SIAM J. MATH. ANAL.
Vol. 42, No. 6, pp. 2626–2651

AVERAGING OF A SINGULAR RANDOM SOURCE TERM IN A DIFFUSION CONVECTION EQUATION*

ALAIN BOURGEAT† AND ANDREY L. PIATNITSKI‡

Abstract. We consider a simplified model for the radionuclides migration in an underground nuclear waste repository, based on a linear partial differential equation of diffusion convection type. This partial differential equation has a source term constituted by a large number of "local" sources spatially periodically distributed and lying on the porous domain median plane. The behavior of each source is spatially homogeneous but their time dependence is uncertain; their release curve (source emission versus space and time) parameters are random both in space and in time. Starting from the mesoscopic model described above, our aim is then to obtain by "upscaling" a model with a deterministic "averaged" source term, describing the global evolution of such a system, and to prove the convergence, estimate (under proper mixing assumptions) the rate of convergence, and characterize the asymptotic behavior of the corrector.

Key words. convection-diffusion, stochastic homogenization, random source term

AMS subject classifications. 35B27, 35K20, 35Q35, 35R60, 60H15, 60H30, 76M50

DOI. 10.1137/080736077

Introduction. Our interest in studying the transport migration of a contaminant in aquifers from a "source site" made of a large number of randomly behaving "local" sources comes initially from the problem of assessing the performance of a long-lived underground nuclear waste repository. An underground waste repository site is made of a large number of sealed packages containing diverse materials contaminated by some radionuclides. Each repository zone is made of a large number of similar storage vaults, each vault containing a group of sealed packages. It is clearly impossible to have an exhaustive description of all the complicated phenomena concerning possible radionuclides leaking into the host geological media. There is both epistemic and aleatoric uncertainty in modeling the complexity of the involved physical phenomena (container corrosion, breaking and alteration of glass packages, the hydrological and mechanical properties of geological or artificial containment materials, and variability of the containers' contents). The source emission versus space and time graph (called the release curve) parameters associated to each package (or local source) should then be considered as random in space and in time.

In order to study the consequence of these uncertainties in the leaking of the packages, we consider a simplified but typical mesoscopic or "local" modeling of the radionuclides migration in a porous media (one-phase saturated). The corresponding equation is a linear diffusion-convection equation, with a high number of parallelepiped-shaped sources, each source with a space and time random behavior

*Received by the editors September 22, 2008; accepted for publication (in revised form) August 3, 2010; published electronically October 21, 2010. This work was partially supported by a grant from the ANDRA/BRGM/CEA/EDF/IRSN CNRS GNR 2439 MoMaS.
http://www.siam.org/journals/sima/42-6/73607.html
†Université de Lyon, Université Lyon1, CNRS, UMR 5208, ICJ, 43, blvd du 11 novembre 1918, F–69200 Villeurbanne Cedex, France (bourgeat@mcs.univ-lyon1.fr). A part of this work was done during this author's visit to Narvik University College.
‡Narvik University College, Postboks 385, 8505 Narvik, Norway, and P.N. Lebedev Physical Institute of RAS, Leninski pr., 53, Moscow 117924, Russia (andrey@sci.lebedev.ru).

(release curve):

$$\partial_t u^\varepsilon - \mathrm{div}(a(x)\nabla u^\varepsilon) + \mathrm{div}(b(x)u^\varepsilon) = f^\varepsilon \quad \text{in } Q\times(0,+\infty); \tag{1}$$

$$u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^\varepsilon - b(x)\cdot n(x)u^\varepsilon + \lambda u^\varepsilon = 0 \quad \text{on } \partial Q, \tag{2}$$

where $\frac{\partial}{\partial n_a}u^\varepsilon = a(x)\nabla u^\varepsilon(x,t)\cdot n(x)$, $n(x)$ is the external normal to ∂Q, and the source density $f^\varepsilon(x,t)$ is the sum of the "local source" densities $f^\varepsilon_{\mathbf{j}}(x,t)$, such that all of these local sources are supported in small parallelepipeds situated periodically along a horizontal plane Σ (see Figure 2); i.e., $f^\varepsilon(x,t) = \sum_{\mathbf{j}\in\varepsilon\mathbb{Z}^2} f^\varepsilon_{\mathbf{j}}(x,t)$. The domain Q comprising the source site is bounded and regular. Moreover, assuming the release curve, $f^\varepsilon_{\mathbf{j}}(\cdot,\cdot)$, of each local source is random, our aim is to construct a mathematical model which characterizes the global evolution of such a system.

According to the above discussion, the form of the equation and the boundary condition in (1)–(2) have been chosen as a simplified but typical model of the radionuclides migration in porous media, as it appears in the context of modeling the long-term behavior of a nuclear waste repository.

The boundary condition imposed in (2) is commonly used in engineering for modeling physical processes that involve diffusive (or molecular) and convective transport within physical systems; see, for instance, [5]. This condition, used in mass transfer, is analogous to the Newton law used in thermal transfer, and states that on the boundary of the domain the concentration obeys Fick's law, i.e., that the flux is proportional to the difference between concentrations inside and outside the domain; λ is then the mass transfer coefficient. Here, for the sake of simplicity, the outside concentration is assumed to be zero.

It should be noted that the approach used in this work applies to any general well-posed initial-boundary problem with a parabolic operator of the form

$$\partial_t u^\varepsilon - \mathrm{div}(a(x)\nabla u^\varepsilon) + b(x)\cdot\nabla u^\varepsilon + c(x)u^\varepsilon = f^\varepsilon \tag{3}$$

with sufficiently regular coefficients. All the results of this work on the limit behavior of solutions on finite time intervals remain valid and can be proved in exactly the same way as in the case of problem (1)–(2). In particular, the boundary conditions in (2) can be replaced with Dirichlet, Neumann, or Fourier boundary conditions.

If the elliptic operator related to (3) with the corresponding boundary condition is dissipative, then the statements on the convergence on the infinite time interval also hold true.

The deterministic case, i.e., the worst possible scenario where all the packages start leaking at the same time with a same deterministic "release curve," was studied in previous papers, with different types of periodic microstructure and different convection regimes; see, for instance, [7], [6], and [8].

In section 1, we define the geometry of the source site and specify the assumptions on the random local sources. In this first section, we study a general "source site" model assuming only statistical homogeneity and ergodicity of the random field $x \mapsto f^\varepsilon(x,t)$ in spatial variables x, and the exponential decay of $f^\varepsilon(x,t)$ in temporal variable t. Then in section 2 we derive an effective (or upscaled) model which characterizes the effective concentration u^0, and we prove the convergence result. The statements of this section are quite simple, but we will provide proofs for presentation completeness. In section 3, assuming additional mixing properties, we estimate the rate of convergence of $(u^\varepsilon - u^0)$ to zero. Namely, we show that if the correlation function or strong mixing coefficient of the random field f^ε decays sufficiently fast, then the expectation of the square of L^2 norm of the discrepancy admits the estimate of order ε^2; see Theorems 2–5 for detailed statements.

In section 4 we study the asymptotic behavior of the corrector $u^1 = (u^\varepsilon - u^0)/\varepsilon$. We show that, under appropriate mixing assumptions on f^ε in spatial variables, the function u^1 converges in law, at any fixed point (x, t), towards a centered Gaussian random variable. Then we prove a similar result for any finite dimensional distribution of the random field $u^1(x, t)$; thus $u^1(x, t)$ weakly converges in law to a Gaussian random field. This is the subject of Theorem 8.

We would like to stress that the results of sections 2–5 also hold for more general source terms concentrated in the vicinity of a surface and possessing statistical homogeneity and good mixing properties. Our analysis does not exploit the specific geometry of the source term assumed in the paper. See section 4 for further discussion.

Similar results are valid for the volume-distributed source terms, but in this case they are not optimal and can be improved.

To the best of our knowledge the equations with oscillating random source terms situated in a small neighborhood of a hypersurface have not been studied in the existing literature. These source terms will naturally appear when, for example, studying various interface problems.

The model equation studied here is derived from the modeling of contaminant transport and migration in aquifers from a nuclear waste underground repository. In this case the above "local model" may be used for all the upscaling stages: from the set of containers to a vault, or from the set of vaults to a repository zone, or finally from the set of repository zones to the entire waste site.

Although this is not the case for the radionuclides migration in an underground nuclear waste repository, we consider in the last section a possible extension of the above results to the case of operators with rapidly oscillating coefficients. We deal with two classical models, namely, operators with random statistically homogeneous coefficients and operators with locally periodic coefficients.

In the random case, obtaining the optimal estimates for the rate of convergence (for a fixed right-hand side) is a well-known challenge. Important progress in this direction has been achieved in [16], where, under certain mixing conditions, it has been shown that the discrepancy admits a polynomial estimate; however, this estimate does not seem sharp. In any case we cannot expect an estimate better than $\sqrt{\varepsilon}$ for the L^2 norm of the discrepancy; thus, the results of sections 3 and 4 cannot hold in this case, and we may only formulate the convergence theorem without specifying the rate of convergence.

To the contrary, in the locally periodic case results identical to those in sections 3 and 4 can be obtained. Actually, in the locally periodic case, in order to avoid the boundary effects, we should assume that the oscillations disappear in the vicinity of the exterior boundary of the domain. Under this same assumption, both Green's function of the original problem and its solution admit a good approximation in terms of Green's function (respectively, a solution) of the effective problem (see [4], [3] for the asymptotic expansion techniques). Therefore, in the locally periodic case, the desired results can be achieved by combining the classical homogenization theorems for Green's functions and our approach in sections 3 and 4.

Notice that in the studied problem the source term has an asymptotically singular structure and converges weakly to a surface measure with time-dependent density. The L^2 norm of the source term tends to infinity as $\varepsilon \to 0$. Due to this, the energy

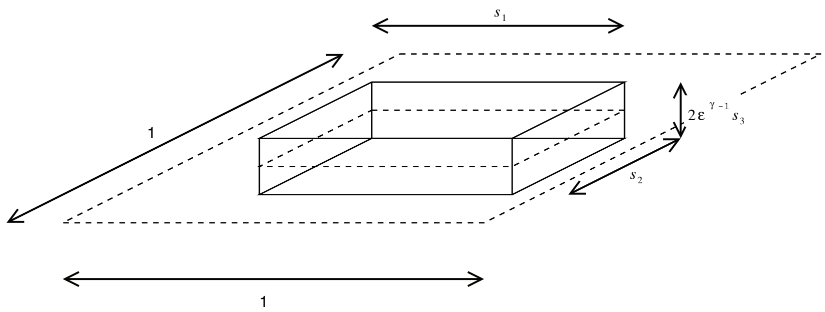

FIG. 1. *Each local source support, in the local variables $y = \frac{x}{\varepsilon}$, is obtained by successive translation of the three-dimensional reference set $K_\varepsilon = [\frac{1-s_1}{2}, \frac{1+s_1}{2}] \times [\frac{1-s_2}{2}, \frac{1+s_2}{2}] \times [-\varepsilon^{\gamma-1}s_3, \varepsilon^{\gamma-1}s_3]$.*

estimates technique fails to provide sharp estimates for the rate of convergence. Instead, we use the representation of a solution of a parabolic problem in terms of the corresponding Green's function and study the limit behavior of this representation, as $\varepsilon \to 0$, exploiting various bounds and asymptotics for Green's function. In particular, our analysis relies on Aronson-type estimates for Green's function. We also need upper bounds for the derivatives of Green's function. These bounds are obtained by combining the Aronson-type and parabolic estimates.

In the case of rapidly oscillating coefficients we also use the results on convergence of Green's function proved by the classical methods of homogenization theory.

1. Definition of the problem.

1.1. Description of the local sources' geometry. We first define a smooth bounded domain $Q \subset \mathbb{R}^3$, with $\mathrm{diam}(Q) = R < \infty$, and such that $Q^+ = \{x \in Q : x_3 > 0\}$ and $Q^- = \{x \in Q : x_3 < 0\}$ are nonempty Lipschitz domains situated on each side of the middle plane $\Sigma = \{x \in Q : x_3 = 0\}$.

Then we describe the geometry of the sources' supports inside this domain. To this end we first denote by ε a small positive number (measuring the typical nondimensionalized length of a source support or the source support's period); then we define, in the rescaled variables $y = x/\varepsilon$, a two- and a three-dimemsional nondimensionalized reference set (see Figure 1):

$$\begin{aligned} K^0 &= \Big[\frac{1-s_1}{2}, \frac{1+s_1}{2}\Big] \times \Big[\frac{1-s_2}{2}, \frac{1+s_2}{2}\Big]; \\ K_\varepsilon &= K^0 \times \varepsilon^{\gamma-1}[-s_3, s_3]; \quad K^\Sigma = \bigcup_{\mathbf{j}\in\mathbb{Z}^2} (K^0 + \mathbf{j}), \end{aligned} \tag{4}$$

with $0 < s_1, s_2, s_3 < 1$ and $\gamma \geq 1$. In the original coordinates x, these reference sets take the form

$$\tilde{B}^0_\varepsilon = \varepsilon K^0 = \Big[\varepsilon\frac{1-s_1}{2}, \varepsilon\frac{1+s_1}{2}\Big] \times \Big[\varepsilon\frac{1-s_2}{2}, \varepsilon\frac{1+s_2}{2}\Big]; \qquad B^0_\varepsilon = \varepsilon K_\varepsilon = \tilde{B}^0_\varepsilon \times \varepsilon^\gamma[-s_3, s_3].$$

We also introduce the sets

$$\tilde{B}^{\mathbf{j}}_\varepsilon = \tilde{B}^0_\varepsilon + \varepsilon\mathbf{j}, \qquad B^{\mathbf{j}}_\varepsilon = B^0_\varepsilon + \varepsilon(\mathbf{j}, 0), \quad \mathbf{j} \in \mathbb{Z}^2,$$

obtained by pertinent translations of the reference sets in (4) with vector $\varepsilon\mathbf{j}$, $\mathbf{j} \in \mathbb{Z}^2$.

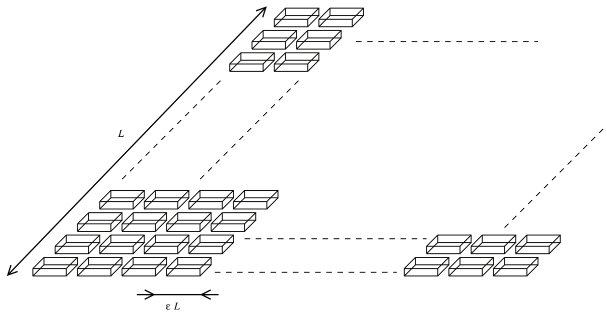

FIG. 2. *The entire three-dimensional "source site" support B_ε is defined by repeating periodically a single three-dimensional source support, $\varepsilon K_\varepsilon$.*

Let $\Pi \subset \Sigma$ be a closed rectangle, $\Pi = [-\beta_1, \beta_1] \times [-\beta_2, \beta_2]$, such that $\Pi \subset Q$. Without loss of generality we may suppose that the origin belongs to Q and that the source supports are located in the vicinity of the rectangle Π. We then define

$$\tilde{B}_\varepsilon = \bigcup_{\mathbf{j} \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi} \tilde{B}_\varepsilon^{\mathbf{j}}, \qquad B_\varepsilon = \bigcup_{\mathbf{j} \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi} B_\varepsilon^{\mathbf{j}}. \tag{5}$$

We assume that B_ε is the support of the source density f^ε in (1), and $\tilde{B}_\varepsilon$ is its projection on Σ (see Figure 2).

In what follows, we will use the notation

$$x' = (x_1, x_2) \in \Sigma;\ x = (x', x_3),\ y = x/\varepsilon, \tag{6}$$

$$\mathbf{z} = [x'/\varepsilon] = ([x_1/\varepsilon], [x_2/\varepsilon]),\ \mathbf{x}' = [x'], \tag{7}$$

with $[\cdot]$ standing for the integer part.

1.2. Description of the random source term. In this subsection we describe the assumptions made on the stochastic properties of the source term f^ε in (1).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space with two-dimensional discrete ergodic dynamical system $T_\mathbf{z}$, $\mathbf{z} \in \mathbb{Z}^2$. Let us recall that $T_\mathbf{z}$ is a collection of measurable maps, $T_\mathbf{z} : \Omega \to \Omega$, $\mathbf{z} \in \mathbb{Z}^2$, which

- preserves the measure $\mathbf{P}$ for all $\mathbf{z} \in \mathbb{Z}^2$;
- possesses the group properties $T_{\mathbf{z}+\mathbf{y}} = T_\mathbf{z} \circ T_\mathbf{y}$ for all $\mathbf{z}, \mathbf{y}$, $T_0 = \mathrm{Id}$;
- is ergodic, i.e., the relation $\mathbf{P}(\mathcal{A})(1 - \mathbf{P}(\mathcal{A})) = 0$ holds for any invariant set $\mathcal{A} \in \mathcal{F}$.

Then, given a random field $\psi(\omega, t)$, $t \in [0, +\infty)$, such that

$$\|\psi(\cdot, t)\|_{L^\infty(\Omega)} \le C \exp(-\Lambda t), \tag{8}$$

we define the random functions $\Phi(x', t)$ and $\phi^\varepsilon(x', t)$ as follows:

$$\Phi(x', t) = \psi(T_{\mathbf{x}'}\omega, t) = \psi(T_{[x']}\omega, t); \quad \phi^\varepsilon(x, t) = \psi(T_\mathbf{z}\omega, t) = \psi(T_{[x'/\varepsilon]}\omega, t); \tag{9}$$

here and in what follows $[x']$ stands for the integer vector $\mathbf{x}'$ with $\mathbf{x}_k$, $k = 1, 2$, being the largest integer which does not exceed x_k. By construction,

$$\phi^\varepsilon(x', t) = \Phi(x'/\varepsilon, t).$$

The source term on the right-hand side of (1) (see also (12) below) is now defined as

$$f^\varepsilon(x,t) = \frac{1}{\varepsilon^\gamma}\mathbb{1}_{B_\varepsilon}(x)\phi^\varepsilon(x',t); \tag{10}$$

and, by construction, the function $x' \mapsto \phi^\varepsilon(x',t)$ is statistically homogeneous and ergodic with respect to the discrete group $T_{[x'/\varepsilon]}$. We also introduce the rescaled source function

$$f(x,t) = \frac{1}{\varepsilon^\gamma}\mathbb{1}_{\epsilon^{-1}B_\varepsilon}(x)\Phi(x',t).$$

Thanks to (8) the source density $f^\varepsilon(x,t)$ satisfies the bound

$$f^\varepsilon(x,t) \le \frac{1}{\varepsilon^\gamma}C\exp(-\Lambda t), \tag{11}$$

with nonrandom constants $\Lambda > 0$ and $C > 0$. It is also clear that

$$\operatorname{supp}(f^\varepsilon) \subset \Sigma^\varepsilon \stackrel{\text{def}}{=} \Sigma \times [-\varepsilon^\gamma, \varepsilon^\gamma].$$

Remark 1. In the context of a geological nuclear waste repository, the general random behavior of the sources, as considered above, could be adapted to the different upscaling stages (from the set of packages to a vault, or from the set of vaults to a repository zone) by adapting the pertinent assumptions on the randomness to each situation.

For example, if the local sources, described by the generic source term Φ in (8), (9) (see also Remark 11), are the altered glass packages of long-lived nuclear waste, then a typical release curve (in mol/year) of a local source is

$$\Phi(.,t) = F_0 e^{-\lambda t} Q_{VA}(t),$$

with the altered glass quantity $Q_{VA}(t)$ given by

$$Q_{VA}(t) = \begin{cases} \alpha(\omega) & \text{if } t_0 < t \le t_0 + M_1/\alpha, \\ \beta(\omega) & \text{if } t_0 + M_1/\alpha < t \le t_0 + M_1/\alpha + M_2/\beta, \\ 0 & \text{otherwise,} \end{cases}$$

where F_0 is the radioactive element initial fraction in the package in mol/gram, and the three parameters $\alpha(\omega), \beta(\omega)$, and $t_0 = t_0(\omega)$ are random.

Remark 2. The results of numerical simulations of a set of altered glass packages in a long-lived nuclear waste repository were presented in [9]. They show total agreement with the theoretical results obtained herein.

1.3. Original mesoscopic equations. Under the definitions and assumptions of sections 1.1 and 1.2 we consider the initial-boundary problem

$$\partial_t u^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) + \operatorname{div}(b(x)u^\varepsilon) = f^\varepsilon \quad \text{in } Q \times (0,\infty); \tag{12}$$

$$u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^\varepsilon - b(x)\cdot n(x)u^\varepsilon + \lambda u^\varepsilon = 0 \quad \text{on } \partial Q \times (0,\infty), \tag{13}$$

where $a(x)$ (the diffusion tensor) is a uniformly positive definite smooth matrix function, $b(x)$ (the convection velocity) is a smooth vector field, and where n_a and n are the external conormal and normal, respectively. We also assume that $\lambda > 0$ so that the studied model is dissipative.

A classical result says (see, for instance, [13, Chapter III, sections 4 and 5]) that for each $\varepsilon > 0$ and each $\omega \in \Omega$, problem (12)–(13) has a unique solution $u^\varepsilon \in L^2_{\mathrm{loc}}(0,+\infty; H^1(Q)) \cap C(0,+\infty; L^2(Q))$.

2. Limit averaged equations; convergence. Our first aim is to show that the limit problem takes the form

$$\partial_t u^0 - \operatorname{div}(a(x)\nabla u^0) + \operatorname{div}(b(x)u^0) = \mathbb{1}_\Pi(x')F(t)\delta_\Sigma(x) \quad \text{in } Q\times(0,\infty); \tag{14}$$

$$u^0\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^0 - b\cdot n u^0 + \lambda u^0 = 0 \quad \text{on } \partial Q\times(0,\infty), \tag{15}$$

where $F(t)$ is the expectation

$$F(t) = 2s_1s_2s_3\mathbf{E}\{\Phi(x',t)\}, \tag{16}$$

which does not depend on x' due to statistical homogeneity of the random field Φ in variable x'; $\delta_\Sigma(x) = dx_1dx_2\delta(x_3)$ is the surface Lebesgue measure with support Σ, and $\mathbb{1}_\Pi(x')$ is the characteristic function of the rectangle Π.

In what follows, for brevity we will use the notation

$$F_\Pi(x,t) = F(t)\mathbb{1}_\Pi(x).$$

Since the surface Lebesgue measure $\mathbb{1}_\Pi(x')\delta_\Sigma(x)$ is an element of $W^{-1,\infty}(Q)$, then the function $\mathbb{1}_\Pi(x')F(t)\delta_\Sigma(x)$ belongs to the space $L^\infty(0,T;W^{-1,\infty}(Q))$ for any $T>0$. Thus, according to [13, Chapter III, section 10], the solution u^0 is an element of $L^2(0,T;H^1(Q))\cap C^\alpha((0,T)\times\overline{Q})$ for some $\alpha\in(0,1)$; here C^α stands for the space of Hölder continuous functions. Considering the smoothness of the coefficients $a(x)$ and $b(x)$ and of the boundary ∂Q, we also derive from the local parabolic estimates (see [13, Chapter III, section 12]) that $u^0(x,t)$ is a smooth function on any compact subset of $Q\times(0,T)$ that does not intersect the set $\{(x,t) : x'\in\Pi,\ x_3=0\}$.

By construction, the function $x'\mapsto f^\varepsilon(x',x_3,t)$ defined in (10) is statistically homogeneous and ergodic in x'. In the rest of this section, with no additional assumption on the function f^ε, except for those already formulated in section 1.2, we prove that u^ε converges almost surely (a.s.) to u^0, as $\varepsilon\to0$, in the $L^2(0,T;H^1(Q))$ norm. To this end we will introduce an auxiliary two-dimensional source term with support on $\Sigma\bigcap\tilde{B}_\varepsilon$ (see (5)):

$$F^\varepsilon(x,t) = 2s_3\mathbb{1}_{\tilde{B}_\varepsilon}(x)\phi^\varepsilon(x',t)\delta_\Sigma(x). \tag{17}$$

We start with two preliminary statements.

LEMMA 1. *Under assumption* (8), *for any fixed* t *the following bound holds:*

$$\|f^\varepsilon(\cdot,t) - F^\varepsilon(\cdot,t)\|_{H^{-1}(Q)} \le C\varepsilon^{\gamma/2}e^{-\Lambda t}$$

with a nonrandom constant C.

Proof. Letting

$$g^\varepsilon(x,t) = \begin{cases} +\displaystyle\int_{-1}^{x_3} f^\varepsilon(x',y,t)dy, & x_3\le 0,\\ -\displaystyle\int_{x_3}^{+1} f^\varepsilon(x',y,t)dy, & x_3>0,\end{cases}$$

we have

$$\frac{\partial}{\partial x_3}g^\varepsilon(x,t) = f^\varepsilon(x,t) - F^\varepsilon(x,t) \qquad\text{and}\qquad |g^\varepsilon(x,t)|\le Ce^{-\Lambda t}.$$

It is also clear that $\mathrm{supp}(g^\varepsilon) \subset B_\varepsilon$. Therefore, for any $\varphi \in C_0^\infty(Q)$,

$$\begin{aligned}
&\Big| \int_Q (f^\varepsilon(x,t) - F^\varepsilon(x,t))\varphi(x)dx \Big| = \Big| \int_Q \frac{\partial}{\partial x_3} g^\varepsilon(x,t)\varphi(x)dx \Big| \\
&\le \Big| \int_Q g^\varepsilon(x,t)\frac{\partial}{\partial x_3}\varphi(x)dx \Big| \le \left(\int_{B_\varepsilon} (g^\varepsilon(x,t))^2 dx \right)^{1/2} \left(\int_Q |\nabla\varphi|^2 dx \right)^{1/2} \\
&\le C\varepsilon^{\gamma/2} \|\varphi\|_{H_0^1(Q)}\, e^{-\Lambda t},
\end{aligned}$$

which implies the statement. □

Moreover, by Birkhoff's ergodic theorem the function $(F^\varepsilon(x,t) - F_\Pi(x,t)\delta_\Sigma(x))$ a.s. converges to 0 weakly in $L^2(\Sigma)$, as $\varepsilon \to 0$, for all $t > 0$. As a consequence, we obtain the following result.

LEMMA 2. *Under the ergodicity assumption on dynamical system T_z (see section 1.2), and under assumption (8), for any $T > 0$ the following limit relation holds true a.s.:*

$$\lim_{\varepsilon\to 0} \|F^\varepsilon - F_\Pi\delta_\Sigma\|_{L^2(0,T;H^{-1}(Q))} = 0. \tag{18}$$

Proof. For any $\varphi \in L^2(0,T;H_0^1(Q))$, denoting by $\langle f, \varphi\rangle$ the duality between $H_0^1(Q)$ and its dual, we have

$$\begin{aligned}
&\Big| \int_0^T \big\langle F^\varepsilon(x,t) - F_\Pi(x,t)\delta_\Sigma(x), \varphi \big\rangle \, dt \Big| \\
&= \Big| \int_0^T \int_\Sigma (F^\varepsilon(x',0,t) - F_\Pi(x',0,t))\varphi(x',0,t)\, dx' dt \Big| \\
&\le \int_0^T \|\varphi(\cdot,t)\|_{H^{1/2}(\Sigma)} \|(F^\varepsilon(\cdot,t) - \mathbb{1}_\Pi F(t)\|_{H^{-1/2}(\Sigma)} dt \\
&\le C\|\varphi\|_{L^2(0,T;H_0^1(Q))} \left(\int_0^T \|(F^\varepsilon(\cdot,t) - F_\Pi(\cdot,t)\|^2_{H^{-1/2}(\Sigma)}\, dt \right)^{1/2};
\end{aligned} \tag{19}$$

here we have also used the fact that Π is a compact subset of Q. Due to the compactness of embedding of $L^2(\Sigma)$ into $H^{-1/2}(\Sigma)$, the a.s. weak convergence of $(F^\varepsilon(x',0,t) - F_\Pi(x',0,t))$ to 0 in $L^2(\Sigma)$ implies that a.s. for all $t \in [0,T]$

$$\lim_{\varepsilon\to 0} \|(F^\varepsilon(\cdot,t) - F_\Pi(\cdot,t))\|_{H^{-1/2}(\Sigma)} = 0.$$

Since $\|(F^\varepsilon(\cdot,t) - F_\Pi(\cdot,t))\|_{L^2(\Sigma)} \le Ce^{-\Lambda t}$, by the Lebesgue theorem we get a.s.

$$\lim_{\varepsilon\to 0} \int_0^T \|(F^\varepsilon(\cdot,t) - F_\Pi(\cdot,t))\|^2_{H^{-1/2}(\Sigma)} dt = 0$$

for any $T > 0$. The desired result now follows from (19). □

Remark 3. Later on, we also use the estimate

$$\|F^\varepsilon(\cdot,t) - F_\Pi(\cdot,t)\delta_\Sigma\|_{H^{-1}(Q)} \le Ce^{-\Lambda t} \;\; \forall(\omega,t) \in \Omega\times(0,\infty), \tag{20}$$

which is an easy consequence of (8).

We now proceed with the convergence result.

THEOREM 1. *Under the assumptions of sections* 1.1, 1.2, *and* 1.3, *we have the convergence*

$$\lim_{\varepsilon\to 0}\|u^\varepsilon - u^0\|_{L^2(0,\infty;H^1(Q))} = 0 \quad a.s.,$$

with u^ε *being a solution of* (12)–(13), *and with* u^0 *being a solution of* (14)–(15).

Proof. Subtracting (14) from (12) we conclude that the difference $(u^\varepsilon - u^0)$ satisfies the equation

$$\begin{aligned}&\partial_t(u^\varepsilon-u^0) - \mathrm{div}(a(x)\nabla(u^\varepsilon-u^0)) - b(x)\nabla(u^\varepsilon-u^0) = f^\varepsilon - F_\Pi\delta_\Sigma \quad \text{in } Q\times(0,\infty);\\ &(u^\varepsilon-u^0)\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}(u^\varepsilon-u^0) + (\lambda - b\cdot n)(u^\varepsilon-u^0) = 0 \quad \text{on } \partial Q\times(0,\infty).\end{aligned} \tag{21}$$

The standard energy estimate for this problem reads (see [13, Chapter III, sections 2 and 5] for the proof)

$$\|u^\varepsilon - u^0\|_{L^2(0,T;H^1(Q))} \le C(T)\|f^\varepsilon - F_\Pi\delta_\Sigma\|_{L^2(0,T;H^{-1}(Q))}; \tag{22}$$

it then follows, from Lemmas 1 and 2, that

$$\lim_{\varepsilon\to 0}\|u^\varepsilon - u^0\|_{L^2(0,T;H^1(Q))} = 0 \quad \text{a.s.} \tag{23}$$

for any $T > 0$.

The proof of convergence on the infinite time interval $(0,\infty)$ relies on the dissipative properties of the studied problem.

First, we consider an auxiliary problem:

$$\begin{aligned}&\partial_t w - \mathrm{div}(a(x)\nabla w) - \mathrm{div}(b(x)w) = 0 \quad \text{in } Q\times(0,\infty),\\ &w\big|_{t=0} = w_0, \qquad \frac{\partial}{\partial n_a}w + (\lambda - b\cdot n)w = 0 \quad \text{on } \partial Q\times(0,\infty);\end{aligned} \tag{24}$$

then we prove the following statement.

LEMMA 3. *A solution of problem* (24) *satisfies the estimate*

$$\|w(\cdot,t)\|_{C(Q)} \le Ce^{-\kappa t}\|w_0\|_{L^2(Q)}, \quad t \ge 1,$$

with some constants $\kappa > 0$ *and* $C > 0$.

Proof of Lemma. Without loss of generality we may assume that $w_0 \ge 0$. Then, by the maximum principle, the solution w is positive for any time. Moreover, by the Harnack inequality (see [2, Theorem E]) and the standard parabolic estimates,

$$\max_{x\in Q} w(x,1) \le C\min_{x\in Q} w(x,1), \qquad \max_{x\in Q} w(x,1) \le \|w_0\|_{L^2(Q)}.$$

Integrating by parts (24) over the set $\{(x,t) : x\in Q,\ 1\le s\le t\}$, we obtain

$$\int_Q w(x,t)dx - \int_Q w(x,1)dx = -\lambda\int_1^t ds\int_{\partial Q} w(x,s)d\sigma,$$

where $d\sigma$ is the surface volume element. It follows from the last two relations and the Harnack inequality that

$$\int_Q w(x,t)dx \le \int_Q w(x,1)dx - c\lambda\int_1^t\int_Q w(x,s)dx,$$

with a constant $c > 0$ which does not depend on w_0. Applying the Gronwall lemma, we conclude that

$$\int_Q w(x,t)dt \le e^{-c\lambda(t-1)} \int_Q w(x,1)dt.$$

Letting $\kappa = c\lambda$, the last estimate reads

$$\int_Q w(x,t)dt \le Ce^{-\kappa t} \int_Q w(x,1)dt.$$

From this estimate the statement of the lemma easily follows by the Harnack inequality. $\square$

Finally, to complete the proof of Theorem 1, we make use of Green's function $G(t,x,y)$ of problems (24) and (21). By Lemma 3 and the Harnack inequality we have

$$G(t,x,y) \le C(t_0)e^{-\kappa t}, \qquad t \ge t_0,$$

where t_0 is a positive number; for instance, we can set $t_0 = 1$. Considering the smoothness of the coefficients and of the domain in (12)–(13), by parabolic estimates (see [13, Chapter III, section 12]) we have

$$\|G(t,\cdot,y)\|_{H^1(Q)} \le Ce^{-\kappa t}, \qquad \|G(t,x,\cdot)\|_{H^1(Q)} \le Ce^{-\kappa t}, \qquad t \ge t_0. \tag{25}$$

Using the integral representation of the solution of (21) in terms of $G(t,x,y)$, and combining (25) and (20) with the estimates of Lemma 1, gives

$$\int_T^\infty \|u^\varepsilon(\cdot,t) - u^0(\cdot,t)\|^2_{H^1(Q)}dt \le Ce^{-2\min(\Lambda,\kappa)T}. \tag{26}$$

Indeed, if for $N \in \mathbb{Z}^+$ we denote $\chi_N^T(t) = \mathbb{1}_{[T+N-2,\,T+N]}(t)$ and introduce $U^{\varepsilon,1}_{T,N}$ and $U^{\varepsilon,2}_{T,N}$ as solutions of problems similar to (21) but with the right-hand side replaced with $(f^\varepsilon - F_\Pi\delta_\Sigma)\chi_N^T$ and $(f^\varepsilon - F_\Pi\delta_\Sigma)(1-\chi_N^T)$, respectively, then, on the interval $(T+N-1, T+N)$, we have $(u^\varepsilon - u^0) = U^{\varepsilon,1}_{T,N} + U^{\varepsilon,2}_{T,N}$. From (20) and (22) we get

$$\left\|U^{\varepsilon,1}_{T,N}\right\|^2_{L^2((T+N-1,T+N)\,;\,H^1(Q))} \le Ce^{-2\Lambda(T+N)}.$$

Estimates (20) and (25) imply that

$$\left\|U^{\varepsilon,2}_{T,N}(\cdot, T+N-1)\right\|^2_{L^2(Q)} \le Ce^{-2\min(\Lambda,\kappa)(T+N)}.$$

By the energy estimate we obtain

$$\left\|U^{\varepsilon,2}_{T,N}\right\|^2_{L^2((T+N-1,T+N)\,;\,H^1(Q))} \le Ce^{-2\min(\Lambda,\kappa)(T+N)}.$$

This yields

$$\left\|u^\varepsilon - u^0\right\|^2_{L^2((T+N-1,T+N)\,;\,H^1(Q))} \le Ce^{-2\min(\Lambda,\kappa)(T+N)}.$$

Taking $N = 1, 2, \ldots$ and summing up the obtained inequalities, we arrive at (26).

Together with the convergence on finite intervals, obtained in (23), estimate (26) implies the desired statement of Theorem 1. $\square$

3. Rate of convergence. In applications the convergence result alone is not of great interest if it is not accompanied by proper estimates of the rate of convergence. In this section, under natural additional assumptions on the ergodic properties of the source term, we provide a number of bounds for the convergence rate. These assumptions will be made on the rate of decay at infinity for the correlation function of the source density or for one of its mixing coefficients; see, for instance, [10] and [14] for the corresponding definitions.

Let us start by recalling the definition of the correlation function of a random field $(x', t) \in \mathbb{R}^2 \times \mathbb{R} \mapsto \Psi(x', t)$:

$$\mathbf{R}(t, s, x', y') \equiv \mathbf{E}\big[(\Psi(x', t) - \mathbf{E}\Psi(x', t))(\Psi(y', s) - \mathbf{E}\Psi(y', s))\big]. \tag{27}$$

Due to the properties of the random field $\Phi(\zeta, t)$ defined in (9), the corresponding correlation function $\mathbf{R}(t, s, x', y')$ takes the form

$$\begin{aligned} &\mathbf{R}(t, s, x', y') = \mathcal{R}(t, s, [x'] - [y']) \\ &= \mathbf{E}\big[(\Phi(0, t) - \mathbf{E}\Phi(0, t))(\Phi([x'] - [y'], s) - \mathbf{E}\Phi([x'] - [y'], s))\big]. \end{aligned} \tag{28}$$

3.1. Mixing assumptions. Ergodicity is a qualitative property; under the sole assumption that $\Phi(x', t)$ is ergodic, it is not possible to estimate effectively the discrepancy $|u^\varepsilon - u^0|$ in Theorem 1.

In this section we introduce a number of conditions on the random field $t \mapsto \Phi(x', t)$ which will be used later on when obtaining error bounds.

- First, we assume that the correlation function of $\Phi(x', t)$ admits the upper bound

$$|\mathbf{R}(t, s, x', y')| \le \exp(-\Lambda \min(s, t)) \bar{R}(x' - y'), \tag{29}$$

  with function $\bar{R}(x' - y')$, which depends only on the difference $(x' - y')$. This bound ensures in particular that there is an estimate uniform in t and s for the correlation of $\Phi(\cdot, t)$ and $\Phi(\cdot, s)$. It should also be noted that the bound (29) is consistent with the previous assumption (8).
- Then, concerning $\bar{R}(y')$, we will assume that at least one of the following conditions holds true:

R0.

$$\bar{R}(y') = 0 \quad \text{if } |y'| > R_0,$$

for some $R_0 > 0$.

R1.

$$\int_{\mathbb{R}^2} \bar{R}(y') dy' < \infty.$$

R2.

$$\bar{R}(y') \le C(1 + |y'|)^{-\nu}, \qquad \nu > 0.$$

Assumptions R0–R2 might be insufficient for obtaining the estimates for higher order moments of the discrepancy $|u^\varepsilon - u^0|$, and for studying the limit law of the normalized difference $(u^\varepsilon - u^0)$. In this case we should impose additional conditions on the mixing coefficients of the random field $\Phi(x', t)$.

- We will assume that the function $\Phi(x',t)$ possesses one of the following mixing properties:

M1. The strong spatial mixing coefficient $\alpha(r)$ of $\Phi(x',t)$ satisfied the inequality

$$\alpha(r) \,<\, C(1+r)^{-\nu_1}, \qquad \nu_1 > 0, \tag{30}$$

where the strong spatial mixing coefficient $\alpha(r)$ is defined as follows:

$$\alpha(r) = \sup_{G_1,\,G_2} \sup_{\substack{\mathcal{E}_1\in\mathcal{F}_{G_1}\\ \mathcal{E}_2\in\mathcal{F}_{G_2}}} |\mathbf{P}(\mathcal{E}_1\cap\mathcal{E}_2) - \mathbf{P}(\mathcal{E}_1)\mathbf{P}(\mathcal{E}_2)|,$$

with $\mathcal{F}_{G_1} = \sigma\{\Phi(y_1',t_1)\,:\, y_1'\in G_1,\, t_1\ge 0\}$, $\mathcal{F}_{G_2} = \sigma\{\Phi(y_2',t_2)\,:\, y_2'\in G_2,\, t_2\ge 0\}$; and the first supremum is taken over all sets $G_1,\, G_2\subset\mathbb{R}^2$ such that $\operatorname{dist}(G_1,G_2)\ge r$.

M2. The maximum spatial correlation coefficient $\beta(r)$ of $\Phi(\cdot,t)$ decays fast enough so that

$$\beta(r) \,<\, C(1+r)^{-\nu_1}, \qquad \nu_1 > 0, \tag{31}$$

with

$$\beta(s) = \sup_{G_1,\,G_2} \sup_{\xi,\,\eta} |\mathbf{E}(\xi\eta)|,$$

where the second supremum is taken over all random variables, ξ and η, which are, respectively, $\mathcal{F}_{G_1}$- and $\mathcal{F}_{G_2}$-measurable and satisfy the conditions $\mathbf{E}\xi = \mathbf{E}\eta = 0$, $\|\xi\|_{L^\infty(\Omega)} = \|\eta\|_{L^\infty(\Omega)} = 1$, and where the first supremum is taken over all sets $G_1,\, G_2\subset\mathbb{R}^2$ such that $\operatorname{dist}(G_1,G_2)\ge r$.

Remark 4. Notice that the condition R0 is fulfilled if the strong mixing coefficient $\alpha(r)$ is equal to 0 for $r\ge R_0$. Also, according to Lemma VIII.3.102 in [10], any one of the conditions M1 or M2 implies the condition R2 with $\nu=\nu_1$.

Replacing the random source density $f^\varepsilon(x,t)$, which is distributed in a small neighborhood of the plane Σ, with a random source density $F^\varepsilon(x,t)$ concentrated on the plane Σ, as specified in (17), we define the auxiliary problem

$$\partial_t\hat u^\varepsilon - \operatorname{div}(a(x)\nabla\hat u^\varepsilon) + \operatorname{div}(b(x)\hat u^\varepsilon) = F^\varepsilon \quad \text{in } Q\times(0,\infty); \tag{32}$$

$$\hat u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}\hat u^\varepsilon - b(x)\cdot n(x)\hat u^\varepsilon + \lambda\hat u^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{33}$$

We proceed by estimating the difference $(u^\varepsilon - \hat u^\varepsilon)$.

LEMMA 4. *Let the above assumptions on the domain geometry and on the coefficients of problem* (12)–(13) *be fulfilled, and assume, moreover, that* (8) *is satisfied. Then the following bounds hold:*

$$\|u^\varepsilon - \hat u^\varepsilon\|_{L^\infty(Q\times(0,T))} \le C\varepsilon^\gamma|\ln\varepsilon|, \tag{34}$$

$$\|u^\varepsilon - \hat u^\varepsilon\|_{L^\infty(0,T;L^p(Q))} \le C(p)\varepsilon^\gamma, \quad 1\le p<\infty. \tag{35}$$

Proof. Denote by $G(t-s,x,y)$ Green's function of problem (12)–(13), and let $\mathcal{K}$ be a compact subset of Q that contains some neighborhood of $\Sigma\cap\Pi$. Using Aronson's estimates for fundamental solutions on finite time intervals [1] and the maximum principle, one can show that on any finite time interval $(0,T]$ and for all $y\in\mathcal{K}$ the function $G(t,x,y)$ admits the upper bound

$$G(t,x,y) \le \frac{C}{t^{3/2}}\exp\left(-c\frac{|x-y|^2}{t}\right), \tag{36}$$

with strictly positive constants C and c that might depend on T. Indeed, we can assume without loss of generality that the matrix $a(\cdot)$ and the vector field $b(\cdot)$ are defined everywhere in $\mathbb{R}^3$ and satisfy the smoothness and ellipticity conditions specified in section 1.3. Denote by $\mathcal{P}(t,x,y)$ the fundamental solution of the corresponding Cauchy problem in $\mathbb{R}^3 \times (0,T)$. Then, according to [1], the following upper bound holds:

$$\mathcal{P}(t,x,y) \le \frac{C}{t^{3/2}} \exp\left(-c\frac{|x-y|^2}{t}\right), \qquad x,\, y \in R^3, \quad t \in (0,T), \tag{37}$$

with the constants $C > 0$ and $c > 0$ which might depend on T. For the difference $G(t,x,y) - \mathcal{P}(t,x,y)$ we have

$$G(0,x,y) - \mathcal{P}(0,x,y) = 0, \qquad x,\, y \in Q,$$

and, for all $y \in \mathcal{K}$ and $x \in \partial Q$,

$$\Big|\frac{\partial}{\partial n_a}(G-\mathcal{P}) - b\cdot n(G-\mathcal{P}) + \lambda(G-\mathcal{P})\Big| \le Ce^{-c_2/t}, \qquad c_2 > 0, \quad C > 0.$$

The energy estimate reads

$$\|G(t,\cdot,y) - \mathcal{P}(t,\cdot,y)\|_{L^2(Q)} \le Ce^{-c_2/t}$$

for all $y \in \mathcal{K}$. Since the coefficients in (12) and the domain Q are smooth, we derive by means of the standard Schauder-type parabolic estimates (see [13, Chapter III, section 12]) that

$$|G(t,x,y) - \mathcal{P}(t,x,y)| \le Ce^{-c_2/t}$$

for all $x \in Q$ and $y \in \mathcal{K}$. Combining this estimate with (37) yields (36).

Applying once again the standard Schauder-type parabolic estimates, we conclude that

$$|\nabla_y G(t,x,y)| \le \frac{C}{t^{3/2}}\frac{|x-y|}{t} \exp\left(-c\frac{|x-y|^2}{t}\right). \tag{38}$$

Clearly, the difference $(f^\varepsilon - F^\varepsilon)$ can be represented as follows:

$$f^\varepsilon(x,t) - F^\varepsilon(x,t) = \frac{\partial}{\partial x_3}\vartheta\Big(\frac{x_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde{B}^\varepsilon}(x')\phi^\varepsilon(x',t),$$

with

$$\vartheta(r) = \begin{cases} r+1, & -1 \le r < 0, \\ r-1, & 0 \le r \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Green's formula and estimates (8), (38), we have

$$\begin{aligned} |u^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)| &= \Big|\int_0^t\int_Q G(t-s,x,y)\frac{\partial}{\partial y_3}\vartheta\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde{B}^\varepsilon}(y')\phi^\varepsilon(x',s)\,dyds\Big| \\ &= \Big|\int_0^t\int_Q \frac{\partial}{\partial y_3}G(t-s,x,y)\vartheta\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde{B}^\varepsilon}(y')\phi^\varepsilon(x',s)\,dyds\Big| \\ &\le \int_0^t\int_Q \frac{C}{(t-s)^2}\frac{|x-y|}{\sqrt{t-s}}\exp\Big(-c\frac{|x-y|^2}{t-s}\Big)\mathbb{1}_{\{|y_3|\le\varepsilon^\gamma\}}\exp(-\Lambda s)\,dsdy. \end{aligned}$$

Integrating first in time and then in space, after straightforward computation, one gets

$$
\begin{aligned}
&\int_0^t \frac{1}{(t-s)^2}\frac{|x-y|}{\sqrt{t-s}}\exp\Big(-c\frac{|x-y|^2}{t-s}\Big)\exp(-\Lambda s)\,ds\\
&\qquad \le \int_0^t \frac{1}{s^2}\frac{|x-y|}{\sqrt{s}}\exp\Big(-c\frac{|x-y|^2}{s}\Big)\,ds\\
&\le \frac{1}{|x-y|^2}\int_0^\infty \frac{1}{s^{5/2}}\exp\Big(-\frac{c}{s}\Big)\,ds = \frac{c_2}{|x-y|^2}\ .
\end{aligned}
\tag{39}
$$

For $|x_3|\le 2\varepsilon^\gamma$ this gives (34), i.e.,

$$
|u^\varepsilon(x,t)-\hat u^\varepsilon(x,t)| \le C\int_0^{2\varepsilon^\gamma} dy_3\int_{|y'|\le R}\frac{1}{|y|^2}\,dy' \le C(Q,\gamma)\varepsilon^\gamma|\ln(\varepsilon)|;
\tag{40}
$$

here and later on $R=\mathrm{diam}(Q)$. For $|x_3|\ge 2\varepsilon^\gamma$ we obtain

$$
|u^\varepsilon(x,t)-\hat u^\varepsilon(x,t)| \le C\varepsilon^\gamma\int_{|y'|\le R}\frac{1}{x_3^2+|y'|^2}\,dy' \le C(Q)\varepsilon^\gamma|\ln(|x_3|)|.
\tag{41}
$$

Finally, (40)–(41) yield

$$
|u^\varepsilon(x,t)-\hat u^\varepsilon(x,t)| \le C(Q,\gamma)\varepsilon^\gamma|\ln(\max\{|x_3|,2\varepsilon^\gamma\})|,
\tag{42}
$$

which implies (34).

Integrating (42) leads to (35). This completes the proof. □

Remark 5. The estimates of Lemma 4 have nothing to do with the randomness of f^ε. We have only used the properties of Green's function $G(t,x,y)$, the structure of the support of f^ε, and the fact that function f^ε is bounded.

Next we denote

$$
\check F^\varepsilon(t) = 2s_3\mathbf{E}\{\Phi(x,t)\}\mathbb{1}_{\tilde B_\varepsilon}(x')\delta_\Sigma(x)
\tag{43}
$$

and consider another deterministic auxiliary problem with a nonrandom source term:

$$
\partial_t\check u^\varepsilon - \mathrm{div}(a(x)\nabla\check u^\varepsilon) + \mathrm{div}(b(x)\check u^\varepsilon) = \check F^\varepsilon \quad \text{in } Q\times(0,\infty);
\tag{44}
$$

$$
\check u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}\check u^\varepsilon - b(x)\cdot n(x)\check u^\varepsilon + \lambda\check u^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty).
\tag{45}
$$

Representing the difference $(\mathbb{1}_{\tilde B_\varepsilon}(x')-s_1s_2)$ in the form $\mathbb{1}_{\tilde B_\varepsilon}(x')-s_1s_2=\varepsilon\mathrm{div}_{x'}\check\theta(x'/\varepsilon)$ with $|\check\theta|\le c$ (one can easily find an explicit formula for $\check\theta$, though we do not use it), and following the lines of the proof of Lemma 4, we obtain the following proposition.

PROPOSITION 1. *Let $\check u^\varepsilon$ be a solution of* (44)–(45) *and u^0 a solution of* (12)–(13)*; then*

$$
\|\check u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(Q))} \le C\varepsilon.
\tag{46}
$$

In view of Lemma 4 and Proposition 1, in order to estimate the discrepancy $\|u^\varepsilon-u^0\|_{L^\infty(0,T;L^2(Q))}$, it suffices to obtain an upper bound for the expression $\|\hat u^\varepsilon - \check u^\varepsilon\|_{L^\infty(0,T;L^2(Q))}$. This is the main and most technical part of this section.

PROPOSITION 2. *Let $\hat u^\varepsilon$ and $\check u^\varepsilon$ be solutions of problems* (32)–(33) *and* (44)–(45), *respectively, and suppose that condition* R0 *is fulfilled; then the following estimate holds:*

$$\mathbf{E}\big\{\|\hat u^\varepsilon - \check u^\varepsilon\|^2_{L^2(0,T;L^2(Q))}\big\} \le C\varepsilon^2. \tag{47}$$

Proof. The difference $U^\varepsilon \equiv (\check u^\varepsilon - \hat u^\varepsilon)$ solves the problem

$$\partial_t U^\varepsilon - \operatorname{div}(a(x)\nabla U^\varepsilon) + \operatorname{div}(b(x)U^\varepsilon) = F^\varepsilon - \check F^\varepsilon \quad \text{in } Q\times(0,\infty); \tag{48}$$

$$U^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}U^\varepsilon - b(x)\cdot n(x)U^\varepsilon + \lambda U^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{49}$$

Our aim is to estimate the expression $\mathbf{E}\big\{\|U^2(t,\cdot)\|^2_{L^2(Q)}\big\}$; to this end, we first obtain a pointwise bound for $\mathbf{E}U^2(t,x)$. Using the notation $F_0^\varepsilon = F^\varepsilon - \check F^\varepsilon$, we have

$$\mathbf{E}\big\{U^2(x,t)\big\} = \mathbf{E}\bigg\{\Big(\int_0^t\int_Q G(t-s,x,y)F_0^\varepsilon(s,y)dyds\Big)^2\bigg\}$$

$$\begin{aligned}
&= \mathbf{E}\bigg\{\int_0^t\int_Q\int_0^t\int_Q G(t-s,x,y)G(t-r,x,\zeta)F_0^\varepsilon(s,y)F_0^\varepsilon(r,\zeta)\,dydsd\zeta dr\bigg\}\\
&\le C\int_0^t\int_{Q'}\int_0^t\int_{Q'} G(t-s,x,(y',0))G(t-r,x,(\zeta',0))\bar R\Big(\frac{|\zeta'-y'|}{\varepsilon}\Big)\,dy'dsd\zeta'dr\\
&= C\int_0^t\int_{Q'}\int_0^t\int_{Q'} G(s,x,(y',0))G(r,x,(\zeta',0))\bar R\Big(\frac{|\zeta'-y'|}{\varepsilon}\Big)\,dy'dsd\zeta'dr\\
&\le C_1\int_0^t\int_{Q'}\int_0^t\int_{Q'} \frac{1}{s^{3/2}r^{3/2}}\exp\Big(-c\frac{x_3^2+|x'-y'|^2}{s}\Big)\\
&\qquad\times\exp\Big(-c\frac{x_3^2+|x'-\zeta'|^2}{r}\Big)\bar R\Big(\frac{|\zeta'-y'|}{\varepsilon}\Big)\,dy'dsd\zeta'dr,
\end{aligned} \tag{50}$$

where we have denoted $Q' = Q\cap\{y_3=0\}$.

Integrating first in s gives

$$\int_0^t \frac{1}{s^{3/2}}\exp\Big(-c\frac{x_3^2+|x'-y'|^2}{s}\Big)\,ds \tag{51}$$

$$= \frac{1}{(x_3^2+|x'-y'|^2)^{1/2}}\int_0^{t/(x_3^2+|x'-y'|^2)}\frac{1}{s^{3/2}}\exp\Big(-\frac{c}{s}\Big)\,ds \le \frac{C_2}{(x_3^2+|x'-y'|^2)^{1/2}},$$

with $C_2 = \int_0^\infty s^{-3/2}\exp(-c/s)ds$. Then substituting (51) in inequality (50), one gets

$$\mathbf{E}\big\{U^2(x,t)\big\} \le C\int_{Q'}\int_{Q'}\frac{\bar R\Big(\frac{|\zeta'-y'|}{\varepsilon}\Big)dy'}{(x_3^2+|x'-y'|^2)^{1/2}}\frac{d\zeta'}{(x_3^2+|x'-\zeta'|^2)^{1/2}}. \tag{52}$$

Without loss of generality we may assume that $0\in Q$. If we denote $Q_0 = \{y'\in\mathbb{R}^2, |y'|<2\mathrm{diam}(Q)\}$ and perform the change of variables $\tilde y' = y'-x'$, $\tilde\zeta' = \zeta'-x'$, then (52) leads to

$$\mathbf{E}\big\{U^2(x,t)\big\} \le C\int_{Q_0}\int_{Q_0}\frac{1}{(x_3^2+|\tilde y'|^2)^{1/2}}\frac{1}{(x_3^2+|\tilde\zeta'|^2)^{1/2}}\bar R\Big(\frac{|\tilde\zeta'-\tilde y'|}{\varepsilon}\Big)\,d\tilde y'd\tilde\zeta'. \tag{53}$$

For brevity denote $\mathbf{Q}_0^{2,\varepsilon} = \{(\tilde{y}', \tilde{\zeta}') \in Q_0 \times Q_0 \; : \; |\tilde{y}' - \tilde{\zeta}'| \le R_0\varepsilon\}$, where R_0 is the constant from assumption R0; then due to R0 inequality (53) implies the estimate

$$\mathbf{E}\big\{U^2(x,t)\big\} \le C \int_{\mathbf{Q}_0^{2,\varepsilon}} \frac{1}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{1}{(x_3^2 + |\tilde{\zeta}'|^2)^{1/2}} \, d\tilde{y}' d\tilde{\zeta}'.$$

In order to achieve an upper bound for the integral on the right-hand side, we divide the integration area into two parts, namely, $\mathbf{Q}_1^{2,\varepsilon} = \mathbf{Q}_0^{2,\varepsilon} \cap \{|\tilde{y}'| < 2R_0\varepsilon, \, |\tilde{\zeta}'| < 2R_0\varepsilon\}$ and $\mathbf{Q}_2^{2,\varepsilon} = \mathbf{Q}_0^{2,\varepsilon} \setminus \mathbf{Q}_1^{2,\varepsilon}$.

The integral over $\mathbf{Q}_1^{2,\varepsilon}$ can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbf{Q}_1^{2,\varepsilon}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{d\tilde{\zeta}'}{(x_3^2 + |\tilde{\zeta}'|^2)^{1/2}} \le \left(\int_{\{|y'| < 2R_0\varepsilon\}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \right)^2 \\
&\qquad = \left(\int_0^{2R_0\varepsilon} \frac{r dr}{(x_3^2 + r^2)^{1/2}} \right)^2 = \left(\int_0^{4R_0^2\varepsilon^2} \frac{ds}{2(x_3^2 + s)^{1/2}} \right)^2 \\
&\qquad \le \left(\int_0^{4R_0^2\varepsilon^2} \frac{ds}{2\sqrt{s}} \right)^2 \le C(R_0)\varepsilon^2;
\end{aligned} \tag{54}$$

in the last inequality here an explicit formula for the corresponding integral has been used.

Denote $d_Q = 2\mathrm{diam}(Q)$. Since for any $(\tilde{y}', \tilde{\xi}') \in \mathbf{Q}_2^{2,\varepsilon}$ we have $R_0\varepsilon \le |\tilde{y}'| \le d_Q$, $|\tilde{\xi}'| \ge R_0\varepsilon$, and $|\tilde{y}' - \tilde{\xi}'| \le R_0\varepsilon$, then for all $(\tilde{y}', \tilde{\xi}') \in \mathbf{Q}_2^{2,\varepsilon}$ and x_3 the following estimate holds:

$$\frac{1}{(x_3^2 + |y'|^2)^{1/2}} \frac{1}{(x_3^2 + |\zeta'|^2)^{1/2}} \le C(R_0) \frac{1}{(x_3^2 + |y'|^2)};$$

hence,

$$\begin{aligned}
&\int_{\mathbf{Q}_2^{2,\varepsilon}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{d\tilde{\zeta}'}{(x_3^2 + |\tilde{\zeta}'|^2)^{1/2}} \le C(R_0) \int_{\mathbf{Q}_2^{2,\varepsilon}} \frac{d\tilde{y}' d\tilde{\zeta}'}{(x_3^2 + |\tilde{y}'|^2)} \\
&\le C(R_0) \int_{\substack{R_0\varepsilon \le |\tilde{y}'| \le d_Q, \\ |\tilde{\xi}' - \tilde{y}'| \le R_0\varepsilon}} \frac{d\tilde{y}' d\tilde{\zeta}'}{(x_3^2 + |\tilde{y}'|^2)} \le C(R_0)\varepsilon^2 \int_{R_0\varepsilon \le |\tilde{y}'| \le d_Q} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)} \\
&= C(R_0)\varepsilon^2 \int_{R_0\varepsilon}^{d_Q} \frac{r dr}{(x_3^2 + r^2)} \le C(R_0, Q)\varepsilon^2 \ln(x_3^2 + \varepsilon^2).
\end{aligned} \tag{55}$$

Combining (54) and (55), we arrive at the desired pointwise upper bound:

$$\mathbf{E}\big\{U^2(x,t)\big\} \equiv \mathbf{E}\big\{(\check{u}^\varepsilon - \hat{u}^\varepsilon)^2\big\} \le C(R_0, Q)\varepsilon^2 \ln(x_3^2 + \varepsilon^2). \tag{56}$$

Now, the estimate (47) is straightforward; we just integrate (56) over Q. ∎

The above statements allow us to estimate the rate of convergence in Theorem 1 if assumption R0 is satisfied.

THEOREM 2. *Let the assumptions of Theorem* 1 *be fulfilled, and assume in addition that condition* R0 *holds true. Then*

$$\mathbf{E}\Big\{ \|u^\varepsilon - u^0\|^2_{L^2(0,T;L^2(Q))} \Big\} \le C(R_0, Q)\varepsilon^2, \tag{57}$$

where u^ε is a solution of the original problem (12)–(13)*, and u^0 is a solution of the upscaled problem* (14)–(15).

Remark 6. Notice that the scaling ε^2 on the right-hand side of (57) is the central limit theorem scaling in dimension two. This scaling is natural in our framework due to the nearly two-dimensional structure of the source term. The limit behavior of a normalized difference $(u^\varepsilon - u^0)/\varepsilon$ and the validity of a full central limit theorem are discussed in section 4.

Remark 7. The estimate for the rate of convergence given by Theorem 2 is better than that obtained with the help of Lemma 1. Indeed, for $\gamma = 1$, using Lemma 1 and an a priori estimate instead of Green's function representation, we would have had in (57) the right-hand side of order ε instead of ε^2.

Proof of Theorem 2. This assertion is a straightforward consequence of the results of Lemma 4, Propositions 1 and 2, and the assumption that $\gamma \geq 1$. ∎

Moreover, taking into account the dissipative properties of the boundary conditions (13) and the bounds (8), (29), and following along the lines of the proof of Proposition 2, one can obtain the estimate

THEOREM 3. *Under the assumptions of Theorem* 2 *the following inequality holds:*

$$\mathbf{E}\Big\{\|u^\varepsilon - u^0\|^2_{L^2((t,\infty);L^2(Q))}\Big\} \leq C(R_0, Q)\exp(-\kappa t)\varepsilon^2, \quad \kappa > 0, \tag{58}$$

with $\kappa > 0$, which depends only on Λ, on the operator in (12)–(13)*, and on the domain Q.*

Our next goal is to relax the mixing assumptions on the source function $f^\varepsilon(x,\omega)$; we want to show that the statement of the last theorem remains valid if the correlation function of $\Phi(\cdot,t)$, or its strong mixing coefficient, satisfies certain polynomial decay conditions.

THEOREM 4. *Suppose that either condition* R2*, with $\nu > 2$, is fulfilled or the strong spatial mixing coefficient $\alpha(s)$ satisfies the upper bound $\alpha(s) \leq C(1+s)^{-\nu_1}$ with $\nu_1 > 2$. Then the following inequality holds:*

$$\mathbf{E}\Big\{\|u^\varepsilon - u^0\|^2_{L^2(0,T;L^2(Q))}\Big\} \leq C(\nu, Q)\varepsilon^2.$$

Proof. Using the results of Lemma 4 and Propositions 1 and 2, as in the proof of Theorem 2, we need only show that the estimate (56) holds. To this end we consider the auxiliary problem (48)–(49) and notice that the upper bounds (52) and (53) are still valid under the assumptions of Theorem 4. It then follows from (53) and the standing assumptions that

$$\begin{aligned}\mathbf{E}\big\{U^2(x,t)\big\} &\leq C\int_{Q_0}\int_{Q_0}\frac{1}{(x_3^2+|\tilde{y}|^2)^{1/2}}\frac{1}{(x_3^2+|\tilde{\zeta}|^2)^{1/2}}\Big(\frac{1}{(1+\varepsilon^{-1}|\tilde{\zeta}-\tilde{y}|)^\nu}\Big)\,d\tilde{y}d\tilde{\zeta}\\ &= C\varepsilon^2\int_{\varepsilon^{-1}Q_0}\int_{\varepsilon^{-1}Q_0}\frac{1}{(X_3^2+|\tilde{y}|^2)^{1/2}}\frac{1}{(X_3^2+|\tilde{\zeta}|^2)^{1/2}}\Big(\frac{1}{(1+|\tilde{\zeta}-\tilde{y}|)^\nu}\Big)\,d\tilde{y}d\tilde{\zeta},\end{aligned}$$

with $\nu > 2$ and $X_3 = x_3/\varepsilon$; for simplicity of notation, here and later on, instead of writing $\tilde{y}'$ and $\tilde{\zeta}'$ (both of them belong to $\mathbb{R}^2$) we write $\tilde{y}$ and $\tilde{\zeta}$.

In order to estimate the latter integral, we divide the domain $\varepsilon^{-1}Q_0 \times \varepsilon^{-1}Q_0$ into three parts, namely,

$$\begin{aligned}&\varepsilon^{-1}Q_0 \times \varepsilon^{-1}Q_0 \equiv Q_1 \cup Q_2 \cup Q_3\\ &\stackrel{\text{def}}{=} \Big\{(\tilde{y},\tilde{\zeta}) : |\tilde{y}| \leq \frac{1}{2}|\tilde{\zeta}|\Big\} \cup \Big\{(\tilde{y},\tilde{\zeta}) : \frac{1}{2}|\tilde{\zeta}| \leq |\tilde{y}| \leq 2|\tilde{\zeta}|\Big\} \cup \Big\{(\tilde{y},\tilde{\zeta}) : |\tilde{y}| \geq 2|\tilde{\zeta}|\Big\};\end{aligned}$$

and we estimate the contribution of each subdomain separately.

In Q_1 we have $|\tilde{\zeta} - \tilde{y}| \geq \frac{1}{2}|\tilde{\zeta}|$, thus

$$\begin{aligned}
&\int_{\varepsilon^{-1}Q_0} \frac{C\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)^{1/2}} \int_{\{2|\tilde{y}|\leq|\tilde{\zeta}|\}} \frac{1}{(X_3^2 + |\tilde{y}|^2)^{1/2}} \frac{1}{(1 + |\tilde{\zeta} - \tilde{y}|)^\nu} \, d\tilde{y} \\
&\leq \int_{\varepsilon^{-1}Q_0} \frac{C\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)^{1/2}} \int_{\{2|\tilde{y}|\leq|\tilde{\zeta}|\}} \frac{1}{(X_3^2 + |\tilde{y}|^2)^{1/2}} \frac{C(\nu)}{(1 + |\tilde{\zeta}|)^\nu} \, d\tilde{y} \\
&\leq \int_{\varepsilon^{-1}Q_0} \frac{C(\nu)\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)^{1/2}(1 + |\tilde{\zeta}|)^\nu} \int_0^{|\tilde{\zeta}|/2} \frac{rdr}{(X_3^2 + r^2)^{1/2}} \\
&\leq \int_{\varepsilon^{-1}Q_0} \frac{C(\nu)\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)^{1/2}(1 + |\tilde{\zeta}|)^\nu} C_1(\nu)(X_3^2 + |\tilde{\zeta}|^2)^{1/2} \leq C_2(\nu)\varepsilon^2.
\end{aligned}$$

The contribution of Q_3 can be estimated in the same way if we exchange the order of integration in the variables $\tilde{y}$ and $\tilde{\zeta}$. It then remains to estimate the integral over Q_2.

We have

$$\begin{aligned}
&\int_{Q_2} \frac{C\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)^{1/2}} \frac{d\tilde{y}}{(X_3^2 + |\tilde{y}|^2)^{1/2}} \frac{1}{(1 + |\tilde{\zeta} - \tilde{y}|)^\nu} \\
&\leq \int_{Q_2} \frac{4C\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)} \frac{d\tilde{y}}{(1 + |\tilde{\zeta} - \tilde{y}|)^\nu} \leq \int_{\varepsilon^{-1}Q_0} \frac{4C\varepsilon^2 d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)} \int_{2\varepsilon^{-1}Q_0} \frac{d\hat{y}}{(1 + |\hat{y}|)^\nu} \\
&\leq C(\nu)\varepsilon^2 \int_{\varepsilon^{-1}Q_0} \frac{d\tilde{\zeta}}{(X_3^2 + |\tilde{\zeta}|^2)} \leq C_3\varepsilon^2(\nu)(1 + |\ln(|x_3|)|).
\end{aligned}$$

Combining the above estimates, we conclude finally that

$$\mathbf{E}\{U^2(x,t)\} \leq C(\nu)\varepsilon^2\big(1 + |\ln(|x_3|)|\big).$$

This yields the desired statement if the assumption R2 with $\nu > 2$ is fulfilled. To complete the proof, we use the fact that condition M1 implies condition R2 with $\nu = \nu_1$. ∎

THEOREM 5. *Assume that at least one of the conditions* R1, R2 *with* $\nu > 2$, *or* M1 *with* $\nu_1 > 2$ *is satisfied; then the discrepancy* $(u^\varepsilon - u^0)$ *admits the estimate*

$$\mathbf{E}\left\{ \|u^\varepsilon - u^0\|^2_{L^2((t,\infty);L^2(Q))} \right\} \leq C\exp(-\kappa t)\varepsilon^2, \quad \kappa > 0.$$

Proof. By following along the lines of the proof of Theorem 4, one can show that under the condition R1 the following upper bound holds:

$$\mathbf{E}\left\{ \|u^\varepsilon - u^0\|^2_{L^2(0,T;L^2(Q))} \right\} \leq C\varepsilon^2.$$

Finally, we combine the above estimates, local in time, and the statement of Lemma 3; considering the exponentially decaying factors in (8) and (29), we arrive at the desired result. ∎

Denote $Q_\delta = \{x \in Q : |x_3| > \delta\}$.

LEMMA 5. *Under the assumptions of Lemma* 4, *for any* $\delta > 0$, *the following two relations hold a.s. for all* $t \geq 0$:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\gamma} \|u^\varepsilon(\cdot,t) - \hat{u}^\varepsilon(\cdot,t)\|_{L^\infty(Q_\delta)} = 0, \tag{59}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\gamma}\|u^\varepsilon(\cdot,t)-\hat u^\varepsilon(\cdot,t)\|_{L^p(Q)}=0,\qquad p\in[1,+\infty). \tag{60}$$

Moreover, the convergence is uniform on finite time intervals.

Proof. In view of (42) the second relation is a consequence of the first one. To prove (59) we make use of the representation

$$f^\varepsilon(x,t)-F^\varepsilon(x,t)=\varepsilon^\gamma\frac{\partial^2}{\partial x_3^2}\vartheta_1\Big(\frac{x_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde B^\varepsilon}(x')f(T_{\mathbf{x}'/\varepsilon}\omega,t),$$

with

$$\vartheta_1(r)=\begin{cases}\frac12(r+1)^2, & -1\le r<0,\\ \frac12(r-1)^2, & 0\le r\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

For sufficiently small $\varepsilon>0$ we have

$$\begin{aligned}|u^\varepsilon(x,t)-\hat u^\varepsilon(x,t)|&=\varepsilon^\gamma\bigg|\int_0^t\int_Q G(t-s,x,y)\frac{\partial^2}{\partial y_3^2}\vartheta_1\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde B^\varepsilon}(y')f(T_{\mathbf{y}'/\varepsilon}\omega,s)\,dyds\bigg|\\ &=\varepsilon^\gamma\bigg|\int_0^t\int_Q \frac{\partial^2}{\partial y_3^2}G(t-s,x,y)\vartheta_1\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{1}_{\tilde B^\varepsilon}(y')f(T_{\mathbf{y}'/\varepsilon}\omega,s)\,dyds\bigg|.\end{aligned}$$

Using upper bound (36) and parabolic Schauder-type estimates (see [13, Chapter III, section 12]) we derive that Green's function satisfies the estimate

$$\Big|\frac{\partial^2}{\partial y_3^2}G(t,x,y)\Big|\le C(\delta)$$

for all x, y, and t such that $|x-y|\ge\delta$ and $t>0$. Then

$$|u^\varepsilon(x,t)-\hat u^\varepsilon(x,t)|\le C(\delta)\varepsilon^{2\gamma}$$

for all $x\in Q_\delta$, $t>0$, and $\varepsilon<\delta$. This implies the desired statement. ◻

Remark 8. In the particular case $\Phi(t,\mathbf{y}')=\varphi(t)\Phi_0(\mathbf{y}')$ with a smooth positive deterministic $\varphi(t)$, making straightforward computations similar to those of Proposition 2 it is easy to see that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbf{E}\big(\|\hat u^\varepsilon-\check u^\varepsilon\|^2_{L^2(0,1;L^2(Q))}\big)=\infty$$

if

$$\int_{\mathbb{R}^2}\mathbf{E}\big((\Phi([y'])-\mathbf{E}\Phi(0))(\Phi(0)-\mathbf{E}\Phi(0))\big)dy'=\infty. \tag{61}$$

If $\nu<2$, then there is a random statistically homogeneous field $\Phi([y'])$ such that condition R2 is fulfilled and relation (61) holds true. In this case the statement of Theorem 4 fails to hold. This shows that the condition $\nu>2$ is sharp.

4. Corrector's limit law. The aim of this section is to study the limit behavior of the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$. We are going to show that it converges weakly in law, as $\varepsilon \to 0$, towards a centered Gaussian random field. This result can be interpreted as a two-dimensional central limit theorem. We first show that $\varepsilon^{-1}(u^\varepsilon(x,t) - u^0(x,t))$ converges to a Gaussian random variable for each (x,t) such that $x_3 \neq 0$, and then we prove the convergence of finite-dimensional distributions of $\varepsilon^{-1}(u^\varepsilon - u^0)$ to the corresponding finite-dimensional distributions of a limit Gaussian random field.

Notice that by Lemma 5 the function $\varepsilon^{-1}(\hat{u}^\varepsilon(x,t) - u^\varepsilon(x,t))$ tends to zero for each $(x,t) \in Q \times (0,+\infty)$ such that $x_3 \neq 0$. It will also be proved later in this section that $\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - u^0(x,t))$ vanishes, as $\varepsilon \to 0$, for each $x \in Q \setminus \Sigma$. The asymptotic behavior of $\varepsilon^{-1}(u^\varepsilon - u^0)$ coincides with that of $\varepsilon^{-1}(\check{u}^\varepsilon - \hat{u}^\varepsilon)$. Bearing this in mind, we mainly focus on the limit behavior of the latter quantity.

Denote

$$\bar{\bar{c}}(t,s) = \lim_{N\to\infty} \frac{1}{N^2} \int_{[0,N]^4} \mathbf{R}(t,s,y,\zeta) \mathbb{1}_{K^\Sigma}(y) \mathbb{1}_{K^\Sigma}(\zeta) dy_1 dy_2 d\zeta_1 d\zeta_2, \tag{62}$$

where K^Σ and $\mathbf{R}(t,s,y,\zeta)$ have been defined in (4) and (28), respectively. It is easy to verify that under condition R1 the above limit exists and admits the upper bound

$$\bar{\bar{c}}(t,s) \le 4 \int_0^\infty \int_0^\infty \bar{R}(y) dy_1 dy_2.$$

Theorem 6. *Assume that one of the conditions* M1 *or* M2 *is fulfilled with* $\nu_1 > 2$. *Then for each* $t > 0$ *and* $x \in Q$, $x_3 \neq 0$, *the normalized difference* $\varepsilon^{-1}(\check{u}^\varepsilon - \hat{u}^\varepsilon)$ *of a solution* $\hat{u}^\varepsilon$ *of* (32)–(33) *and a solution* $\check{u}^\varepsilon$ *of* (44)–(45) *converges in law towards a Gaussian random variable with zero mean and variance*

$$\sigma^2(t,x) = \int_0^t \int_0^t \int_\Pi G(t-s,x,(y',0)) G(t-r,x,(y',0)) \bar{\bar{c}}(s,r) dy' ds dr. \tag{63}$$

Proof. We follow the classical scheme of proof of the central limit theorem; namely, we are going to show that the characteristic functions $\mathbf{E} \exp\left(\frac{i\lambda}{\varepsilon}(\check{u}^\varepsilon - \hat{u}^\varepsilon)\right)$ converge, as $\varepsilon \to 0$, to the function $\exp(-\sigma^2\lambda^2)$ with σ^2 defined in (63).

First, let δ be a sufficiently small positive number, say $\delta = \frac{1}{2}(\nu_1 - 2)/(\nu_1 + 2)$; then we define

$$L_\varepsilon = \left[\varepsilon^{\frac{1+\delta}{2}}/\varepsilon\right], \quad N_\varepsilon = L_\varepsilon^2, \tag{64}$$

where, as above, the symbol $[\cdot]$ stands for the integer part. Clearly, $N_\varepsilon = \varepsilon^{-(1-\delta)} + O\big(\varepsilon^{-\frac{1-\delta}{2}}\big)$.

We introduce also, for any $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$, the following two sets:

$$\begin{aligned} J_{\mathbf{z}} = \{\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi \, : \\ L_\varepsilon z_1 + 1 \le j_1 \le L_\varepsilon(z_1+1);\ L_\varepsilon z_2 + 1 \le j_2 \le L_\varepsilon(z_2+1)\}; \end{aligned} \tag{65}$$

$$S_{\mathbf{z}} = \{x' \in \mathbb{R}^2 \cap \Pi \, : \, \varepsilon L_\varepsilon z_1 \le x_1 \le \varepsilon L_\varepsilon(z_1+1);\ \varepsilon L_\varepsilon z_2 \le x_2 \le \varepsilon L_\varepsilon(z_2+1)\}. \tag{66}$$

Next, we introduce the four sets of indices:

$$I^{00} = \{\mathbf{z} = (z_1,z_2) \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi \, : \, z_1 \text{ and } z_2 \text{ are even}\}, \tag{67}$$

$$I^{01} = \{\mathbf{z} = (z_1,z_2) \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi \, : \, z_1 \text{ is even, } z_2 \text{ is odd}\}, \tag{68}$$

$$I^{10} = \{\mathbf{z} = (z_1,z_2) \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi \, : \, z_1 \text{ is odd, } z_2 \text{ is even}\}, \tag{69}$$

$$I^{11} = \{\mathbf{z} = (z_1,z_2) \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi \, : \, z_1 \text{ and } z_2 \text{ are odd}\}. \tag{70}$$

Clearly, all four sets do not intersect and their union coincides with $\mathbb{Z}^2 \cap \varepsilon^{-1}\Pi$.

Finally, we introduce a family of random variables:

$$X_{\mathbf{j}} = X^{\varepsilon}_{\mathbf{j}}(x,t) = \int_0^t \int_{\Sigma} G(t-s,x,(y',0))\big(\psi(T_{\mathbf{j}}\omega,t) - \mathbf{E}\psi(\cdot,t)\big)\mathbb{1}_{\tilde{B}^{\mathbf{j}}_{\varepsilon}}(y')dy'dt; \qquad \mathbf{j} \in \mathbb{Z}^2 \cap \varepsilon^{-1}\Pi, \tag{71}$$

and

$$\eta^{\varepsilon}_{\mathbf{z}} = \frac{1}{\varepsilon}\sum_{\mathbf{j}\in J_{\mathbf{z}}} X_{\mathbf{j}}; \qquad \theta^{\varepsilon}_{lm} = \sum_{\mathbf{z}\in I^{lm}} \eta^{\varepsilon}_{\mathbf{z}}, \quad l,m = 0,1. \tag{72}$$

Notice that $\varepsilon^{-1}(\check{u}^{\varepsilon} - \hat{u}^{\varepsilon}) = \theta^{\varepsilon}_{00} + \theta^{\varepsilon}_{01} + \theta^{\varepsilon}_{10} + \theta^{\varepsilon}_{11}$.

With all of this preliminary notation, we are now able to prove the following statement.

LEMMA 6. *Under either of conditions* M1 *or* M2 *with* $\nu_1 > 2$*, the random vector* $(\theta^{\varepsilon}_{00}, \theta^{\varepsilon}_{10}, \theta^{\varepsilon}_{01}, \theta^{\varepsilon}_{11})$ *converges in law, as* $\varepsilon \to 0$*, to a Gaussian vector* $(\theta^{0}_{00}, \theta^{0}_{10}, \theta^{0}_{01}, \theta^{0}_{11})$ *consisting of four independent identically distributed Gaussian random variables with zero average and variance* $\frac{1}{4}\sigma^2(x,t)$*,* σ^2 *being defined in* (63)*.*

Proof. First we show that $\theta^{\varepsilon}_{00}$ converges in law to a Gaussian random variable with zero mean and variance $\frac{1}{4}\sigma^2(x,t)$. To this end we study the limit behavior of the characteristic function of $\theta^{\varepsilon}_{00}$:

$$\Theta(\lambda) = \mathbf{E}\exp(i\lambda\theta^{\varepsilon}_{00}).$$

Let $\mathbf{z}^1$ be one of the elements of I^{00}. Clearly, for any $x \in S_{\mathbf{z}^1}$ and $y \in \bigcup_{\mathbf{z}\in I^{00}\setminus\{\mathbf{z}^1\}} S_{\mathbf{z}}$, it holds that $\left|\frac{x}{\varepsilon} - \frac{y}{\varepsilon}\right| \ge \varepsilon^{-\frac{1-\delta}{2}}$. Then, from conditions M1 or M2, and due to Lemma VIII.3.102 in [10], one has

$$\left|\mathbf{E}\bigg(\prod_{\mathbf{z}\in I^{00}} \exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})\bigg) - \mathbf{E}\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}^1})\mathbf{E}\bigg(\prod_{\mathbf{z}\in I^{00}\setminus\{\mathbf{z}^1\}} \exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})\bigg)\right| \le C(\varepsilon^{-\frac{1-\delta}{2}})^{-\nu_1};$$

here we have also used the fact that $|\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}^1})| = 1$ and $\left|\prod_{\mathbf{z}}\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})\right| = 1$.

Iterating this procedure and considering the inequality $|I^{00}| \le c\varepsilon^{-1-\delta}$, we end up with the relation

$$\left|\mathbf{E}\bigg(\prod_{\mathbf{z}\in I^{00}} \exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})\bigg) - \bigg(\prod_{\mathbf{z}\in I^{00}} \mathbf{E}\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})\bigg)\right| \le c\varepsilon^{-1-\delta}\varepsilon^{\nu_1\frac{1-\delta}{2}} = c\varepsilon^{\frac{1}{4}(\nu_1-2)}; \tag{73}$$

the last equality here is a consequence of our choice of δ; indeed, if $\delta = \frac{\nu_1-2}{2(\nu_1+2)}$, then $-1-\delta+\nu_1\frac{1-\delta}{2} = \frac{\nu_1-2}{4}$.

Our next aim is to compute the expectation $\mathbf{E}\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}})$, $\mathbf{z} \in I^{00} \cap \varepsilon^{-1}\Pi$. From (64), (65), (66), and (72) we deduce that $|\eta^{\varepsilon}_{\mathbf{z}}| \le c\varepsilon^{\delta}$ for any $\mathbf{z} \in I^{00}$. Therefore, by the Taylor formula,

$$\left|\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}}) - 1 - i\lambda\eta^{\varepsilon}_{\mathbf{z}} + \frac{\lambda^2}{2}(\eta^{\varepsilon}_{\mathbf{z}})^2\right| \le c\varepsilon^{\delta}(\eta^{\varepsilon}_{\mathbf{z}})^2.$$

This yields

$$\left|\mathbf{E}\exp(i\lambda\eta^{\varepsilon}_{\mathbf{z}}) - 1 + \frac{\lambda^2}{2}\mathbf{E}(\eta^{\varepsilon}_{\mathbf{z}})^2\right| \le c(\lambda)\varepsilon^{\delta}\mathbf{E}(\eta^{\varepsilon}_{\mathbf{z}})^2.$$

Taking into account the definition of $\eta_{\mathbf{z}}^\varepsilon$ in (72), after straightforward computations we obtain

$$\mathbf{E}(\eta_{\mathbf{z}}^\varepsilon)^2 = \varepsilon^{1+\delta}\int_0^t\int_0^t G(t-s,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))G(t-r,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))\bar{\bar{c}}(s,r)dsdr + o(\varepsilon^{1+\delta}), \tag{74}$$

where $\bar{\bar{c}}(s,r)$ is defined in (62). Indeed,

$$\mathbf{E}(\eta_{\mathbf{z}}^\varepsilon)^2 = \int_{S_{\mathbf{z}}}\int_{S_{\mathbf{z}}}\int_0^t\int_0^t G(t-s,x,(y',0))G(t-r,x,(\zeta',0)) \times \mathbf{R}\Big(s,r,\frac{y'}{\varepsilon},\frac{\zeta'}{\varepsilon}\Big)\mathbb{1}_{\tilde{B}_\varepsilon}(\zeta')\mathbb{1}_{\tilde{B}_\varepsilon}(y')d\zeta'dy'dsdr.$$

Since $\nu_1 > 2$, assumption R1 is satisfied. Also, for $x_3 \neq 0$, the function $G(t-s,x,(y',0))$ is continuously differentiable in s and y'. Therefore,

$$\begin{aligned}&\int_{S_{\mathbf{z}}}\int_{S_{\mathbf{z}}} G(t-s,x,(y',0))G(t-r,x,(\zeta',0))\mathbf{R}\Big(s,r,\frac{y'}{\varepsilon},\frac{\zeta'}{\varepsilon}\Big)\mathbb{1}_{\tilde{B}_\varepsilon}(\zeta')\mathbb{1}_{\tilde{B}_\varepsilon}(y')d\zeta'dy' \\ &= \int_{S_{\mathbf{z}}}\int_{S_{\mathbf{z}}} G(t-s,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))G(t-r,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))\mathbf{R}\Big(s,r,\frac{y'}{\varepsilon},\frac{\zeta'}{\varepsilon}\Big)\mathbb{1}_{\tilde{B}_\varepsilon}(\zeta')\mathbb{1}_{\tilde{B}_\varepsilon}(y')d\zeta'dy' \\ &\quad + o(\varepsilon^{\frac{1+\delta}{2}}) = \varepsilon^{\frac{1+\delta}{2}} G(t-s,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))G(t-r,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))\bar{\bar{c}}(s,r) + o(\varepsilon^{\frac{1+\delta}{2}})\end{aligned}$$

for all s and r, $0 \le s,r \le t$, and the relation (74) follows. Passing to the limit, as $\varepsilon \to 0$, in (73) yields

$$\begin{aligned}&\lim_{\varepsilon\to 0}\mathbf{E}\Big(\prod_{\mathbf{z}\in I^{00}}\exp(i\lambda\eta_{\mathbf{z}}^\varepsilon)\Big) \\ &= \lim_{\varepsilon\to 0}\prod_{\mathbf{z}\in I^{00}}\Big(1-\frac{\lambda^2}{2}\varepsilon^{1+\delta}\int_0^t\int_0^t G(t-s,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0)) \\ &\qquad G(t-r,x,(\varepsilon^{\frac{1+\delta}{2}}\mathbf{z},0))\bar{\bar{c}}(s,r)dsdr + o(\varepsilon^{1+\delta})\Big) \\ &= \exp\Big(-\frac{\lambda^2}{8}\int_0^t\int_0^t\int_\Pi G(t-s,x,(y',0))G(t-r,x,(y',0))\bar{\bar{c}}(s,r)dsdrdy'\Big) \\ &= \exp(-(\lambda^2/8)(\sigma(x,t))^2;\end{aligned}$$

the factor 1/8 appears in the last line here because the set I^{00} contains only the integer vectors with even coordinates. Thus, θ_{00}^ε converges in law towards a Gaussian random variable with zero mean and variance $\frac{1}{4}\sigma^2(x,t)$. Similarly, one can show that the random variables θ_{01}^ε, θ_{10}^ε, and θ_{11}^ε converge in law to the same limit.

Making use of our mixing assumptions, M1 or M2, it is straightforward to check that the vector $(\theta_{01}^\varepsilon, \theta_{01}^\varepsilon, \theta_{10}^\varepsilon, \theta_{11}^\varepsilon)$ converges in law to a vector consisting of independent random variables. This implies that $\theta_{00}^\varepsilon + \theta_{01}^\varepsilon + \theta_{10}^\varepsilon + \theta_{11}^\varepsilon$ converges in law to a Gaussian random variable with zero mean and variance $(\sigma(x,t))^2$. This completes the proof of Lemma 6. ◻

The statement of Theorem 6 is now an immediate consequence of the formula $\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)) = \theta_{00}^\varepsilon + \theta_{01}^\varepsilon + \theta_{10}^\varepsilon + \theta_{11}^\varepsilon$. ◻

The result of the last theorem can be generalized to the finite-dimensional distributions of the normalized difference $\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t))$.

THEOREM 7. *Under the assumptions of Theorem* 6*, for any finite set* (x^1,t^1), $\dots,(x^N,t^N)$, $x_3^j \neq 0$*, the random vector* $\big\{(\varepsilon^{-1}(\hat{u}^\varepsilon(x^j,t^j)-\check{u}^\varepsilon(x^j,t^j)))\big\}$, $j=1,\dots,N$, *converges in law towards a centered Gaussian vector with the covariance matrix* $\{\sigma_{ij}\}$ *given by*

$$\sigma_{ij} = \int_0^{t^i}\int_0^{t^j}\int_\Pi G(t^i-s,x^i,(y',0))G(t^j-r,x^j,(y',0))\bar{\bar{c}}(s,r)dy'drds. \tag{75}$$

Proof. The proof is exactly the same as that of Theorem 6. ∎

We proceed with estimating the difference $(u^0-\check{u}^\varepsilon)$. In order to improve the estimate (46), we will suppose that the ε-periodic structure is consistent with the rectangle $\Pi = [-\beta_1,\beta_1]\times[-\beta_2,\beta_2]$; i.e., we add the following assumption.

C1. Both numbers β_1 and β_2 are integer multipliers of ε.

LEMMA 7. *Under assumption* C1*, for any* $t \geq 0$ *and* $x \in (Q\setminus\Sigma)$*, the following relation holds:*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}|\check{u}^\varepsilon(x,t)-u^0(x,t)| = 0.$$

Proof. Denote $H(y',t) = 2s_3\mathbf{E}\{\Phi(\cdot,t)\}\mathbb{1}_{K^\Sigma}(y') - F(t)$, with $F(t)$ defined in (16). Then $H(y',t)$ is a periodic function in y' having zero average for each $t\geq 0$. The difference $(\check{u}^\varepsilon(x,t)-u^0(x,t))$ can now be represented as

$$\check{u}^\varepsilon(x,t)-u^0(x,t) = \int_0^t\int_\Pi G(t-s,x,(y',0))H\left(\frac{y'}{\varepsilon},t\right)dy'ds.$$

Let $\zeta(y')$ be a periodic solution to the problem $\Delta_{y'}\zeta(y',t) = H(y',t)$, and denote $h(y',t) = \nabla_{y'}\zeta(y',t)$. Then $H(\frac{y'}{\varepsilon},t) = \varepsilon\mathrm{div}_{y'}h(\frac{y'}{\varepsilon},t)$; substituting this last relation into the above integral and integrating by parts, we obtain the desired bound. ∎

Combining the statements of Lemma 7, Proposition 2, and Theorem 7, we arrive at the last and main result of this study.

THEOREM 8. *Let condition* C1 *be fulfilled, and assume that at least one of the conditions* M1 *or* M2 *holds true with* $\nu_1 > 2$*. Then for any finite set* $(x^1,t^1),(x^2,t^2)$, $\dots,(x^N,t^N)$ *with* $x_3^j\neq 0$*, the random vector* $\big\{(\varepsilon^{-1}(u^\varepsilon(x^j,t^j)-u^0(x^j,t^j)))\big\}$, $j=1,\dots,N$*, converges in law towards a centered Gaussian vector with the covariance matrix* $\{\sigma_{ij}\}$ *introduced in* (75).

Remark 9. Without the assumption C1 the statement of Lemma 7 might fail to hold and the difference $(\check{u}^\varepsilon - u^0)$ might be of order ε.

Remark 10. The statement of the last theorem shows in particular that the estimates for the rate of convergence obtained in the previous section are optimal. Indeed, it follows from Theorem 8 that for any $(x,t)\in Q\times(0,T)$ we have

$$\mathbf{E}(|u^\varepsilon(x,t)-u^0(x,t)|^2) = \sigma^2(x,t)\varepsilon^2(1+o(1)),$$

where $o(1)$ tends to zero as $\varepsilon\to 0$. Since, except for some degenerate cases, $\sigma^2(x,t) > 0$ for all (x,t), then by the Fatou lemma

$$\lim_{\varepsilon\to 0}\mathbf{E}\Big(\frac{1}{\varepsilon^2}\|u^\varepsilon-u^0\|^2_{L^2(Q\times(0,T))}\Big) \geq \int_0^T\int_Q\sigma^2(x,t)dxdt > 0.$$

This shows that the estimate of Theorem 2 and other estimates of section 2 are optimal.

Remark 11. In problem (12)–(13) the right-hand side f^ε has a specific periodic geometry and is statistically homogeneous with respect to a discrete grid. There is no essential change in the results if we suppose that

$$f^\varepsilon = \frac{1}{2\varepsilon^\gamma}\mathbb{1}_\Pi(x')\mathbb{1}_{[-\varepsilon^\gamma,\varepsilon^\gamma]}(x_3)\Phi(x'/\varepsilon,t), \qquad \gamma > 1,$$

with $\Phi(y',t)$ being a random field bounded statistically homogeneously in $y' \in \mathbb{R}^2$; in this case the corresponding dynamical system has a continuous arguments $y' \in \mathbb{R}^2$. If the random field $\phi^\varepsilon(x',t) = \Phi(x'/\varepsilon,t)$ possesses either of the conditions M1 or M2 with $\nu_1 > 2$, then the statement of Theorem 8 holds true with

$$\sigma_{ij} = \int_0^{t^i}\int_0^{t^j}\int_\Pi G(t^i-s,x^i,(y',0))G(t^j-r,x^j,(y',0))\left(\int_{\mathbb{R}^2}\widehat{R}(s,r,z')dz'\right)dy'drds,$$

where $\widehat{R}(s,r,z') = \mathbf{E}\big((\Phi(y',s)-\mathbf{E}\Phi(y',s))(\Phi(0,r)-\mathbf{E}\Phi(0,r))\big)$.

5. Operators with oscillating coefficients. Homogenization. Although this is not the case of the application we have in mind (modeling a geological nuclear waste repository), we consider in this section a possible extension of the above results to the case of operators with rapidly oscillating coefficients. The most interesting case corresponds to the two characteristic lengths (oscillations and period of the sources) being of same order; it leads to the following local problem:

$$\partial_t u^\varepsilon - \operatorname{div}\left(a\left(x,\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + \operatorname{div}\left(b\left(x\frac{x}{\varepsilon}\right)u^\varepsilon\right) = f^\varepsilon \quad \text{in } Q\times(0,\infty); \tag{76}$$

$$u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^\varepsilon - b(x)\cdot n(x)u^\varepsilon + \lambda u^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{77}$$

All the assumptions on the right-hand side f^ε and on the domain are the same as in sections 1–4. Concerning the coefficients, we suppose one of the following assumptions holds.

- **O1**. For each $x \in Q$ the functions $a_{ij}(x,y) = a_{ij}(x,y,\omega)$ and $b_i(x,y) = b_i(x,y,\omega)$ are statistically homogeneous ergodic random fields of $y \in \mathbb{R}^3$. The realizations $a_{ij}(x,y,\omega)$ and $b_i(x,y,\omega)$ are smooth in x and y.
- **O2**. $a_{ij}(x,y)$ and $b_i(x,y)$ are smooth periodic in y functions on $\overline{Q}\times\mathbb{R}^3$.

We also assume the uniform ellipticity conditions: there is $\Lambda > 0$ such that

$$a(x,y) \geq \Lambda\mathbf{I}, \qquad |a_{ij}(x,y)| \leq \Lambda^{-1}, \quad |b_i(x,y)| \leq \Lambda^{-1},$$

where $\mathbf{I}$ stands for the unit 3×3 matrix.

In this section we consider only the case of a finite time interval; the results for the infinite interval follow from the dissipativity as in the previous sections.

We will consider two types of oscillating coefficients: random and periodic. We first consider the operators with randomly oscillating coefficients. Simple arguments show that we cannot expect a precision better than that of order $\sqrt{\varepsilon}$. However, this precision has not been achieved. To the best of our knowledge, the best result in this direction was proved in [16], where it was shown that if the uniform mixing coefficient decays polynomially, then the discrepancy admits an estimate by some positive power of ε. This exponent is defined in a rather implicit way and can be quite small.

This means that the arguments of sections 3 and 4 in this case are irrelevant. We justify only the convergence.

THEOREM 9. *Assume that* $\tilde a(x,\omega)$ *and* $\tilde b(x,\omega)$ *are elements of* $C(\overline{Q}; L^\infty(\Omega))$, *and suppose that* $a(x,y) = \tilde a(x,\mathcal{T}_y\omega)$, $b(x,y) = \tilde b(x,\mathcal{T}_y\omega)$, *where* $\mathcal{T}.$ *is an ergodic three-dimensional dynamical system. Then a solution* u^ε *converges in* $L^2(0,T;L^2(Q))$ *to a solution of the problem*

$$\partial_t u^0 - \operatorname{div}(a^{\hom}(x)\nabla u^0) + \operatorname{div}(b^{\hom}(x)u^0) = \mathbb{1}_\Pi(x')F(t)\delta_\Sigma(x) \quad \text{in } Q\times(0,\infty); \tag{78}$$

$$u^0\big|_{t=0} = 0, \qquad a^{\hom}\nabla u^0\cdot n - b^{\hom}\cdot n u^0 + \lambda u^0 = 0 \quad \text{on } \partial Q\times(0,\infty), \tag{79}$$

with $F(t)$ *defined in* (16)*; here* $a^{\hom}(x)$ *and* $b^{\hom}(x)$ *are the coefficients of the homogenized equation (see* [12], [15], [11]*).*

Proof. Denote by U^ε a solution of the following initial-boundary problem:

$$\partial_t U^\varepsilon - \operatorname{div}\left(a\left(x,\frac{x}{\varepsilon}\right)\nabla U^\varepsilon\right) + \operatorname{div}\left(b\left(x\frac{x}{\varepsilon}\right)U^\varepsilon\right) = F_\Pi\delta_\Sigma \quad \text{in } Q\times(0,\infty); \tag{80}$$

$$U^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}U^\varepsilon - b(x)\cdot n(x)U^\varepsilon + \lambda U^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{81}$$

With the help of the standard energy estimate it is straightforward to check that

$$\|u^\varepsilon - U^\varepsilon\|_{L^2(0,T;H^1(Q))} \le C(T)\|f^\varepsilon - F_\Pi\delta_\Sigma\|_{L^2(0,T;H^{-1}(Q))},$$

with a constant $C(T)$ which depends only on the ellipticity constants and does not depend on ε. By Lemma 2 this yields

$$\|u^\varepsilon - U^\varepsilon\|_{L^2(0,T;H^1(Q))} \longrightarrow 0, \tag{82}$$

as $\varepsilon\to 0$. The fact that U^ε converges to u^0, as $\varepsilon\to0$, in $L^2(0,T;L^2(Q))$ is a classical homogenization result. Combined with (82) this implies the desired statement. ∎

Remark 12. Notice that the above convergence result follows from the facts that

– the operator in (12)–(13) admits a.s. homogenization;

– the right-hand side in (12) converges a.s. in the $L^2(0,T;H^{-1}(Q))$ norm.

This means in particular that the result remains valid in the case when the operator coefficients and the right-hand side have different stationarities and different scaling factors (microscopic length scales), and that the homogenization and the upscaling processes do not interact.

We proceed with the case of periodic homogenization. Here, in order to avoid boundary effects, we assume that $a(x,y)$ and $b(x,y)$ do not depend on y in a sufficiently small neighborhood of ∂Q.

As was already announced, in the periodic case the results similar to those of sections 3 and 4 hold. We illustrate this by proving the statement similar to that of Theorem 8. Other results can be obtained in an analogous way. First we introduce some notation. Denote by $G^{\hom}(t,x,y)$ Green's function of the limit problem (78)–(79) and by $\chi(x,y)$ a periodic in y solution of the cell problem

$$\operatorname{div}_y a(x,y)(\nabla_y\chi(x,y) + \mathrm{I}) = 0.$$

It is well known that this problem has a unique up to an additive constant solution $\chi(x,y)$; in order to fix the choice of the additive constant, we assume that the average of χ in y is equal to zero.

Using a classical technique of asymptotic expansion (see [4], [3]) and the standard parabolic estimates, it is straightforward to deduce that for all $x \in Q$ and $y \in Q$, $x \neq y$, and $t \geq 0$ the following bound holds:

$$\left| G^{\varepsilon}(t,x,y) - G^{\text{hom}}(t,x,y) - \varepsilon\chi\Big(x, \frac{x}{\varepsilon}\Big) \cdot \nabla_x G^{\text{hom}}(t,x,y) \right| \leq C\varepsilon^2, \tag{83}$$

where $G^{\varepsilon}(t,x,y)$ is Green's function of problem (76)–(77), and the constant C depends only on the distance $|x-y|$. Also we introduce

$$\sigma_{ij}^{\text{hom}} = \int_0^{t^i} \int_0^{t^j} \int_{\Pi} G^{\text{hom}}(t^i - s, x^i, (y', 0)) G^{\text{hom}}(t^j - r, x^j, (y', 0)) \bar{\bar{c}}(s,r) dy' dr ds, \tag{84}$$

with $\bar{\bar{c}}(s,r)$ defined in (62).

THEOREM 10. *Let the above condition* O2 *on $a(x,y)$ and $b(x,y)$ and assumption* C1 *be fulfilled, and assume that at least one of the conditions* M1 *or* M2 *holds true with $\nu_1 > 2$. Then for any finite set $(x^1,t^1), (x^2,t^2), \ldots, (x^N,t^N)$ with $x_3^j \neq 0$, the random vector $\big\{(\varepsilon^{-1}(u^{\varepsilon}(x^j,t^j) - u^0(x^j,t^j)))\big\} - \chi\big(x, \frac{x}{\varepsilon}\big) \cdot \nabla u^0(x^j,t^j)$, $j = 1, \ldots, N$, converges in law towards a centered Gaussian vector with the covariance matrix $\{\sigma_{ij}^{\text{hom}}\}$ introduced in* (84).

Proof. The proof relies on bound (83) and then follows along the lines of the proof of Theorem 8. $\square$

REFERENCES

[1] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc., 73 (1967), pp. 890–896.

[2] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa (3), 22 (1968), pp. 607–694.

[3] N. Bakhvalov and G. Panasenko, *Homogenisation: Averaging Processes in Periodic Media. Mathematical Problems in the Mechanics of Composite Materials*, Math. Appl. (Soviet Ser.) 36, Kluwer Academic, Dordrecht, The Netherlands, 1989.

[4] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis of Periodic Structures*, North–Holland, Amsterdam, 1978.

[5] R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd ed., John Wiley & Sons, New York, 2007.

[6] A. Bourgeat, O. Gipouloux, and E. Marušić-Paloka, *Mathematical modeling of an underground waste disposal site by upscaling*, Math. Methods Appl. Sci., 27 (2004), pp. 381–403.

[7] A. Bourgeat and E. Marušić-Paloka, *A homogenized model of an underground waste repository including a disturbed zone*, Multiscale Model. Simul., 3 (2005), pp. 918–939.

[8] A. Bourgeat, E. Marušić-Paloka, and A. Piatnitski, *Scaling up of an underground nuclear waste repository including a possibly damaged zone*, Asymptot. Anal., 67 (2010), pp. 147–165.

[9] O. Gipouloux and F. Smai, *Scaling up of an underground waste disposal model with random source terms*, Internat. J. Multiscale Comput. Engrg., 6 (2008), pp. 309–325.

[10] J. Jacod and A. Shiryayev, *Limit Theorem for Stochastic Processes*, Springer-Verlag, Berlin, 2003.

[11] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, New York, 1994.

[12] S. M. Kozlov, *The averaging of random operators*, Math. USSR Sb., 109 (1979), pp. 188–202.

[13] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. 23, AMS, Providence, RI, 1968.

[14] R. Liptser and A. Shiryayev, *Theory of Martingales*, Math. Appl. (Soviet Ser.) 49, Kluwer Academic, Dordrecht, The Netherlands, 1989.

[15] G. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, in Random Fields. Rigorous Results in Statistical Mechanics and Quantum Field Theory, Colloq. Math. Soc. János Bolyai 27, North–Holland, Amsterdam, New York, 1981, pp. 835–873.

[16] V. Yurinski, *Averaging of symmetric diffusion in random medium*, Sibirskii Mat. Zh., 27 (1986), pp. 167–180.