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HOMOGENIZATION OF ELLIPTIC DIFFERENCE OPERATORS*

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Abstract. We develop some aspects of general homogenization theory for second order elliptic difference operators and consider several models of homogenization problems for random discrete elliptic operators with rapidly oscillating coefficients. More precisely, we study the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast checker-board structures. Then we compare, for various discretization methods, the effective coefficients obtained with the homogenized coefficients for corresponding differential operators.

Key words. random media, homogenization, H-convergence, difference operator, percolation, random walk

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1. Introduction.

We develop some aspects of general H-convergence and homogenization theory for second order elliptic difference operators and consider several homogenization problems for random discrete elliptic operators with rapidly oscillating coefficients. More precisely, we study the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast checker-board structures. Then we compare, for various discretization methods, the effective coefficients obtained with the homogenized coefficients for corresponding differential operators.

Many results can also be formulated in terms of the central limit theorem for random walks in random statistically homogeneous media.

Originally, G- and H-convergence of differential operators and Γ-convergence of the corresponding functionals were introduced by Spagnolo [27], De Giorgi [7], [8], and Murat and Tartar [22]. Then these notions were developed and generalized essentially in the works of Bensoussan, Lions, and Papanicolaou [4], Tartar [26], Murat [21], Jikov et al. [28], G. Dal Maso [18], and many others. This resulted in the appearance of advanced homogenization theory.

In recent years, significant progress has been achieved in the homogenization theory of random differential operators. We refer to the original works of Kozlov [13] and Papanicolaou and Varadhan [24], and to the book by Jikov, Kozlov, and Oleinik [11] wherein an additional bibliography can be found. In particular, in case of random high-contrast checker-board structures, the asymptotics of effective diffusion have been constructed in Jikov, Kozlov, and Oleinik [11]. Berlyand and Golden in [5] have improved this result in a special case.

In contrast with differential operators, the homogenization theory of difference operators is not so well developed. There are only a few mathematical works on this subject, among them Künnemann [17], Kozlov [14], [15], and Krasniansky [16]. In

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[17] it is proved that the central limit theorem holds for symmetric random walks in random ergodic statistically homogeneous media. Then, many interesting results for various kinds of random walks in random media were obtained in Kozlov [14]. The first homogenization results for difference schemes were formulated and proved in Kozlov [15]. We also mention the work Bricmont and Kupiainen [6] where the central limit theorem was obtained for a class of nonsymmetric random walks.

Perhaps the difference operators with rapidly oscillating coefficients did not attract the attention of mathematicians because these operators did not appear in the classical difference schemes approximation approach (see, for example, Quarteroni and Valli [25]): the fast oscillation of coefficients of difference schemes would contradict the regularity and even the measurability of coefficients of the initial differential equations.

On the other hand, many modern practical and numerical applications involve various homogenization problems for discrete operators with rapidly oscillating coefficients. For instance, when discretizing microinhomogeneous media, due to the natural restrictions, it is not possible to keep the size of the numerical grid much smaller than the typical size of inhomogeneity (the microscopic length scale) of the medium. This leads to the appearance of difference operators with rapidly oscillating coefficients (see, for instance, McCarthy [19], Nœtinger [23]). The most important question here is, How far could the effective coefficients of a difference scheme diverge from ones of corresponding differential operators? The first successful attempt to answer this question was done by Avellaneda, Hou, and Papanicolaou [2] where it was shown that, in the multidimensional case, the finite difference approach does not provide the right homogenized coefficients unless the ratio of the size of a discretization mesh to the microscopic length scale goes to 0.

In the present work we show that the effective coefficients of the difference schemes approximating a family of elliptic PDEs with rapidly oscillating coefficients depend essentially on the discretization method.

The paper is divided into two parts. The first one is devoted to H-convergence and homogenization of difference operators.

Earlier homogenization problems for difference operators were investigated by Kozlov in [15] where a number of homogenization results for difference schemes were obtained. In the present work we extend further the homogenization theory of discrete operators and prove a number of basic statements such as convergence of solutions of the Neumann problem, convergence of energies and of arbitrary solutions, Γ-convergence, and some others. To this end we mainly use the discrete analogue of the compensated compactness technique originally introduced in Murat [21] and Tartar [26] for functions of continuous arguments. Namely, we prove a version of compensated compactness lemma, adapted to difference operators, and then apply it systematically in our considerations in combination with the method of correctors and variational techniques.

For the sake of completeness we also formulate some technical results from Kozlov [15] and give another proof of the homogenization theorem for random difference operators. An additional reason for this is the fact that we use a more general definition of ellipticity than that in [15].

It should be noted that although some basic ideas here have been borrowed from homogenization theory of differential equations, still the peculiarities of difference operators such as the big dimension of difference gradient, the irreducibility and ellipticity conditions in the case of boundary-value problems, and the asymptotic nature of difference schemes, create additional difficulties in studying these operators and make the generalization of homogenization theory to difference operators nontrivial.

In the second part of the paper, we discretize high-contrast two-dimensional checker-board structures, find the asymptotics of effective diffusion, and show that different discretization methods lead to different asymptotics.

1.1. Difference elliptic operators. Let $Q \subset \mathbb{R}^d$ be a smooth bounded domain and let $Q_\varepsilon = Q \cap \varepsilon\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the standard integer lattice in $\mathbb{R}^d$ and $\varepsilon > 0$. We consider the discrete Dirichlet problem in Q_ε:

(1.1)
$$A_\varepsilon u^\varepsilon(x) = \sum_{z,z' \in \Lambda} \partial^\varepsilon_{-z}\left(a^\varepsilon_{zz'}(x)\partial^\varepsilon_{z'} u^\varepsilon(x)\right) = f^\varepsilon(x) \quad \text{in } Q_\varepsilon\,, \qquad u^\varepsilon(x) = 0 \quad \text{on } \partial Q^\Lambda_\varepsilon\,.$$

Here Λ is a fixed finite subset of $\mathbb{Z}^d$ symmetric with respect to 0, the matrix $\mathcal{A}^\varepsilon = \{a^\varepsilon_{zz'}\}$ is symmetric, $\partial Q^\Lambda_\varepsilon$ is the boundary of Q_ε defined by

$$\partial Q^\Lambda_\varepsilon \stackrel{\triangle}{=} (Q_\varepsilon + \varepsilon\Lambda) \setminus Q_\varepsilon = \{x + \varepsilon z \,|\, x \in Q_\varepsilon,\, z \in \Lambda\} \setminus Q_\varepsilon\,,$$

and ∂^ε_z is the standard difference derivative: $(\partial^\varepsilon_z v)(x) \stackrel{\triangle}{=} \frac{1}{\varepsilon}\left(v(x+\varepsilon\, z) - v(x)\right)$. For any $v^\varepsilon : Q_\varepsilon \mapsto \mathbb{R}$, we introduce the following norm (the $L^2(Q_\varepsilon)$-norm): $\|v^\varepsilon\|^2_{L^2(Q_\varepsilon)} \stackrel{\triangle}{=} \varepsilon^d \sum_{x\in Q_\varepsilon} |v^\varepsilon(x)|^2$. We say that a function v^ε defined on $\varepsilon\mathbb{Z}^d$ belongs to the space $W^{1,2}_0(Q_\varepsilon)$ if $v(x) = 0$ for $x \notin Q_\varepsilon$. We define the norm on the space $W^{1,2}_0(Q_\varepsilon)$ as follows: $\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)} = \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{i=1}^d |\partial^\varepsilon_{\pm e_i} v^\varepsilon(x)|^2$, where $\{e_i\}_{i=1,\dots,d}$ is the standard basis in $\mathbb{R}^d$ and $\overline{Q_\varepsilon} \stackrel{\triangle}{=} Q_\varepsilon + \varepsilon\Lambda = Q_\varepsilon \cup \partial Q^\Lambda_\varepsilon$; $W^{-1,2}(Q_\varepsilon)$ is the dual space to $W^{1,2}_0(Q_\varepsilon)$.

In the summation in (1.1), we can consider only the elements from the set $\Lambda \setminus \{0\}$, as the contribution of the element $\{0\}$ is null.

Definition 1.1. *We say that the family of problems* (1.1) *(or, simply, problem* (1.1)*) is* uniformly elliptic *if there are $c_1,\, c_2 > 0$ and $\varepsilon_0 > 0$ such that, for any $v^\varepsilon \in W^{1,2}_0(Q_\varepsilon)$ and any $\varepsilon < \varepsilon_0$,*

$$|a^\varepsilon_{zz'}(x)| \le c_1, \tag{1.2}$$

$$c_2\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)} \le \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\partial_{z'} v^\varepsilon(x)\partial_z v^\varepsilon(x)\,. \tag{1.3}$$

Remark 1.2. *The uniform boundedness of the matrix $\mathcal{A}^\varepsilon$ implies the following upper bound:*

$$\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\partial_{z'} v^\varepsilon(x)\partial_z v^\varepsilon(x) \le c(\Lambda)\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)}\,.$$

Indeed, it suffices to represent z as a sum $z = z^1 + z^2 + \cdots + z^N$ with $|z^i| = 1$ for all $i = 1 \dots N$. Then,

$$\partial^\varepsilon_z v^\varepsilon(x) = \sum_{k=1}^N \partial^\varepsilon_{z^k} v^\varepsilon(x + z^1 + \cdots + z^{k-1})$$

and the required bound is the consequence of the finiteness of Λ*.*

In what follows we always assume the uniform ellipticity conditions (1.2)–(1.3) to hold.

It should be noted that, in general, the uniform ellipticity condition (1.3) is rather implicit. For instance, it neither requires the positiveness of the matrix $\{a^\varepsilon_{zz'}(x)\}$ nor follows from the estimate

$$c_3|\xi|^2 \le \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)(\xi,z)(\xi,z') \le c_4|\xi|^2, \quad \xi\in\mathbb{R}^d,\ c_3>0\,, \tag{1.4}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^d$. One can easily see this by considering the one-dimensional problem with

$$a^\varepsilon_{zz'}(x) = \begin{cases} 1/2 & \text{if } z=z',\ |z|=2, \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, (1.3) is not satisfied although (1.4) holds.

In order to ensure the uniform ellipticity of problem (1.1) one should combine estimates such as (1.4) with a proper irreducibility condition. Below we show that for two important particular classes of difference operators commonly used in applications, the ellipticity conditions can be easily verified.

Suppose we are given a family of functions $p^\varepsilon_z(x)$, $x\in Q_\varepsilon$, $z\in\Lambda$, possessing the following properties:

1. positiveness: $p^\varepsilon_z(x)\ge 0$, $\sum_{z\in\Lambda} p^\varepsilon_z(x)=1$ for each $x\in Q_\varepsilon$,
2. $p^\varepsilon_{\pm e_i}(x)\ge\delta>0$, $i=1,\dots,d$,
3. symmetry: $p^\varepsilon_z(x)=p^\varepsilon_{-z}(x+\varepsilon z)$.

Then, the family of problems

$$u^\varepsilon(x)=\sum_{z\in\Lambda} p^\varepsilon_z(x)\,u^\varepsilon(x+\varepsilon z)+\varepsilon^2 f^\varepsilon(x) \quad \text{in } Q_\varepsilon\,, \qquad u^\varepsilon(x)=0 \quad \text{on } \partial Q^\Lambda_\varepsilon\,, \tag{1.5}$$

can be easily rewritten in the form (1.1) with

$$a^\varepsilon_{zz'}(x) = \begin{cases} p^\varepsilon_z(x) & \text{if } z=z',\ z\ne 0, \\ 0 & \text{otherwise.}\end{cases} \tag{1.6}$$

Proposition 1.3. *Let* $\{p^\varepsilon_z(x)\}$ *possess the abovementioned properties* (1), (2), *and* (3)*. Then problem* (1.5) *is uniformly elliptic.*

Proof. Summing by parts, one can show after simple calculations that

$$\begin{aligned}\delta\varepsilon^{-d}\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)} &= \delta\sum_{x\in\overline{Q_\varepsilon}}\sum_{i=1}^d|\partial^\varepsilon_{\pm e_i}v^\varepsilon|^2 \\ &\le \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\partial_{z'}v^\varepsilon(x)\partial_z v^\varepsilon(x) \\ &\le C\sum_{x\in\overline{Q_\varepsilon}}\sum_{z\in\Lambda}|\partial^\varepsilon_z v^\varepsilon|^2 \le c(\Lambda)\varepsilon^{-d}\|v^\varepsilon\|^2_{W^{1,2}_0(Q_\varepsilon)}\end{aligned}$$

uniformly in ε. This yields the desired result. □

Assumption (2) can be relaxed as follows:

(2′) For some $N > 0$ and $\delta > 0$ and for any $x', x'' \in Q_\varepsilon$, $|x' - x''| = \varepsilon$, there is a finite sequence of vectors $y^1, y^2, \dots, y^k \in \Lambda$, $k \le N$, such that $x'' = x' + \varepsilon \sum_{j=1}^k y^j$ and $p^\varepsilon_{y^j}(x' + \varepsilon \sum_{i=1}^{j-1} y^i) \ge \delta$.

Another important class of uniformly elliptic operators is formed by matrices $\{a^\varepsilon_{zz'}(x)\}$ that satisfy the estimate

$$\sum_{z,z' \in \Lambda} a^\varepsilon_{zz'}(x)\, \eta_z\, \eta_{z'} \ge c \sum_{i=1}^{d} |\eta_{\pm e_i}|^2 , \qquad \eta \in \mathbb{R}^{|\Lambda|} ,$$

uniformly in ε and $x \in Q_\varepsilon$; here we assume that all the vectors $\pm e_i$, $i = 1, 2, \dots, d$, are elements of Λ; if it is not the case, the right-hand side (RHS) of the latter formula does not make sense.

Clearly, the uniform ellipticity implies the coerciveness of problem (1.1) and we have the following statement.

Proposition 1.4. *Let problem* (1.1) *be uniformly elliptic and* $f^\varepsilon \in L^2(Q_\varepsilon)$*. Then there exists a unique solution* $u^\varepsilon \in W_0^{1,2}(Q_\varepsilon)$ *and the estimate*

$$\|u^\varepsilon\|_{W_0^{1,2}(Q_\varepsilon)} \le c \|f^\varepsilon\|_{L^2(Q_\varepsilon)}$$

holds uniformly in ε*.* Henceforth we usually suppose that $f^\varepsilon(\cdot)$ is a discretization of a given function $f \in L^2(Q)$.

We also define the norm on the space $W^{1,2}(Q_\varepsilon)$ by

$$\|v^\varepsilon\|^2_{W^{1,2}(Q_\varepsilon)} = \varepsilon^d \sum_{x \in \overline{Q_\varepsilon}} \sum_{i=1}^{d} |\bar\partial^\varepsilon_{\pm e_i} v^\varepsilon(x)|^2 + \|v^\varepsilon\|^2_{L^2(Q^\varepsilon)} ,$$

where we use the notation

$$\bar\partial^\varepsilon_z \varphi(x) = \begin{cases} \partial^\varepsilon_z \varphi(x) & \text{if } x + \varepsilon z \in \overline{Q_\varepsilon} , \\ 0 & \text{otherwise.} \end{cases}$$

2. Tools for discrete operators analysis.

2.1. Compensated compactness lemma. One of the main tools in the homogenization of differential operators is the so-called compensated compactness lemma (see Murat [21] and Tartar [26]), which gives a sufficient condition for passing to the limit in the inner product of two weakly converging sequences of vector functions. In this section, we prove the discrete version of this result that serves the case of functions defined on a grid.

First of all, we introduce the discrete divergence as follows: for any vector function $q \in \big(L^2(Q_\varepsilon)\big)^{|\Lambda|}$,

$$\operatorname{div}^\varepsilon_\Lambda q(x) \stackrel{\triangle}{=} \sum_{z \in \Lambda} \partial^\varepsilon_{-z} q_z(x).$$

It should be emphasized that the above divergence operator depends on the choice of the set Λ.

Lemma 2.1. *Let* q^ε *and* v^ε *be sequences of vector functions from* $\big(L^2(Q_\varepsilon)\big)^{|\Lambda|}$ *such that*

$$q^\varepsilon \xrightarrow[\varepsilon \to 0]{} q^0 \text{ weakly in } L^2(Q_\varepsilon) , \operatorname{div}^\varepsilon_\Lambda q^\varepsilon \xrightarrow[\varepsilon \to 0]{} f^0 \text{ in } W^{-1,2}(Q_\varepsilon) ,$$
$$v^\varepsilon \xrightarrow[\varepsilon \to 0]{} v^0 \text{ weakly in } L^2(Q_\varepsilon) , v^\varepsilon_z(x) = \partial^\varepsilon_z u^\varepsilon(x) \text{ for some } u^\varepsilon \in W^{1,2}(Q_\varepsilon) .$$

Then, the sequence $(q^\varepsilon\, v^\varepsilon)$ *converges* $\star$*-weakly to* $q^0\, v^0$*:* $\quad q^\varepsilon\, v^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} q^0\, v^0\,.$

Proof. According to Kozlov [15, Proposition 3], the weak convergence of q^ε in $L^2(Q_\varepsilon)$ implies the following weak convergence in $W^{-1,2}(Q_\varepsilon)$:

$$\mathrm{div}^\varepsilon_\Lambda q^\varepsilon \xrightarrow[\varepsilon\to 0]{} \sum_{z\in\Lambda} \frac{\partial}{\partial z} q^0_z = \sum_{z\in\Lambda} z\,\cdot\,\nabla q^0_z;$$

here the standard notation $\frac{\partial}{\partial z} f(x) = z\,\cdot\,\nabla_x f(x)$ for the derivative along arbitrary vector z has been used. Thus, $\sum_{z\in\Lambda} z\,\cdot\,\nabla q^0_z = f^0$, and we have

$$\lim_{\varepsilon\to 0} \|\mathrm{div}^\varepsilon_\Lambda (q^\varepsilon - q^0)\|_{W^{-1,2}(Q_\varepsilon)} = 0\,.$$

From now on, the notation like q^0 or v^0 is used both for the functions of continuous argument and for their discretization (see Appendix A). Using the representation $q^\varepsilon\, v^\varepsilon = (q^\varepsilon - q^0)\, v^\varepsilon + q^0\, v^\varepsilon$ and taking into account the $\star$-weak convergence of $q^\varepsilon\, v^0$ to $q^0\, v^0$, one can assume, without loss of generality, that $q^0 = 0$. Also, under the proper choice of additive constant, $\sum_{x\in Q_\varepsilon} u^\varepsilon(x) = 0$. Then, by the Poincaré inequality, the sequence u^ε is uniformly bounded in the $W^{1,2}$-norm. For any $\varphi \in \mathcal{C}^\infty_0(Q)$ we get

$$\begin{aligned}
\varepsilon^d \sum_{x\in Q_\varepsilon} q^\varepsilon(x)\, v^\varepsilon(x)\, \varphi(x) \; &= \varepsilon^d \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q^\varepsilon_z(x)\, \partial^\varepsilon_z u^\varepsilon(x)\, \varphi(x) \\
&= \varepsilon^d \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} \{ q^\varepsilon_z(x)\, \partial^\varepsilon_z (u^\varepsilon(x)\, \varphi(x)) - q^\varepsilon_z(x)\, u^\varepsilon(x)\, \partial^\varepsilon_z \varphi(x)\} + \tau(\varepsilon)
\end{aligned}$$

with $\lim_{\varepsilon\to 0} \tau(\varepsilon) = 0$ (see Appendix B). Summing by parts in the latter expression leads to

$$\begin{aligned}
&\varepsilon^d \sum_{x\in Q_\varepsilon} q^\varepsilon(x)\, v^\varepsilon(x)\, \varphi(x) \\
&\qquad = \varepsilon^d \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} \left\{ \partial^\varepsilon_{-z} q^\varepsilon_z(x)\, u^\varepsilon(x)\, \varphi(x) - q^\varepsilon_z(x)\, u^\varepsilon(x)\, \partial^\varepsilon_z \varphi(x) \right\} + \tau(\varepsilon) \\
&\qquad = \varepsilon^d \sum_{x\in Q_\varepsilon} (\mathrm{div}^\varepsilon_\Lambda\, q^\varepsilon(x)\ , u^\varepsilon\, \varphi) - \varepsilon^d \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q^\varepsilon_z(x)\, u^\varepsilon(x)\, \partial^\varepsilon_z \varphi(x) + \tau(\varepsilon)\,.
\end{aligned}$$

Since u^ε is uniformly bounded in $W^{1,2}(Q_\varepsilon)$ and $\mathrm{div}^\varepsilon_\Lambda\, q^\varepsilon$ converges to 0 in the $W^{-1,2}$-norm, the first term in the RHS goes to 0 as $\varepsilon \to 0$. The second term goes to 0 because $q^\varepsilon_z\, \partial^\varepsilon_z \varphi$ converges to 0 in $L^2(Q_\varepsilon)$ weakly. Finally, for any $\varphi \in \mathcal{C}^\infty_0(Q)$, $\lim_{\varepsilon\to 0} \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q^\varepsilon_z(x)\, v^\varepsilon_z(x)\, \varphi(x) = 0\,.$ $\square$

2.2. *H*-convergence and homogenization. In this section, we give the definitions of the H-convergence and the homogenization of discrete operators and then study the main properties of this convergence (see Spagnolo [27], Murat and Tartar [22] for the relevant definitions in case of differential operators).

Consider a family of uniformly elliptic discrete Dirichlet problems,

$$A_\varepsilon u^\varepsilon = \mathrm{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\, \partial^\varepsilon_{z'} u^\varepsilon \right) = f^\varepsilon\,, \quad u^\varepsilon \in W^{1,2}_0(Q_\varepsilon)\,, \tag{2.1}$$

and denote by $\mathcal{A}^\varepsilon(x)$ the matrices of the coefficients $\{a^\varepsilon_{zz'}(x)\}$. Let $\mathcal{A}(x) = \{a_{zz'}(x)\}$, $x \in Q$, be a $|\Lambda| \times |\Lambda|$ matrix.

DEFINITION 2.2 (H-convergence). *We say that the matrix $\mathcal{A}^\varepsilon$ H-converges to $\mathcal{A}$ ($\mathcal{A}^\varepsilon \xrightarrow[\varepsilon\to0]{H} \mathcal{A}$) if, for any sequence $f^\varepsilon \in W^{-1,2}(Q_\varepsilon)$ such that $f^\varepsilon \xrightarrow[\varepsilon\to0]{} f$ in $W^{-1,2}(Q_\varepsilon)$, we have*

$$\begin{aligned} u^\varepsilon &\xrightarrow[\varepsilon\to0]{} u^0 && \text{weakly in } W_0^{1,2}(Q_\varepsilon)\,, \\ s^\varepsilon = \sum_{z\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_z u^\varepsilon &\xrightarrow[\varepsilon\to0]{} s^0 = \sum_{z\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z}u^0 && \text{weakly in } L^2(Q_\varepsilon)\,, \end{aligned}$$

where u^0 is the solution of the limit Dirichlet problem,

$$\sum_{z,z'\in\Lambda} -\frac{\partial}{\partial z}\left(a_{zz'}(x)\,\frac{\partial}{\partial z'}u^0\right) = f, \quad u^0 \in W_0^{1,2}(Q).$$

The homogenization is a particular case of H-convergence. Given a matrix-valued function $\mathcal{A}^1(x) = \{a^1_{zz'}(x)\}$, z, $z' \in \Lambda$, $x \in \mathbb{Z}^d$, we define the sequence $\mathcal{A}^\varepsilon$ as follows: $\mathcal{A}^\varepsilon(x) = \mathcal{A}^1(x/\varepsilon)$, $\quad x \in Q_\varepsilon$. Suppose that the corresponding family of problems (defined in (2.1)) is uniformly elliptic.

DEFINITION 2.3. *The constant matrix $\mathcal{A}$ is the* homogenized matrix *for $\mathcal{A}^\varepsilon(x) = \{a^\varepsilon_{zz'}(x)\}$ if, for any sequence $f^\varepsilon \in W^{-1,2}(Q_\varepsilon)$ such that $f^\varepsilon \xrightarrow[\varepsilon\to0]{} f$ in $W^{-1,2}(Q)$, the solutions u^ε of the Dirichlet problems*

$$\operatorname{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon\right) = f^\varepsilon,\ \ u^\varepsilon \in W_0^{1,2}(Q_\varepsilon)\,,$$

converge to the solution u^0 of the limit Dirichlet problem

$$-\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z} a_{zz'} \frac{\partial}{\partial z'} u^0 = f\,,\ \ u^0 \in W_0^{1,2}(Q), \tag{2.2}$$

in the following sense:

$$\begin{aligned} u^\varepsilon &\xrightarrow[\varepsilon\to0]{} u^0 && \text{weakly in } W_0^{1,2}(Q)\,, \\ \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon &\xrightarrow[\varepsilon\to0]{} \sum_{z\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z}u^0 && \text{weakly in } L^2(Q)\,. \end{aligned}$$

REMARK 2.4. *The dimension of the difference gradient of functions defined on Q_ε is equal to $|\Lambda|$ and does not coincide with the dimension of the standard gradient of functions defined on Q. This is the reason we write the limit equation in the definitions above in a nonstandard form. This allows us to define the convergence of streams. Of course, one can easily transform the limiting equation to the standard form*

$$\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z} a_{zz'}(x) \frac{\partial}{\partial z'} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \check{a}_{ij}(x) \frac{\partial}{\partial x_j}\,, \quad \check{a}_{ij}(x) = \sum_{z,z'\in\Lambda} (z, e_i) a_{zz'}(x)(z', e_j)\,.$$

One of the remarkable properties of H- and G-convergences of differential operators is the compactness of a family of uniformly elliptic operators; see, for example, Murat and Tartar [22], Zhikov et al. [28]. We proceed by quoting the compactness result for a family of uniformly elliptic difference operators.

Proposition 2.5 (see Kozlov [15, section 2]). *Any uniformly elliptic sequence of problems defined in* (2.1) *contains an H-convergent subsequence. The limit problem involves a second order uniformly elliptic operator in divergence form:*

$$A\,u = -\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z}\left(a_{zz'}(x)\,\frac{\partial}{\partial z'}\,u\right) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial}{\partial x_j}u\right).$$

In the subsections below we prove a number of general results on H-convergence and homogenization of difference operators that are not exhibited in the existing literature.

2.2.1. Convergence of arbitrary solutions. One of the significant properties of H-convergence is the fact that the H-limit operator depends only on the original sequence of operators and does not depend on the type of boundary conditions and on the domain. In a general form, this can be formulated as follows.

Theorem 2.6 (convergence of arbitrary solutions). *Let a sequence of uniformly elliptic operators A_ε H-converge in a domain Q to the limit operator A, and suppose that a sequence of functions $w^\varepsilon \in W^{1,2}(Q_\varepsilon)$ satisfies the conditions*

$$\begin{aligned} & w^\varepsilon \xrightarrow[\varepsilon\to 0]{} w^0 \quad \textit{weakly in } W^{1,2}(Q_\varepsilon)\,, \\ & \mathrm{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\left(g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon\right)\right) = f\,, \end{aligned} \tag{2.3}$$

where $g \in \left(L^2(Q)\right)^{|\Lambda|}$ and $f \in W^{-1,2}(Q)$ do not depend on ε. Then, w^0 satisfies the homogenized equation

$$-\sum_{z,z'\in\Lambda} \frac{\partial}{\partial z}\left[a_{zz'}\left(g_{z'} + \frac{\partial}{\partial z'}w^0\right)\right] = f\,,$$

and the streams do converge in $L^2(Q_\varepsilon)$ weakly:

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\left(g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} \sum_{z'\in\Lambda} a_{zz'}\left(g_{z'} + \frac{\partial}{\partial z'}w^0\right).$$

Proof. Under the conditions of the theorem, the streams are uniformly bounded in $L^2(Q_\varepsilon)$. Thus, taking a proper subsequence, we have $\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\left(g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon\right) \xrightarrow[\varepsilon\to 0]{}$ $\xi_z$weakly in $L^2(Q_\varepsilon)$. Passing to the limit in (2.3), one can easily check that $-\sum_{z\in\Lambda} \frac{\partial}{\partial z}\xi_z$ $= f$. We have to prove the relation $\xi_z = \sum_{z'\in\Lambda} a_{zz'}\left(g_{z'} + \frac{\partial}{\partial z'}w^0\right)$. Let u^0 be an arbitrary function from $W^{1,2}_0(Q)$. Denote by u^ε the solution of the Dirichlet problem,

$$\mathrm{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon\right) = \sum_{z,z'\in\Lambda} \frac{\partial}{\partial z}\left(a_{zz'}\,\frac{\partial}{\partial z'}\,u^0\right),$$

and consider the following identity:

$$\sum_{z\in\Lambda}\left(g_z + \partial^\varepsilon_z w^\varepsilon\right) \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon = \sum_{z\in\Lambda} \partial^\varepsilon_z u^\varepsilon \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\left(g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon\right). \tag{2.4}$$

By the definition of H-convergence, we have

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z'} u^0 \quad \text{weakly in } L^2(Q_\varepsilon)\,,$$

while the limiting relation

$$\sum_{z\in\Lambda} (g_z + \partial^\varepsilon_z w^\varepsilon) \xrightarrow[\varepsilon\to 0]{} \sum_{z\in\Lambda} \left(g_z + \frac{\partial}{\partial z} w^0\right) \quad \text{weakly in } L^2(Q_\varepsilon)$$

is an evident consequence of the weak convergence of w^ε. Now, passing to the limit on the left-hand side (LHS) of (2.4), with the help of Lemma 2.1 we obtain

$$\sum_{z\in\Lambda} (g_z + \partial^\varepsilon_z w^\varepsilon) \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \sum_{z\in\Lambda} \left(g_z + \frac{\partial}{\partial z} w^0\right) \sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z'} u^0\,.$$

The fact that g_z does not depend on ε has also been used here.

Similarly, passing to the limit on the RHS of (2.4) gives

$$\sum_{z\in\Lambda} \partial^\varepsilon_z u^\varepsilon \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,(g_{z'} + \partial^\varepsilon_{z'} w^\varepsilon) \xrightarrow[\varepsilon\to 0]{\star} \sum_{z\in\Lambda} \frac{\partial}{\partial z} u^0\,\xi_z\,.$$

Finally, considering the fact that u^0 is arbitrary function from $W_0^{1,2}(Q)$, we deduce

$$\xi_z = \sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\left(g_{z'} + \frac{\partial}{\partial z'} w^0\right)\,. \qquad \square$$

Corollary 2.7 (local property of H-convergence). *If $A_\varepsilon \xrightarrow[\varepsilon\to 0]{H} A$ in a domain Q, then $A_\varepsilon \xrightarrow[\varepsilon\to 0]{H} A$ in any subdomain $Q_1 \subset Q$.*

2.2.2. Convergence of energies. In this section, we address a family of Dirichlet problems with nonhomogeneous boundary conditions:

$$\operatorname{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'} u^\varepsilon\right) = f\,,\ \ u^\varepsilon - u^0 \in W_0^{1,2}(Q_\varepsilon)\,, \tag{2.5}$$

where $u^0 \in W^{1,2}(\mathbb{R}^d)$ and $f \in W^{-1,2}(Q)$ are fixed given functions.

We suppose that the family $\{A_\varepsilon\}$ is uniformly elliptic and H-converges to the limit operator A. Then, one can assume without loss of generality that the function u^0 satisfies the equation $A\,u^0 = f$ in the domain Q.

In order to show the uniform boundedness of $\{u^\varepsilon\}$ in $W^{1,2}(\overline{Q_\varepsilon})$, we replace u^ε by $u^\varepsilon - u_0$ in (2.5), multiply the resulting equation by $u^\varepsilon - u_0$, and then sum over $\overline{Q}_\varepsilon$. After summation by parts we get

$$\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z\big(u^\varepsilon(x) - u_0(x)\big)\,\partial^\varepsilon_{z'}(u^\varepsilon(x) - u_0(x)) = \sum_{x\in\overline{Q_\varepsilon}} f(x)\big(u^\varepsilon(x) - u_0(x)\big)$$

$$- \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z\big(u^\varepsilon(x) - u_0(x)\big)\,\partial^\varepsilon_{z'} u_0(x).$$

This implies the required boundedness.

By Theorem 2.6 (convergence of arbitrary solution), any weak limiting point of the sequence $\{u^\varepsilon\}$ coincides with u^0 in Q. Hence, the whole family $\{u^\varepsilon\}$ converges to u^0 in $W^{1,2}(\overline{Q_\varepsilon})$ weakly.

Proposition 2.8 (convergence of energies). *Let $A_\varepsilon \xrightarrow[\varepsilon\to 0]{H} A$ and let u^ε be the solution of problem* (2.5). *Then the following limit relation holds true:*

$$\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'} u^\varepsilon(x) \xrightarrow[\varepsilon\to 0]{} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.$$

Proof. By (2.5) we have

$$\begin{aligned}
\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} &\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z (u^\varepsilon-u^0)(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x)\\
&= \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x)\\
&\quad - \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x) + \tau(\varepsilon)\\
&= \varepsilon^d \sum_{x\in Q_\varepsilon} f(x)\,(u^\varepsilon-u^0)(x) - \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x)\\
&\quad + \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x) \;+\; \tau(\varepsilon)\,;
\end{aligned}$$

here and afterwards $\tau(\varepsilon)$ stands for a generic function that vanishes as $\varepsilon\to 0$. On the other hand,

$$\begin{aligned}
\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} &\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z (u^\varepsilon-u^0)(x)\,\partial^\varepsilon_{z'}(u^\varepsilon-u^0)(x)\\
&= \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'} u^\varepsilon(x)\\
&\quad - 2\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x)\\
&\quad + \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x) + \tau(\varepsilon)\,.
\end{aligned}$$

After subtraction we find

$$\begin{aligned}
\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'} u^\varepsilon(x) &- \varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\partial^\varepsilon_{z'}u^\varepsilon(x)\\
&- \varepsilon^d \sum_{x\in Q_\varepsilon} f(x)\,(u^\varepsilon-u^0)(x) + \tau(\varepsilon) = 0\,.
\end{aligned} \tag{2.6}$$

Passing to the limit in the last relation, and taking into account the weak convergence of $u^\varepsilon-u^0$ to 0 in $W^{1,2}_0(Q_\varepsilon)$ and the weak convergence of the streams $a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon$ in

$L^2(Q_\varepsilon)$, we obtain

$$\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_{z'} u^\varepsilon(x) \xrightarrow[\varepsilon\to 0]{} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,. \qquad \square$$

In fact, the result on convergence of energies can be formulated in more "local" form, as follows.

PROPOSITION 2.9. *Under the assumptions of Theorem* 2.6 *one has*

(2.7)
$$\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\partial_z w^\varepsilon(x)(\partial_{z'}w^\varepsilon(x)+g_{z'}(x)) \xrightarrow[\varepsilon\to 0]{\star} \sum_{z,z'\in\Lambda} a_{zz'}(x)\frac{\partial}{\partial z}w^0(x)\Big(\frac{\partial}{\partial z'}w^0(x)+g_{z'}(x)\Big).$$

Proof. In the expression

$$\sum_{z,z'\in\Lambda} \partial_z w^\varepsilon(x) a^\varepsilon_{zz'}(x)(\partial_{z'}w^\varepsilon(x)+g_{z'}(x))\,,$$

the streams $a^\varepsilon_{zz'}(x)(\partial_{z'}w^\varepsilon(x)+g_{z'}(x))$ converge weakly in $L^2(Q_\varepsilon)$ (by Theorem 2.6) to the limit stream $a_{zz'}(x)\Big(\frac{\partial}{\partial z'}w^0(x)+g_{z'}(x)\Big)$, and the family $\partial_z w^\varepsilon(x)$ converges to $\frac{\partial}{\partial z}w^0(x)$ weakly by the assumption of Theorem 2.6. Now, the desired statement follows from Lemma 2.1. $\square$

REMARK 2.10. *In the case of elliptic differential equations, H-convergence of operators implies weak L^1-convergence of the corresponding energy functions. This result relies on the Meyers estimates of the gradient of solutions; see Meyers* [20].

For the difference operators the Meyers-type estimates have not been obtained, so the weak L^1-convergence of energies is an open question.

2.2.3. Neumann problem. The notion of the H-limit operator has been expressed in terms of the operators of the corresponding Dirichlet problems. But, as was already mentioned in the previous section, we can also consider other boundary value problems. In this section, the Neumann problem is investigated.

DEFINITION 2.11. *Let $f\in\big(L^2(Q)\big)^{|\Lambda|}$. We say that $u^\varepsilon\in W^{1,2}(Q_\varepsilon)$ is a solution of the Neumann problem for the equation*

$$\mathrm{div}^\varepsilon_\Lambda\left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon\right) = \sum_{z\in\Lambda}\partial^\varepsilon_{-z} f^\varepsilon_z$$

if the relation

$$\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z\varphi^\varepsilon(x)\,\bar\partial^\varepsilon_{z'}u^\varepsilon(x) = \sum_{x\in\overline{Q_\varepsilon}}\sum_{z\in\Lambda} f^\varepsilon_z(x)\,\bar\partial^\varepsilon_z\varphi^\varepsilon(x) \tag{2.8}$$

holds true for any $\varphi\in W^{1,2}(Q)$; here we use the notation

$$\bar\partial^\varepsilon_z\varphi = \begin{cases} \partial^\varepsilon_z\varphi & \textit{if}\ \ x+\varepsilon z\in\overline{Q_\varepsilon},\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, the functions u^ε are defined up to an additive constant. To fix the choice of the constant, we assume that $\sum_{x\in\overline{Q_\varepsilon}} u^\varepsilon(x)=0\,.$

In order to study the Neumann problem, we should modify the definition of uniform ellipticity and impose a slightly stronger condition because Definition 1.1 above does not ensure the coerciveness of problem (2.8).

DEFINITION 2.12. *We say that the family of operators* $\{A_\varepsilon\}$ *is* N-elliptic *in a domain* Q_ε *if the inequality*

$$\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z\varphi(x)\bar\partial^\varepsilon_{z'}\varphi(x) \ge c\sum_{x\in\overline{Q_\varepsilon}}\sum_{i=1}^{d}\left(\bar\partial^\varepsilon_{\pm e_i}\varphi(x)\right)^2, \quad c>0, \tag{2.9}$$

holds for any φ.

It should be noted that N-ellipticity implies the uniform ellipticity in the same domain Q and that, under the condition of Proposition 1.3, the family of operators is always N-elliptic.

Example. To clarify the difference between the uniform ellipticity and N-ellipticity we provide below a simple one-dimensional example which shows that due to "boundary effects," a uniformly elliptic operator is not necessary N-elliptic.

Let Q be an open interval $(0,1)$, and suppose $\Lambda=\{0,\pm1,\pm2,\pm3\}$. If we set

$$p_{\pm1}(0)=\frac12,\quad p_z(0)=0 \text{ if } z\ne\pm1;$$

$$p_{-1}(1)=\frac12,\quad p_0(1)=\frac12,\quad p_z(1)=0 \text{ if } z\ne -1,\ 0;$$

$$p_{\pm3}(2)=\frac12,\quad p_z(2)=0 \text{ if } z\ne\pm3,$$

and extend this function periodically with period 3, then for $\varepsilon=1/n$ with integer $n>3$ we have

$$Q_\varepsilon=\left\{\frac1n,\frac2n,\dots,\frac{n-1}{n}\right\},\quad \overline{Q}_\varepsilon=\left\{\frac{-2}{n},\frac{-1}{n},0,\frac1n,\frac2n,\dots,\frac{n-1}{n},1,\frac{n+1}{n},\frac{n+2}{n}\right\}.$$

Consider the following test function:

$$\varphi^\varepsilon(x)=\begin{cases}1 & \text{if } x=-\frac2n,-\frac3n,\\ 0 & \text{otherwise.}\end{cases}$$

For this function the LHS of (2.9) is equal to zero while the RHS is strictly positive. Thus (2.9) cannot hold. On the other hand, one can easily verify that this problem is uniformly elliptic.

PROPOSITION 2.13. *Suppose that a family of* N*-elliptic operators* $\{A_\varepsilon\}$ H*-converges to the operator* A *in the domain* Q*. Then the solutions* u^ε *of problem* (2.8) *converge, as* $\varepsilon\to0$*, in* $W^{1,2}(Q_\varepsilon)$ *to the solution of the limit Neumann problem: for any* $\varphi\in W^{1,2}(Q)$,

$$\int_Q\left(\sum_{z,z'\in\Lambda}a_{zz'}(x)\frac{\partial}{\partial z}\varphi(x)\,\frac{\partial}{\partial z'}u^0(x)\right)dx=\int_Q\left(\sum_{z\in\Lambda}f_z(x)\frac{\partial}{\partial z}\varphi(x)\right)dx\,.$$

Moreover, the streams also converge.

Proof. Using the Poincaré inequality, we derive from the N-ellipticity the uniform coerciveness of problem (2.8). Thus, the family u^ε is uniformly bounded in $W^{1,2}(Q_\varepsilon)$. By Theorem 2.6, any limit point w^0 of the family u^ε satisfies the H-limit equation and

$$\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\,\partial^\varepsilon_{z'}u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial}{\partial z'}w^0 \quad \text{weakly in } L^2(Q_\varepsilon)\,.$$

So, for any $\varphi\in W^{1,2}(Q)$, passing to the limit in (2.8), we get

$$\int_Q \left(\sum_{z,z'\in\Lambda} a_{zz'}(x)\frac{\partial}{\partial z}\varphi(x)\,\frac{\partial}{\partial z'}w^0(x)\right)dx = \int_Q\left(\sum_{z\in\Lambda} f_z(x)\frac{\partial}{\partial z}\varphi(x)\right)dx\,.$$

Moreover, $\int_Q w^0(x)\,dx = 0$. $\square$

2.2.4. Γ-convergence. The results proved in this section exhibit the relation between the H-convergence of operators and a special kind of convergence of corresponding quadratic forms, so-called Γ-convergence, that was introduced originally in De Giorgi [8].

PROPOSITION 2.14. *Let A_ε be a N-elliptic family of operators in a domain Q. Then, $A_\varepsilon \xrightarrow[\varepsilon\to 0]{H} A$ in Q if and only if the following conditions are satisfied:*

1. *For any $u^0\in W^{1,2}(Q)$ and for any sequence $w^\varepsilon\in W^{1,2}(Q_\varepsilon)$ such that $w^\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0$ weakly in $W^{1,2}(Q_\varepsilon)$, the following inequality holds:*

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} & a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z w^\varepsilon(x)\,\bar\partial^\varepsilon_{z'}w^\varepsilon(x)\\ &\geq \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.\end{aligned}$$

2. *For any $u^0\in W^{1,2}(Q)$, there exists a sequence $u^\varepsilon\in W^{1,2}(Q_\varepsilon)$ such that $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0$ weakly in $W^{1,2}(Q)$, $u^\varepsilon - u^0\in W^{1,2}_0(Q)$, and*

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda} & a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z u^\varepsilon(x)\,\bar\partial^\varepsilon_{z'}u^\varepsilon(x)\\ &= \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.\end{aligned}$$

Proof. Suppose that $A_\varepsilon \xrightarrow[\varepsilon\to 0]{H} A$.

1. Consider the Neumann problem (2.8), with $f_z = \left(\sum_{z'\in\Lambda} a_{zz'}\,\frac{\partial u^0}{\partial z'}\right)_{z\in\Lambda}$, where u^0 is the solution of the H-limit Neumann problem. The solution u^ε of (2.8) provides the minimum in the following variational problem: $E = \inf_{v\in W^{1,2}(Q_\varepsilon)} J^\varepsilon(v)$, where

$$J^\varepsilon(v) = \varepsilon^d\sum_{x\in\overline{Q_\varepsilon}}\sum_{z,z'\in\Lambda}\left[a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z v(x)\,\bar\partial^\varepsilon_{z'}v(x) - 2a_{zz'}(x)\,\bar\partial^\varepsilon_z v(x)\,\frac{\partial}{\partial z'}u^0(x)\right].$$

For any sequence $\{w^\varepsilon\}$ such that $w^\varepsilon \to u^0$ weakly in $W^{1,2}(Q_\varepsilon)$, we have

$$J^\varepsilon(w^\varepsilon) \geq J^\varepsilon(u^\varepsilon)\,. \tag{2.10}$$

Then, by Proposition (2.13), $\partial_z^\varepsilon u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \frac{\partial}{\partial z}u^0$ weakly in $L^2(Q_\varepsilon)$ and, therefore,

$$\begin{aligned}
\varepsilon^d \sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} & a^\varepsilon_{zz'}(x)\,\bar\partial^\varepsilon_z w^\varepsilon(x)\,\bar\partial^\varepsilon_{z'} w^\varepsilon(x) \\
&= J^\varepsilon(w^\varepsilon) + 2\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\bar\partial^\varepsilon_z w^\varepsilon(x)\,\frac{\partial}{\partial z'}u^0(x) \\
&\geq J^\varepsilon(u^\varepsilon) + 2\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\bar\partial^\varepsilon_z w^\varepsilon(x)\,\frac{\partial}{\partial z'}u^0(x) \\
&= -\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\bar\partial^\varepsilon_z u^\varepsilon(x)\,\frac{\partial}{\partial z'}u^0(x) \\
&\quad +2\sum_{x\in\overline{Q_\varepsilon}} \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\bar\partial^\varepsilon_z w^\varepsilon(x)\,\frac{\partial}{\partial z'}u^0(x) \\
&\xrightarrow[\varepsilon\to 0]{} \int_Q \sum_{z,z'\in\Lambda} a_{zz'}(x)\,\frac{\partial}{\partial z}u^0(x)\,\frac{\partial}{\partial z'}u^0(x)\,dx\,.
\end{aligned}$$

Equation (2.8) has also been used here. Now, taking the infimum limit in both sides of (2.10), we obtain the required inequality.

2. It is the statement of Proposition 2.8.

The remaining part of the proposition is an easy consequence of the uniqueness of the H-limit. □

Remark 2.15. *The statements of Propositions* 2.8 *and* 2.14 *remain valid if we replace the sums over $x \in \overline{Q_\varepsilon}$ by the sums over $x \in Q_\varepsilon$.*

2.3. Description of the random environment. In this section we introduce random difference elliptic operators with statistically homogeneous rapidly oscillating coefficients.

Let $(\Omega, \mathcal{F}, \mu)$ be a standard probability space, where $\mathcal{F}$ is a σ-algebra of subsets of Ω and μ is a probability measure. Let $\{T_x : \Omega \mapsto \Omega;\, x \in \mathbb{Z}^d\}$ be a group of $\mathcal{F}$-measurable transformations which preserve the measure μ:

1. $T_x : \Omega \mapsto \Omega$ is $\mathcal{F}$–measurable for all $x \in \mathbb{Z}^d$,
2. $\mu(T_x\mathcal{B}) = \mu(\mathcal{B})$, for any $\mathcal{B} \in \mathcal{F}$ and $x \in \mathbb{Z}^d$,
3. $T_0 = I\,,\ T_x \circ T_y = T_{x+y}$.

In what follows we assume that the group T_x is ergodic. That is, any $f \in L^1(\Omega)$ such that $f(T_x\,\omega) = f(\omega)$ μ-a.s for each $x \in \mathbb{Z}^d$ is equal to a constant μ-a.s.

Let Λ be a finite subset of $\mathbb{Z}^d$. Given a matrix-valued $\mathcal{F}$-measurable function $\{a_{zz'}(\omega)\}$, $z, z' \in \Lambda$, with values in the space of symmetric $|\Lambda| \times |\Lambda|$ matrices, we define a family of difference operators A^ε with the coefficients

$$a^\varepsilon_{zz'}(x) = a_{zz'}(T_{x/\varepsilon}\omega), \quad x \in \varepsilon\mathbb{Z}^d,\ z, z' \in \Lambda\,. \tag{2.11}$$

We suppose here that $\pm e_i \in \Lambda,\ i = 1, \dots, d$, and that

$$\sum_{z,z' \in \Lambda} a_{zz'}(\omega)\eta_z\eta_{z'} \geq c \sum_{i=1}^{d} |\eta_{\pm e_i}|^2 , \quad \eta \in \mathbb{R}^{|\Lambda|} . \tag{2.12}$$

$$|a_{zz'}(\omega)| \leq c_1, \quad z, z' \in \Lambda. \tag{2.13}$$

It is easy to see that these inequalities imply the N-ellipticity and the uniform ellipticity of the corresponding family A_ε in any regular domain Q.

In applications, especially in those related to random walks, we usually deal with the following particular case of the above construction.

Let $\{q(\omega, z),\ z \in \mathbb{Z}^d\}$ be a family of random variables such that μ-a.s,

1. $\sum\limits_{z \in \mathbb{Z}^d} q(\omega, z) = 1$,
2. $q(T_x\omega, z) = q(T_{x+z}\omega, -z)$,
3. $q(\omega, z) \geq 0, \quad q(\omega, \pm e_i) \geq \delta > 0,\ i = 1, \dots, d$ (ellipticity condition).

We introduce a family of transition probabilities as follows:

$$p_z(x) = q(T_x\omega, z),$$

where the argument ω, treated as a realization of the medium, is omitted. The important characteristic of a family of transition probabilities is the structure of its support:

$$\Lambda = \left\{ z \in \mathbb{Z}^d \mid \operatorname*{ess\,sup}_{\Omega} p_z(x) \neq 0 \right\} .$$

In all the models considered below, the set Λ is finite.

Now, if we denote $p_z^\varepsilon(x) = p_z(\varepsilon^{-1}x),\ x \in Q_\varepsilon,\ z \in \Lambda$, then due to the assumptions on $q(\omega, x)$, problem (1.5) is uniformly and N-elliptic.

It is convenient to define the "ω-divergence" operator:

for any random variable $v \in L^2(\Omega)$, $\quad \operatorname{div}_\omega v(\omega) \overset{\triangle}{=} \sum\limits_{z \in \Lambda} v(T_{-z}\omega) - v(\omega)$.

We will use it in the following analysis.

2.4. Homogenization of random operators. This section is devoted to homogenization of the random difference operators introduced in the preceding section. The first proof of the homogenization theorem for such operators was obtained in [15], where the "corrector technique" was used. Here we give another proof of the theorem, which relies on the compensated compactness lemma.

2.4.1. Auxiliary problem. Let us define the following subspaces of $\left(L^2(\Omega)\right)^{|\Lambda|}$ (see Kozlov [15]):

$L^2_{pot}(\Omega, \Lambda)$ is the closure of the set
$\left\{ v \in (L^2(\Omega))^{|\Lambda|} ;\, v_z(\omega) = u(T_z\omega) - u(\omega) \text{ for some } u \in L^\infty(\Omega) \right\}$,
$L^2_{sol}(\Omega, \Lambda)$ is the closure of the set: $\left\{ v \in (L^2(\Omega))^{|\Lambda|} ;\, \operatorname{div}_\omega v = 0 \right\}$.

For $\lambda \in \mathbb{R}^{|\Lambda|}$ we denote by $\mathcal{V}^2_{pot,\lambda}(\Omega, \Lambda)$ the closed set $\left\{ v + \lambda ;\, v \in L^2_{pot}(\Omega, \Lambda) \right\}$.

Consider the following auxiliary problem: given $\lambda\in\mathbb{R}^{|\Lambda|}$, find $v\in\mathcal{V}^2_{pot,\lambda}(\Omega,\,\Lambda)$ such that

$$\operatorname{div}_\omega\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,v_{z'}(\omega)\right)=0\,. \tag{2.14}$$

In order to prove the existence and uniqueness of the solution of this problem we introduce the operator

$$\begin{array}{rccl} A_{pot}\,: & L^2_{pot}(\Omega,\,\Lambda) & \mapsto & L^2_{pot}(\Omega,\,\Lambda)\\ & (v_z)_{z\in\Lambda} & \mapsto & \Pi_{pot}\left(\displaystyle\sum_{z'\in\Lambda} a_{zz'}(\omega)\,v_{z'}(\omega)\right), \end{array}$$

where Π_{pot} is the orthogonal projection onto the subspace $L^2_{pot}(\Omega,\,\Lambda)$.

In view of the Weyl decomposition (see Kozlov [15]) $\left(L^2(\Omega)\right)^{|\Lambda|}=L^2_{pot}(\Omega,\Lambda)\oplus L^2_{sol}(\Omega,\Lambda)$, we can rewrite the problem (2.14) in the following form: given $\lambda\in\mathbb{R}^{|\Lambda|}$, find $v\in L^2_{pot}(\Omega)$ such that

$$A_{\rm pot}v=\Pi_{pot}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,\lambda_{z'}\right).$$

The operator A_{pot} is coercive. Indeed, for any $v\in L^2_{pot}(\Omega,\,\Lambda)$, we have

$$\begin{aligned}(A_{pot}v,v)&=\sum_{z\in\Lambda}\left(\Pi_{pot}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,v_{z'}(\omega)\right),\,v_z(\omega)\right)_{L^2(\Omega)}\\ &=\sum_{z\in\Lambda}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,v_{z'}(\omega),\,\Pi_{pot}\left(v_z(\omega)\right)\right)_{L^2(\Omega)}\\ &=\sum_{z\in\Lambda}\left(\sum_{z'\in\Lambda} a_{zz'}(\omega)\,v_{z'}(\omega),\,v_z(\omega)\right)_{L^2(\Omega)}.\end{aligned}$$

According to hypothesis (2.12), this implies $(A_{\rm pot}v,v)\geq c\,E\left[\sum_{i=1}^n|v_{\pm e_i}|^2\right]$, where E stands for the expectation with respect to the measure μ. On the other hand, for any v of the form $v_z(\omega)=u(T_z\omega)-u(\omega),\ u\in L^2(\Omega)$, we have

$$\begin{aligned}\|v\|^2_{(L^2(\Omega))^{|\Lambda|}}&=E\left[\sum_{z\in\Lambda}|v_z(\omega)|^2\right]\;=\;E\left[\sum_{z\in\Lambda}|u(T_z\omega)-u(\omega)|^2\right]\\ &=E\left[\sum_{z\in\Lambda}\left|\sum_{i=0}^{N(z)-1}u(T_{\zeta_{i+1}}\omega)-u(T_{\zeta_i}\omega)\right|^2\right],\end{aligned}$$

where $\zeta_0=0$, $\zeta_{N(z)}=z$, $|\zeta_{i+1}-\zeta_i|=1$, and $N(z)\leq d\,\mathrm{diam}(\Lambda)$. Therefore,

$$\|v\|^2_{(L^2(\Omega))^{|\Lambda|}}\leq E\left[d(\mathrm{diam}(\Lambda))^2|\Lambda|\sum_{i=1}^d(v_{\pm e_i}(\omega))^2\right]\leq c_1(d,\Lambda)\,E\left[\sum_{i=1}^d(v_{\pm e_i}(\omega))^2\right].$$

By definition, the said set of $v(\omega)$ is dense in $L^2_{pot}(\Omega,\Lambda)$, and by the continuity arguments, the latter estimate holds for any $v\in L^2_{pot}(\Omega,\Lambda)$.

Thus, $(A_{\rm pot}v,v)\geq c_2(d,\Lambda)\,\|v\|^2_{(L^2(\Omega))^{|\Lambda|}}$, and the desired existence and uniqueness follow from the Lax–Milgram lemma.

2.4.2. Homogenization. In this section, we study the family of random operators $\{A_\varepsilon\}$ with statistically homogeneous coefficients given by (2.11). The homogenization theorem for such operators was originally proved in [15]. We give another proof based on the compensated compactness lemma, which seems to be easier and shorter. The main result here is the following theorem.

THEOREM 2.16. *Let the coefficients of A_ε be given by* (2.11), *and suppose the condition* (2.12) *is fulfilled. Then, a.s., the family $\{A_\varepsilon\}$ admits homogenization and the limit matrix $\mathcal{A}^0$ does not depend on ω.*

Proof. For a fixed $f \in W^{-1,2}(Q)$, consider the following Dirichlet problems:

$$\operatorname{div}^\varepsilon_\Lambda \left(\sum_{z'\in\Lambda} a^\varepsilon_{zz'} \partial^\varepsilon_{z'} u^\varepsilon \right) = f\,, \quad u^\varepsilon \in W^{1,2}_0(Q_\varepsilon)\,. \tag{2.15}$$

Since u^ε and $\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\, \partial^\varepsilon_{z'} u^\varepsilon$ are uniformly bounded, respectively, in $W^{1,2}(Q_\varepsilon)$ and $(L^2(Q_\varepsilon))^{|\Lambda|}$, we have

$$\begin{aligned} &u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0 \quad \text{weakly in } W^{1,2}_0(Q_\varepsilon)\,,\\ &s^\varepsilon \xrightarrow[\varepsilon\to 0]{} s^0 \quad \text{weakly in } \left(L^2(Q_\varepsilon)\right)^{|\Lambda|}\,, \end{aligned}$$

where s^ε_z stands for $\sum_{z'\in\Lambda} a^\varepsilon_{zz'}\ \partial^\varepsilon_{z'} u^\varepsilon$.

Let $v_z(\omega)$ solve the auxiliary problem (2.14). If we denote $v^\varepsilon(x) \triangleq v\left(T_{x/\varepsilon}\,\omega\right)$, $q^\varepsilon \triangleq v^\varepsilon \mathcal{A}^\varepsilon$, i.e., $\forall z \in \Lambda$, $q^\varepsilon_z = \sum_{z'\in\Lambda} v^\varepsilon_{z'}\, a^\varepsilon_{zz'}$, then, the identity

$$\sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} s^\varepsilon_z(x)\, v^\varepsilon_z(x) = \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} q^\varepsilon_z(x)\, \partial^\varepsilon_z u^\varepsilon(x) \tag{2.16}$$

obviously holds. We introduce a constant matrix $\mathcal{A}$ to satisfy the relation $E(q^\varepsilon) = \lambda\, \mathcal{A}^0$. This matrix is well defined because q^ε is a linear functional of λ. By the Birkhoff ergodic theorem, we have

$$\begin{aligned} &v^\varepsilon \xrightarrow[\varepsilon\to 0]{} E(v^\varepsilon) = \lambda \quad \text{weakly in } L^2(Q_\varepsilon) \text{ a.s.},\\ &q^\varepsilon \xrightarrow[\varepsilon\to 0]{} E(q^\varepsilon) = \lambda\, \mathcal{A} \quad \text{weakly in } L^2(Q_\varepsilon) \text{ a.s.} \end{aligned}$$

It follows from (2.14) and the definition of $\operatorname{div}_\omega$ and $\operatorname{div}^\varepsilon_\Lambda$ that for almost all realizations we have $\operatorname{div}^\varepsilon_\Lambda q^\varepsilon = 0$, while the fact that $v - \lambda \in L^2_{pot}(\Omega, \Lambda)$ implies a.s. the relation $v^\varepsilon_z = \partial^\varepsilon_z \theta^\varepsilon$ for some (in general not statistically homogeneous) functions θ^ε. Also, from (2.15) we have $\operatorname{div}^\varepsilon_\Lambda s^\varepsilon = f$. By Lemma 2.1,

$$\sum_{z\in\Lambda} s^\varepsilon_z\, v^\varepsilon_z \xrightarrow[\varepsilon\to 0]{\star} \sum_{z\in\Lambda} s^0_z\, \lambda_z$$

and

$$\sum_{z\in\Lambda} q^\varepsilon_z\, \partial^\varepsilon_z u^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \sum_{z,z'\in\Lambda} \lambda_z\, a^0_{zz'}\, \frac{\partial}{\partial z'} u^0\,,$$

or, equivalently,

$$\sum_{z,z'\in\Lambda} v^\varepsilon_{z'}\, a^\varepsilon_{zz'}\, \partial^\varepsilon_z u^\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \sum_{z,z'\in\Lambda} \lambda_z\, a^0_{zz'}\, \frac{\partial}{\partial z'} u^0;$$

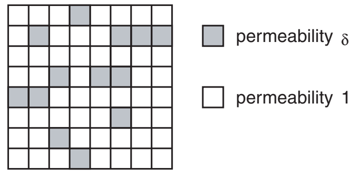

FIG. 3.1. *Example of a realization of the random medium.*

we have also used here Proposition 3 from [15]. Hence, passing to the limit in (2.16) and bearing in mind the fact that λ is an arbitrary vector, we find

$$s_z^0 = \sum_{z' \in \Lambda} a_{zz'}^0 \frac{\partial}{\partial z'} u^0 .$$

Since $\sum_{z\in\Lambda} \frac{\partial}{\partial z} s_z^0 = f$, the function u^0 is the solution of the homogenized problem and $\mathcal{A}^0$ is the limit matrix. ∎

3. Asymptotic behavior of the effective coefficient. In this second part of the work, we consider the difference operators obtained by discretizing a random two-dimensional high-contrast checker-board structure, as various discretization procedures are applied. For each discretization method, we find the asymptotics of the effective coefficient. The results obtained in this section rely essentially on the fine results from the percolation theory, such as channel property and related statements. For the reader's convenience, we formulate these results and provide necessary definitions in section 3.1.

To define the random media, we split the plane $\mathbb{R}^2$ into regular squares $\{[-\frac{1}{2}, \frac{1}{2}]^2 + j\}$, $j \in \mathbb{Z}^2$, and assign a value of permeability, independently at each square, as follows:

$$\kappa(y) \triangleq \begin{cases} \delta & \text{with probability} \quad p \\ 1 & \text{with probability} \quad 1-p \end{cases} , \quad y \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2 + j, \ j \in \mathbb{Z}^2,$$

where δ is a small strictly positive parameter (see Figure 3.1). Then, we consider the grid $\mathbb{Z}^2$, fix a finite set $\Lambda \subset \mathbb{Z}^2$, and define the transition probabilities $\{p_z(x);\, x \in \mathbb{Z}^2,\, z \in \Lambda\}$ to be a function of $\{\kappa(x+z)\}$, $z \in \Lambda$. Finally, we define the coefficients of operator A_ε in terms of $\{p_z(x)\}$ by (1.6).

Henceforth, we suppose that the properties (1), (2), and (3) in section 1.1 are satisfied. It then follows from the independence of $\kappa(j)$ for different $j \in \mathbb{Z}^2$ that the family $\{p_z(x)\}$ is ergodic. Now, the following assertion is a direct consequence of Theorem 2.16 (see also Kozlov [15, section 2]).

PROPOSITION 3.1.

1. *The operators A_ε G-converge as $\varepsilon \to 0$ to an elliptic operator with constant nonrandom coefficients $\mathcal{A} = \{a_{zz'}\}_{z,z' \in \Lambda\setminus(0,0)}$.*
2. *The limit matrix is isotropic: $\mathcal{A} = a^\delta(p)\, I$ (I is the identity matrix).*

We call $a^\delta(p)$ the effective coefficient and study its asymptotics as $\delta \to 0$ for various $p \in [0, 1]$.

3.1. Some results from percolation theory. In this section, we quote and discuss briefly several results from percolation theory. We consider the so-called site percolation model (see Grimmett [10]) and, following the tradition, say black and

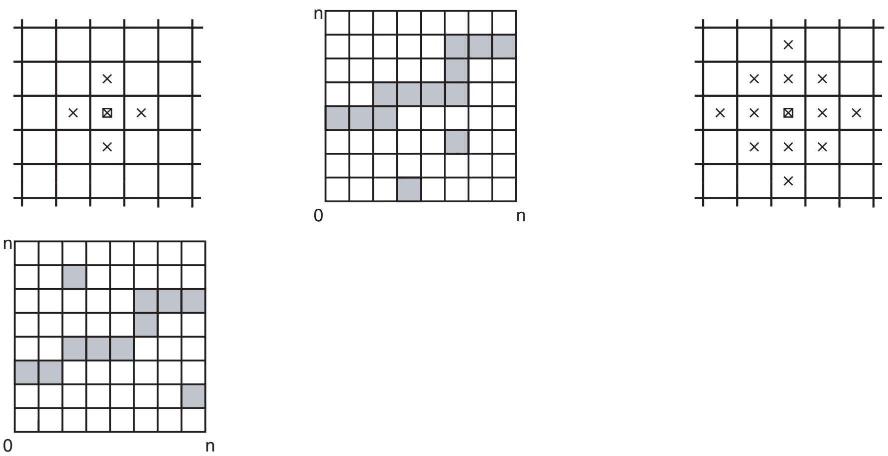

FIG. 3.2. *The neighbor squares and black channels in the cases $\gamma = 1$ (left) and $\gamma = \sqrt{2}$ (right).*

white squares instead of "δ" and "1" squares, respectively. All the squares are enumerated by the coordinates of their centers and the distance $\mathrm{dist}(i, j)$ between squares i and j, $(i, j \in \mathbb{Z}^2)$, is defined as the Euclidean distance $|i - j|$.

DEFINITION 3.2.

- *Two black squares i and j are* γ-connected *if* $\mathrm{dist}(i, j) \leq \gamma$. *As soon as the value of γ is fixed, we just refer to* connected *squares or* neighbor *squares.*
- *Consider the random subgraph containing only the black squares. The connected components of this graph are called* black clusters.
- *A finite set of black squares forms a* black channel *if the squares can be enumerated in such a way that any two successive squares in this enumeration are γ-connected (see Figure* 3.2 *for examples).*

Similarly, we define γ-connected white squares, white clusters and white channels. When the probability p varies, the geometric properties of the black clusters are modified. The more p increases, the bigger are the sizes of the clusters, and they eventually form the unique infinite cluster (see, for example, Grimmett [10]). Below, some basic constructions of percolation theory are presented.

The probability space is introduced as follows. As sample space, we take $K = \Pi_{s\in\mathbb{Z}^2}\{\delta, 1\}$. Each point of K: $\kappa = (\kappa(s); s \in \mathbb{Z}^2)$ is called a configuration. We take $\mathcal{G}$ to be the σ-field of subsets of K generated by the finite dimensional cylinders. And, for each $p \in [0, 1]$, we define the probability measure P_p as the product measure on $(K, \mathcal{G})$ such that the random variables $\kappa(s), s \in \mathbb{Z}^2$ are independent and satisfy $P_p(\kappa(x) = \delta) = p$.

In what follows we identify the probability space $(K, \mathcal{G}, P_p)$ with the general probability space $(\Omega, \mathcal{F}, P)$ defined above.

Let $|C|$ be the cardinal of the cluster which contains the origin. The cluster-size distribution is given by $\theta_n(p) = P_p(|C| = n),\ n \in \mathbb{N}^*$. The probability $\theta(p) = P_p(|C| = +\infty)$ that the origin belongs to the infinite cluster is called the *percolation probability.* There exists a *critical probability* $p_c(\gamma)$, also called the *percolation threshold*, such that

$$\begin{cases} \theta(p) = 0 & \text{if} \quad p \leq p_c(\gamma)\,, \\ \theta(p) > 0 & \text{if} \quad p > p_c(\gamma)\,. \end{cases}$$

TABLE 3.1
Evolution of the number of infinite cluster with respect to p.

<table>
<tr><th>p</th><th>0</th><th>$p_c(2)$</th><th>$p_c(\sqrt{2})$</th><th>$p_c(1)$</th><th>$1-p_c(2)$</th><th>1</th></tr>
<tr><td>$\gamma=1$</td><td colspan="2">White</td><td colspan="2">No infinite cluster</td><td colspan="2">Black</td></tr>
<tr><td>$\gamma=\sqrt{2}$</td><td colspan="2">White</td><td colspan="2">Black and White</td><td colspan="2">Black</td></tr>
<tr><td>$\gamma=2$</td><td>White</td><td colspan="4">Black and White</td><td>Black</td></tr>
</table>

Thus, for each fixed γ, the *critical probability* is $p_c(\gamma) \stackrel{\triangle}{=} \sup\{p : \theta(p) = 0\}$.

Figure 3.2 shows the sets of neighbor squares, with respect to the marked square, in the cases $\gamma = 1$ and $\gamma = \sqrt{2}$, and it emphasizes the difference between the structures of channels.

In Table 3.1, we can see, for three different values of γ, the presence of white and black clusters with respect to the values of p. The following relation holds: $p_c(1) + p_c(\sqrt{2}) = 1$, while $p_c(1) \sim 0.59$ and $p_c(\sqrt{2}) \sim 0.41$ (see Kesten [12]).

Moreover, according to Aizenman and Grimmett [1], $p_c(2) < p_c(\sqrt{2})$.

3.1.1. The channel property. Denote by $N(n)$ the number of mutually non-intersecting black channels joining the left and the right sides of the box $[0, n]^2$.

PROPOSITION 3.3 (see Kesten [12, section 11]). *Let $\gamma = 1$ or $\gamma = \sqrt{2}$. If $p > p_c(\gamma)$, then for almost all $\kappa \in K$ the inequality*

$$N(n) \geq c(p)\, n\,, \quad c(p) > 0,$$

holds for any $n \geq n_0(\kappa)$

REMARK 3.4. *In fact, this result holds true for any value of γ (see Golden and Kozlov* [9]*).*

REMARK 3.5. *For all $\gamma \geq \sqrt{2}$, the percolation models admit the coexistence of the channels of both colors (see Figure* 3.2*).* The geometry of the white and black subgraphs is rather different in subcritical and supercritical zones. In this connection, it is interesting to study carefully what happens near $p_c(\gamma)$.

PROPOSITION 3.6 (see Kesten [12, section 11]). *There exist some strictly positive constants c_1, c_2, c_3, δ_1, δ_2 such that, for $p > p_c(\gamma)$,*

$$P_p\left(N(n) \geq c_1\,(p - p_c(\gamma))^{\delta_1} n\right) \geq 1 - c_2\,(n+1)\, e^{-c_3\, n\, (p-p_c(\gamma))^{\alpha_2}}\,.$$

By the Borel–Cantelli lemma, we have

$$c(p) \geq c_1\,(p - p_c(\gamma))^{\delta_1}\,. \tag{3.1}$$

REMARK 3.7. *One can easily check that all the channels can be chosen to be no longer than $\theta(p)\, n$.*

3.2. Behavior of the effective coefficient. In this section, for the checkerboard model introduced above, we consider several discrete models characterized by

- the set of admissible jumps, i.e., the set Λ;
- the corresponding transition probabilities $\{p_z\}_{z\in\Lambda}$.

In all these models, the distribution of $\{p_z\}_{z\in\Lambda}$ will be invariant with respect to rotations at the angle $\pi/2$. This symmetry implies the isotropy of the effective tensor, and thus there is only one scalar effective coefficient $a^\delta(p)$ to be determined.

For each model, we study the limit behavior of the effective coefficient as $\delta \to 0$.

3.2.1. Harmonic mean. We begin by considering the "harmonic mean" model. Namely, we assume that

$$\Lambda = \{\pm(1,0),\ \pm(0,1),\ (0,0)\}$$

and define the transition probabilities as the harmonic mean of the values of $\kappa(\cdot)$ at the corresponding points:

$$p_z(x) = \begin{cases} \dfrac{1}{4}\,\dfrac{2\,\kappa(x)\,\kappa(x+z)}{(\kappa(x)+\kappa(x+z))} & \text{if } z \in \Lambda \setminus \{(0,0)\}\,, \\ 1 - \sum_{z\in\Lambda\setminus\{(0,0)\}} p_z(x) & \text{if } z = (0,0)\,, \\ 0 & \text{if } z \notin \Lambda\,. \end{cases}$$

Clearly, the family $\{p_z(x)\}$ satisfies the conditions (1), (2), and (3) in section 1.1, and moreover, its distribution is isotropic.

REMARK 3.8. *The choice of the harmonic mean is natural in the framework of the finite volume approach. Indeed, with this choice for the coefficients, we* conserve the fluxes. *This conservation is violated under another choices (see explanations in McCarthy* [19]*).*

The asymptotic behavior of the effective coefficient $a^\delta(p)$ as $\delta \to 0$ is described by the following statement.

THEOREM 3.9. *The effective coefficient $a^\delta(p)$ satisfies, for small δ, the following inequalities:*

$$\begin{array}{ll} 0 < c_1(p) \le a^\delta(p) \le 1 & \textit{if}\ \ 0 \le p < p_c(\sqrt{2}), \\ \delta \le a^\delta(p) \le c_2(p)\,\delta,\ c_2(p) > 0 & \textit{if}\ \ p_c(\sqrt{2}) < p \le 1. \end{array}$$

This means, in particular, that $a^\delta(p)$ does not vanish as $\delta \to 0$ if $p < p_c(\sqrt{2})$.

Proof.

1. *Case* $0 \le p < p_c(\sqrt{2})$.

   Consider the percolation model with $\gamma = 1$. By Proposition 3.3, for $0 \le p < 1 - p_c(1)$ there are at least $N(n) = c(p)\,n$ mutually nonintersecting white channels joining the left and the right sides of the square $[0,n]^2$. We denote by C_k the kth channel, $1 \le k \le N(n)$.

   Define on the space $\left(L^2(\Omega)\right)^{|\Lambda|}$ the following seminorm:

   $$\|\phi\|^2 \triangleq E\left\{\sum_{z\in\Lambda} p_z(\omega)\,(\phi_z(\omega))^2\right\}, \tag{3.2}$$

   where E is the expectation related to the measure μ. In fact, under the assumptions of the theorem, it is a norm, but we will not use this fact.

   Let $P_1(z) = z_1$ be the projection onto the first coordinate of vector z. According to Kozlov ([14, Chapter II, section 2], the effective coefficient $a^\delta(p)$ can be calculated as follows:

   $$a^\delta(p) = \inf_{\varphi\in L^2_{pot}(\Omega,\Lambda)} \|P_1(z) - \varphi\|^2\,, \tag{3.3}$$

   where the subspace $L^2_{pot}(\Omega,\Lambda)$ has been defined in section 2.4.1 of this paper. Denote by $\mathcal{H}$ the linear set $\mathcal{H} = \{\varphi_z(\omega) = \tilde\varphi(T_z\omega) - \tilde\varphi(\omega)\,;\ \tilde\varphi \in L^\infty(\Omega)\}$. This set $\mathcal{H}$ is dense in $L^2_{pot}(\Omega,\Lambda)$ (see section 2.4.1) and the functional $\varphi \to \|z_1 - \varphi\|$ is continuous in $L^2_{pot}(\Omega,\Lambda)$. Therefore, the infimum over $L^2_{pot}(\Omega,\Lambda)$ in (3.3) can be replaced by the infimum over $\mathcal{H}$.

Let φ belong to $\mathcal{H}$: there exists $\tilde\varphi \in L^\infty(\Omega)$ such that $\varphi_z(\omega) = \tilde\varphi(T_z\,\omega) - \tilde\varphi(\omega)$. Then,

$$\|P_1(z) - \varphi\|^2 = E\left\{\sum_{z\in\Lambda} p_z(\omega)(z_1 - (\tilde\phi(T_z\,\omega) - \tilde\phi(\omega)))^2\right\}.$$

Since T_x is ergodic, by the Birkhoff theorem we have for almost all realizations

$$\begin{aligned}\|P_1(z) - \phi\|^2 &= \lim_{n\to+\infty} \frac{1}{n^2} \sum_{x\in\mathbb{Z}^2\cap[0,n]^2} \sum_{z\in\Lambda} p_z(T_x\,\omega)\,\left(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\right)^2 \\ &= \lim_{n\to+\infty} \frac{1}{n^2} \sum_{x\in\mathbb{Z}^2\cap[0,n]^2} \sum_{z\in\Lambda} p_z(x)\,\left(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\right)^2. \qquad (3.4)\end{aligned}$$

Our goal now is to construct a uniformly positive lower bound for $a^\delta(p)$. To this end, on the RHS of the last formula, we first take into account only the points x located inside the channels:

$$\|P_1(z) - \phi\|^2 \ge \liminf_{n\to+\infty} \frac{1}{n^2} \sum_{x\in C}\sum_{z\in\Lambda} p_z(x)\,\left(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\right)^2,$$

where C stands for the union of white channels. Then, we enumerate the points x along each channel in such a way that any consecutive numbers correspond to neighbor points, and we replace the inner sum over $z\in\Lambda(x)$ by the sum over z such that $x+z$ belong to the same channel as x and have greater index than x. Denote this latter set of z by $\lambda(x)$, and notice that for each x from the union of white channels $\lambda(x)$ is not empty and consists of only one element. For $z\in\lambda(x)$, we clearly have $p_z(x) = 1/4$. Hence,

$$\|P_1 - \varphi\|^2 \ge \liminf_{n\to+\infty} \frac{1}{4\,n^2} \sum_{x\in C}\sum_{z\in\lambda(x)} \left(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\right)^2,$$

If we denote $S(x) = \sum_{z\in\lambda(x)}(z_1 - \tilde e(T_{x+z}\,\omega) + \tilde e(T_x\,\omega))$, and enumerate the channels $C = \cup_{k=1}^{N(n)} C_k$, then, for the kth channel, we have

$$\begin{aligned}\sum_{x\in C_k} S(x) &= \sum_{x\in C_k}\sum_{z\in\lambda(x)} (z_1 - \tilde\phi(T_{x+z}\omega) + \tilde\phi(T_x\omega)) \\ &= n + \sum_{x\in C_k}\sum_{z\in\lambda(x)} (-\tilde\phi(T_{x+z}\omega) + \tilde\phi(T_x\omega)) \\ &= n + \tilde\phi(T_{x_s(C_k)}\omega) - \tilde\phi(T_{x_f(C_k)}\,\omega) \ge n - c\,,\end{aligned}$$

where $c = 2\,\|\tilde\phi\|_{L^\infty(\Omega)}$, and $x_s(C_k)$ and $x_f(C_k)$ are, respectively, the starting and final points of kth channel. Summing up over the channels, we obtain

$$\sum_{x\in C} S(x) \ge (n-c)\,N(n)\,. \qquad (3.5)$$

By the Cauchy inequality, taking into account Remark 3.7, we get

$$\sum_{x\in C} S(x)^2 \ge \frac{\left(\sum_{x\in C} S(x)\right)^2}{\theta(p)\,n\,N(n)}\,.$$

In view of (3.5) this implies

$$\sum_{x\in C} S(x)^2 \geq \frac{(n-c)^2\, N(n)^2}{\theta(p)\, n\, N(n)}\,, \tag{3.6}$$

and

$$\|P_1(z)-\phi\|^2 \geq \lim_{n\to+\infty} \frac{1}{4\,n^2}\frac{c(p)}{\theta(p)}\,(n-c)^2 \geq c_1 > 0\,. \tag{3.7}$$

Hence: $a^\delta(p) \geq c_1 > 0$. The upper bound $a^\delta(p) \leq 1$ is obvious and, finally,

$$0 < c_1 \leq a^\delta(p) \leq 1\,.$$

2. *Case* $p_c(\sqrt{2}) < p \leq 1$.
Consider the percolation model with $\gamma = \sqrt{2}$. There are at least $c(p)\,n$ non-intersecting black channels C_k, $k = 1, 2, \ldots, N(n)$, joining the left and the right sides of the square $[0, n]^2$ (see Proposition 3.3).
Let us denote $\varepsilon = 1/n$ and define functions w^ε on $\varepsilon\mathbb{Z}^2 \cap [0,1]^2$ as follows:
   - $w^\varepsilon(\cdot, 0) = 0$, $w^\varepsilon(\cdot, 1) = 1$ (boundary conditions),
   - $w^\varepsilon(x) = \frac{(k-1/2)}{N(n)}$ for $x \in \varepsilon C_k$,
   - $w^\varepsilon(x) = \frac{k}{N(n)}$ for x from the set bounded by εC_k and εC_{k+1}.

Here, we suppose without loss of generality that the channels do not intersect the bottom and top faces of the square. The above function w^ε has been designed to possess the following properties :
   - In the area situated between any two consecutive channels C_k and C_{k+1}, this function is equal to a constant, the constants are different in distinct areas.
   - At each channel C_k the function w^ε makes a jump. The values of jumps are uniformly distributed on the channels so that the total increment of w^ε, as x_2 varies from 0 to 1, is equal to one.

By the definition and according to Proposition 3.3, the sequence w^ε is uniformly bounded in $W^{1,2}(Q_\varepsilon)$ and uniformly Lipschitz continuous; moreover, the Lipschitz constant is less than or equal to $c^{-1}(p)$. Thus, for a proper subsequence, we have

$$w^\varepsilon \xrightarrow[\varepsilon\to 0]{} u_0 \quad \text{weakly in } W^{1,2}(Q_\varepsilon)\,,$$
$$\sup_{x\in Q} |w^\varepsilon - u_0| \xrightarrow[\varepsilon\to 0]{} 0\,,$$

where $u_0 \in W^{1,2}(Q)$, $u_0(\cdot, 0) = 0$, $u_0(\cdot, 1) = 1$, and

$$|u_0(x^1) - u_0(x^2)| \leq c^{-1}(p)|x^1 - x^2|\,, \quad x^1,\, x^2 \in [0,1]^2\,.$$

Consider the expression

(3.8)
$$J^\varepsilon(w^\varepsilon) = \varepsilon^2 \sum_{x\in Q_\varepsilon} \sum_{z,z'\in\Lambda} a^\varepsilon_{zz'}(x)\, \partial^\varepsilon_z w^\varepsilon(x)\, \partial^\varepsilon_{z'} w^\varepsilon(x) = \varepsilon^2 \sum_{x\in Q_\varepsilon} \sum_{z\in\Lambda} p^\varepsilon_z(x)\, (\partial^\varepsilon_z w^\varepsilon(x))^2\,.$$

It follows from the definitions of w^ε and $p^\varepsilon_z(x)$ that

$$J^\varepsilon(w^\varepsilon) \leq c^{-2}(p)\, \delta\,. \tag{3.9}$$

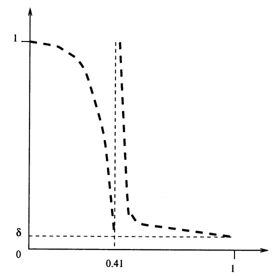

FIG. 3.3. *Illustration of Theorem* 3.10. *The behavior of* $a^\delta(p)$.

Moreover, by Proposition 2.14, $\liminf_{\varepsilon\to 0} J^\varepsilon(w^\varepsilon) \geq a^\delta(p) \int_{[0,1]^2} |\nabla u_0(x)|^2 dx \geq a^\delta(p)$. Combining the last two estimates, we get the desired inequality $a^\delta(p) \leq c^{-2}(p)\,\delta$. The lower bound $a^\delta(p) \geq \delta$ is evident. □

The next result describes the behavior of the effective coefficient $a^\delta(p)$ for p from a neighborhood of the critical point $p_c(\sqrt{2})$.

THEOREM 3.10. *In the vicinity of* $p_c(\sqrt{2})$, *the following inequalities hold:*

$$c_1\,(p_c(\sqrt{2}) - p)^{\alpha_1} \leq a^\delta(p) \quad \text{if } p < p_c(\sqrt{2}),$$
$$a^\delta(p) \leq \frac{c_2}{(p - p_c(\sqrt{2}))^{\alpha_2}}\,\delta \quad \text{if } p_c(\sqrt{2}) < p,$$

where c_1, c_2, α_1, *and* α_2 *are strictly positive constants.*

Figure 3.3 illustrates this result.

Proof. It is sufficient to substitute the estimate (3.1) in (3.7) and (3.9). The required estimates are now straightforward. □

3.2.2. Comparison with the behavior in continuous media. The asymptotic behavior of the effective coefficient described in the previous section (section 3.2.1) differs essentially from that obtained for the case of differential equations (see Jikov, Kozlov, and Oleinik [11, Chapter 9]). One of the reasons for this disagreement is the fact that we ignore the streams through the neighborhoods of vertices of the checker-board structure.

Here we modify the model of the previous section by involving the streams along the "diagonal directions," so that the asymptotic behavior of the effective coefficient as $\delta \to 0$ in this new model is similar to that obtained for the corresponding differential operator.

Let us begin by describing the scheme of discretization. We set

$$\Lambda = \{(0,0), \pm e_1, \pm e_2, \pm(e_1+e_2), \pm(e_1-e_2)\}, \qquad e_1 \stackrel{\triangle}{=} (1,0),\ e_2 \stackrel{\triangle}{=} (0,1),$$

(so, at each step, a trajectory of the corresponding random walk can choose one of the eight nearest points of $\mathbb{Z}^2$ or keep the same position).

In order to assign the values for $p_z(x)$, $|z| = \sqrt{2}$, we consider auxiliary periodic checker-board structure with a cell of periodicity shown in Figure 3.4. The effective coefficient of this medium is equal to $\sqrt{\delta}$ (see Jikov, Kozlov, and Oleinik [11, section 7.2]). This gives us an idea that, for the combination of squares shown in Figure 3.4, the coefficient $p_z(x)$ with $z = (e_1 + e_2)$, should be of order $\sqrt{\delta}$.

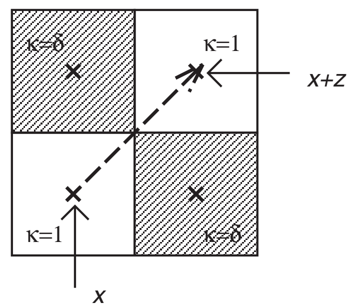

FIG. 3.4.

Inspired by these heuristic arguments, we define the transition probabilities by

$$p_z(x) = \begin{cases} \frac{1}{8}\min\left(\frac{2\,\kappa(x)\,\kappa(x+z)}{\kappa(x)+\kappa(x+z)},\ \sqrt{\frac{\kappa(x+z_1e_1)+\kappa(x+z_2e_2)}{2}\,\frac{\kappa(x)+\kappa(x+z)}{2}}\right) & \text{if } |z| = \sqrt{2}\,, \\ \dfrac{1}{8}\dfrac{2\,\kappa(x)\,\kappa(x+z)}{\kappa(x)+\kappa(x+z)} & \text{if } |z| = 1\,, \\ 1 - \sum_{z\in\Lambda, z\neq(0,0)} p_z(x) & \text{if } z = (0,0)\,, \\ p_z(x) = 0 & \text{if } z \notin \Lambda\,. \end{cases}$$

The following theorem describes the asymptotic behavior of the effective coefficient $a^\delta(p)$.

THEOREM 3.11. *The effective coefficient $a^\delta(p)$ satisfies, for small δ, the estimates*

$$\begin{array}{ll} 0 < c_1(p) \le a^\delta(p) \le 1 & \text{if } \ 0 \le p < p_c(\sqrt{2}), \\ c_2(p)\sqrt{\delta} \le a^\delta(p) \le c_3(p)\sqrt{\delta} & \text{if } \ p_c(\sqrt{2}) < p < 1 - p_c(\sqrt{2}), \\ \delta \le a^\delta(p) \le c_4(p)\,\delta & \text{if } \ 1 - p_c(\sqrt{2}) < p \le 1, \end{array}$$

where $c_1(p)$, $c_2(p)$, $c_3(p)$, and $c_4(p)$ are strictly positive.

Thus, the effective coefficient is uniformly positive when $p < p_c(\sqrt{2})$, is of order $\sqrt{\delta}$ when p is between $p_c(\sqrt{2})$ and $1 - p_c(\sqrt{2})$, and is of order δ when $p > 1 - p_c(\sqrt{2})$.

Proof. The cases $0 \le p < p_c(\sqrt{2})$ and $1 - p_c(\sqrt{2}) < p \le 1$ can be studied exactly in the same way as in Theorem 3.9. □

Now, we proceed with the case $p_c(\sqrt{2}) < p < 1 - p_c(\sqrt{2})$.

Consider the percolation model with $\gamma = \sqrt{2}$. Again, for sufficiently large n, there are at least $c(p)\,n$ mutually nonintersecting black $\sqrt{2}$-channels and white $\sqrt{2}$-channels joining the left and the right sides of the square $[0,n]^2$ (see Figure 3.5).

Lower bound. We consider the infinite white cluster. In order to obtain the lower bound for $a^\delta(p)$, we follow part (1) of the proof of Theorem 3.9. We point out that, along each white channel, if both x and $x+z$ belong to the channel and $|z| \le \sqrt{2}$, then $p_z(x) \ge \sqrt{\delta}/8$. Indeed, in this case $\kappa(x) = \kappa(x+z) = 1$ and, by the definition, $p_z(x)$ takes on one of the following values: $\frac{1}{8}\,,\ \frac{1}{8}\sqrt{\frac{1+\delta}{2}}\,,\ \frac{1}{8}\sqrt{\delta}\,.$

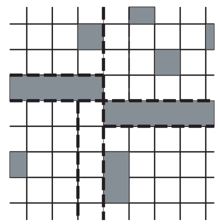

FIG. 3.5. *Intersection between a black and a white channel;* $p \in]\, p_c(\sqrt{2}),\, 1 - p_c(\sqrt{2})[$.

From (3.4), (3.6) and the above estimate of $p_z(x)$, we get

$$\begin{aligned}\|P_1(z) - \varphi\|^2 &= \lim_{n\to+\infty} \frac{1}{n^2} \sum_{x\in\mathbb{Z}^2\cap[0,n]^2} \sum_{z\in\Lambda(x)} p_z(T_x\,\omega)\, \Big(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\Big)^2 \\ &\geq \liminf_{n\to+\infty} \frac{\sqrt{\delta}}{8\,n^2} \sum_{x\in C^w} \sum_{z\in\Lambda(x)} \Big(z_1 - \tilde\phi(T_{x+z}\,\omega) + \tilde\phi(T_x\,\omega)\Big)^2 \\ &\geq \lim_{n\to+\infty} \frac{\sqrt{\delta}}{8\,n^2} \frac{c(p)}{\theta(p)} (n-c)^2 \geq c\sqrt{\delta}\,,\end{aligned}$$

where symbol C^w stands for the union of white channels. By virtue of (3.3), the last inequality implies the required lower bound.

Upper bound. We consider the infinite black cluster and the $N(n) = c(p)\, n$ black channels C_k^b, $k = 1, 2, \ldots, N(n)$ in the square $[0,\, n]^2$.

The upper bound $a^\delta(p) \leq c_3(p)\sqrt{\delta}$ can be established with the help of the following auxiliary functions:

- $w^\varepsilon(\cdot, 0) = 0\,,\ w^\varepsilon(\cdot, 1) = 1$;
- $w^\varepsilon(x) = \frac{(k-1/2)}{N(n)}$ for $x \in \varepsilon C_k^b$;
- $w^\varepsilon(x) = \frac{k}{N(n)}$ for x from the set bounded by εC_k^b and εC_{k+1}^b,

where $\varepsilon = 1/n$. Direct calculations show that $J^\varepsilon(w^\varepsilon) \leq c^{-2}(p)\sqrt{\delta}$; indeed, by the definition of $\{p_z(x)\}$, we have $p_z(x) \leq \delta/8$ if x belongs to a black channel, and $p_z(x) \leq \frac{\sqrt{\delta}}{8}$ if x and $x+z$ are situated at the opposite banks of a black channel. If we denote by u_0 an accumulating point of w^ε, then we have by Proposition 2.14

$$c(p)^{-2}\,\sqrt{\delta} \geq \lim_{\varepsilon\to 0} J^\varepsilon(w^\varepsilon) \geq a^\delta(p) \int_{[0,\,1]^2} |\nabla u_0(x)|^2\; dx \geq a^\delta(p)\,.$$

Comparing these results with Jikov, Kozlov, and Oleinik [11, Chapter 9, Theorem 9.5] shows that the discrete operators considered in this section adopt the asymptotic properties of the corresponding differential operators.

3.2.3. Geometric mean. We modify here the scheme of discretization of section 3.2.1 by taking the geometric mean in the definition of transition probabilities instead of the harmonic mean:

$$p_z(x) = \begin{cases} \dfrac{1}{4}\sqrt{\kappa(x)\,\kappa(x+z)} & \text{if } z \in \Lambda \setminus \{(0,0)\}\,, \\ 1 - \sum_{z\in\Lambda\setminus\{(0,0)\}} p_z(x) & \text{if } z = (0,0)\,, \\ 0 & \text{if } z \notin \Lambda\,, \end{cases}$$

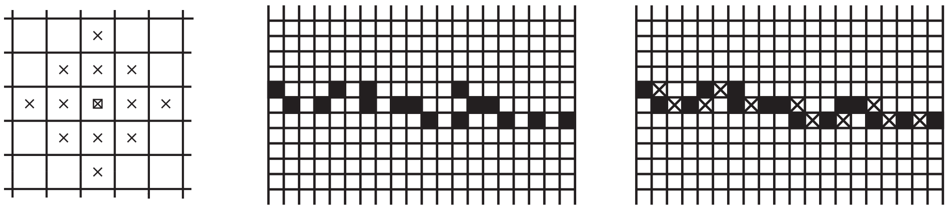

FIG. 3.6. $\gamma = 2$. *The neighbor squares (left), a black channel C_k (center), and one of its possible modifications $\tilde{C}_k$ (right).*

the set Λ being the same as in section 3.2.1 (i.e., with displacements toward the four nearest neighbors). Then, the asymptotic behavior of the effective coefficient $a^\delta(p)$ is described by the following statement.

THEOREM 3.12. *The effective coefficient $a^\delta(p)$ satisfies, for small δ, the estimates:*

$$\begin{array}{ll} 0 < c_1(p) \le a^\delta(p) \le 1 & \text{if } \ 0 \le p < p_c(\sqrt{2}), \\ c_2(p)\sqrt{\delta} \le a^\delta(p) \le c_3(p)\sqrt{\delta} & \text{if } \ p_c(\sqrt{2}) < p < 1 - p_c(2), \\ \delta \le a^\delta(p) \le c_4(p)\,\delta & \text{if } \ 1 - p_c(2) < p \le 1, \end{array}$$

where $c_1(p)$, $c_2(p)$, $c_3(p)$, and $c_4(p)$ are strictly positive.

Proof.

1. In the case $0 \le p < p_c(\sqrt{2})$, we need to justify only the lower bound. It can be done exactly in the same way as in Theorem 3.9. Another way to obtain the lower bound is to notice that for $|z| \ne 0$ the coefficients $p_z(x)$ under consideration majorate the respective coefficients defined as the harmonic mean. By virtue of the convergence of energy result and Theorem 3.9 this implies the desired lower bound.
2. In order to obtain the upper bound for $p_c(\sqrt{2}) < p < 1 - p_c(2)$ one can apply the technique developed in the part (2) of the proof of Theorem 3.9.
   To justify the lower bound in the case $p_c(\sqrt{2}) < p < 1 - p_c(2)$, we consider the percolation model with $\gamma = 2$ (see Remark 3.4). Here we encounter an additional difficulty: for $p \in]1 - p_c(\sqrt{2})\,, 1 - p_c(2)[$ the white 2-channels are not connected in a usual sense.
   We proceed as follows. For each channel C_k we introduce its 1-neighborhood:
   $$C_k^+ = \{x \in \mathbb{Z}^2 \,:\, |x - j| \le 1 \text{ for some } j \in C_k\}\,.$$
   It is easily seen that C_k^+ contains a sequence of squares $\{x_i\}$ denoted by $\tilde{C}_k$, which joins the left and the right sides of the square $[0, n]^2$ and has the following properties:
   - $|x_{i+1} - x_i| = 1$ for any consecutive x_i and x_{i+1};
   - $p_z(x) \ge \sqrt{\delta}/4$ for any x and z such that $x,\, z + z \in \tilde{C}_k$ and $|z| = 1$

   (see Figure 3.6) These sets $\tilde{C}_k$ are connected in a usual sense and consist in general of both white and black squares. Clearly, the number $\tilde{N}(n)$ of mutually nonintersecting sets $\tilde{C}_k$ still satisfies the estimate $\tilde{N}(n) \ge \tilde{c}(p)\, n$, $\tilde{c}(p) > 0$, for sufficiently large n. Then, one can use $\tilde{C}_k$ instead of C_k and argue like in part (1) of the proof of Theorem 3.9.
3. The upper bound in the case $1 - p_c(2) < p \le 1$ requires slightly different arguments than above. Consider the percolation model with $\gamma = 2$, and for each white cluster $\mathbb{C}$ denote by $\mathbb{C}^+$ the 1-neighborhood of $\mathbb{C}$:
   $$\mathbb{C}^+ = \{x \in \mathbb{Z}^2 \,:\, |x - j| \le 1 \text{ for some } j \in \mathbb{C}\}\,.$$

Let $\mathbb{C}^+(0)$ be the set $\mathbb{C}^+$ containing 0, and denote by $W(0)$ the size of $\mathbb{C}^+(0)$. If 0 does not belong to the 1-neighborhood of the union of white clusters, then $\mathbb{C}^+(0)$ is empty and $W(0)=0$.
We introduce the following sequence of random variables $\tilde\varphi^N(\omega)\in L^\infty(\Omega)$:

$$\tilde\varphi^N=\begin{cases} -\min\limits_{j\in\mathbb{C}^+(0)} j_1 & \text{if } 1\le W(0)\le N\\ 0 & \text{otherwise,}\end{cases}$$

and put $\varphi_z^N(\omega)=\tilde\varphi^N(T_z\omega)-\tilde\varphi^N(\omega)\,,\ z\in\Lambda$. It is clear that $|\varphi_z^N(\omega)|\le 2N$. According to Kesten [12, Theorem 5.1], the estimate

$$P_p\{W(0)>n\}\le c\,\exp(-c(p)\,n)\,,\quad c(p)>0\,, \tag{3.10}$$

holds for all $p>1-p_c(2)$. Therefore, by the definition of φ_z^N, we have

$$P_p\{\varphi_z^N\ge n\}\le c\,\exp(-c_1(p)\,n)\,,\quad c_1(p)>0\,,\quad n=1,2,\dots,2N\,. \tag{3.11}$$

The random variables φ_z^N and p_z possess the following properties:
- if both 0 and z belong to $\mathbb{C}^+(0)$ and $W(0)\le N$, then $P_1(z)-\varphi_z^N=0\,$;
- if at least one of them does not belong to $\mathbb{C}^+(0)$, then $p_z=\delta/4\,$.

In combination with (3.10) and (3.11), this implies

$$\begin{aligned} a^\delta(p)&\le\|P_1(z)-\varphi_z^N\|=E\sum_{z\in\Lambda}p_z(z_1-\varphi_z^N)^2\\ &\le c\,\delta\sum_{k=1}^{2N}k\,\exp(-c_1(p)k)+c\,\exp(-c(p)N)\\ &\le\bar c\,\delta+c\,\exp(-c(p)N)\,,\end{aligned}$$

where $\bar c$ does not depend on N. Passing to the limit as $N\to\infty$ gives $a^\delta(p)\le\bar c\,\delta$. ∎

3.2.4. Arithmetic mean. This section deals with another modification of the scheme of section 3.2.1. Namely, the transition probabilities are defined as the corresponding arithmetic means

$$p_z(x)=\begin{cases}\dfrac14\dfrac{\kappa(x)+\kappa(x+z)}{2} & \text{if } z\in\Lambda\setminus\{(0,0)\}\,,\\ 1-\sum_{z\in\Lambda\setminus\{(0,0)\}}p_z(x) & \text{if } z=(0,0)\,,\\ 0 & \text{if } z\notin\Lambda\,,\end{cases}$$

while the set Λ remains the same as in section 3.2.1.

Theorem 3.13. *The effective coefficient $a^\delta(p)$ satisfies, for small δ, the estimates*

$$\begin{aligned} &0<c_1(p)\le a^\delta(p)\le 1 &&\text{if }\ 0\le p<1-p_c(2),\\ &\delta\le a^\delta(p)\le c_2(p)\,\delta &&\text{if }\ 1-p_c(2)<p\le 1,\end{aligned}$$

where $c_1(p)$ and $c_2(p)$ are strictly positive.

Proof. The first estimate relies on the channel property of the percolation model corresponding to $\gamma=2$. As in the preceding theorem, we enlarge the white 2-channels to make them connected, and note that along each modified channel the transition probabilities are uniformly positive: $p_z(x)\ge(1+\delta)/8$ if $z\in\Lambda$ and x and $x+z$ belong to a modified channel. As above, this implies the lower bound $a^\delta(p)\ge c_1(p)>0$.

The proof of the second estimate is exactly the same as that of the last estimate in the preceding theorem. ∎

REMARK 3.14. *The statements of Theorems* 3.9–3.11 *remain unchanged if we assume that the size of mesh $h(\varepsilon)$ of a grid is less than ε while $h(\varepsilon)/\varepsilon$ is a constant.*

Appendices.

Appendix A. Convergence of discrete functions. Let f^ε be an arbitrary function defined in the discrete domain $Q_\varepsilon = \varepsilon\mathbb{Z}^d \cap Q$, and let $\tilde{f}^\varepsilon$ be the piecewise-constant interpolation of f^ε:

$$\tilde{f}^\varepsilon(x) = f^\varepsilon(y) \quad \text{if } y \in Q_\varepsilon \text{ and } x \in y + \left[\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}\right]^d .$$

DEFINITION A.1. *We say that a family of functions $f^\varepsilon \in L^2(Q_\varepsilon)$ converges strongly (resp., weakly) to the function $f \in L^2(Q)$ as $\varepsilon \to 0$ if $\tilde{f}^\varepsilon$ converges strongly (resp., weakly) to f in $L^2(Q)$. For this convergence we use the notation*

$$f^\varepsilon \xrightarrow[\varepsilon\to 0]{} f \quad \textit{in } L^2(Q_\varepsilon) \quad (\textit{resp., weakly in } L^2(Q_\varepsilon))\,.$$

Similarly, one can define the $W^{1,2}(Q)$-convergence of discrete functions with $\tilde{f}^\varepsilon$ being the piecewise linear interpolation of f^ε (instead of the piecewise constant one).

The convergence in $W^{-1,2}(Q)$ can be defined in terms of duality. Namely, we say that $f^\varepsilon \in W^{-1,2}(Q_\varepsilon)$ converges to $f \in W^{-1,2}(Q)$ strongly (resp., weakly) if for any sequence $g^\varepsilon \in W_0^{1,2}(Q_\varepsilon)$ and $g \in W_0^{1,2}(Q)$ such that $g^\varepsilon \to g$ weakly (resp., strongly) in $W^{1,2}(Q)$, we have

$$\langle f^\varepsilon, g^\varepsilon\rangle \xrightarrow[\varepsilon\to 0]{} \langle f, g\rangle\,.$$

DEFINITION A.2. *Let $w^\varepsilon \in L^2(Q_\varepsilon)$ and $w^0 \in L^2(Q)$. The sequence w^ε* converges $\star$-weakly *to w^0 if for any $\varphi \in \mathcal{C}_0^\infty(Q)$,*

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x\in Q_\varepsilon} w^\varepsilon(x)\,\varphi(x) = \int_Q w^0(x)\,\varphi(x)\,dx\,.$$

Appendix B. The derivative of a product of discrete functions.

PROPOSITION B.1. *Let f and g belong to $W^{1,2}(Q_\varepsilon)$. Then,*

$$\sum_{z\in Q_\varepsilon} |\partial_z^\varepsilon(fg) - f\partial_z^\varepsilon g - g\partial_z^\varepsilon f| \le \varepsilon \sum_{z\in Q_\varepsilon} |\partial_z^\varepsilon f||\partial_z^\varepsilon g|\,.$$

Proof. We have

$$\begin{aligned}\varepsilon\,\partial_z^\varepsilon(f(x)\,g(x)) &= f(x+\varepsilon\,z)\,g(x+\varepsilon\,z) - f(x)\,g(x)\\ &= g(x)\,(f(x+\varepsilon\,z)-f(x)) + f(x+\varepsilon\,z)\,(g(x+\varepsilon\,z)-g(x))\\ &= \varepsilon\,[g(x)\,\partial_z^\varepsilon f(x) + f(x+\varepsilon\,z)\,\partial_z^\varepsilon g(x)]\,.\end{aligned}$$

We have $f(x+\varepsilon\,z) = f(x) + \varepsilon\partial_z^\varepsilon f(x)$. Therefore,

$$\partial_z^\varepsilon(f(x)\,g(x)) = g(x)\,\partial_z^\varepsilon f(x) + f(x)\,\partial_z^\varepsilon g(x) + \varepsilon\partial_z^\varepsilon f(x)\partial_z^\varepsilon g(x),$$

and the desired estimate immediately follows. ∎

Appendix C. The Friedrichs and Poincaré inequalities.

This appendix is devoted to the Friedrichs and Poincaré inequalities for grid functions. In fact, in order to prove the propositions below, one can follow the same ideas as in the case of the continuous argument. For this reason, we omit the proof.

PROPOSITION C.1. *Let Q be a bounded domain with piecewise smooth boundary and denote the discretization of Q by Q_ε. Then, for any $v^\varepsilon \in W_0^{1,2}(Q_\varepsilon)$ the following inequality holds:*

$$\|v^\varepsilon\|^2_{L^2(Q_\varepsilon)} \le c(Q)\,\varepsilon^d \sum_{x\in Q_\varepsilon}\sum_{i=1}^{d}(\partial^\varepsilon_{\pm e_i} v^\varepsilon(x))^2 . \tag{C.1}$$

PROPOSITION C.2. *Let Q be a smooth bounded domain. Then, for all sufficiently small ε and for any $v^\varepsilon \in W^{1,2}(Q_\varepsilon)$ such that $\sum_{x\in Q_\varepsilon} v^\varepsilon(x) = 0$, the following inequality is satisfied:*

$$\sum_{x\in Q_\varepsilon} |v^\varepsilon(x)|^2 \le C(q)\,\varepsilon^d \sum_{x\in Q_\varepsilon}\sum_{i=1}^{d}|\bar\partial^\varepsilon_{\pm e_i} v^\varepsilon(x)|^2 . \tag{C.2}$$

REMARK C.3. *The statement of Proposition C.2 remains valid for the domain $\overline{Q}_\varepsilon$.*

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