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C. Constanda
M.E. Pérez
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Homogenization of a Convection–Diffusion Equation in a Thin Rod Structure

G. Panasenko,[1] I. Pankratova,[2] and A. Piatnitski[2]

[1] Université de Saint Étienne, France; grigory.panasenko@univ-st-etienne.fr
[2] Narvik University College, Norway; iripan@hin.no, andrey@sci.lebedev.ru

26.1 Introduction

This chapter is devoted to the homogenization of a stationary convection–diffusion model problem in a thin rod structure. More precisely, we study the asymptotic behavior of solutions to a boundary value problem for a convection–diffusion equation defined in a thin cylinder that is the union of two nonintersecting cylinders with a junction at the origin. We suppose that in each of these cylinders the coefficients are rapidly oscillating functions that are periodic in the axial direction, and that the microstructure period is of the same order as the cylinder diameter. On the lateral boundary of the cylinder we assume the Neumann boundary condition, while at the cylinder bases the Dirichlet boundary conditions are posed.

Similar problems for the elasticity system have been intensively studied in the existing literature. We quote here the works [KoPa92], [MuSi99], [Naz82], [Naz99], [TuAg86], [TrVi87], [Ve95]. The contact problem of two heterogeneous bars was considered in [Pa94-I], [Pa96-II], [Past02]. Elliptic equations in divergence form have been addressed, for example, in [BaPa89] and [Pa05]. In contrast to the divergence-form operators, in the case of the convection-diffusion equation the asymptotic behavior of solutions depends crucially on the direction of what is called the effective convection, which is introduced in Section 26.2. In this chapter we only consider the case when in each of the two cylinders (being the constituents of the rod) the effective convection is directed from the end of the cylinder towards the junction.

The asymptotic expansion of a solution includes the interior expansion, the boundary layers in the neighborhoods of the cylinder ends, and the interior boundary layer in the vicinity of the junction. Note that the leading term of the asymptotics is described in terms of a pair of first order ordinary differential equations. The construction of the interior expansions follows the classical scheme. The analysis of boundary layers in the neighborhoods of the cylinder ends relies on the results obtained in [PaPi09]. In order to build the interior boundary layer we study a qualitative problem for the convection–diffusion

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equation in an infinite cylinder. This is done in Section 26.7. As far as the authors are aware, no one has studied a convection–diffusion equation with first order terms in an infinite cylinder. In the case under consideration, when in each of the two cylinders the effective convection is directed from the end of the cylinder towards the junction, we prove the existence of a solution for such a problem and discuss its qualitative properties. In other cases the situation is much more difficult (especially in the case when effective convections occur in opposite directions) and outside the scope of the present work.

26.2 Problem Statement

Let Q be a bounded $C^{2,\alpha}$ domain in $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$ with points $x' = (x_2, ..., x_d)$. Denote $G_\varepsilon = [-1,1] \times (\varepsilon Q) \subset \mathbb{R}^d$ a thin rod with the lateral boundary $\Gamma_\varepsilon = [-1,1] \times \partial(\varepsilon Q); x = (x_1, x')$. We study the homogenization of a scalar elliptic equation with periodically oscillating coefficients

$$\begin{cases} A^\varepsilon u^\varepsilon \equiv -\operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) - \dfrac{1}{\varepsilon}\left(b^\varepsilon(x), \nabla u^\varepsilon\right) = \dfrac{1}{\varepsilon} f(x_1), & x \in G_\varepsilon, \\ B^\varepsilon u^\varepsilon \equiv \dfrac{\partial u^\varepsilon}{\partial n_{a^\varepsilon}} = g(x_1), & x \in \Gamma_\varepsilon, \\ u^\varepsilon(-1, x') = \varphi^-\left(\dfrac{x'}{\varepsilon}\right), \quad u^\varepsilon(1, x') = \varphi^+\left(\dfrac{x'}{\varepsilon}\right), & x' \in \varepsilon Q, \end{cases} \tag{26.1}$$

where the matrix-valued function $a^\varepsilon(x)$ and the vector field $b^\varepsilon(x)$ are given by $a^\varepsilon(x) = a(x/\varepsilon)$, $b^\varepsilon(x) = b(x/\varepsilon)$, and $\varepsilon > 0$ is a small parameter. In (26.1) $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^d$; $\partial u^\varepsilon/\partial n_{a^\varepsilon} = (a^\varepsilon \nabla u^\varepsilon, n)$ is the co-normal derivative of u^ε, and n is an external unit normal. Throughout the chapter we denote

$$\mathbb{G} = (-\infty, +\infty) \times Q, \quad \Gamma = (-\infty, +\infty) \times \partial Q;$$

$$G_\alpha^\beta = (\alpha, \beta) \times Q, \quad -\infty \le \alpha \le \beta \le +\infty.$$

We suppose the following conditions to hold:

(H1) The coefficients $a_{ij}(y) \in C^{1,\alpha}(\mathbb{G})$ and $b_j(y) \in C^\alpha(\mathbb{G})$ are periodic outside some compact set $K \Subset G_{-1}^1$. More precisely,

$$a_{ij}(y) = \begin{cases} a_{ij}^+(y), & y_1 > 1, \\ \tilde{a}_{ij}(y), & |y_1| \le 1, \\ a_{ij}^-(y), & y_1 < -1; \end{cases} \qquad b(y) = \begin{cases} b_j^+(y), & y_1 > 1, \\ \tilde{b}_j(y), & |y_1| \le 1, \\ b_j^-(y), & y_1 < -1; \end{cases}$$

where $a^\pm(y)$ and $b^\pm(y)$ are periodic in y_1. Without loss of generality, we assume that the period is equal to 1.

(H2) The matrices $a^\pm(y)$ are symmetric.

(H3) We assume that $a^{\pm}(y)$ satisfy the uniform ellipticity condition; that is, there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$\Lambda\,|\xi|^2 \leq \sum_{i,j=1}^{d} a^{\pm}_{ij}(y)\,\xi_i\,\xi_j, \quad \forall \xi \in \mathbb{R}^d. \tag{26.2}$$

(H4) $\varphi^{\pm}(y') \in H^{1/2}(Q)$.

(H5) Functions $f(x_1)$ and $g(x_1)$ are supposed to be smooth, namely, $f(x_1) \in C^2(G_\varepsilon)$ and $g(x_1) \in C^2(\Gamma_\varepsilon)$.

The goal of this work is to study the asymptotic behavior of $u^\varepsilon(x)$, as $\varepsilon \to 0$. As was noted in the Introduction, in contrast to the case of an operator in divergence form, the situation turns out to depend crucially on the signs of the effective fluxes $\bar{b}_1^{\pm}$, the constants which are defined in terms of the kernel of the adjoint periodic operators and coefficients of the equation. When constructing boundary layer functions, we consider only one case: $\bar{b}_1^{+} < 0$, $\bar{b}_1^{-} > 0$.

26.3 Formal Asymptotic Expansion

In the sequel we use the following notation:

$$G_\varepsilon^{+} = \{x = (x_1, x') \in G_\varepsilon : x_1 > \varepsilon\}, \quad G_\varepsilon^{-} = \{x = (x_1, x') \in G_\varepsilon : x_1 < -\varepsilon\};$$
$$A_y^{\pm} v \equiv -\mathrm{div}_y\,(a^{\pm}(y)\nabla_y v) - (b^{\pm}(y), \nabla_y v), \quad y \in Y;$$
$$B_y^{\pm} v \equiv \frac{\partial v}{\partial n_{a^{\pm}}} = \sum_{i,j=1}^{d} a^{\pm}_{ij}(y)\,\partial_{y_j} v\, n_i, \quad y \in Y,$$

where $Y = \mathfrak{S}_1 \times Q$, with $\mathfrak{S}_1$ a unit circle, denotes the cell of periodicity. In what follows we identify y_1-periodic functions with functions defined on Y. Notice that $\partial Y = \mathfrak{S}_1 \times \partial Q$.

In each half-cylinder G_ε^{+} and G_ε^{-} the inner asymptotic expansion of a solution to equation (26.1) has the form (see, for example, [BaPa89], [BLP78])

$$\begin{aligned} u_\infty^{\pm} &= v_0^{\pm}(x_1) + \varepsilon\big[N_1^{\pm}\big(\tfrac{x}{\varepsilon}\big)\,(v_0^{\pm})'(x_1) + v_1^{\pm}(x_1) + q_1^{\pm}\big(\tfrac{x}{\varepsilon}\big)g(x_1)\big] \\ &+ \varepsilon^2\big[N_2^{\pm}\big(\tfrac{x}{\varepsilon}\big)\,(v_0^{\pm})''(x_1) + N_1^{\pm}\big(\tfrac{x}{\varepsilon}\big)\,(v_1^{\pm})'(x_1) + v_2^{\pm}(x_1) + q_2^{\pm}\big(\tfrac{x}{\varepsilon}\big)g'(x_1)\big]. \end{aligned} \tag{26.3}$$

The leading term of the asymptotics, $v_0^{\pm}$, satisfies a first order ordinary differential equation

$$\bar{b}_1^{\pm}(v_0^{\pm})'(x_1) = f(x_1) + g(x_1)\int_{\partial Y} p^{\pm}(y)d\sigma_y, \tag{26.4}$$

where

$$\bar{b}_1^\pm = \int\limits_Y \left(a_{i1}^\pm(y)\partial_{y_i}p^\pm(y) - b_1^\pm(y)p^\pm(y)\right) dy$$

is called the effective axial drift; and $p^\pm(y)$ belong to the kernels of adjoint periodic operators defined on Y:

$$\begin{cases} -\mathrm{div}\,(a^\pm(y)\nabla p^\pm) + \mathrm{div}\,(b^\pm p^\pm) = 0, & y \in Y, \\ \dfrac{\partial p^\pm}{\partial n_{a^\pm}} - (b^\pm, n)\, p^\pm = 0, & y \in \partial Y. \end{cases}$$

Throughout the chapter we will assume that

(H6) $\bar{b}_1^- > 0$ *and* $\bar{b}_1^+ < 0$.

Notice that since $f(x_1), g(x_1) \in C^2([-1,1])$, then $v_0^+(x_1) \in C^3(\varepsilon, 1)$, $v_0^-(x_1) \in C^3(-1, -\varepsilon)$.

One can see that necessarily the functions $N_1^\pm$ and $q_1^\pm$ satisfy the problems

$$\begin{cases} A_y^\pm N_1^\pm = \partial_{y_i} a_{i1}^\pm + b_1^\pm + \bar{b}_1^\pm, & y \in Y, \\ B_y^\pm N_1^\pm = -a_{i1}^\pm n_i, & y \in \partial Y; \end{cases} \qquad \begin{cases} A_y^\pm q_1^\pm = -\displaystyle\int_{\partial Y} p^\pm d\sigma_y, & y \in Y, \\ B_y^\pm q_1^\pm = 1, & y \in \partial Y. \end{cases} \tag{26.5}$$

Obviously, by the definition of $\bar{b}_1^\pm$, the compatibility conditions for (26.5) are satisfied; thus, these problems are uniquely (up to an additive constant) solvable. Since we assumed that $a_{ij}(y) \in C^{1,\alpha}(\overline{\mathbb{G}})$ and $b_j(y) \in C^\alpha(\overline{\mathbb{G}})$, then $N_1^\pm(y)$ and $q_1^\pm(y)$ belong to $C^{2,\alpha}(\overline{Y})$ (see, for example, [GiTr98], [LaUr68]).

The equation for $v_1^\pm$ reads

$$\bar{b}_1^\pm\, (v_1^\pm)'(x_1) = h_2^\pm\, (v_0^\pm)''(x_1) + \overline{q_1^\pm}\, g'(x_1), \tag{26.6}$$

where $h_2^\pm$ and $\overline{q_1^\pm}$ are constants given by the following expressions:

$$h_2^\pm = \int_Y \left(a_{11}^\pm p^\pm - a_{i1}^\pm N_1^\pm(y)\partial_{y_i}p^\pm + b_1^\pm N_1^\pm p^\pm + a_{1j}^\pm \partial_{y_j} N_1^\pm p^\pm\right) dy;$$
$$\overline{q_1^\pm} = \int_Y \left(- a_{i1}^\pm q_1^\pm \partial_{y_i} p^\pm + b_1^\pm q_1^\pm p^\pm + a_{1j}^\pm \partial_{y_j} q_1^\pm p^\pm\right) dy.$$

Let us note that $v_1^\pm(x_1)$, as a solution of (26.6), has continuous derivatives in $\overline{Y}$ up to the second order.

One can see that $N_2^\pm$ and $q_2^\pm$ satisfy the problems

$$\begin{cases} A_y^\pm N_2^\pm = a_{11}^\pm + \partial_{y_i}(a_{i1}^\pm N_1^\pm) + b_1^\pm N_1^\pm + a_{1j}^\pm \partial_{y_j} N_1^\pm - h_2^\pm, & y \in Y, \\ B_y^\pm N_2^\pm = -a_{i1}^\pm N_1^\pm n_i, & y \in \partial Y; \end{cases} \tag{26.7}$$

$$\begin{cases} A_y^\pm q_2^\pm = \partial_{y_i}(a_{i1}^\pm q_1^\pm) + b_1^\pm q_1^\pm + a_{1j}^\pm \partial_{y_j} q_1^\pm - \overline{q_1^\pm}, & y \in Y, \\ B_y^\pm q_1^\pm = -a_{i1}^\pm q_1^\pm n_i, & y \in \partial Y. \end{cases} \tag{26.8}$$

The compatibility conditions are satisfied and problems (26.7)–(26.8) are uniquely solvable. The smoothness of the coefficients and the properties of the functions $N_1^\pm$, $q_1^\pm$ imply that $N_2^\pm(y), q_2^\pm(y) \in C^{2,\alpha}(\overline{Y})$.

The equation for $v_2^\pm(x_1)$ is the following:

$$\bar{b}_1^\pm\,(v_2^\pm)'(x_1) = h_3^\pm\,(v_0^\pm)^{(3)}(x_1) + h_2^\pm\,(v_1^\pm)''(x_1) + \overline{q_2^\pm}\,g''(x_1), \tag{26.9}$$

where

$$h_3^\pm = \int_Y \left(a_{11}^\pm N_1^\pm p^\pm - a_{i1}^\pm N_2^\pm \partial_{y_i} p^\pm + b_1^\pm N_2^\pm p^\pm + a_{1j}^\pm \partial_{y_j} N_2^\pm p^\pm\right) dy;$$

$$\overline{q_2^\pm} = \int_Y \left(a_{11}^\pm q_1^\pm p^\pm - a_{i1}^\pm q_2^\pm \partial_{y_i} p^\pm + b_1^\pm q_2^\pm p^\pm + a_{1j}^\pm \partial_{y_j} q_2^\pm p^\pm\right) dy.$$

The function $v_2^\pm$ as a solution of (26.9) is a $C^1(\overline{Y})$ function.

Note that the infinite number of terms in series (26.3) can be constructed. Interested readers can find in [Pa05] the description of the general method for such a construction together with some applications and examples.

26.4 Boundary Layers Near the Rod Ends

The asymptotic series (26.3) does not satisfy the boundary conditions on the bases of the rod, which is why we introduce the boundary layer functions in the neighborhoods of $S_{\pm 1} = \{x \in G_\varepsilon : x_1 = \pm 1,\ x' \in \varepsilon Q\}$:

$$\begin{aligned} v_{bl}^\pm(x) \equiv \left[w_0^\pm\left(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}_0^\pm\right] + \varepsilon\left[w_1^\pm\left(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}_1^\pm\right] \\ + \varepsilon^2\left[w_2^\pm\left(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}_2^\pm\right]. \end{aligned} \tag{26.10}$$

Here $w_0^\pm(y)$ are the solutions of homogeneous problems in semi-infinite cylinders $G_{-\infty}^0$ and $G_0^{+\infty}$, respectively,

$$\begin{cases} A_y^+ w_0^+(y) = 0, & y \in G_{-\infty}^0, \\ B_y^+ w_0^+ = 0, & y \in \Gamma_{-\infty}^0, \\ w_0^+(0, y') = \varphi^+(y'), \end{cases} \qquad \begin{cases} A_y^- w_0^-(y) = 0, & y \in G_0^{+\infty}, \\ B_y^- w_0^- = 0, & y \in \Gamma_0^{+\infty}, \\ w_0^-(0, y') = \varphi^-(y'). \end{cases} \tag{26.11}$$

As was proved in [PaPi09] (see Theorem 5.1), under assumptions **(H1)**–**(H6)**, problems (26.11) possess unique solutions stabilizing to constants $\hat{w}_0^\pm$ at an exponential rate, as $y_1 \to \mp\infty$. As boundary conditions for $v_0^\pm$ we choose $v_0^\pm(\pm 1) = \hat{w}_0^\pm$.

The functions $w_1^\pm$ satisfy the following problems:

$$\begin{cases} A_y^+ w_1^+(y) = 0, \quad y \in G_{-\infty}^0, \\ B_y^+ w_1^+ = 0, \quad y \in \Gamma_{-\infty}^0, \\ w_1^+(0, y') = -N_1^+(\delta, y')\,(v_0^+)'(1) \\ -q_1^+(\delta, y')\,g(1), \end{cases} \qquad \begin{cases} A_y^- w_1^-(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^- w_1^- = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_1^-(0, y') = -N_1^-(-\delta, y')\,(v_0^-)'(-1) \\ -q_1^-(-\delta, y')\,g(-1), \end{cases}$$

for some fixed $\delta \in [0,1)$ (δ is a fractional part of ε^{-1}). Taking into account that $\bar{b}_1^+ < 0$, $\bar{b}_1^- > 0$, one can see that $w_1^\pm$ stabilize to uniquely defined constants which we denote by $\hat{w}_1^\pm$ (see [PaPi09]). Then we take the constant $\hat{w}_1^\pm$ as boundary conditions for $v_1^\pm(x_1)$ as $x_1 = \pm 1$: $v_1^\pm(\pm 1) = \hat{w}_1^\pm$.

Turning back to (26.10), $w_2^\pm$ solve the problems

$$\begin{cases} A_y^\pm w_2^\pm = 0, \quad y \in G_{-\infty}^0 \ (y \in G_0^{+\infty}), \\ B_y^\pm w_2^\pm = 0, \quad y \in \Gamma_{-\infty}^0 \ (y \in \Gamma_0^{+\infty}), \\ w_2^\pm(0,y') = -N_2^\pm(\pm\delta, y')\,(v_0^\pm)''(\pm 1) \\ -N_1^\pm(\pm\delta, y')\,(v_1^\pm)'(\pm 1) - q_2^\pm(\pm\delta, y')\,g'(\pm 1). \end{cases} \tag{26.12}$$

$w_2^\pm$ tend to constants $\hat{w}_2^\pm$, as $y_1 \to \mp\infty$. As before, the existence and uniqueness of solutions and the property of the exponential stabilization to constants are ensured by Theorem 5.1 in [PaPi09]. Now we can choose a boundary condition for the functions $v_2^\pm$ as $x_1 = \pm 1$: $v_2^\pm(1) = \hat{w}_2^\pm$.

26.5 Boundary Layer in the Middle of the Rod

Before constructing the boundary layer functions in the middle of the rod, let us extend $v_0^+(x_1)$ (keeping the same notation) to $(-\infty, \varepsilon)$ as a solution of equation (26.4) satisfying the boundary condition $v_0^+(1) = \hat{w}_0^+$. In the same way we can extend v_1^+, v_2^+ to $(-\infty,\varepsilon)$, and v_0^-, v_1^-, v_2^- to $(-\varepsilon, +\infty)$ as solutions to corresponding ordinary differential equations. Periodic in y_1 functions $N_k^\pm$ and $q_k^\pm$, $k = 1,2,3$, we regard as defined everywhere in $\mathbb{G} = \mathbb{R} \times Q$.

Obviously, it suffices to match the formal asymptotic series u_∞^+, defined by (26.3) in G_∞^+, with zero in the vicinity of $S_0^\varepsilon = \{x \in G_\varepsilon : x_1 = 0\}$. Then, in the same way we can match u_∞^- with zero, and, summing up the obtained expressions, arrive at the final boundary layer corrector in the neighborhood of S_0^ε. In order to do this, we are looking for a "corrected" solution in the form

$$\begin{aligned} v_\varepsilon^\pm(x) = {} & \chi_0^\pm(y)\, v_0^\pm(x_1) + \varepsilon\, N_1^\pm(y)\, \phi^\pm(y)\,(v_0^\pm)'(x_1) + \varepsilon\, \chi_{1,1}^\pm(y)\,(v_0^\pm)'(x_1) \\ & + \varepsilon\, q_1^\pm(y)\, \phi^\pm(y) g(x_1) + \varepsilon\, \chi_{1,2}^\pm\, g(x_1) + \varepsilon\, \chi_1^\pm(y)\, v_1^\pm(x_1) \\ & + \varepsilon^2 N_2^\pm(y)\, \phi^\pm(y)\,(v_0^\pm)''(x_1) + \varepsilon^2 \chi_{2,1}^\pm(y)\,(v_0^\pm)''(x_1) \\ & + \varepsilon^2 N_1^\pm(y)\, \phi^\pm(y)\,(v_1^\pm)'(x_1) + \varepsilon^2 \chi_{2,2}^\pm(y)\,(v_1^\pm)'(x_1) \\ & + \varepsilon^2 q_2^\pm(y)\, \phi^\pm(y) g'(x_1) + \varepsilon^2 \chi_{2,3}^\pm(y)\, g'(x_1) + \varepsilon^2 \chi_2^\pm(y)\, v_2^\pm(x_1), \quad y = x/\varepsilon, \end{aligned} \tag{26.13}$$

where the functions $\chi_1^\pm(y)$, $\chi_{1,1}^\pm(y)$, $\chi_{1,2}^\pm(y)$, $\chi_{2,1}^\pm(y)$, $\chi_{2,2}^\pm(y)$, $\chi_{2,3}^\pm(y)$, and $\chi_2^\pm(y)$ are to be determined; $\phi^+(y) = \phi^+(y_1)$ is a smooth cut-off function such that $\phi^+(y) = 0$ if $y_1 < -1$ and $\phi^+(y) = 1$ if $y_1 > 1$, $\phi^- = 1 - \phi^+$.

Substituting (26.13) into (26.1) and collecting power-like terms related to different powers of ε, one gets equations for the unknown functions. Due to lack of space, we do not produce the calculations here.

$$\begin{cases} A_y\, \chi_m^{\pm} = 0, & y \in \mathbb{G}, \\ B_y\, \chi_m^{\pm} = 0, \quad y \in \Gamma, & m = 0, 1, 2. \end{cases} \tag{26.14}$$

$$\begin{cases} A_y\chi_{1,1}^{\pm} = -A_y(N_1^{\pm}(y)\phi^{\pm}(y)) + a_{1j}(y)\partial_{y_j}\chi_0^{\pm}(y) \\ \qquad + \partial_{y_i}\big(a_{i1}\chi_0^{\pm}(y)\big) + b_1(y)\chi_0^{\pm}(y) - \bar{b}_1^{\pm}\,\phi^{\pm}(y), & y \in \mathbb{G}; \\ B_y\chi_{1,1}^{\pm} = -a_{i1}\,\chi_0^{\pm}\,n_i - a_{ij}\,\partial_{y_j}(N_1^{\pm}\,\phi^{\pm})\,n_i, & y \in \Gamma; \end{cases} \tag{26.15}$$

$$\begin{cases} A_y\chi_{1,2}^{\pm} = -A_y\big(q_1^{\pm}(y)\phi^{\pm}(y)\big) - \phi^{\pm}(y)\displaystyle\int_{\partial Y} p^{\pm}(y)\,d\sigma_y, & y \in \mathbb{G}, \\ B_y\chi_{1,2}^{\pm} = -a_{ij}\,\partial_{y_j}\big(q_1^{\pm}(y)\,\phi^{\pm}(y)\big)\,n_i + \phi^{\pm}(y), & y \in \Gamma; \end{cases} \tag{26.16}$$

Problems (26.14)–(26.16), stated in the infinite cylinder $\mathbb{G}$, were derived by formal calculations which, of course, do not imply the solvability of these problems. Theorem 2, proved in Section 26.7, guarantees the existence of solutions to problems (26.14)–(26.16) in proper classes and, moreover, gives an additional qualitative information about the solutions. Indeed, we can choose $\chi_m^{\pm}$, $m = 0, 1, 2$, such that

$$\begin{aligned} \chi_m^{+} \underset{y_1 \to +\infty}{\longrightarrow} 1, \quad \chi_m^{+} \underset{y_1 \to -\infty}{\longrightarrow} 0; \\ \chi_m^{-} \underset{y_1 \to +\infty}{\longrightarrow} 0, \quad \chi_m^{-} \underset{y_1 \to -\infty}{\longrightarrow} 1, \quad m = 0, 1, 2. \end{aligned} \tag{26.17}$$

Such a choice of $\chi_0^{\pm}$ and definitions of $N_1^{\pm}(y)$ and $\phi^{\pm}(y)$ ensure the existence of solutions $\chi_{1,1}^{\pm}$ of problem (26.15), which stabilize to the constants at infinity. For the functions $\chi_{1,1}^{\pm}$ we assign zeros at infinity: $\chi_{1,1}^{\pm} \to 0$, $y_1 \to \pm\infty$.

Similarly, taking into account (26.5) and the definition of $\phi^{\pm}$, one can see that problems (26.16) are solvable. We also choose zeros as constants at infinity for $\chi_{1,2}^{\pm}$: $\chi_{1,2}^{\pm} \to 0$, $y_1 \to \pm\infty$.

In much the same way, we see that there exist $\chi_{2,1}^{\pm}$, $\chi_{2,2}^{\pm}$, $\chi_{2,3}^{\pm}$ stabilizing to zero, as $y_1 \to \pm\infty$, which solve the following problems:

$$\begin{cases} A_y\,\chi_{2,1}^{+} = -A_y(N_2^{+}\,\phi^{+}) + a_{11}\,\chi_0^{+} + a_{1j}\,\partial_{y_j}(N_1^{+}\,\phi^{+}) + \partial_{y_i}(a_{i1}\,N_1^{+}\,\phi^{+}) \\ + b_1\,N_1^{+}\,\phi^{+} + a_{1j}\,\partial_{y_j}\chi_{1,1}^{+} + \partial_{y_i}(a_{i1}\,\chi_{1,1}^{+}) + b_1\,\chi_{1,1}^{+} - h_2^{+}\,\phi^{+}, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,1}^{+} = -B_y\,(N_2^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_{1,1}^{+} - a_{i1}\,n_i\,N_1^{+}\,\phi^{+}, \quad y \in \Gamma; \end{cases} \tag{26.18}$$

$$\begin{cases} A_y\,\chi_{2,2}^{+} = -A_y(N_1^{+}\,\phi^{+}) + a_{1j}\,\partial_{y_j}\chi_1^{+} \\ \qquad + \partial_{y_i}(a_{i1}\chi_1^{+}) + b_1\,\chi_1^{+} - \bar{b}_1^{+}\,\phi^{+}, & y \in \mathbb{G}, \\ B_y\,\chi_{22}^{+} = -B_y(N_1^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_1^{+}, & y \in \Gamma; \end{cases} \tag{26.19}$$

$$\begin{cases} A_y\,\chi_{2,3}^{+} = -A_y(q_2^{+}\,\phi^{+}) + a_{1j}\,\partial_{y_j}(q_1^{+}\,\phi^{+}) + \partial_{y_i}(a_{i1}\,q_1^{+}\,\phi^{+}) + b_1\,q_1^{+}\,\phi^{+} \\ + a_{1j}\,\partial_{y_j}\chi_{1,2}^{+} + \partial_{y_i}(a_{i1}\,\chi_{1,2}^{+}) + b_1\,\chi_{1,2}^{+} - \overline{q_1^{+}}\,\phi^{+}, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,3}^{+} = -B_y\,(q_2^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_{1,2}^{+} - a_{i1}\,n_i\,q_1^{+}\,\phi^{+}, \quad y \in \Gamma. \end{cases} \tag{26.20}$$

Finally, taking into account the constructed inner formal asymptotic expansion and boundary layer correctors in the neighborhoods of $S_{\pm 1}$ and S_0, we arrive at the asymptotic solution of problem (26.1):

$$u_\infty^\varepsilon(x) \equiv v_\varepsilon^+(x) + v_{bl}^+(x) + v_\varepsilon^-(x) + v_{bl}^-(x), \tag{26.21}$$

where v_ε^+, v_ε^-, and $v_{bl}^\pm$ are defined by (26.13) and (26.10).

Remark 1. Adding the boundary layer functions $v_{bl}^\pm$ to the inner expansions $u_\infty^\pm$ makes it possible to satisfy the boundary conditions on the bases of the rod G_ε with an accuracy up to the third order in ε. Representing (26.21) as the sum of the inner expansions and the boundary layer functions

$$\begin{aligned} u_\infty^\varepsilon &= u_\infty^+(x) + (v_\varepsilon^+(x) - u_\infty^+(x)) + v_{bl}^+(x) \\ &+ u_\infty^-(x) + (v_\varepsilon^-(x) - u_\infty^-(x)) + v_{bl}^-(x), \end{aligned}$$

we make $(v_\varepsilon^\pm - u_\infty^\pm)$ exponentially small (but not vanishing) on $S_\pm^\varepsilon$, as well as v_{bl}^+ on S_{-1}^ε and v_{bl}^- on S_{+1}^ε. In order to satisfy exactly the boundary conditions, one can replace (26.21) with

$$\begin{aligned} \tilde{u}_\infty^\varepsilon &= u_\infty^+(x) + (v_\varepsilon^+(x) - u_\infty^+(x))\, \phi_1(x) + v_{bl}^+(x)\, \phi_1^+(x) \\ &+ u_\infty^-(x) + (v_\varepsilon^-(x) - u_\infty^-(x))\, \phi_1(x) + v_{bl}^-(x)\, \phi_1^-(x), \end{aligned}$$

where $\phi_1(x) = 1$ if $|x_1| < 1/3$ and $\phi_1(x) = 0$ otherwise;

$$\phi_1^+(x) = \begin{cases} 1, & x_1 > 2/3, \\ 0, & x_1 < 1/3. \end{cases} \qquad \phi_1^-(x) = \begin{cases} 1, & x_1 < -2/3, \\ 0, & x_1 > -1/3. \end{cases}$$

Substituting $\tilde{u}_\infty^\varepsilon$ into (26.1), it is straightforward to check that the presence of the cut-off functions results in the appearance of additional exponentially small (with respect to ε^{-1}) terms on the right-hand side. Later on we will prove a priori estimates (26.23) and (26.24) which ensure that the exponentially small perturbation of the right-hand side leads to the exponentially small perturbation of the solution, and, thus, is negligible in any polynomial in ε expansion. To simplify the notation, we deal with (26.21) neglecting the discrepancies on $S_{\pm 1}^\varepsilon$ which are exponentially small with respect to ε^{-1}.

26.6 Justification of the Procedure

Theorem 1. *Let the conditions* **(H1)**–**(H6)** *hold true. Then the approximate solution u_∞^ε given by formula* (26.21) *satisfies the estimates*

$$\begin{aligned} \|\nabla u_\infty^\varepsilon - \nabla u^\varepsilon\|_{L^2(G_\varepsilon)} &\le C\, \varepsilon^{3/2}\, \varepsilon^{(d-1)/2}, \\ \|u_\infty^\varepsilon - u^\varepsilon\|_{L^2(G_\varepsilon)} &\le C\, \varepsilon^{3/2}\, \varepsilon^{(d-1)/2}, \end{aligned} \tag{26.22}$$

where $u^\varepsilon(x)$ is the exact solution to problem (26.1).

Proof. First we obtain an a priori estimate for a solution to the problem

$$\begin{cases} A^\varepsilon u^\varepsilon = f^\varepsilon(x), & x \in G_\varepsilon, \\ B^\varepsilon u^\varepsilon = g^\varepsilon(x), & x \in \Gamma_\varepsilon, \\ u^\varepsilon(\pm 1, x') = 0, & x' \in \varepsilon Q \end{cases}$$

in terms of $f^\varepsilon(x)$ and $g^\varepsilon(x)$ (for the moment we do not specify the particular structure of these functions). While proving Theorem 2 in Section 26.7, we will show that the following estimates hold true:

$$\|\nabla u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\sqrt{\varepsilon}\,\|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\sqrt{\varepsilon}\,\|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{26.23}$$

Making use of the Friedrichs inequality for the function u^ε in G_ε, we obtain

$$\|u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\sqrt{\varepsilon}\,\|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\sqrt{\varepsilon}\,\|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{26.24}$$

Estimation of the $L^2(G_\varepsilon)$-norm of $A^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ and the $L^2(\Gamma_\varepsilon)$-norm of $B^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ will complete the justification procedure. Due to lack of space, we have to drop these estimates and leave them to the reader.

$$\big\|A^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)\big\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2};$$

$$\big\|B^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)\big\|_{L^2(\Gamma_\varepsilon)} \le C\,\varepsilon^2\,\varepsilon^{(d-2)/2}; \tag{26.25}$$

Taking into account (26.23)–(26.25) we get (26.22).

Remark 2. The estimates (26.23)–(26.24) imply that we can take $f(x_1) \in L^2(G_\varepsilon)$ and $g(x_1) \in L^2(\Gamma_\varepsilon)$.

26.7 Existence of a Solution in an Infinite Cylinder

We consider the following boundary value problem:

$$\begin{cases} A_{\#}\, u \equiv -\mathrm{div}\,(a(x)\,\nabla\, u(x)) - (b(x), \nabla\, u(x)) = f(x), & x \in \mathbb{G}, \\ B_{\#}\, u \equiv \dfrac{\partial\, u}{\partial\, n_a} = g(x), & x \in \Gamma. \end{cases} \tag{26.26}$$

We assume that
(H5)$'$ The functions $f \in C(\bar{\mathbb{G}})$ and $g \in C(\Gamma)$ are such that

$$\|f\|_{L^2(G_n^{n+1})} \le Ce^{-\gamma_1 n}, \quad \|g\|_{\Gamma^2(\Gamma_n^{n+1})} \le Ce^{-\gamma_1 n}, \quad \gamma_1 > 0, n \in \mathbb{R}.$$

The goal of this section is to show that in the case $\bar{b}_1^+ < 0$, $\bar{b}_1^- > 0$, problem (26.26) possesses a bounded (in a proper sense) solution, which stabilizes to constants, as $|x_1| \to \infty$.

Definition 1. *A weak solution $u(x)$ of problem* (26.26) *is said to be bounded if*

$$\|u\|_{L^2(G_n^{n+1})} \le C,$$

with a constant C independent of n.

The following theorem contains the main result of the section.

Theorem 2. *Let conditions* $(\mathbf{H1}) - (\mathbf{H3}), (\mathbf{H5})', (\mathbf{H6})$ *be fulfilled. Then for any constants K_∞^+ and K_∞^- there exists a bounded solution $u(x)$ of problem* (26.26) *such that it converges at the exponential rate to these constants, as $x_1 \to \pm\infty$,*

$$\|u - K_\infty^-\|_{L^2(G_{-\infty}^{-n})} \le C\,(1 + K_\infty^-)\,e^{-\gamma n},$$

$$\|u - K_\infty^+\|_{L^2(G_n^{+\infty})} \le C\,(1 + K_\infty^+)\,e^{-\gamma n}, \quad \gamma > 0,$$

and the following estimates hold:

$$\|u\|_{L^2(G_n^{n+1})} \le C\,\big(\|(1 + \sqrt{|x_1|})\,f\|_{L^2(\mathbb{G})} + \|(1 + \sqrt{|x_1|})\,g\|_{L^2(\Gamma)}\big);$$

$$\|\nabla\,u\|_{L^2(\mathbb{G})} \le C\,\big(\|(1 + \sqrt{|x_1|})\,f\|_{L^2(\mathbb{G})} + \|(1 + \sqrt{|x_1|})\,g\|_{L^2(\Gamma)}\big).$$

Proof. Let us consider the following sequence of auxiliary boundary value problems in a growing family of finite cylinders:

$$\begin{cases} A_\#\,u_k = f(x), & x \in G_{-k}^k, \\ B_\#\,u_k = g(x), & x \in \Gamma_{-k}^k, \\ u_k(-k, x') = u_k(k, x') = 0, & x' \in Q. \end{cases} \tag{26.27}$$

Without loss of generality, we assume that $f(x) > 0$ and $g(x) > 0$. Moreover, we assume that the functions f and g are equal to zero in the half-cylinder $G_{-\infty}^0$; that is, $\operatorname{supp} f, \operatorname{supp} g \subset G_0^{+\infty}$. The case when the supports of f and g belong to $G_{-\infty}^0$ can be considered similarly. Due to the regularity assumptions $(\mathbf{H1}), (\mathbf{H5})'$, the maximum principle and the boundary point lemma are valid (see, e.g., [GiTr98]), and, consequently, a negative minimum cannot be attended in the internal part of G_{-k}^k and its lateral boundary; that is, $u_k \ge 0$ in G_{-k}^k.

In the cylinder G_{-k}^{-1} the function $u_k(x)$ is a solution of a homogeneous equation. Since $u_k(-k, x') = 0$ and $\bar{b}_1^- > 0$, we have the following estimate:

$$u_k(x) \le \|u_k\|_{L^\infty(S_{-1})}\,e^{\gamma\,x_1}, \quad x \in G_{-k}^{-1},\ \gamma > 0.$$

The proof of this fact can be found in [PaPi09], Section 5, Theorem 5.5.

For the nonnegative function $u_k(x)$, the Harnack inequality is valid in the fixed domain G_{-1}^0 with a constant α which depends only on $d, |Q|$, and Λ; that is,

$$u_k(x) \le \alpha\,\min_{G_{-1}^0} u_k(x)\,e^{\gamma\,x_1}.$$

Obviously, there exists $\xi > 1$, independent of k, such that

$$u_k(-\xi, x') < \frac{1}{2} \min_{G^0_{-1}} u_k(x). \tag{26.28}$$

In $G^k_{-\xi}$, due to the linearity of the problem, we can represent u_k as a sum $v_k + w_k$, where v_k is a solution of the homogeneous equation with nonzero Dirichlet boundary condition $v_k(-\xi, x') = u_k(-\xi, x')$; and w_k is a solution of the nonhomogeneous equation with functions f and g on the right-hand side and homogeneous Dirichlet boundary conditions on the bases. In view of the maximum principle and (26.28) we obtain an estimate for $v_k(x)$,

$$v_k(x) \le \frac{1}{2} \min_{G^0_{-1}} u_k(x), \quad x \in G^k_{-\xi}. \tag{26.29}$$

One can prove (see [PaPi09], Lemma 7.2, estimates (7.10), (7.11)) that the following estimate for w_k holds:

$$\|w_k\|_{L^2(G_N^{N+1})} \le C\,\|(1+\sqrt{x_1})\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma_0^{+\infty})}. \tag{26.30}$$

In this way, taking into account (26.29) and (26.30), one can see that

$$\min_{G^0_{-1}} u_k(x) \le \|u_k\|_{L^2(G^0_{-1})} \le \frac{1}{2} \min_{G^0_{-1}} u_k(x) + \|w_k\|_{L^2(G^0_{-1})}.$$

It follows from the last inequality that

$$\min_{G^0_{-1}} u_k(x) \le C\,\|(1+\sqrt{x_1})\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma_0^{+\infty})}. \tag{26.31}$$

In view of the Harnack inequality and (26.31), $u_k(-1, x') \le C$. Then, by the maximum principle,

$$u_k(x) \le C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}, \quad x \in G^{-1}_{-k}.$$

Combining the last estimate with (26.29)–(26.31) and recalling the relation $u_k = v_k + w_k$, we see that

$$\|u_k\|_{L^2(G_N^{N+1})} \le C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}, \quad N \in \mathbb{Z},$$

where the constant C does not depend on k. Standard elliptic estimates imply

$$\|\nabla u_k\|_{L^2(G_N^{N+1})} \le C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}.$$

Thus, up to a subsequence, $u_k(x)$ converges in $H^1_{loc}(\mathbb{G})$ to some function $u(x)$, as $k \to \infty$. Passing to the limit in the integral identity, one can see that $u(x)$ solves problem (26.26). The existence of a bounded solution to problem (26.1) is proved. The result on the exponential stabilization to a constant at $+\infty$ and $-\infty$ of a solution to problem (26.26) follows from the similar results for equations stated in a semi-infinite cylinder (see [PaPi09], Theorem 7.6).

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