\partial x_j} \left(M_i(x,\xi) \frac{\partial u_0(x)}{\partial x_i} \right)\Big|_{\xi=x/\varepsilon} \right)$$
$$-\varepsilon \operatorname{div}_x \left(\frac{\partial}{\partial x_j} \left(M_i(x,\xi) \frac{\partial u_0(x)}{\partial x_i} \right) \right)\Big|_{\xi=x/\varepsilon},$$

application of the Stokes formula yields

$$\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial M_i(x,\xi)}{\partial \xi_j} \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\Big|_{\xi=x/\varepsilon} v(x)\,dx + \int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j} \frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon} v(x)\,dx$$

$$= \varepsilon \int_{S_\varepsilon} (\nabla_x u_1(x,\xi)\Big|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{\Gamma_1^\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)v(x)\,ds$$

$$+\varepsilon \int\limits_{\Gamma_2} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \mathcal{N}) v(x)\, ds + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}. \tag{42}$$

Using (41) and the boundary condition in (15), we estimate the expression

$$\Big| \int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon(x, x/\varepsilon)\nabla v(x)\, dx + \varepsilon \int\limits_{S_\varepsilon} q(x,x/\varepsilon) z_\varepsilon(x,x/\varepsilon) v(x)\, ds + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x},\hat{x}/\varepsilon) z_\varepsilon(x,x/\varepsilon) v(x)\, ds \Big|$$

$$\le \varepsilon \Big| \varepsilon \int\limits_{S_\varepsilon} q(x,x/\varepsilon) u_1(x,x/\varepsilon) v(x)\, ds + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x},\hat{x}/\varepsilon) u_1(x,x/\varepsilon) v(x)\, ds \Big|$$

$$+ \Big| \varepsilon \int\limits_{S_\varepsilon} q(x,x/\varepsilon) u_0(x) v(x)\, ds - \frac{1}{|\square \cap \omega|} \int\limits_{\Omega'_\varepsilon} Q(x) u_0(x) v(x)\, dx \Big| + \Big| \varepsilon \int\limits_{\Omega^\varepsilon} \Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon} v(x)\, dx \Big|$$

$$+ \Big| \int\limits_{S_\varepsilon} \left(\frac{\partial u_0(x)}{\partial n_\varepsilon} + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right)\Big|_{\xi=x/\varepsilon} \right) v(x)\, ds \Big|$$

$$+ \Big| \int\limits_{\Gamma_1^\varepsilon} p(\hat{x},\hat{x}/\varepsilon) u_0(x) v(x)\, ds - \int\limits_{\Gamma_1} P(\hat{x}) u_0(x) v(x)\, d\hat{x} \Big| + \Big| \int\limits_{\Gamma_1} G(\hat{x}) v(x) d\hat{x} - \int\limits_{\Gamma_1^\varepsilon} g(\hat{x},\hat{x}/\varepsilon) v(x)\, ds \Big|$$

$$+ \Big| \int\limits_{\Gamma_1^\varepsilon} \left(\frac{\partial u_0(x)}{\partial \nu_\varepsilon} + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\Big|_{\xi=x/\varepsilon} \nu_\varepsilon^j \right) \right) v(x)\, ds$$

$$- \int\limits_{\Gamma_1} \sum_{i=1}^d \left\langle \delta_{id} + \frac{\partial M_i(x,\xi)}{\partial \xi_d} \right\rangle \frac{\partial u_0(x)}{\partial x_i} v(x)\, d\hat{x} \Big| + \Big| \int\limits_{Q_\varepsilon \cap \Pi_\varepsilon} \frac{1}{\varepsilon} \Delta_\xi u_1(x,\xi)|_{\xi=x/\varepsilon} v\, dx \Big|$$

$$+ \Big| \int\limits_{\Pi_\varepsilon} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(\left[\delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right] \frac{\partial u_0(x)}{\partial x_i} \right)\Big|_{\xi=x/\varepsilon} v(x)\, dx \Big|$$

$$+ \Big| \int\limits_{\Omega'_\varepsilon} \sum_{i,j=1}^d \Big(\frac{1}{|\square \cap \omega|} \frac{\partial}{\partial x_j} \left[\left\langle \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} \right]$$

$$- \frac{\partial}{\partial x_j} \left[\frac{\partial M_i(x,\xi)}{\partial \xi_j} \frac{\partial u_0(x)}{\partial x_i} \right] \Big)\Big|_{\xi=x/\varepsilon} v(x)\, dx - \varepsilon \int\limits_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\, ds \Big|$$

$$+ \Big| \int\limits_{\Omega\setminus\Omega'} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(\left[\left\langle \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle - \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right] \frac{\partial u_0(x)}{\partial x_i} \right)\Big|_{\xi=x/\varepsilon} v(x)\, dx$$

$$+ \int\limits_{\Omega\setminus\Omega'} Q(x) u_0(x) v(x)\, dx \Big| + \Big| \int\limits_{\Pi_\varepsilon} f(x) v(x)\, dx \Big| + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}. \tag{43}$$

Lemmas 1 and 6 imply that $I_7 \le C_1\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}$, $I_9 \le C_2\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}$, $I_5 \le C_7\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}$, and $I_6 \le C_6\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}$. Since $Q(x) \equiv 0$ in $\Omega\setminus\Omega'$ and $M_i(x,\xi) \equiv 0$ for $x \in \Omega\setminus\Omega'$, we have $I_{11} = 0$. Estimate I_2. According to Lemma 7.

$$I_2 = \left| \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) u_0(x) v(x)\, ds - \frac{1}{|\square \cap \omega|} \int\limits_{\Omega'_\varepsilon} Q(x) u_0(x) v(x)\, dx \right|$$
$$\leq C_{15} \varepsilon \|u_0\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}.$$

It is clear that I_1 and I_3 admit the estimate

$$|I_1| + |I_3| \leq C_{16} \varepsilon \|v\|_{H^1(\Omega^\varepsilon)}.$$

The identity $I_4 \equiv 0$ follows from the boundary conditions of (6). By Lemma 8, we may assume that the function $f(x)$ equals zero in the layer Π_ε. Then $I_{12} = 0$. Estimate the integral I_{10}. Using (14), we can easily verify that

$$\int\limits_{\square \cap \omega} \left[\frac{1}{|\square \cap \omega|} \frac{\partial}{\partial x_j} \left\langle \delta_{ij} + \frac{\partial M_i(x, \xi)}{\partial \xi_j} \right\rangle - \frac{\partial}{\partial x_j} \left(\delta_{ij} - \frac{\partial M_i(x, \xi)}{\partial \xi_j} \right) \right] d\xi - U_i(x) = 0.$$

Applying the technique of the proof of Lemma 7, we can demonstrate that the preceding relation implies the inequality

$$|I_{10}| \leq C_{17} \varepsilon \left\| \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} \right\|_{H^1(\Omega_1^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}.$$

Here we have used smoothness of $f(x)$, i.e., the fact that $f(x) \in C^1(\Omega_1^\varepsilon)$. Finally, Lemma 9 yields the estimate

$$I_8 \leq C_{14} \sqrt{\varepsilon} \|v\|_{H^1(\Omega^\varepsilon)}.$$

Inserting $v = u_0 + \varepsilon u_1 - u_\varepsilon$ in (43) and using the above estimates and Lemma 4, we arrive at (17). The theorem is proven.

§ 5. A Boundary Layer Corrector

The aim of this section is to construct a boundary layer corrector which helps us to improve the asymptotic behavior of $u_\varepsilon(x)$. We fix the asymptotic expansion of the solution $u_\varepsilon(x)$ to the problem (1):

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon v_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \dots.$$

The function $u_0(x)$ is defined in the problem (15),

$$u_1(x, x/\varepsilon) = \sum_{i=1}^{d} \frac{\partial u_0(x)}{\partial x_i} M_i(x, x/\varepsilon),$$

and $M_i(x, \xi)$ satisfy the problem (9). We construct a boundary layer function $v_1(x, x/\varepsilon)$ of the shape

$$v_1(x, x/\varepsilon) = N(x, x/\varepsilon) + N_0(x, x/\varepsilon) u_0(x) + \sum_{i=1}^{d} N_i(x, x/\varepsilon) \frac{\partial u_0(x)}{\partial x_i}. \tag{45}$$

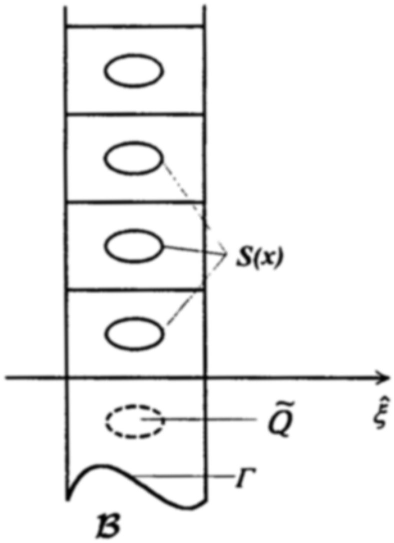

Fig. 4

All functions $N(x,\xi)$, $N_0(x,\xi)$, and $N_i(x,\xi)$, $i=0,1,\dots,d$, are defined in

$$\mathcal{B} = \{\xi \mid \hat{\xi} \in T^{d-1},\ F(\hat{x},\hat{\xi}) < \xi_d < \infty\} \setminus \{\xi | \xi_d > 0; \chi_Q(\xi) = 1\};$$

moreover, we identify 1-periodic functions in $\hat{\xi}$ with functions defined on the standard $(d-1)$-dimensional torus $T^{d-1} = \mathbf{R}^{d-1}/\mathbf{Z}^{d-1}$. Also, we suppose that $N(x,\xi)$, $N_0(x,\xi)$, and $N_i(x,\xi)$, $i=0,1,\dots,d$, decrease exponentially as $\xi_d \to +\infty$. To find $N(x,\xi)$, $N_0(x,\xi)$, and $N_i(x,\xi)$, we insert the formal asymptotic series

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + \varepsilon(N(x,x/\varepsilon) \\ &+ N_0(x,x/\varepsilon)u_0(x) + \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}(N_i(x,x/\varepsilon) \\ &+ M_i(x,x/\varepsilon))) + \varepsilon^2 u_2(x,x/\varepsilon) + \dots \end{aligned} \tag{46}$$

in (1) and equate the terms of the same order in ε. We arrive at the equations

$$\Delta_\xi N(x,\xi) = 0, \quad \Delta_\xi N_0(x,\xi) = 0, \quad \Delta_\xi N_i(x,\xi) = -\Delta_\xi M_i(x,\xi).$$

Observe that $\Delta_\xi M_i(x,\xi)$ has compact support in $\mathcal{B}$ which lies in $\widetilde{Q} = \{\xi | \xi \in \mathcal{B}, \xi_d < 0, \chi_Q(\xi) = 1\}$ (see Fig. 4). Similarly, inserting (46) in the boundary conditions of (1), we obtain the following formal equality on Γ_1^ε:

$$\begin{aligned} 0 &= \frac{\partial u_\varepsilon(x)}{\partial \nu_\varepsilon} + p(\hat{x},\hat{x}/\varepsilon)u_\varepsilon(x) - g(\hat{x},\hat{x}/\varepsilon) \cong \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\nu_\varepsilon^i + \varepsilon(\nabla_x N(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon) \\ &+ (\nabla_\xi N(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon) + \varepsilon(\nabla_x N_0(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)u_0(x) + (\nabla_\xi N_0(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)u_0(x) \\ &+ \varepsilon N_0(x,\xi)|_{\xi=x/\varepsilon}\frac{\partial u_0(x)}{\partial \nu_\varepsilon} + \varepsilon\sum_{i=1}^d \left(\nabla_x \frac{\partial u_0(x)}{\partial x_i}, \nu_\varepsilon\right) N_i(x,x/\varepsilon) \\ &+ \varepsilon\sum_{i=1}^d \left(\nabla_x\left(\frac{\partial u_0(x)}{\partial x_i}M_i(x,\xi)\right), \nu_\varepsilon\right)\Bigg|_{\xi=x/\varepsilon} + \varepsilon\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(x,\xi)}{\partial x_j}\Bigg|_{\xi=x/\varepsilon}\nu_\varepsilon^j \\ &+ \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(x,\xi)}{\partial \xi_j}\Bigg|_{\xi=x/\varepsilon}\nu_\varepsilon^j + \varepsilon\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial x_j}\Bigg|_{\xi=x/\varepsilon}\nu_\varepsilon^j \\ &+ \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial \xi_j}\Bigg|_{\xi=x/\varepsilon}\nu_\varepsilon^j + p(\hat{x},\hat{x}/\varepsilon)u_0(x) - g(\hat{x},\hat{x}/\varepsilon) + O(\varepsilon) \\ &= \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\left(\sum_{j=1}^d \frac{\partial N_i(x,\xi)}{\partial \xi_j}\nu_\varepsilon^j + \nu_\varepsilon^i + \sum_{j=1}^d \frac{\partial M_i(x,\xi)}{\partial \xi_j}\nu_\varepsilon^j - \frac{\langle \delta_{id} + \frac{\partial M_i(x,\xi)}{\partial \xi_d}\rangle}{\sqrt{1 + (\nabla_\xi F(\hat{x},\hat{\xi}))^2}}\right)\Bigg|_{\xi=x/\varepsilon} \\ &+ \left(\sum_{i=1}^d \frac{\partial N(x,\xi)}{\partial \xi_i}\nu_\varepsilon^i - g(\hat{x},\hat{x}/\varepsilon) + \frac{G(\hat{x})}{\sqrt{1 + (\nabla_\xi F(\hat{x},\hat{\xi}))^2}}\right)\Bigg|_{\xi=x/\varepsilon} \end{aligned}$$

$$+u_0(x)\left(\sum_{i=1}^{d}\frac{\partial N_0(x,\xi)}{\partial\xi_i}\nu_\varepsilon^i+p(\hat{x},\hat{x}/\varepsilon)-\frac{P(\hat{x})}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}\right)\Bigg|_{\xi=x/\varepsilon}+\dots.$$

Here we have also used the fact that

$$\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\right\rangle+P(\hat{x})u_0(x)=G(\hat{x})+O(\varepsilon)$$

for all $x\in\Gamma_1^\varepsilon$. To remove the terms that involve the functions N, N_0, and N_i in the above formula, we must choose the following boundary conditions on Γ:

$$\frac{\partial N(x,\xi)}{\partial\nu}=g(\hat{x},\hat{\xi})-\frac{G(\hat{x})}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}},$$

$$\frac{\partial N_0(x,\xi)}{\partial\nu}=\frac{P(\hat{x})}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}-p(\hat{x},\hat{\xi}),$$

$$\frac{\partial N_i(x,\xi)}{\partial\nu}=\frac{\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\rangle}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}-\nu_\varepsilon^i-\frac{\partial M_i(x,\xi)}{\partial\nu}.$$

Arguing similarly, we obtain the homogeneous Neumann conditions on the boundary of the "holes." Finally, the functions $N(x,\xi)$, $N_0(x,\xi)$, and $N_i(x,\xi)$ satisfy the following problems in $\mathcal{B}$:

$$\begin{gathered}\Delta_\xi N(x,\xi)=0\quad\text{in}\quad\mathcal{B},\\ \frac{\partial N(x,\xi)}{\partial\nu}=g(\hat{x},\hat{\xi})-\frac{G(\hat{x})}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}\quad\text{on}\quad\Gamma,\\ \frac{\partial N(x,\xi)}{\partial\tilde{n}}=0\quad\text{on}\quad S(x),\end{gathered}\tag{47}$$

$$\begin{gathered}\Delta_\xi N_0(x,\xi)=0\quad\text{in}\quad\mathcal{B},\\ \frac{\partial N_0(x,\xi)}{\partial\nu}=\frac{P(\hat{x})}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}-p(\hat{x},\hat{\xi})\quad\text{on}\quad\Gamma,\\ \frac{\partial N_0(x,\xi)}{\partial\tilde{n}}=0\quad\text{on}\quad S(x),\end{gathered}\tag{48}$$

$$\begin{gathered}\Delta_\xi N_i(x,\xi)=-\Delta_\xi M_i(x,\xi)\quad\text{in}\quad\mathcal{B},\\ \frac{\partial N_i(x,\xi)}{\partial\nu}=\frac{\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\rangle}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}-\nu_\varepsilon^i-\frac{\partial M_i(x,\xi)}{\partial\nu}\quad\text{on}\quad\Gamma,\\ \frac{\partial N_i(x,\xi)}{\partial\tilde{n}}=0\quad\text{on}\quad S(x).\end{gathered}\tag{49}$$

We consider these problems in the space of functions with bounded Dirichlet integrals in $\mathcal{B}$.

Applying the technique of [26, 27], we can verify that each of the problems (47)–(49) has a unique solution vanishing as $\xi_d\to+\infty$. Moreover, these solutions decrease exponentially.

Theorem 2. *Suppose that $f(x) \in C^1(\mathbf{R}^d)$. $F(\hat{x},\hat{\xi})$, $g(\hat{x},\hat{\xi})$, and $p(\hat{x},\hat{\xi})$ are the above-defined smooth functions 1-periodic in $\hat{\xi}$, and $q(x,\xi)$ is a smooth function 1-periodic in ξ. Then the following estimate is valid for all sufficiently small ε:*

$$\|u_0 + \varepsilon u_1 + \varepsilon v_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq K_2\varepsilon, \tag{50}$$

where K_2 is independent of ε, u_0 is a solution to the problem (15), u_1 is a solution to the problem (6), and v_1 is the boundary layer corrector given by (45) with the functions $N(x,\xi)$, $N_0(x,\xi)$, and $N_i(x,\xi)$ satisfying the respective equations (47)–(49).

PROOF. To estimate $\|u_0 + \varepsilon u_1 + \varepsilon v_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)}$, we insert the expression

$$w_\varepsilon(x,x/\varepsilon) = u_0(x) + \varepsilon u_1(x,x/\varepsilon) + \varepsilon v_1(x,x/\varepsilon) - u_\varepsilon(x)$$

in (1) and after simple transformations obtain the following equality in Ω^ε:

$$\Delta_x(w_\varepsilon(x,x/\varepsilon)) = \Delta_x u_0(x) + \varepsilon\Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon} + 2(\nabla_x, \nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon})$$
$$+\frac{1}{\varepsilon}\Delta_\xi u_1(x,\xi)|_{\xi=x/\varepsilon} + \varepsilon\Delta_x v_1(x,\xi)|_{\xi=x/\varepsilon} + 2(\nabla_x,\nabla_\xi v_1(x,\xi)|_{\xi=x/\varepsilon}) + \frac{1}{\varepsilon}\Delta_\xi v_1(x,\xi)|_{\xi=x/\varepsilon} - \Delta_x u_\varepsilon(x). \tag{51}$$

Using the relations (36), (37), and

$$2(\nabla_x, \nabla_\xi v_1(x,\xi)) = 2\sum_{i=1}^d \frac{\partial^2 N(x,\xi)}{\partial\xi_i\partial x_i} + 2\sum_{i=1}^d \frac{\partial^2 N_0(x,\xi)}{\partial\xi_i\partial x_i}u_0(x)$$
$$+2\sum_{i=1}^d \frac{\partial N_0(x,\xi)}{\partial\xi_i}\frac{\partial u_0(x)}{\partial x_i} + 2\sum_{i,j=1}^d \frac{\partial^2 N_i(x,\xi)}{\partial\xi_j\partial x_j}\frac{\partial u_0(x)}{\partial x_i} + 2\sum_{i,j=1}^d \frac{\partial N_i(x,\xi)}{\partial\xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j},$$

we rewrite (51) in Ω_1^ε as follows:

$$\Delta_x(w_\varepsilon(x,x/\varepsilon)) = \varepsilon\Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon} + 2\sum_{i,j=1}^d \frac{\partial M_i(x,\xi)}{\partial\xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\bigg|_{\xi=x/\varepsilon}$$
$$+2\sum_{i,j=1}^d \frac{\partial^2 M_i(x,\xi)}{\partial x_j\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon} + \varepsilon\Delta_x v_1(x,\xi)|_{\xi=x/\varepsilon} + 2\sum_{i=1}^d \frac{\partial^2 N(x,\xi)}{\partial x_i\partial\xi_i}\bigg|_{\xi=x/\varepsilon}$$
$$+2\sum_{i=1}^d \frac{\partial^2 N_0(x,\xi)}{\partial x_i\partial\xi_i}u_0(x)\bigg|_{\xi=x/\varepsilon} + 2\sum_{i=1}^d \frac{\partial N_0(x,\xi)}{\partial\xi_i}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon}$$
$$+2\sum_{i,j=1}^d \frac{\partial N_i(x,\xi)}{\partial\xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\bigg|_{\xi=x/\varepsilon} + 2\sum_{i,j=1}^d \frac{\partial^2 N_i(x,\xi)}{\partial x_j\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon}$$
$$+\Delta_x u_0(x) - \frac{1}{|\square\cap\omega|}\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right) + \frac{1}{|\square\cap\omega|}Q(x)u_0(x). \tag{52}$$

Similarly, insertion of $w_\varepsilon(x,x/\varepsilon)$ in the boundary conditions on S_ε yields

$$\frac{\partial w_\varepsilon(x,x/\varepsilon)}{\partial n_\varepsilon} = -(\nabla_x u_\varepsilon(x), n_\varepsilon) + (\nabla_x u_0(x), n_\varepsilon) + \varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)$$

$$+(\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon) + \varepsilon(\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon) + (\nabla_\xi v_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)$$
$$= \varepsilon q(x, x/\varepsilon)u_\varepsilon(x) + (\nabla_x u_0(x), n_\varepsilon) + \varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)$$
$$+ \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\tilde{n}^j(x,\xi)\right)\bigg|_{\xi=x/\varepsilon} + \varepsilon(\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon}, n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon}$$
$$+\varepsilon(\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon) + \frac{\partial N(x,\xi)}{\partial \tilde{n}}\bigg|_{\xi=x/\varepsilon} + u_0(x)\frac{\partial N_0(x,\xi)}{\partial \tilde{n}}\bigg|_{\xi=x/\varepsilon}$$
$$+ \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial N_i(x,\xi)}{\partial \xi_j}\tilde{n}^j(x,\xi)\right)\bigg|_{\xi=x/\varepsilon} + \varepsilon(\nabla_\xi v_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon}.$$

On Γ_1^ε we obtain

$$\frac{\partial w_\varepsilon(x,x/\varepsilon)}{\partial \nu_\varepsilon} = -(\nabla_x u_\varepsilon(x), \nu_\varepsilon) + (\nabla_x u_0(x), \nu_\varepsilon) + \varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)$$
$$+(\nabla_\xi u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon) + \varepsilon(\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon) + (\nabla_\xi v_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)$$
$$= p(\hat{x}, \hat{x}/\varepsilon)u_\varepsilon(x) - g(\hat{x}, \hat{x}/\varepsilon) + (\nabla_x u_0(x), \nu_\varepsilon) + \varepsilon(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)$$
$$+\left\{ \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\nu_\varepsilon^j\right) + \varepsilon(\nabla_x v_1(x,\xi), \nu_\varepsilon)\right.$$
$$\left.+\frac{\partial N(x,\xi)}{\partial \nu_\varepsilon} + u_0(x)\frac{\partial N_0(x,\xi)}{\partial \nu_\varepsilon} + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial N_i(x,\xi)}{\partial \xi_j}\nu_\varepsilon^j\right)\right\}\bigg|_{\xi=x/\varepsilon}$$

Finally, on Γ_2 we have the trivial boundary condition

$$\frac{\partial w_\varepsilon(x,x/\varepsilon)}{\partial \mathcal{N}} = 0.$$

Multiplying (52) by $v(x)$ and integrating over Ω^ε, we now obtain

$$\int_{\Omega^\varepsilon} \Delta_x(w_\varepsilon(x,x/\varepsilon))v(x)\,dx = \varepsilon\int_{\Omega^\varepsilon} \Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon}v(x)\,dx$$
$$+\varepsilon\int_{\Omega^\varepsilon} \Delta_x v_1(x,\xi)|_{\xi=x/\varepsilon}v(x)\,dx + 2\int_{\Omega^\varepsilon}\sum_{i,j=1}^d \frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\bigg|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+2\int_{\Omega^\varepsilon}\sum_{i,j=1}^d \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon} v(x)\,dx + 2\int_{\Omega^\varepsilon}\sum_{i=1}^d \frac{\partial^2 N(x,\xi)}{\partial x_i \partial \xi_i}\bigg|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+2\int_{\Omega^\varepsilon}\sum_{i,j=1}^d \frac{\partial^2 N_i(x,\xi)}{\partial x_j \partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon} v(x)\,dx + 2\int_{\Omega^\varepsilon}\sum_{i=1}^d \frac{\partial^2 N_0(x,\xi)}{\partial x_i \partial \xi_i}u_0(x)\bigg|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+2\int_{\Omega^\varepsilon}\sum_{i,j=1}^d \frac{\partial N_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\bigg|_{\xi=x/\varepsilon} v(x)\,dx + 2\int_{\Omega^\varepsilon}\sum_{i=1}^d \frac{\partial N_0(x,\xi)}{\partial \xi_i}\frac{\partial u_0(x)}{\partial x_i}\bigg|_{\xi=x/\varepsilon} v(x)\,dx$$
$$+\int_{\Omega^\varepsilon}\Delta_x u_0(x)v(x)\,dx + \int_{\Pi_\varepsilon} f(x)v(x)\,dx - \frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon}\sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right)v(x)\,dx$$
$$+\frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon} Q(x)u_0(x)v(x)\,dx. \tag{53}$$

On the other hand, using the Green's formula, we find that

$$\int_{\Omega^\varepsilon} \Delta_x(w_\varepsilon(x,x/\varepsilon))v(x)\,dx = \int_{S_\varepsilon} \frac{\partial w_\varepsilon}{\partial n_\varepsilon} v(x)\,ds + \int_{\Gamma_1^\varepsilon} \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} v(x)\,ds + \int_{\Gamma_2} \frac{\partial w_\varepsilon}{\partial \mathcal{N}} v(x)\,ds$$

$$-\int_{\Omega^\varepsilon} \nabla w_\varepsilon \nabla v(x)\,dx = \varepsilon \int_{S_\varepsilon} q(x,x/\varepsilon)u_\varepsilon(x)v(x)\,ds + \int_{S_\varepsilon} \frac{\partial u_0(x)}{\partial n_\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{S_\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon} \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right) \bigg|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{S_\varepsilon} (\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi v_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon} \sum_{i=1}^d \left(\frac{\partial N(x,\xi)}{\partial \xi_i} \tilde{n}^i(x,\xi) \right) \bigg|_{\xi=x/\varepsilon} v(x)\,ds + \int_{S_\varepsilon} \sum_{i=1}^d u_0(x) \left(\frac{\partial N_0(x,\xi)}{\partial \xi_i} \tilde{n}^i(x,\xi) \right) \bigg|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon} \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial N_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right) \bigg|_{\xi=x/\varepsilon} v(x)\,ds + \int_{\Gamma_1^\varepsilon} p(\hat{x},\hat{x}/\varepsilon)u_\varepsilon(x)v(x)\,ds - \int_{\Gamma_1^\varepsilon} g(\hat{x},\hat{x}/\varepsilon)v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon} \frac{\partial u_0}{\partial \nu_\varepsilon} v(x)\,ds + \varepsilon \int_{\Gamma_1^\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon} \nu_\varepsilon)v(x)\,ds + \int_{\Gamma_1^\varepsilon} \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\bigg|_{\xi=x/\varepsilon} \nu_\varepsilon^j \right) v(x)\,ds$$

$$+\varepsilon \int_{\Gamma_1^\varepsilon} (\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, \nu_\varepsilon)v(x)\,ds + \int_{\Gamma_1^\varepsilon} \sum_{i=1}^d \left(\frac{\partial N(x,\xi)}{\partial \xi_i}\bigg|_{\xi=x/\varepsilon} \nu_\varepsilon^i \right) v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon} \sum_{i=1}^d u_0(x) \left(\frac{\partial N_0(x,\xi)}{\partial \xi_i}\bigg|_{\xi=x/\varepsilon} \nu_\varepsilon^i \right) v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon} \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial N_i(x,\xi)}{\partial \xi_j}\bigg|_{\xi=x/\varepsilon} \nu_\varepsilon^j \right) v(x)\,ds - \int_{\Omega^\varepsilon} \nabla w_\varepsilon(x,x/\varepsilon)\nabla v(x)\,dx. \tag{54}$$

It follows from (53) and (54) that

$$\int_{\Omega^\varepsilon} \nabla w_\varepsilon(x,x/\varepsilon)\nabla v(x)\,dx = \varepsilon \int_{S_\varepsilon} q(x,x/\varepsilon)u_\varepsilon(x)v(x)\,ds + \int_{S_\varepsilon} \frac{\partial u_0(x)}{\partial n_\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{S_\varepsilon} (\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon} \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right) \bigg|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\varepsilon \int_{S_\varepsilon} (\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon}, n_\varepsilon)v(x)\,ds + \varepsilon \int_{S_\varepsilon} (\nabla_\xi v_1(x,\xi), n'_\varepsilon(x,\xi))|_{\xi=x/\varepsilon} v(x)\,ds$$

$$+\int_{S_\varepsilon}\sum_{i=1}^{d}\left(\frac{\partial N(x,\xi)}{\partial \xi_i}\tilde n^i(x,\xi)\right)\Big|_{\xi=x/\varepsilon} v(x)\,ds+\int_{S_\varepsilon}\sum_{i=1}^{d}u_0(x)\left(\frac{\partial N_0(x,\xi)}{\partial \xi_i}\tilde n^i(x,\xi)\right)\Big|_{\xi=x/\varepsilon} v(x)\,ds$$

$$\cdot\int_{S_\varepsilon}\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial N_i(x,\xi)}{\partial \xi_j}\tilde n^j(x,\xi)\right)\Big|_{\xi=x/\varepsilon} v(x)\,ds+\int_{\Gamma_1^\varepsilon}p(\hat x,\hat x/\varepsilon)u_\varepsilon(x)v(x)\,ds-\int_{\Gamma_1^\varepsilon}g(\hat x,\hat x/\varepsilon)v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon}\frac{\partial u_0}{\partial \nu_\varepsilon}v(x)\,ds+\varepsilon\int_{\Gamma_1^\varepsilon}(\nabla_x u_1(x,\xi)|_{\xi=x/\varepsilon},\nu_\varepsilon)v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\Big|_{\xi=x/\varepsilon}\nu_\varepsilon^j\right)v(x)\,ds+\varepsilon\int_{\Gamma_1^\varepsilon}(\nabla_x v_1(x,\xi)|_{\xi=x/\varepsilon},\nu_\varepsilon)v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon}\sum_{i=1}^{d}\left(\frac{\partial N(x,\xi)}{\partial \xi_i}\Big|_{\xi=x/\varepsilon}\nu_\varepsilon^i\right)v(x)\,ds+\int_{\Gamma_1^\varepsilon}\sum_{i=1}^{d}u_0(x)\left(\frac{\partial N_0(x,\xi)}{\partial \xi_i}\Big|_{\xi=x/\varepsilon}\nu_\varepsilon^i\right)v(x)\,ds$$

$$+\int_{\Gamma_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial N_i(x,\xi)}{\partial \xi_j}\Big|_{\xi=x/\varepsilon}\nu_\varepsilon^j\right)v(x)\,ds$$

$$-\varepsilon\int_{\Omega^\varepsilon}\Delta_x u_1(x,\xi)|_{\xi=x/\varepsilon}v(x)\,dx-2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\Big|_{\xi=x/\varepsilon}v(x)\,dx$$

$$-2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial^2 M_i(x,\xi)}{\partial x_j\partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx-\varepsilon\int_{\Omega^\varepsilon}\Delta_x v_1(x,\xi)|_{\xi=x/\varepsilon}v(x)\,dx$$

$$-2\int_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial^2 N(x,\xi)}{\partial x_i\partial \xi_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx-2\int_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial N_0(x,\xi)}{\partial \xi_i}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx$$

$$-2\int_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial^2 N_0(x,\xi)}{\partial x_i\partial \xi_i}u_0(x)\Big|_{\xi=x/\varepsilon}v(x)\,dx-2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial N_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\Big|_{\xi=x/\varepsilon}v(x)\,dx$$

$$-2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial^2 N_i(x,\xi)}{\partial x_j\partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx-\int_{\Omega^\varepsilon}\Delta_x u_0(x)v(x)\,dx$$

$$+\frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)v(x)\,dx$$

$$-\frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon}Q(x)u_0(x)v(x)\,dx-\int_{\Pi_\varepsilon}f(x)v(x)\,dx. \qquad (55)$$

Recalling (42), the boundary conditions in (15), and (47)–(49), from the last relation we derive the estimate

$$\Big|\int_{\Omega^\varepsilon}\nabla w_\varepsilon(x,x/\varepsilon)\nabla v(x)\,dx+\varepsilon\int_{S_\varepsilon}q(x,x/\varepsilon)w_\varepsilon(x,x/\varepsilon)v(x)\,ds+\int_{\Gamma_1^\varepsilon}p(\hat x,\hat x/\varepsilon)w_\varepsilon(x,x/\varepsilon)v(x)\,ds\Big|$$

$$
\leq \varepsilon \Bigg| \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) u_1(x, x/\varepsilon) v(x)\, ds + \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) v_1(x, x/\varepsilon) v(x)\, ds
$$

$$
+ \int\limits_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) u_1(x, x/\varepsilon) v(x)\, ds + \int\limits_{\Gamma_1^\varepsilon} p(\hat{x}, \hat{x}/\varepsilon) v_1(x, x/\varepsilon) v(x)\, ds \Bigg|
$$

$$
+ \Bigg| \varepsilon \int\limits_{S_\varepsilon} q(x, x/\varepsilon) u_0(x) v(x)\, ds - \frac{1}{|\Box \cap \omega|} \int\limits_{\Omega'_\varepsilon} Q(x) u_0(x) v(x)\, dx \Bigg|
$$

$$
+ \Bigg| \varepsilon \int\limits_{S_\varepsilon} (\nabla_x v_1(x,\xi)|_{\xi = x/\varepsilon}, n_\varepsilon) v(x)\, ds - \varepsilon \int\limits_{\Omega^\varepsilon} \Delta_x u_1(x,\xi)|_{\xi = x/\varepsilon} v(x)\, dx
$$

$$
+ \varepsilon \int\limits_{\Gamma_2} (\nabla_x v_1(x,\xi)|_{\xi = x/\varepsilon}, \mathcal{N}) v(x)\, ds + \int\limits_{\Gamma_2} (\nabla_\xi v_1(x,\xi)|_{\xi = x/\varepsilon}, \mathcal{N}) v(x)\, ds
$$

$$
+ \varepsilon \int\limits_{\Gamma_1^\varepsilon} (\nabla_x v_1(x,\xi)|_{\xi = x/\varepsilon}, \nu_\varepsilon) v(x)\, ds - \varepsilon \int\limits_{\Omega^\varepsilon} \Delta_x v_1(x,\xi)|_{\xi = x/\varepsilon} v(x)\, dx \Bigg|
$$

$$
+ \Bigg| \int\limits_{S_\varepsilon} \left(\frac{\partial u_0(x)}{\partial n_\varepsilon} + \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \tilde{n}^j(x,\xi) \right) \Bigg|_{\xi = x/\varepsilon} \right) v(x)\, ds \Bigg|
$$

$$
+ \Bigg| \int\limits_{\Gamma_1^\varepsilon} \frac{P(\hat{x})}{\sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2}} \Bigg|_{\xi = x/\varepsilon} u_0(x) v(x)\, ds - \int\limits_{\Gamma_1} P(\hat{x}) u_0(x) v(x)\, d\hat{x}
$$

$$
+ \int\limits_{\Gamma_1} G(\hat{x}) v(x)\, d\hat{x} - \int\limits_{\Gamma_1^\varepsilon} \frac{G(\hat{x})}{\sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2}} \Bigg|_{\xi = x/\varepsilon} v(x)\, ds
$$

$$
+ \int\limits_{\Gamma_1^\varepsilon} \left(\frac{\partial u_0(x)}{\partial \nu_\varepsilon} + \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i} \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} \Bigg|_{\xi = x/\varepsilon} \nu_\varepsilon^j \right) \right) v(x)\, ds
$$

$$
+ \int\limits_{\Gamma_1^\varepsilon} \sum_{i=1}^{d} \frac{\partial u_0(x)}{\partial x_i} \left[\frac{\langle \delta_{id} + \frac{\partial M_i(x,\xi)}{\partial \xi_d} \rangle}{\sqrt{1 + (\nabla_\xi F(\hat{x}, \hat{\xi}))^2}} - \nu_\varepsilon^i - \frac{\partial M_i(x,\xi)}{\partial \nu_\varepsilon} \right] \Bigg|_{\xi = x/\varepsilon} v(x)\, ds
$$

$$
- \int\limits_{\Gamma_1} \sum_{i=1}^{d} \left\langle \delta_{id} + \frac{\partial M_i(x,\xi)}{\partial \xi_d} \right\rangle \frac{\partial u_0(x)}{\partial x_i} v(x)\, d\hat{x} \Bigg| \Bigg| \int\limits_{\Pi_\varepsilon} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(\left[\delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right] \frac{\partial u_0(x)}{\partial x_i} \right) \Bigg|_{\xi = x/\varepsilon} v(x)\, dx
$$

$$
+ 2 \int\limits_{\Pi_\varepsilon} \sum_{i,j=1}^{d} \frac{\partial^2 N_i(x,\xi)}{\partial x_j \partial \xi_j} \frac{\partial u_0(x)}{\partial x_i} \Bigg|_{\xi = x/\varepsilon} v(x) dx + 2 \int\limits_{\Pi_\varepsilon} \sum_{i,j=1}^{d} \frac{\partial N_i(x,\xi)}{\partial \xi_j} \Bigg|_{\xi = x/\varepsilon} \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} v(x) dx \Bigg|
$$

$$
+ \Bigg| \int\limits_{\Omega'_\varepsilon} \sum_{i,j=1}^{d} \left(\frac{1}{|\Box \cap \omega|} \frac{\partial}{\partial x_j} \left[\left\langle \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} \right] \right.
$$

$$
\left. - \frac{\partial}{\partial x_j} \left[\frac{\partial M_i(x,\xi)}{\partial \xi_j} \frac{\partial u_0(x)}{\partial x_i} \right] \right) \Bigg|_{\xi = x/\varepsilon} v(x)\, dx - \varepsilon \int\limits_{S_\varepsilon} (\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi)) \Bigg|_{\xi = x/\varepsilon} v(x)\, ds \Bigg|
$$

$$+\left|\int\limits_{\Omega\setminus\Omega'}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left[\left\langle\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle-\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right]\frac{\partial u_0(x)}{\partial x_i}\right)\Big|_{\xi=x/\varepsilon}v(x)\,dx\right.$$

$$\left.+\int\limits_{\Omega\setminus\Omega'}Q(x)u_0(x)v(x)\,dx\right|+\left|\int\limits_{\Pi_\varepsilon}f(x)v(x)\,dx\right|$$

$$+\left|2\int\limits_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial^2 N(x,\xi)}{\partial x_i\partial\xi_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx+2\int\limits_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial^2 N_0(x,\xi)}{\partial x_i\partial\xi_i}u_0(x)\Big|_{\xi=x/\varepsilon}v(x)\,dx\right.$$

$$+2\int\limits_{\Omega^\varepsilon}\sum_{i=1}^{d}\frac{\partial N_0(x,\xi)}{\partial\xi_i}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx+2\int\limits_{\Omega_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial^2 N_i(x,\xi)}{\partial x_j\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=x/\varepsilon}v(x)\,dx$$

$$\left.+2\int\limits_{\Omega_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial N_i(x,\xi)}{\partial\xi_j}\Big|_{\xi=x/\varepsilon}\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}v(x)\,dx\right|+O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}$$

$$=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9+I_{10}+O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}.$$

Since $Q(x)\equiv 0$ in $\Omega\setminus\Omega'$ and $M_i(x,\xi)\equiv 0$ for $x\in\Omega\setminus\Omega'$, we have $I_8=0$. We now estimate the term I_2 on the right-hand side. By Lemma 7, we have

$$I_2=\left|\varepsilon\int\limits_{S_\varepsilon}q(x,x/\varepsilon)u_0(x)v(x)\,ds-\frac{1}{|\square\cap\omega|}\int\limits_{\Omega'_\varepsilon}Q(x)u_0(x)v(x)\,dx\right|\le C_{18}\varepsilon\|u_0\|_{H^1(\Omega^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)}.$$

Clearly, the terms I_1 and I_3 can be estimated as follows:

$$|I_1|+|I_3|\le C_{19}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

The identity $I_4\equiv 0$ ensues from the boundary conditions of the problem (6). Lemma 8 enables us to assume that the function $f(x)$ vanishes in the layer Π_ε. Consequently, $I_9\equiv 0$. The integral I_7 has been already estimated in § 4:

$$|I_7|\le C_{20}\varepsilon\left\|\frac{\partial^2 u_0(x)}{\partial x_i\partial x_j}\right\|_{H^1(\Omega_1^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)}.$$

Recalling the boundary condition of the problem (15) and Lemma 5, we infer that

$$|I_5|=\left|\int\limits_{\Gamma_1^\varepsilon}\frac{\left(\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\rangle+P(\hat{x})u_0(x)-G(\hat{x})\right)}{\sqrt{1+(\nabla_\xi F(\hat{x},\hat{\xi}))^2}}\Big|_{\xi=x/\varepsilon}v(x)\,ds\right.$$

$$\left.-\int\limits_{\Gamma_1}\left(\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left\langle\delta_{id}+\frac{\partial M_i(x,\xi)}{\partial\xi_d}\right\rangle+P(\hat{x})u_0(x)-G(\hat{x})\right)v(x)\,d\hat{x}\right|\le C_{21}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

Similarly, using the technique of the proof of Lemma 8, we obtain

$$I_6\le C_{22}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

Finally, owing to the exponential decay of $N_0(x,\xi)$ and $N_i(x,\xi)$ as $\xi_d\to+\infty$, we conclude that

$$I_{10}\le C_{23}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

Letting $v=u_0+\varepsilon u_1+\varepsilon v_1-u_\varepsilon$ and recalling all above estimates and Lemma 4, we arrive at (50). The theorem is proven.

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