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Homogenization of a class of quasilinear elliptic equations in high-contrast fissured media

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The aim of the paper is to study the asymptotic behaviour of the solution of a quasilinear elliptic equation of the form

$$-\operatorname{div}(a^\varepsilon(x)|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + g(x)|u^\varepsilon|^{p-2}u^\varepsilon = S^\varepsilon(x) \quad \text{in } \Omega,$$

with a high-contrast discontinuous coefficient $a^\varepsilon(x)$, where ε is the parameter characterizing the scale of the microstucture. The coefficient $a^\varepsilon(x)$ is assumed to degenerate everywhere in the domain Ω except in a thin connected microstructure of asymptotically small measure. It is shown that the asymptotical behaviour of the solution u^ε as $\varepsilon \to 0$ is described by a homogenized quasilinear equation with the coefficients calculated by local energetic characteristics of the domain Ω.

1. Introduction

In this paper, we study the homogenization of the following quasilinear elliptic problem:

$$-\operatorname{div}(a^\varepsilon(x)|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + g(x)|u^\varepsilon|^{p-2}u^\varepsilon = S^\varepsilon(x) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and ε is a parameter tending to zero. We assume that $a^\varepsilon(x)$ does not degenerate along a thin connected microstructure $\Omega_{\mathrm{f}}^\varepsilon$ (called the fracture part), while in the complement to the fracture part $\Omega_{\mathrm{m}}^\varepsilon$ (called the matrix part) $a^\varepsilon(x)$ is a positive function vanishing asymptotically as $\varepsilon \to 0$. The rate of degeneration will be specified later.

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The basic set of equations considered here arises, for example, from compressible flows in porous media, and non-Newtonian flow, etc., through thin fissures. This problem is closely related to the so-called double-porosity homogenization models widely discussed in the mathematical literature (see, for example, [17]). The linear double-porosity model was first studied in [4], and was then revisited in the mathematical literature by many other authors (see, for example, [17] for a review). Nonlinear models were treated in [10, 24]. Then a general non-periodic model and a random model were considered in [6] and [7], respectively. Note also that the homogenization of nonlinear elliptic equations is a long-standing problem and a number of methods have been developed. There is an extensive literature on this subject. We will not attempt a review of the literature here, but merely mention a few references (see, for example, [9, 11, 12, 14, 15, 21, 22] and the references therein).

In contrast with the works mentioned above, where the measure of the fracture part remains uniformly positive, in our model we will assume that the measure of the fracture part is asymptotically small. This problem, in the linear case, was considered in [1, 2, 5, 25]; the singular double-porosity model was studied in [8].

In this paper we deal with a quasilinear elliptic problem in a domain with an asymptotically small fissure part. Following the approach introduced in [18], instead of a classical periodicity assumption, we impose certain conditions on the so-called *local energetic characteristics* associated with the boundary-value problem (1.1). These characteristics include a penalization term. Following the scheme developed in [1,25], we obtain the homogenization result by combining the local characteristics method with an appropriate extension condition from the fracture part to the whole domain Ω.

Since the measure of the fracture part is small, we cannot use the usual notions of convergence and compactness. Instead we introduce the convergence and compactness adapted to the singularity of the fracture measure.

The homogenized equation takes the form

$$-\partial_{x_i} a_i(x, \nabla u) + B(x)|u|^{p-2}u = \rho(x)S(x) \quad \text{in } \Omega,$$

where the functions a_i, $i = 1, 2, \dots, n$, and $B(x)$ are defined in terms of the local characteristics mentioned above.

The paper is organized as follows. In § 2 all necessary mathematical notation is defined, the microscopic problem is formulated and the general assumptions are stated. In § 3 we introduce the notions of convergence and compactness in domains of asymptotically degenerating measure. The main result of the paper is formulated in § 4. It will then be proved in §§ 5 and 6. Finally, in § 7 we present two- and three-dimensional periodic examples. In these examples the coefficients of the homogenized problem are calculated explicitly.

2. Statement of the problem and assumptions

In this section, we describe a microscopic model for a quasilinear elliptic equation in high-contrast fissured media. Let $\Omega = \Omega_{\mathrm{f}}^{\varepsilon} \cup \bar{\Omega}_{\mathrm{m}}^{\varepsilon}$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with piecewise smooth boundary $\partial\Omega$, and let

$$\operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{2.1}$$

Moreover, throughout the paper we assume that the set $\Omega_{\mathrm{f}}^{\varepsilon}$ is asymptotically distributed in a regular way in the domain Ω, i.e. there exists a constant $C > 0$ such that, for any ball $V(y,r) = \{x : |x-y| < r\}$ of radius r centred at $y \in \Omega$ and $\varepsilon > 0$ small enough ($\varepsilon \leqslant \varepsilon_0(r)$),

$$C^{-1} r^n \geqslant \mu^{\varepsilon} \operatorname{meas}(\Omega_{\mathrm{f}}^{\varepsilon} \cap V(x,r)) \geqslant C r^n, \tag{2.2}$$

where

$$\mu^{\varepsilon} = \frac{\operatorname{meas} \Omega}{\operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}}. \tag{2.3}$$

We consider the variational problem

$$\mu^{\varepsilon} \int_{\Omega} \{a^{\varepsilon}(x) |\nabla u^{\varepsilon}|^p + g(x)|u^{\varepsilon}|^p - pS^{\varepsilon}(x) u^{\varepsilon}\} \, \mathrm{d}x \to \inf, \quad u^{\varepsilon} \in W^{1,p}(\Omega), \tag{2.4}$$

where $p > 1$, g is a smooth positive function in $\bar{\Omega}$ such that $g(x) \geqslant C > 0$,

$$S^{\varepsilon}(x) = \mathbf{1}_{\mathrm{f}}^{\varepsilon}(x) S(x) \tag{2.5}$$

with $S \in L^{p'}(\Omega)$, where p' satisfies $(1/p) + (1/p') = 1$ and $\mathbf{1}_{\mathrm{f}}^{\varepsilon} = \mathbf{1}_{\mathrm{f}}^{\varepsilon}(x)$ is the characteristic function of the set $\Omega_{\mathrm{f}}^{\varepsilon}$, $a^{\varepsilon} = a^{\varepsilon}(x)$ is a measurable function such that

$$0 < a_0 \leqslant a^{\varepsilon}(x) \leqslant a_0^{-1} \qquad \text{in } \Omega_{\mathrm{f}}^{\varepsilon}, \tag{2.6}$$

$$0 < a_1(\varepsilon) \leqslant a^{\varepsilon}(x) \leqslant a_2(\varepsilon) \quad \text{in } \Omega_{\mathrm{m}}^{\varepsilon}, \tag{2.7}$$

with

$$\mu^{\varepsilon} a_2(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{2.8}$$

It is known (see, for example, [19]) that, for any $\varepsilon > 0$, there exists a unique solution $u^{\varepsilon} \in W^{1,p}(\Omega)$ of the variational problem (2.4), and that u^{ε} solves the Neumann boundary-value problem for the corresponding Euler equation:

$$-\operatorname{div}(a^{\varepsilon}(x)|\nabla u^{\varepsilon}|^{p-2} \nabla u^{\varepsilon}) + g(x)|u^{\varepsilon}|^{p-2} u^{\varepsilon} = S^{\varepsilon}(x) \quad \text{in } \Omega.$$

3. Convergence in domains of degenerating measure

Due to the vanishing measure of the fissure part, we should define the convergence of sequences according to the singularity of the fissure measure. Assume that the family of domains $\Omega_{\mathrm{f}}^{\varepsilon} \subset \Omega$, $\varepsilon > 0$, satisfies (2.1) and (2.2). In this section, following [25] (see also [20] or [27] for similar considerations), we introduce the concept of convergence in domains $\Omega_{\mathrm{f}}^{\varepsilon}$ as $\varepsilon \to 0$.

We adopt the following notation: $\|\cdot\|_{\Omega}$ and $\|\cdot\|_{1,\Omega}$ are the norms in the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, $1 < p < +\infty$, respectively; $\operatorname{Lip}(M,\Omega)$ is the class of continuous functions u in Ω such that $|u(x)| \leqslant M$ and $|u(x) - u(y)| \leqslant M|x-y|$ for any $x, y \in \Omega$.

Definition 3.1. A sequence of functions $\{u^{\varepsilon} \in L^p(\Omega_{\mathrm{f}}^{\varepsilon})\}$ is said to $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-converge to a function $u \in L^p(\Omega)$ if there exists an approximating sequence

$$\{u_M \in \operatorname{Lip}(M,\Omega),\ M = 1,2,\dots\}$$

converging strongly in $L^p(\Omega)$ to u as $M \to \infty$, and

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}} \|u^{\varepsilon} - u_M\|^p_{\Omega_{\mathrm{f}}^{\varepsilon}} = 0. \tag{3.1}$$

Note that under condition (2.2) the limiting function u in definition 3.1 is independent of the approximating sequence $\{u_M,\ M = 1, 2, \dots\}$.

REMARK 3.2. If in definition 3.1 u is a smooth function in Ω, then (3.1) may be rewritten as follows:

$$\limsup_{\varepsilon\to 0} \frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}} \|u^{\varepsilon} - u\|^p_{\Omega_{\mathrm{f}}^{\varepsilon}} = 0. \tag{3.2}$$

In a natural way one defines the notion of compactness with respect to the $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-convergence.

DEFINITION 3.3. A sequence $\{u^{\varepsilon} \in L^p(\Omega_{\mathrm{f}}^{\varepsilon})\}$ is a $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-compact set if one can extract from any its subsequence a $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-convergent subsequence.

In what follows, we mainly deal with sequences of functions $u^{\varepsilon} \in W^{1,p}(\Omega_{\mathrm{f}}^{\varepsilon})$ such that

$$\|u^{\varepsilon}\|^p_{1,\Omega_{\mathrm{f}}^{\varepsilon}} \leqslant C \operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}. \tag{3.3}$$

From now on, C is a generic constant independent of ε. Furthermore, in this paper we restrict ourselves to domains $\Omega_{\mathrm{f}}^{\varepsilon}$ satisfying the so-called 'strong connectedness' condition (the SC condition).

DEFINITION 3.4. A family of domains $\Omega_{\mathrm{f}}^{\varepsilon}$ is said to satisfy the SC condition if for any sequence $\{u^{\varepsilon} \in C^1(\Omega_{\mathrm{f}}^{\varepsilon})\}$ satisfying (3.3), and any M, $M = 1, 2, \dots$, there exists a family of subsets $\mathcal{G}_M^{\varepsilon} \subset \Omega_{\mathrm{f}}^{\varepsilon}$ such that $u^{\varepsilon} \in \operatorname{Lip}(M, \Omega_{\mathrm{f}}^{\varepsilon} \setminus \mathcal{G}_M^{\varepsilon})$ and

$$\operatorname{meas} \mathcal{G}_M^{\varepsilon} = \frac{\phi(M)}{M^p} \operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}, \quad \|u^{\varepsilon}\|^p_{\mathcal{G}_M^{\varepsilon}} = \phi(M) \operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}$$

for all ε, $\varepsilon \leqslant \varepsilon_0(M)$, where $\lim_{M\to\infty} \phi(M) = 0$.

We now formulate a sufficient condition for $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-compactness in the class of domains $\Omega_{\mathrm{f}}^{\varepsilon}$ satisfying (2.2) and the SC condition. The following theorem holds (see [3, 27]).

THEOREM 3.5. *Let $\Omega_{\mathrm{f}}^{\varepsilon} \subset \Omega$ be a family of domains satisfying the SC condition. Then, any sequence $\{u^{\varepsilon} \in W^{1,p}(\Omega_{\mathrm{f}}^{\varepsilon})\}$ satisfying (3.3) is a $D^p_{\Omega_{\mathrm{f}}^{\varepsilon}}$-compact set.*

REMARK 3.6. In the proof of theorem 3.5 we construct the sequence $\{u_M^{\varepsilon}\}$ such that $u_M^{\varepsilon}(x) = u^{\varepsilon}(x)$ for $x \in \Omega_M^{\varepsilon}$, where $\Omega_M^{\varepsilon} = \Omega_{\mathrm{f}}^{\varepsilon} \setminus \bar{\mathcal{G}}_M^{\varepsilon}$. The functions u_M^{ε} satisfy the Lipschitz condition and it follows from Witney's theorem [26] that these functions can be extended from Ω_M^{ε} to the whole Ω. This means that the SC condition could be formulated as an extension condition for the functions defined in the domain $\Omega_{\mathrm{f}}^{\varepsilon}$ to the domain Ω with some distortion on the set $\mathcal{G}_M^{\varepsilon}$ whose measure is small with respect to the measure of the set $\Omega_{\mathrm{f}}^{\varepsilon}$.

4. Formulation of the main result

In this section we introduce the local energy characteristics of the sets $\Omega_{\mathrm{f}}^{\varepsilon}$ and $\Omega_{\mathrm{m}}^{\varepsilon}$ associated with the variational problem (2.4), and formulate the main result of the paper. We study the asymptotic behaviour of u^{ε} solutions of the variational problem (2.4) as $\varepsilon \to 0$. The classical periodicity assumption is here substituted by an abstract one covering a variety of concrete behaviours such as the periodicity, the almost periodicity, and many more besides. For this, we assume that $\Omega_{\mathrm{f}}^{\varepsilon} \subset \Omega$ is a disperse medium, i.e. the following assumptions hold.

ASSUMPTION 4.1. There exists a continuous function $\rho(x) > 0$ in $\bar{\Omega}$ such that

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \mu^{\varepsilon} h^{-n} \operatorname{meas}[\Omega_{\mathrm{f}}^{\varepsilon} \cap K_h^x] = \rho(x),$$

for any open cube K_h^x centred at $x \in \Omega$ with lengths equal to $h > 0$.

ASSUMPTION 4.2. The family of domains $\Omega_{\mathrm{f}}^{\varepsilon}$, $\varepsilon > 0$, satisfies the SC condition (see definition 3.4).

Instead of the classical periodicity assumption on the microstructure of the disperse media, we impose certain conditions on local energy characteristics of the subdomains $\Omega_{\mathrm{f}}^{\varepsilon}$ and $\Omega_{\mathrm{m}}^{\varepsilon}$.

For $z \in \Omega$ we define:

(i) the functional associated with the energy in $\Omega_{\mathrm{f}}^{\varepsilon}$, by

$$E^{\varepsilon,h}(z;\boldsymbol{q}) = \inf_{v^{\varepsilon}} \mu^{\varepsilon} \int_{K_h^z \cap \Omega_{\mathrm{f}}^{\varepsilon}} \{a^{\varepsilon}(x)|\nabla v^{\varepsilon}|^p + h^{-p-\gamma}|v^{\varepsilon} - (x-z,\boldsymbol{q})|^p\}\,\mathrm{d}x, \tag{4.1}$$

where $p > \gamma > 0$, $\boldsymbol{q} = \{q_1, q_2, \dots, q_n\} \in \mathbb{R}^n$, and where the infimum is taken over $v^{\varepsilon} \in W^{1,p}(K_h^z \cap \Omega_{\mathrm{f}}^{\varepsilon})$;

(ii) the functional associated with the exchange between the matrix and the fissure system, by

$$b^{\varepsilon,h}(z) = \inf_{w^{\varepsilon}} \mu^{\varepsilon} \int_{K_h^z} \{a^{\varepsilon}(x)|\nabla w^{\varepsilon}|^p + g(x)\mathbf{1}_m^{\varepsilon}(x)|w^{\varepsilon}|^p + h^{-p-\gamma}\mathbf{1}_{\mathrm{f}}^{\varepsilon}(x)|w^{\varepsilon}-1|^p\}\,\mathrm{d}x, \tag{4.2}$$

where $\mathbf{1}_m^{\varepsilon} = \mathbf{1}_m^{\varepsilon}(x)$, $\mathbf{1}_{\mathrm{f}}^{\varepsilon} = \mathbf{1}_{\mathrm{f}}^{\varepsilon}(x)$ are the characteristic functions of the sets $\Omega_{\mathrm{m}}^{\varepsilon}$ and $\Omega_{\mathrm{f}}^{\varepsilon}$, respectively, and where the infimum is taken over $w^{\varepsilon} \in W^{1,p}(K_h^z)$.

We make the following further assumptions.

ASSUMPTION 4.3. For any $x \in \Omega$ there exist the limits

$$\lim_{h\to 0}\liminf_{\varepsilon\to 0} h^{-n}E^{\varepsilon,h}(x;\boldsymbol{q}) = \lim_{h\to 0}\limsup_{\varepsilon\to 0} h^{-n}E^{\varepsilon,h}(x;\boldsymbol{q}) = A(x,\boldsymbol{q}),$$

with the function $A(x,\boldsymbol{q})$ such that $A(x,\cdot) \in C^{2+\beta}(\mathbb{R}^n)$, $\beta > 0$ and $A(\cdot,\boldsymbol{q}) \in C(\Omega)$. Moreover,

$$C_1|\boldsymbol{q}|^{p-2}|\xi|^2 \geqslant A_{p_ip_j}\xi_i\xi_j \geqslant C_2|\boldsymbol{q}|^{p-2}|\xi|^2, \quad C_1, C_2 > 0, \tag{4.3}$$

$$A(x,\boldsymbol{q}) \geqslant C_3(|\boldsymbol{q}|^p - 1), \qquad C_3 > 0. \tag{4.4}$$

ASSUMPTION 4.4. For any $x \in \Omega$ there exist the limits

$$\lim_{h\to 0}\liminf_{\varepsilon\to 0} h^{-n} b^{\varepsilon,h}(x) = \lim_{h\to 0}\limsup_{\varepsilon\to 0} h^{-n} b^{\varepsilon,h}(x) = b(x),$$

where $b \in C(\Omega)$.

We are now in a position to formulate the main result of this paper.

THEOREM 4.5. *Let assumptions 4.1–4.4 hold. The solution u^ε of problem (2.4) then $D^p_{\Omega^\varepsilon_{\mathrm{f}}}$-converges to u the solution of the problem*

$$J_{\mathrm{hom}}[u] \equiv \int_\Omega \{A(x,\nabla u) + B(x)|u|^p - p\rho(x)S(x)u\}\,\mathrm{d}x \to \inf, \quad u \in W^{1,p}(\Omega), \tag{4.5}$$

where $B(x) = g(x)\rho(x) + b(x)$.

5. Preliminary results

In this section we construct a convenient approximation for the solution of the variational problem (2.4) in the subdomains $\Omega^\varepsilon_{\mathrm{m}}, \Omega^\varepsilon_{\mathrm{f}} \subset \Omega$. To this end, we first introduce the following notation.

Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h' := h - h^{1+\gamma/p}$, $\varepsilon \ll h \ll 1$. Let us cover the domain Ω by the cubes K^α_h of length $h > 0$ centred at the points x^α. We associate with this covering a partition of unity $\{\varphi_\alpha\} : 0 \leqslant \varphi_\alpha(x) \leqslant 1$; $\varphi_\alpha(x) = 0$ for $x \notin K^\alpha_h$, $\varphi_\alpha(x) = 1$ for $x \in K^\alpha_h \setminus \bigcup_{\beta\neq\alpha} K^\beta_h$, $\sum_\alpha \varphi_\alpha(x) = 1$ for $x \in \Omega$, $|\nabla\varphi_\alpha(x)| \leqslant Ch^{-1-\gamma/p}$.

Denote by $K^\alpha_{h'}$ and Π^α_h the cube of side length h' centred at the point x^α, and the set $K^\alpha_h \setminus K^\alpha_{h'}$, respectively.

LEMMA 5.1. *Imagine that assumptions 4.1 and 4.4 hold. Then, for each $h > 0$, there exist sets $\mathcal{B}^\varepsilon_h \subset \Omega^\varepsilon_{\mathrm{f}}$ and functions Y^ε_h such that:*

(i) $0 \leqslant Y^\varepsilon_h(x) \leqslant 1$ *in* Ω;

(ii) $Y^\varepsilon_h(x) = 1$ *in* $\Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{B}^\varepsilon_h$;

(iii) $\limsup_{\varepsilon\to 0} \mu^\varepsilon \operatorname{meas} \mathcal{B}^\varepsilon_h = O(h^\gamma)$ *as* $h \to 0$;

(iv) *for any function $w \in C^1(\Omega)$, we have*

$$\limsup_{\varepsilon\to 0} \mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla Y^\varepsilon_h|^p + g(x)|Y^\varepsilon_h|^p\}|w|^p\,\mathrm{d}x \leqslant \int_\Omega B(x)|w|^p\,\mathrm{d}x + o(1), \tag{5.1}$$

as $h \to 0$.

Proof of lemma 5.1. Let $w^{\varepsilon,\alpha}_h$ be a minimizer of the functional in (4.2) with $z = x^\alpha$. It follows from assumption 4.4 that, as $h \to 0$,

$$\limsup_{\varepsilon\to 0} \mu^\varepsilon \int_{\Pi^\alpha_h} \{a^\varepsilon(x)|\nabla w^{\varepsilon,\alpha}_h|^p + g(x)\mathbf{1}^\varepsilon_m(x)|w^{\varepsilon,\alpha}_h|^p\}\,\mathrm{d}x = o(h^n), \tag{5.2}$$

$$\limsup_{\varepsilon\to 0} \mu^\varepsilon \int_{\Pi^\alpha_h} \mathbf{1}^\varepsilon_{\mathrm{f}}(x)|w^{\varepsilon,\alpha}_h - 1|^p\,\mathrm{d}x = o(h^{n+p+\gamma}). \tag{5.3}$$

In addition, using assumptions 4.1 and 4.4, we obtain

$$\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{K_h^\alpha}\{a^\varepsilon(x)|\nabla w_h^{\varepsilon,\alpha}|^p+g(x)|w_h^{\varepsilon,\alpha}|^p\}\,\mathrm{d}x\leqslant h^nB(x^\alpha)+o(h^n)\quad\text{as } h\to 0. \tag{5.4}$$

Besides, since $w_h^{\varepsilon,\alpha}$ minimizes the functional in (4.2) we obtain $0\leqslant w_h^{\varepsilon,\alpha}(x)\leqslant 1$ and

$$\limsup_{\varepsilon\to 0}\mu^\varepsilon\operatorname{meas}B_h^{\varepsilon,\alpha}\leqslant Ch^{n+\gamma}, \tag{5.5}$$

where $B_h^{\varepsilon,\alpha}=\{x\in K_h^\alpha\cap\Omega_{\mathrm{f}}^\varepsilon : w_h^{\varepsilon,\alpha}\leqslant 1-h\}$. The inequality (5.5) means that the measure of the set $B_h^{\varepsilon,\alpha}$, where the function $w_h^{\varepsilon,\alpha}$ minimizing the functional in (4.2) is not close to 1, is small with respect to the measure of the set $\Omega_{\mathrm{f}}^\varepsilon\cap K_h^\alpha$.

Let us introduce the function

$$W_h^{\varepsilon,\alpha}=\begin{cases}1, & \text{if}\quad w_h^{\varepsilon,\alpha}\geqslant 1-h,\\ (1-h)^{-1}w_h^{\varepsilon,\alpha}, & \text{otherwise.}\end{cases} \tag{5.6}$$

It is clear that $|W_h^{\varepsilon,\alpha}-1|\leqslant|w_h^{\varepsilon,\alpha}-1|$. One can easily show that the function $W_h^{\varepsilon,\alpha}$ satisfy the estimates (5.2)–(5.4). We set

$$\mathcal{B}_h^\varepsilon=\bigcup_\alpha B_h^{\varepsilon,\alpha},\qquad Y_h^\varepsilon(x)=\sum_\alpha W_h^{\varepsilon,\alpha}(x)\varphi_\alpha(x).$$

Then, using the properties of the functions $W_h^{\varepsilon,\alpha}$ and $\{\varphi_\alpha\}$ and taking into account the estimate (5.5), it is easy to show that the functions $Y_h^\varepsilon(x)$ and the sets $\mathcal{B}_h^\varepsilon$ satisfy lemma 5.1(i)–(iv). This completes the proof of lemma 5.1. □

Lemma 5.2. *Let assumptions 4.2 and 4.3 hold and let w be a smooth function in Ω. Then, for any $h>0$ and $M\in\{1,2,\dots\}$, there exist functions $W_{Mh}^\varepsilon\in W^{1,p}(\Omega)$, such that:*

(i) $|W_{Mh}^\varepsilon(x)-w(x)|\leqslant CMh$ *in* Ω;

(ii) $|W_{Mh}^\varepsilon(x)-W_{Mh}^\varepsilon(y)|\leqslant CM|x-y|$ *for any* $x,y\in\Omega$;

(iii) *the inequality*

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon}a^\varepsilon(x)|\nabla W_{Mh}^\varepsilon|^p\,\mathrm{d}x\leqslant\int_\Omega A(x,\nabla w)\,\mathrm{d}x$$

holds.

Proof of lemma 5.2. Let w be a smooth function in Ω and let $v_h^{\varepsilon,\alpha}$ be a minimizer of the functional in (4.1) for $z=x^\alpha$ and $\boldsymbol{q}=\nabla w(x^\alpha)$. Since $v_h^{\varepsilon,\alpha}$ minimizes the functional in (4.1), we have

$$\sup_{x\in K_h^\alpha\cap\Omega_{\mathrm{f}}^\varepsilon}|v_h^{\varepsilon,\alpha}(x)|\leqslant\tfrac{1}{2}h. \tag{5.7}$$

Besides, by virtue of assumption 4.3, for ε small enough ($\varepsilon \leqslant \hat{\varepsilon}(h)$), we have

$$\mu^\varepsilon \int_{\Pi_h^\alpha \cap \Omega_{\mathrm{f}}^\varepsilon} |\nabla v_h^{\varepsilon,\alpha}|^p \,\mathrm{d}x = o(h^n), \tag{5.8}$$

$$\mu^\varepsilon \int_{\Pi_h^\alpha \cap \Omega_{\mathrm{f}}^\varepsilon} |v_h^{\varepsilon,\alpha} - (x - x^\alpha, \nabla w(x^\alpha))|^p \,\mathrm{d}x = o(h^{n+p+\gamma}), \tag{5.9}$$

$$\mu^\varepsilon \int_{K_h^\alpha \cap \Omega_{\mathrm{f}}^\varepsilon} a^\varepsilon(x) |\nabla v_h^{\varepsilon,\alpha}|^p \,\mathrm{d}x \leqslant h^n A(x^\alpha, \nabla w(x^\alpha)) + o(h^n) \tag{5.10}$$

as $h \to 0$. The estimates (5.8)–(5.10) are uniform with respect to x^α on any compact subset of Ω.

Let us consider the function

$$w_h^\varepsilon(x) = \sum_\alpha \{w(x) + v_h^{\varepsilon,\alpha}(x) - (x - x^\alpha, \nabla w(x^\alpha))\}\varphi_\alpha(x). \tag{5.11}$$

Then, using (5.8)–(5.10) and the properties of the functions φ_α, we obtain

$$\mu^\varepsilon \int_{\Omega_{\mathrm{f}}^\varepsilon} a^\varepsilon(x) |\nabla w_h^\varepsilon|^p \,\mathrm{d}x \leqslant \int_\Omega A(x, \nabla w) \,\mathrm{d}x + o(1) \tag{5.12}$$

as $\varepsilon \leqslant \hat{\varepsilon}(h)$ and $h \to 0$. Besides, according to (5.7) and (5.11), we have

$$\sup_{x \in \Omega_{\mathrm{f}}^\varepsilon} |w_h^\varepsilon(x) - w(x)| \leqslant Ch. \tag{5.13}$$

Since the domains $\Omega_{\mathrm{f}}^\varepsilon$ satisfy the SC condition (see definition 3.4), for any $M = 1, 2, \ldots$, there exist sets Q_{Mh}^ε and functions $W_{Mh}^\varepsilon \in \mathrm{Lip}(M, \Omega)$ such that $W_{Mh}^\varepsilon = w_h^\varepsilon$ in $\Omega_{\mathrm{f}}^\varepsilon \setminus Q_{Mh}^\varepsilon$, and

$$\limsup_{M \to \infty} M^p \limsup_{h \to 0} \limsup_{\varepsilon \to 0} \mu^\varepsilon \operatorname{meas} Q_{Mh}^\varepsilon = 0. \tag{5.14}$$

Now the statements of the lemma follow from (5.12), (5.13) and the properties of the functions W_{Mh}^ε. Lemma 5.2 is proved. □

LEMMA 5.3. *Let assumptions 4.1 and 4.4 hold. For any function $w \in W^{1,p}(\Omega)$, there then exists a sequence $\{w^\varepsilon \in C^1(\Omega)\}$ that $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converges to w and such that*

$$\limsup_{\varepsilon \to 0} I^\varepsilon[w^\varepsilon] \equiv \limsup_{\varepsilon \to 0} \mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla w^\varepsilon|^p + g(x)|w^\varepsilon|^p\} \,\mathrm{d}x \leqslant C_0 \|w\|_{1,\Omega}^p, \tag{5.15}$$

where $C_0 > 0$ is independent of w.

Proof of lemma 5.3. The proof of the lemma will be given in two steps.

STEP 1. Let us first assume that $w \in C^\infty(\Omega)$. We then construct a sequence $\{w^\varepsilon\}$ as follows. Consider the function $w_h^\varepsilon(x) = w(x) Y_h^\varepsilon(x)$, where $Y_h^\varepsilon(x)$ is defined in lemma 5.1. According to lemma 5.1, there exist sets $\mathcal{B}_h^\varepsilon \subset \Omega_{\mathrm{f}}^\varepsilon$ with $\operatorname{meas} \mathcal{B}_h^\varepsilon \leqslant Ch^\gamma \operatorname{meas} \Omega_{\mathrm{f}}^\varepsilon$ (here ε is sufficiently small; $\varepsilon \leqslant \tilde{\varepsilon}(h)$), such that $w_h^\varepsilon = w$ in $\Omega_{\mathrm{f}}^\varepsilon \setminus \mathcal{B}_h^\varepsilon$ and

$$\int_{\mathcal{B}_h^\varepsilon} |w_h^\varepsilon - w|^p \,\mathrm{d}x \leqslant \int_{\mathcal{B}_h^\varepsilon} |w|^p \,\mathrm{d}x. \tag{5.16}$$

Consider now the integral $I^\varepsilon[w_h^\varepsilon]$ (see (5.15)). Using (2.6) it is easy to show that

$$I^\varepsilon[w_h^\varepsilon] \leqslant a_0^{-1}\mu^\varepsilon \int_{\Omega_{\mathrm{f}}^\varepsilon} |\nabla w|^p \,\mathrm{d}x + 2^{p-1}\mu^\varepsilon \int_{\Omega} \{a^\varepsilon(x)|\nabla Y_h^\varepsilon|^p + g(x)|Y_h^\varepsilon|^p\}|w|^p \,\mathrm{d}x \\ + 2^{p-1}\mu^\varepsilon \int_{\mathcal{B}_h^\varepsilon \cup \Omega_{\mathrm{m}}^\varepsilon} a^\varepsilon(x)|Y_h^\varepsilon|^p|\nabla w|^p \,\mathrm{d}x. \quad (5.17)$$

Moreover, it follows from condition (2.7) and lemma 5.1 that

$$\limsup_{\varepsilon\to 0} \mu^\varepsilon \int_{\mathcal{B}_h^\varepsilon \cup \Omega_{\mathrm{m}}^\varepsilon} a^\varepsilon(x)|Y_h^\varepsilon|^p|\nabla w|^p \,\mathrm{d}x = O(h^\gamma)$$

as $h \to 0$. On the other hand, assumption 4.1 and equation (2.2) imply the convergence

$$\mu^\varepsilon \int_{\Omega_{\mathrm{f}}^\varepsilon} |\nabla w|^p \,\mathrm{d}x \to \int_{\Omega} \rho(x)|\nabla w|^p \,\mathrm{d}x, \quad \text{as } \varepsilon \to 0. \quad (5.18)$$

Define a sequence $\{\hat\varepsilon_j\}$, $\hat\varepsilon_j \downarrow 0$, such that for each $h_j = 1/j$, it holds that $\hat\varepsilon_j \leqslant \hat\varepsilon(h_j)$ and set

$$w^\varepsilon = w_h^\varepsilon|_{h=1/j} \quad \text{when } \hat\varepsilon_j \geqslant \varepsilon > \hat\varepsilon_{j+1}.$$

It follows from (5.17), (5.18) and lemma 5.1(iv) that $\{w^\varepsilon\}$ satisfies (5.15) with $C_0 = \max\{a_0^{-1} \max_{x\in\Omega} \rho(x), 2^{m-1} \max_{x\in\Omega} B(x)\}$. From the definition of the function w^ε and (5.16) we also get

$$\lim_{\varepsilon\to 0} \frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^\varepsilon} \int_{\Omega_{\mathrm{f}}^\varepsilon} |w^\varepsilon - w|^p \,\mathrm{d}x = 0. \quad (5.19)$$

This means that the sequence $\{w^\varepsilon\}$ $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converges to the function w (see remark 3.2).

Clearly, the functions w^ε can be approximated by smooth functions in such a way that, for the approximation sequence, (5.15) and (5.19) hold true.

STEP 2. Now consider an arbitrary function $w \in W^{1,p}(\Omega)$, and approximate w by the smooth functions $w_M \in \operatorname{Lip}(M, \Omega)$, $M = 1, 2, \dots$, such that

$$\|w - w_M\|_{1,\Omega} \leqslant \frac{1}{k(M)} \quad \text{with } k(M) \to +\infty \text{ as } M \to +\infty. \quad (5.20)$$

According to step 1, there is a sequence $\{w_M^\varepsilon\}$ such that

$$\frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^\varepsilon} \int_{\Omega_{\mathrm{f}}^\varepsilon} |w_M^\varepsilon - w_M|^p \,\mathrm{d}x \leqslant \frac{1}{M}, \quad (5.21)$$

and

$$I^\varepsilon[w_M^\varepsilon] \leqslant C_1\|w\|_{1,\Omega}^p,$$

where ε is sufficiently small, $\varepsilon \leqslant \delta(M)$, C_1 does not depend on ε and M. Here $\delta(M) \to 0$ as $M \to \infty$. Moreover, in view of (2.2), $\delta(M)$ can be chosen in such a way that, as $\varepsilon \leqslant \delta(M)$,

$$\frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^\varepsilon} \int_{\Omega_{\mathrm{f}}^\varepsilon} |w_M - w_L|^p \,\mathrm{d}x \leqslant C_2\|w_M - w_L\|_{\Omega}^p, \quad \text{for all } L = 1, \dots, M-1, \quad (5.22)$$

with a constant C_2 independent of ε and M. We choose a sequence $\{\hat\delta_j\}_{j=1,2,\dots}$, $\delta_j \downarrow 0$, such that $\delta_j \leqslant \delta(j)$ and set

$$w^\varepsilon = w^\varepsilon_M \quad \text{for } \varepsilon \in]\hat\delta_{M+1}, \hat\delta_M].$$

It is easy to see that the sequence $\{w^\varepsilon\}$ satisfies (5.15). Moreover,

$$\frac{1}{\operatorname{meas} \Omega^\varepsilon_{\mathrm{f}}} \int_{\Omega^\varepsilon_{\mathrm{f}}} |w^\varepsilon - w_{M_0}|^p \,\mathrm{d}x \leqslant \frac{2}{\operatorname{meas} \Omega^\varepsilon_{\mathrm{f}}} \int_{\Omega^\varepsilon_{\mathrm{f}}} |w^\varepsilon_M - w_M|^p \,\mathrm{d}x + 2C_2 \|w_M - w_{M_0}\|^p_\Omega$$

for any M_0 and $\varepsilon \in]\hat\delta_{M+1}, \hat\delta_M]$. Therefore, according to (5.20)–(5.22), the sequence $\{w^\varepsilon\}$ $D^p_{\Omega^\varepsilon_{\mathrm{f}}}$-converges to w. Lemma 5.3 is proved. □

LEMMA 5.4. *Let assumptions 4.1–4.3 hold, and let $\{v^\varepsilon \in C^1(\Omega^\varepsilon_{\mathrm{f}})\}$ be a sequence satisfying (3.3). There then exists a family of continuous functions $\{v^\varepsilon_M\}$, $M = 1, 2, \dots,$ in Ω such that the following conditions apply.*

(i) $\lim\limits_{M\to\infty} M^p \limsup\limits_{\varepsilon\to 0} \mu^\varepsilon \operatorname{meas}\{x \in \Omega^\varepsilon_{\mathrm{f}} : v^\varepsilon(x) \neq v^\varepsilon_M(x)\} = 0.$

(ii) $v^\varepsilon_M \in \mathrm{Lip}(CM, \Omega)$ *with a constant* $C > 0$ *independent of* ε *and* M.

(iii) *For any* M, *there exists a subsequence* $\{v^{\varepsilon_j}_M\}$, $\varepsilon_j \to 0$, *converging uniformly in* Ω *to a function* $v_M \in \mathrm{Lip}(CM, \Omega)$.

(iv) *For any sequence of sets* $\mathcal{Q}^\varepsilon_M \subset \Omega^\varepsilon_{\mathrm{f}}$ *such that*

$$\lim_{M\to\infty} M^p \limsup_{\varepsilon\to 0} \mu^\varepsilon \operatorname{meas} \mathcal{Q}^\varepsilon_M = 0, \tag{5.23}$$

we have

$$\liminf_{\varepsilon=\varepsilon_j\to 0} \left\{ \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{Q}^\varepsilon_M} a^\varepsilon(x) |\nabla v^\varepsilon|^p \,\mathrm{d}x - \int_\Omega A(x, \nabla v_M) \,\mathrm{d}x \right\} = o(1), \tag{5.24}$$

as $M \to \infty$. *Moreover,*

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \mu^\varepsilon \|v^\varepsilon\|^p_{\mathcal{Q}^\varepsilon_M} = 0. \tag{5.25}$$

Proof of lemma 5.4. Using 4.2 we have that there is a set $\mathcal{G}^\varepsilon_M$ such that

$$\mu^\varepsilon \operatorname{meas} \mathcal{G}^\varepsilon_M = M^{-p}\phi(M) \quad \text{and} \quad \mu^\varepsilon \|v^\varepsilon\|^p_{\mathcal{G}^\varepsilon_M} = \phi(M)$$

for $\varepsilon \leqslant \varepsilon_0(M)$ with $\phi(M) \to 0$ as $M \to \infty$ and $v^\varepsilon \in \mathrm{Lip}(M, \Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{G}^\varepsilon_M)$. This implies that (5.25) holds. Moreover, according to Witney's theorem [26], the functions v^ε can be extended to Ω so that $v^\varepsilon_M \in \mathrm{Lip}(CM, \Omega)$ and $v^\varepsilon = v^\varepsilon_M$ in $\Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{G}^\varepsilon_M$, where $C > 0$ is independent of ε and M. Then, for any fixed M, $\{v^\varepsilon_M\}$ is a compact set in $C(\bar\Omega)$. Therefore, there is a subsequence $\{v^{\varepsilon_j}_M\}$, $\varepsilon_j \to 0$, converging uniformly in Ω to a function $v_M \in \mathrm{Lip}(CM, \Omega)$. Thus, the family $\{v^\varepsilon_M\}$ satisfies assertions (i)–(iii) of lemma 5.4.

It remains to prove lemma 5.4(iv). Let $v_{M\delta}$ be a smooth function in Ω such that

$$\|v_{M\delta} - v_M\|_{1,\Omega} < \delta. \tag{5.26}$$

We want to construct a sequence $\{v^\varepsilon_{M\delta}\}$ satisfying

$$\limsup_{\varepsilon\to 0} \mu^\varepsilon \|v^\varepsilon_{M\delta}\|^p_{1,\Omega^\varepsilon_{\mathrm{f}}} \leqslant C\delta^p \quad \text{and} \quad \limsup_{\varepsilon\to 0} \mu^\varepsilon \|v^\varepsilon_{M\delta} - (v_{M\delta} - v_M)\|^p_{\Omega^\varepsilon_{\mathrm{f}}} = 0. \tag{5.27}$$

For this we introduce a sequence of smooth functions w_k such that, for any $k = 1, 2, \dots,$

$$\int_\Omega |\nabla w_k|^p \,\mathrm{d}x \leqslant \delta^p + \frac{1}{k} \quad \text{and} \quad \int_\Omega |w_k - (v_{M\delta} - v_M)|^p \,\mathrm{d}x \leqslant \frac{1}{k}.$$

Then, since w_k, $|\nabla w_k|$, $v_{M\delta}$ and v_M are continuous functions in Ω, from (2.2) and assumption 4.1 we obtain

$$\lim_{\varepsilon\to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} |\nabla w_k|^p \,\mathrm{d}x = \int_\Omega |\nabla w_k|^p \rho(x)\,\mathrm{d}x,$$

$$\lim_{\varepsilon\to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} |w_k - (v_{M\delta} - v_M)|^p \,\mathrm{d}x = \int_\Omega |w_k - (v_{M\delta} - v_M)|^p \rho(x)\,\mathrm{d}x.$$

This implies that there exists a sequence $\{\hat\varepsilon_k\}$, $\hat\varepsilon_k \downarrow 0$, such that

$$\mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} |\nabla w_k|^p \,\mathrm{d}x \leqslant C\left(\delta^p + \frac{1}{k}\right), \quad \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} |w_k - (v_{M\delta} - v_M)|^p \,\mathrm{d}x \leqslant \frac{C}{k},$$

as $\varepsilon < \hat\varepsilon_k$. For $\hat\varepsilon_{k+1} \leqslant \varepsilon < \hat\varepsilon_k$ we set $v^\varepsilon_{M\delta} = w_k$. Then the sequence $\{v^\varepsilon_{M\delta}\}$ satisfies (5.27).

In order to prove (5.24) we first note that

$$\int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)|\nabla v^\varepsilon_M|^p \,\mathrm{d}x = \int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)|\nabla(v^\varepsilon_M + v^\varepsilon_{M\delta})|^p \,\mathrm{d}x + \int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)\{|\nabla v^\varepsilon_M|^p - |\nabla(v^\varepsilon_M + v^\varepsilon_{M\delta})|^p\}\,\mathrm{d}x.$$

Then, according to (5.27) we have

$$\liminf_{\varepsilon\to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)|\nabla v^\varepsilon_M|^p \,\mathrm{d}x \geqslant \liminf_{\varepsilon\to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)|\nabla(v^\varepsilon_M + v^\varepsilon_{M\delta})|^p \,\mathrm{d}x + \xi_1(\delta), \tag{5.28}$$

where $\xi_1(\delta) \to 0$ as $\delta \to 0$.

Now we cover the domain Ω by cubes $K^\alpha_h = \{x \in \Omega : |x_l - y^\alpha_l| \leqslant h/2\}$ with non-intersecting interiors. Consider an arbitrary cube K^α_h such that $K^\alpha_h \cap \partial\Omega = \emptyset$. We set

$$\psi^\varepsilon_{M\delta}(x) = v^\varepsilon_M(x) + v^\varepsilon_{M\delta}(x) - v_{M\delta}(y^\alpha).$$

The sequence $\{v^{\varepsilon_j}_M\}$ converges uniformly in Ω to v_M as $\varepsilon_j \to 0$. Then (5.27) implies that

$$\int_{\Omega^\varepsilon_{\mathrm{f}}\cap K^\alpha_h} a^\varepsilon(x)|\nabla\psi^\varepsilon_{M\delta}|^p \,\mathrm{d}x$$
$$= \int_{\Omega^\varepsilon_{\mathrm{f}}\cap K^\alpha_h} \{a^\varepsilon(x)|\nabla\psi^\varepsilon_{M\delta}|^p + h^{-p-\gamma}|\psi^\varepsilon_{M\delta} - (\nabla v_{M\delta}(y^\alpha), x - y^\alpha)|^p\}\,\mathrm{d}x - \xi^\alpha_2(\varepsilon, h, M)$$

with $\limsup_{\varepsilon=\varepsilon_j\to 0}\xi_2^\alpha(\varepsilon,h,M)=O(h^{n+p-\gamma})$ as $h\to 0$. Therefore, it follows from assumption 4.3 that

$$\liminf_{\varepsilon=\varepsilon_j\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon}a^\varepsilon(x)|\nabla(v_M^\varepsilon+v_{M\delta}^\varepsilon)|^p\,\mathrm{d}x=\liminf_{\varepsilon=\varepsilon_j\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon}a^\varepsilon(x)|\nabla\psi_{M\delta}^\varepsilon|^p\,\mathrm{d}x$$
$$\geqslant\int_\Omega A(x,\nabla v_{M\delta})\,\mathrm{d}x. \tag{5.29}$$

Now, passing to the limit in (5.29) as $\delta\to 0$ and using (5.26), (5.28) we obtain

$$\liminf_{\varepsilon=\varepsilon_j\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon}a^\varepsilon(x)|\nabla v_M^\varepsilon|^p\,\mathrm{d}x\geqslant\int_\Omega A(x,\nabla v_M)\,\mathrm{d}x. \tag{5.30}$$

Finally, it is easy to see that

$$\liminf_{\varepsilon=\varepsilon_j\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon\setminus\mathcal{G}_M^\varepsilon}a^\varepsilon(x)|\nabla v^\varepsilon|^p\,\mathrm{d}x\geqslant\liminf_{\varepsilon=\varepsilon_j\to 0}\mu^\varepsilon\int_{\Omega_{\mathrm{f}}^\varepsilon}a^\varepsilon(x)|\nabla v_M^\varepsilon|^p\,\mathrm{d}x-\xi(M), \tag{5.31}$$

where

$$\xi(M)=\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\mathcal{G}_M^\varepsilon\cup\mathcal{Q}_M^\varepsilon}a^\varepsilon(x)|\nabla v_M^\varepsilon|^p\,\mathrm{d}x=o(1)\quad\text{as } M\to\infty.$$

Assertion (iv) of Lemma 5.4 follows from (5.30) and (5.31). This completes the proof of Lemma 5.4. □

6. Proof of theorem 4.5

We begin this section by obtaining *a priori* estimates for the minimizer of problem (2.4):

$$J^\varepsilon[u^\varepsilon]\equiv\mu^\varepsilon\int_\Omega\{a^\varepsilon(x)|\nabla u^\varepsilon|^p+g(x)|u^\varepsilon|^p-pS^\varepsilon(x)u^\varepsilon\}\,\mathrm{d}x\to\inf,\quad u^\varepsilon\in W^{1,p}(\Omega), \tag{6.1}$$

Since $J^\varepsilon[u^\varepsilon]\leqslant J^\varepsilon[0]=0$, by virtue of the Young inequality and (2.5) we have

$$\mu^\varepsilon\int_\Omega\{a^\varepsilon(x)|\nabla u^\varepsilon|^p+g(x)|u^\varepsilon|^p\}\,\mathrm{d}x\leqslant C_1\mu^\varepsilon\|S\|_{\Omega_{\mathrm{f}}^\varepsilon}^{p'}\leqslant C_2, \tag{6.2}$$

where the constants C_1 and C_2 do not depend on ε. It then follows from (6.2) that

$$\mu^\varepsilon\|u^\varepsilon\|_{1,\Omega_{\mathrm{f}}^\varepsilon}^p\leqslant C. \tag{6.3}$$

Hence, $\{u^\varepsilon\}$ is a $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-compact set and one can extract a subsequence (still denoted by $\{u^\varepsilon\}$) $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converging to a function $u\in L^p(\Omega)$. Let us show that $u=u(x)$ is a solution of the variational problem (4.5). This will be done in two steps.

6.1. Step 1. Upper bound

Let $w=w(x)$ be an arbitrary smooth function in Ω and let Y_h^ε, W_{Mh}^ε, $\mathcal{B}_h^\varepsilon$ be the same as in lemmas 5.1 and 5.2. We set

$$\vartheta_{Mh}^\varepsilon(x)=Y_h^\varepsilon(x)W_{Mh}^\varepsilon(x).$$

It is clear that $\vartheta_{Mh}^\varepsilon\in W^{1,p}(\Omega)$.

First we prove that

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0} J^\varepsilon[\vartheta^\varepsilon_{Mh}] \leqslant J_{\mathrm{hom}}[w], \tag{6.4}$$

where

$$J_{\mathrm{hom}}[w] = \int_\Omega \{A(x,\nabla w) + B(x)|w|^p - p\rho(x)S(x)w\}\,\mathrm{d}x \tag{6.5}$$

with $B(x) = (g\rho + b)(x)$. We have

$$J^\varepsilon[\vartheta^\varepsilon_{Mh}] = \mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla\vartheta^\varepsilon_{Mh}|^p + g(x)|\vartheta^\varepsilon_{Mh}|^p - pS^\varepsilon(x)\vartheta^\varepsilon_{Mh}\}\,\mathrm{d}x. \tag{6.6}$$

Consider the third term in (6.6). It follows from assumption 4.1, (2.5), lemmas 5.1(i), (ii), and 5.2(i) that

$$\lim_{M\to\infty}\lim_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_\Omega S^\varepsilon(x)\vartheta^\varepsilon_{Mh}(x)\,\mathrm{d}x = \int_\Omega S(x)w(x)\rho(x)\,\mathrm{d}x. \tag{6.7}$$

Consider the second term in (6.6). We have

$$\mu^\varepsilon\int_\Omega g(x)|\vartheta^\varepsilon_{Mh}|^p\,\mathrm{d}x = \mu^\varepsilon\int_\Omega g(x)|Y^\varepsilon_h|^p|w|^p\,\mathrm{d}x + \mu^\varepsilon\int_\Omega g(x)|Y^\varepsilon_h|^p\{|W^\varepsilon_{Mh}|^p - |w|^p\}\,\mathrm{d}x. \tag{6.8}$$

By lemmas 5.1(iv) and 5.2(i) we obtain

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_\Omega g(x)|Y^\varepsilon_h|^p\{|W^\varepsilon_{Mh}|^p - |w|^p\}\,\mathrm{d}x = 0. \tag{6.9}$$

For the first term in (6.6) we have

$$\begin{aligned}\mu^\varepsilon\int_\Omega a^\varepsilon(x)|\nabla\vartheta^\varepsilon|^p\,\mathrm{d}x = \mu^\varepsilon&\int_{\Omega^\varepsilon_{\mathrm{f}}\setminus\mathcal{B}^\varepsilon_h} a^\varepsilon(x)|\nabla W^\varepsilon_{Mh}|^p\,\mathrm{d}x\\ &+\mu^\varepsilon\int_{\mathcal{B}^\varepsilon_h} a^\varepsilon(x)|\nabla Y^\varepsilon_h W^\varepsilon_{Mh} + \nabla W^\varepsilon_{Mh}Y^\varepsilon_h|^p\,\mathrm{d}x\\ &+\mu^\varepsilon\int_{\Omega^\varepsilon_{\mathrm{m}}} a^\varepsilon(x)|\nabla Y^\varepsilon_h W^\varepsilon_{Mh} + \nabla W^\varepsilon_{Mh}Y^\varepsilon_h|^p\,\mathrm{d}x.\end{aligned} \tag{6.10}$$

Assertion (iii) of lemma 5.2 implies that the following limit relation holds:

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\Omega^\varepsilon_{\mathrm{f}}\setminus\mathcal{B}^\varepsilon_h} a^\varepsilon(x)|\nabla W^\varepsilon_{Mh}|^p\,\mathrm{d}x \leqslant \int_\Omega A(x,\nabla w)\,\mathrm{d}x. \tag{6.11}$$

It is clear that

$$\begin{aligned}\mu^\varepsilon&\int_{\mathcal{B}^\varepsilon_h} a^\varepsilon|\nabla Y^\varepsilon_h W^\varepsilon_{Mh} + \nabla W^\varepsilon_{Mh}Y^\varepsilon_h|^p\,\mathrm{d}x\\ &= \mu^\varepsilon\int_{\mathcal{B}^\varepsilon_h} a^\varepsilon|\nabla Y^\varepsilon_h|^p|w|^p\,\mathrm{d}x\\ &\quad+\mu^\varepsilon\int_{\mathcal{B}^\varepsilon_h} a^\varepsilon|\nabla Y^\varepsilon_h|^p\{|W^\varepsilon_{Mh}|^p - |w|^p\}\,\mathrm{d}x\\ &\quad+\mu^\varepsilon\int_{\mathcal{B}^\varepsilon_h} a^\varepsilon\{|\nabla\vartheta^\varepsilon_{Mh}|^p - |\nabla Y^\varepsilon_h W^\varepsilon_{Mh}|^p\}\,\mathrm{d}x.\end{aligned} \tag{6.12}$$

Lemmas 5.1(iv) and 5.2(i) imply that

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\mathcal{B}_h^\varepsilon} a^\varepsilon(x)|\nabla Y_h^\varepsilon|^p\{|W_{Mh}^\varepsilon|^p-|w|^p\}\,\mathrm{d}x=0. \tag{6.13}$$

Assertions (i), (iii) and (iv) of lemma 5.1 and (i) and (ii) of lemma 5.2 imply that

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\mathcal{B}_h^\varepsilon} a^\varepsilon(x)\{|\nabla\vartheta_{Mh}^\varepsilon|^p-|\nabla Y_h^\varepsilon W_{Mh}^\varepsilon|^p\}\,\mathrm{d}x=0. \tag{6.14}$$

For the third term on the right-hand side of (6.10) we have

$$\begin{aligned}\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon|\nabla Y_h^\varepsilon W_{Mh}^\varepsilon+\nabla W_{Mh}^\varepsilon Y_h^\varepsilon|^p\,\mathrm{d}x&\\ =\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon|\nabla Y_h^\varepsilon|^p|w|^p\,\mathrm{d}x&\\ +\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon|\nabla Y_h^\varepsilon|^p\{|W_{Mh}^\varepsilon|^p-|w|^p\}\,\mathrm{d}x&\\ +\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon\{|\nabla\vartheta_{Mh}^\varepsilon|^p-|\nabla Y_h^\varepsilon W_{Mh}^\varepsilon|^p\}\,\mathrm{d}x&\end{aligned} \tag{6.15}$$

and by the condition (2.7) and lemmas 5.1(i), (iv), and 5.2(i), (ii), we obtain

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon(x)|\nabla Y_h^\varepsilon|^p\{|W_{Mh}^\varepsilon|^p-|w|^p\}\,\mathrm{d}x=0 \tag{6.16}$$

and

$$\limsup_{M\to\infty}\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\mu^\varepsilon\int_{\Omega_\mathrm{m}^\varepsilon} a^\varepsilon(x)\{|\nabla\vartheta_{Mh}^\varepsilon|^p-|\nabla Y_h^\varepsilon W_{Mh}^\varepsilon|^p\}\,\mathrm{d}x=0. \tag{6.17}$$

We now obtain (6.4) from (6.7)–(6.17) and assertion (iv) of lemma 5.1. Since u^ε minimizes the functional J^ε, it follows from (6.4) that

$$\limsup_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon]\leqslant J_{\mathrm{hom}}[w] \tag{6.18}$$

for any smooth function w. By density arguments, (6.18) also holds for any function $w\in W^{1,p}(\Omega)$.

6.2. Step 2. Lower bound

Let $\{u^\varepsilon\}$ be a sequence of solutions of the variational problem (6.1) which $D^p_{\Omega_\mathrm{f}^\varepsilon}$-converges to a function u. We want to show that

$$\liminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon]\geqslant J_{\mathrm{hom}}[u]. \tag{6.19}$$

According to lemma 5.4 there exists a family of functions

$$\{u_M^\varepsilon\in \mathrm{Lip}(CM,\Omega),\ M=1,2,\dots\}$$

such that

$$\lim_{M\to\infty}\limsup_{\varepsilon\to 0}\mu^\varepsilon\|u^\varepsilon-u_M^\varepsilon\|^p_{\Omega_\mathrm{f}^\varepsilon}=0. \tag{6.20}$$

Moreover, for any M, there is a subsequence $\{u_M^{\varepsilon_j}\}$, $\varepsilon_j \to 0$, converging uniformly in Ω to a function u_M, and

$$\liminf_{\varepsilon=\varepsilon_j\to 0} \mu^\varepsilon \int_{\Omega_{\mathrm{f}}^\varepsilon} a^\varepsilon(x)|\nabla u^\varepsilon|^p \,\mathrm{d}x \geqslant \limsup_{M\to\infty} \int_\Omega A(x,\nabla u_M)\,\mathrm{d}x - \xi(M), \tag{6.21}$$

where $\xi(M) \to 0$ as $M \to \infty$. Since the sequence $\{u^\varepsilon\}$ $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converges to u and $\{u_M^{\varepsilon_j}\}$ converges uniformly to u_M, (6.20) implies that the functions u_M converge in $L^p(\Omega)$ to u as $M \to \infty$. In addition, it follows from (6.21) and (4.4) that the sequence $\{u_M\}$ is bounded in $W^{1,p}(\Omega)$. Thus, $u \in W^{1,p}(\Omega)$.

Let us approximate u by smooth functions $u_\delta(x)$, $\delta > 0$, in Ω,

$$\|u_\delta - u\|^p_{1,\Omega} \leqslant \delta, \tag{6.22}$$

and set $w_\delta(x) = u_\delta(x) - u(x)$. By virtue of lemma 5.3 there exists a sequence $\{w_\delta^\varepsilon\}$ that $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converges to w_δ and satisfies the bound

$$\limsup_{\varepsilon\to 0} I^\varepsilon[w_\delta^\varepsilon] \leqslant C\delta, \tag{6.23}$$

where C does not depend on u_δ, and the functional I^ε is defined by (5.15). We set

$$u_\delta^\varepsilon = w_\delta^\varepsilon + u^\varepsilon. \tag{6.24}$$

The sequence $\{u_\delta^\varepsilon\}$ $D^p_{\Omega_{\mathrm{f}}^\varepsilon}$-converges to u_δ and according to (6.23),

$$\lim_{\varepsilon\to 0} J^\varepsilon[u_\delta^\varepsilon] \leqslant \liminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] + i^\varepsilon(\delta), \tag{6.25}$$

where $i^\varepsilon(\delta) \to 0$ as $\delta \to 0$ (by passing, if necessary, to a subsequence we can assume that the limit on the right-hand side of (6.25) exists).

Since $u_\delta(x)$ is a smooth function, from remark 3.2 we may deduce that

$$\lim_{\varepsilon\to 0} \mu^\varepsilon \|u_\delta^\varepsilon - u_\delta\|^p_{\Omega_{\mathrm{f}}^\varepsilon} = 0. \tag{6.26}$$

Inequality (6.25) and lemma 5.4 imply the existence of functions

$$u_{\delta M}^\varepsilon \in \mathrm{Lip}(CM,\Omega), \quad M = 1,2,\dots,$$

and sets $\mathcal{Q}_M^\varepsilon$ such that $u_{\delta M}^\varepsilon(x) = u_\delta^\varepsilon(x)$ for $x \in \Omega_{\mathrm{f}}^\varepsilon \setminus \mathcal{Q}_M^\varepsilon$ and

$$\mu^\varepsilon \operatorname{meas} \mathcal{Q}_M^\varepsilon = M^{-p}\phi(M), \quad \mu^\varepsilon \|u_\delta^\varepsilon\|^p_{\mathcal{Q}_M^\varepsilon} = \phi(M)$$

for $\varepsilon \leqslant \hat{\varepsilon}(M)$ and $\phi(M) \to 0$ as $M \to \infty$. Moreover, for any $M \in \{1,2,\dots\}$ fixed, one can extract a subsequence $\{u_{\delta M}^{\varepsilon_j}\}$, $\varepsilon_j \to 0$, converging uniformly in Ω to a function $u_{\delta M} \in \mathrm{Lip}(CM,\Omega)$. At the same time, due to condition (2.2), functions $u_{\delta M}$ converge in $L^p(\Omega)$ to u_δ as $M \to \infty$, since

$$\begin{aligned}\int_\Omega &|u_{\delta M} - u_\delta|^p \,\mathrm{d}x \\ &\leqslant C \limsup_{\varepsilon=\varepsilon_j\to 0} \mu^\varepsilon \int_{\Omega_{\mathrm{f}}^\varepsilon} \{|u_{\delta M} - u_{\delta M}^\varepsilon|^p + |u_{\delta M}^\varepsilon - u_\delta^\varepsilon|^p + |u_\delta^\varepsilon - u_\delta|^p\}\,\mathrm{d}x \xrightarrow[M\to\infty]{} 0.\end{aligned}$$

It follows from (6.26) that there exist a sequence $\{r^\varepsilon > 0\}$, $r^\varepsilon \to 0$, and sets $\mathcal{B}^\varepsilon_M$ such that

$$\lim_{\varepsilon\to 0} \mu^\varepsilon \operatorname{meas} \mathcal{B}^\varepsilon_M = 0 \quad \text{and} \quad |u^\varepsilon_{\delta M}(x) - u_\delta(x)| \leqslant r^\varepsilon \text{ in } \Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{Z}^\varepsilon_M,$$

where $\mathcal{Z}^\varepsilon_M = \mathcal{Q}^\varepsilon_M \cup \mathcal{B}^\varepsilon_M$. Let us define the functions

$$v^\varepsilon_{\delta M}(x) = \begin{cases} u_\delta(x) + r^\varepsilon, & \text{if } u^\varepsilon_{\delta M}(x) > u_\delta(x) + r^\varepsilon, \\ u^\varepsilon_{\delta M}(x), & \text{if } |u^\varepsilon_{\delta M}(x) - u_\delta(x)| \leqslant r^\varepsilon, \\ u_\delta(x) - r^\varepsilon, & \text{if } u^\varepsilon_{\delta M}(x) < u_\delta(x) - r^\varepsilon. \end{cases} \tag{6.27}$$

Clearly, $v^\varepsilon_{\delta M} \in \operatorname{Lip}(CM, \Omega)$. Moreover, the functions $v^\varepsilon_{\delta M}$ converge uniformly in Ω to u_δ as $\varepsilon \to 0$.

We set $V^\varepsilon_{\delta M} = u^\varepsilon_\delta - v^\varepsilon_{\delta M}$ and consider the left-hand side of the inequality (6.25). Since $v^\varepsilon_{\delta M}(x) = u^\varepsilon_\delta(x)$ for $x \in \Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{Z}^\varepsilon_M$, we have

$$\begin{aligned} I^\varepsilon[u^\varepsilon_\delta] = \mu^\varepsilon \biggl(& \int_{\Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x) |\nabla V^\varepsilon_\delta|^p \,\mathrm{d}x + \int_{\Omega^\varepsilon_{\mathrm{m}}} \{a^\varepsilon(x)|\nabla u^\varepsilon_\delta|^p + g(x)|u^\varepsilon_\delta|^p\} \,\mathrm{d}x \biggr) \\ & + \mu^\varepsilon \biggl(\int_{\Omega^\varepsilon_{\mathrm{f}} \setminus \mathcal{Z}^\varepsilon_M} a^\varepsilon(x) |\nabla u^\varepsilon_\delta|^p \,\mathrm{d}x + \int_{\Omega^\varepsilon_{\mathrm{f}}} g(x)|u^\varepsilon_\delta|^p \,\mathrm{d}x \biggr) \\ & + \mu^\varepsilon \biggl(\int_{\Omega^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_M} a^\varepsilon(x) |\nabla u^\varepsilon_\delta|^p \,\mathrm{d}x - \int_{\Omega^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_M} a^\varepsilon(x) |\nabla (u^\varepsilon_\delta - v^\varepsilon_{\delta M})|^p \,\mathrm{d}x \biggr) \\ \equiv {} & \theta_1 + \theta_2 + \theta_3. \end{aligned} \tag{6.28}$$

Consider the first term on the right-hand side of (6.28). First we define $\Omega_\zeta \subset \Omega$:

$$\Omega_\zeta = \{x \in \Omega : |u_\delta(x)| > 2\zeta\},$$

where $\zeta > 0$. Let us cover Ω_ζ by cubes K^α_h of length h centred at x^α with non-intersecting interiors. For ε and h sufficiently small, we have $|v^\varepsilon_{\delta M}| > \zeta$ in K^α_h. One can show that for $x \in \Omega^\varepsilon_{\mathrm{f}} \cap K^\alpha_h$ we have

$$\begin{aligned} & (1 + A_1 h^{p/p-1}) a^\varepsilon(x) |\nabla (u^\varepsilon_\delta - v^\varepsilon_{\delta M})|^p \\ & \qquad \geqslant a^\varepsilon(x) |v^\varepsilon_{\delta M}|^p \biggl| \nabla \biggl(\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}} \biggr) \biggr|^p - a_0^{-1} A_2 \biggl(1 + \frac{1}{h^p} \biggr) |u^\varepsilon_\delta - v^\varepsilon_{\delta M}|^p \frac{|\nabla v^\varepsilon_{\delta M}|^p}{|v^\varepsilon_{\delta M}|^p}, \end{aligned} \tag{6.29}$$

where A_1 and A_2 are positive constants independent of ε, δ and M. In a similar way, for $x \in \Omega^\varepsilon_{\mathrm{f}} \cap K^\alpha_h$ we have

$$\begin{aligned} & (1 + A_1 h^{p/p-1}) a^\varepsilon(x) |\nabla u^\varepsilon_\delta|^p \\ & \qquad \geqslant a^\varepsilon(x) |v^\varepsilon_{\delta M}|^p \biggl| \nabla \biggl(\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}} \biggr) \biggr|^p - a_2(\varepsilon) A_2 \biggl(1 + \frac{1}{h^p} \biggr) |u^\varepsilon_\delta|^p \frac{|\nabla v^\varepsilon_{\delta M}|^p}{|v^\varepsilon_{\delta M}|^p}, \end{aligned} \tag{6.30}$$

where $a_2(\varepsilon)$ is defined in (2.7).

Now we make use of (6.26) and of the definition of the function $v^\varepsilon_{\delta M}(x)$ and its properties. For any $K^\alpha_h \subset \Omega_\zeta$, we obtain

$$\mu^\varepsilon \int_{K^\alpha_h \cap \Omega^\varepsilon_{\mathrm{f}}} a^\varepsilon(x)|\nabla V^\varepsilon_{\delta M}|^p \,\mathrm{d}x + \mu^\varepsilon \int_{K^\alpha_h \cap \Omega^\varepsilon_{\mathrm{m}}} \{a^\varepsilon(x)|\nabla u^\varepsilon_\delta|^p + g(x)|u^\varepsilon_\delta|^p\} \,\mathrm{d}x$$
$$\geqslant |u_\delta(x^\alpha)|^p \mu^\varepsilon \biggl\{ \int_{K^\alpha_h} a^\varepsilon(x)\biggl|\nabla\biggl(\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}}\biggr)\biggr|^p \,\mathrm{d}x + \int_{K^\alpha_h \cap \Omega^\varepsilon_{\mathrm{m}}} g(x)\biggl|\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}}\biggr|^p \,\mathrm{d}x \biggr\} + o(h^n), \tag{6.31}$$

for ε small enough ($\varepsilon \leqslant \hat{\varepsilon}(h)$) and $h \to 0$. Assumption 4.4 implies that

$$\liminf_{\varepsilon\to 0} \mu^\varepsilon \biggl\{ \int_{K^\alpha_h} a^\varepsilon(x)\biggl|\nabla\biggl(\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}}\biggr)\biggr|^p \,\mathrm{d}x + \int_{K^\alpha_h \cap \Omega^\varepsilon_{\mathrm{m}}} g(x)\biggl|\frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}}\biggr|^p \,\mathrm{d}x \biggr\} \geqslant h^n b(x^\alpha) + o(h^n) \tag{6.32}$$

as $h \to 0$. It now follows from (6.31) and (6.32) that

$$\liminf_{\varepsilon\to 0} \theta_1 \geqslant \int_{\Omega_\zeta} b(x)|u_\delta|^p \,\mathrm{d}x. \tag{6.33}$$

Taking into account the definition of Ω_ζ and passing to the limit as $\zeta \to 0$ in (6.33), we get

$$\liminf_{\varepsilon\to 0} \theta_1 \geqslant \int_{\Omega} b(x)|u_\delta|^p \,\mathrm{d}x. \tag{6.34}$$

In order to estimate θ_2 from below in (6.28) we argue as follows. Using lemma 5.4, (6.26) and assumption 4.1, we obtain

$$\liminf_{\varepsilon=\varepsilon_j\to 0} \theta_2 \geqslant \int_{\Omega} A(x,\nabla u_{\delta M}) \,\mathrm{d}x + \int_{\Omega} g(x)|u_\delta|^p \rho(x) \,\mathrm{d}x + o(1) \tag{6.35}$$

as $M \to \infty$. Since the first term on the right-hand side of (6.35) is a weakly lower semi-continuous functional in $W^{1,p}(\Omega)$, and functions $u_{\delta M}$ converge in $L^p(\Omega)$ to u_δ as $M \to \infty$, we have

$$\liminf_{M\to\infty} \liminf_{\varepsilon=\varepsilon_j\to 0} \theta_2 \geqslant \int_{\Omega} \{A(x,\nabla u_\delta) + g(x)\rho(x)|u_\delta|^p\} \,\mathrm{d}x. \tag{6.36}$$

Finally, we consider the third term on the right-hand side of (6.28). Using (6.27) and (2.6) we get

$$|\theta_3| \leqslant C_1 \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_M} |\nabla u_\delta| \{|\nabla u^\varepsilon_\delta|^{p-1} + |\nabla u_\delta|^{p-1}\} \,\mathrm{d}x,$$

where C_1 is a constant independent of ε, δ and M. Since $u_\delta(x)$ is a smooth function in Ω, we finally get

$$|\theta_3| \leqslant C_2 \mu^\varepsilon \int_{\Omega^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_M} \{1 + |\nabla u^\varepsilon_\delta|^{p-1}\} \,\mathrm{d}x, \tag{6.37}$$

where C_2 is a constant independent of ε and M. It is now easy to see that the definition of the function u^ε_δ, (6.24), (6.3), (6.23), the estimate for the measure of $\mathcal{Z}^\varepsilon_M$, and Hölder's inequality yield

$$\lim_{M\to\infty}\limsup_{\varepsilon\to 0}|\theta_3| = 0. \tag{6.38}$$

Thus, it follows from (6.34), (6.36), (6.38), (2.5) and assumption 4.1 that

$$\liminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon_\delta] \geqslant J_{\mathrm{hom}}[u_\delta]. \tag{6.39}$$

This inequality, (6.22), and (6.25) immediately yield (6.19).

Inequalities (6.18), (6.19) mean that if a subsequence of solutions of problem (6.1) $D^p_{\Omega^\varepsilon_{\mathrm{f}}}$-converges to a function $u = u(x)$, then u minimizes the functional J_{hom} in $W^{1,p}(\Omega)$, i.e. u is a solution of the problem (4.5). Since $b(x) \geqslant 0$, this problem has a unique solution and the whole sequence of solutions of problem (6.1) $D^p_{\Omega^\varepsilon_{\mathrm{f}}}$-converges to the function u. This completes the proof of theorem 4.5.

7. Periodic examples

As an application of the previous general result, we now give two examples of fissured media, where the distribution of the fracture part is specified.

Theorem 4.5 of § 4 provides sufficient conditions for the existence of the homogenized problem (4.5). The goals of this section are to prove that, for appropriate periodic examples, all the conditions of theorem 4.5 are satisfied and to compute the coefficients of the homogenized problem (4.5) explicitly. We will study the following variational problem:

$$\mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla u^\varepsilon|^p + g|u^\varepsilon|^p - pS^\varepsilon(x)u^\varepsilon\}\,\mathrm{d}x \to \inf, \quad u^\varepsilon \in W^{1,p}(\Omega), \tag{7.1}$$

where $p \geqslant 2$ and

$$a^\varepsilon(x) = \alpha_{\mathrm{f}}\mathbf{1}^\varepsilon_{\mathrm{f}}(x) + \alpha_m\varepsilon^\theta\mathbf{1}^\varepsilon_m(x), \quad S^\varepsilon(x) = \mathbf{1}^\varepsilon_{\mathrm{f}}(x)S(x), \tag{7.2}$$

with $S \in L^{p'}(\Omega)$, g, α_{f} and α_m are strictly positive constants and $\theta > 0$ is a parameter.

In the following subsections we study a periodic thin connected microstructure $\Omega^\varepsilon_{\mathrm{f}}$ of two different types.

7.1. Two-dimensional periodic example

Let $\Omega = \Omega^\varepsilon_{\mathrm{f}} \cup \bar{\Omega}^\varepsilon_{\mathrm{m}}$ be a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. We define the set $\Omega^\varepsilon_{\mathrm{f}}$ as follows. Let $\mathcal{P}^\varepsilon \subset \mathbb{R}^2$ be the simplest lattice structure consisting of two ε-periodic systems of thin strips oriented in the coordinate directions. The width of the strips is equal to d_ε, given by

$$d_\varepsilon = d\varepsilon^{\theta/p}, \quad d > 0, \ \theta > p \geqslant 2. \tag{7.3}$$

This case describes the critical thickness of the fissures when the exchange process between the matrix and the fissures is not negligible. We set $\Omega^\varepsilon_{\mathrm{f}} = \Omega \cap \mathcal{P}^\varepsilon$. Then $\Omega^\varepsilon_{\mathrm{m}}$ is made of periodically (with period ε) distributed squares $\mathcal{M}^\varepsilon_i$ with centres at $x^{i,\varepsilon} \in \Omega$.

Let us formulate the homogenization result for this example.

THEOREM 7.1. *Let $\{u^\varepsilon\}$ be the sequence of solutions of problem (7.1), (7.2). $\{u^\varepsilon\}$ then $D^p_{\Omega^\varepsilon_{\mathrm{f}}}$-converges to u the solution of the problem:*

$$\int_\Omega \{\tfrac{1}{2}\alpha_{\mathrm{f}}(|u_{x_1}|^p + |u_{x_2}|^p) + B|u|^p - pS(x)u\}\,\mathrm{d}x \to \inf, \quad u \in W^{1,p}(\Omega), \tag{7.4}$$

where

$$B = g + \frac{2(\alpha_m)^{1/p}}{d}\left(\frac{g}{p-1}\right)^{(p-1)/p}. \tag{7.5}$$

7.1.1. *Proof of theorem 7.1*

We must verify assumptions 4.1–4.4 and calculate the functions $\rho(x)$, $A(x,\boldsymbol{q})$, and $b(x)$. For this example, the main difficulty is the verification of assumption 4.4.

First, it is easy to see that $\operatorname{meas}\Omega^\varepsilon_{\mathrm{f}} = 2d_\varepsilon\varepsilon^{-1}\operatorname{meas}\Omega + o(1)$ as $\varepsilon \to 0$ and, consequently,

$$\mu^\varepsilon = \frac{\varepsilon}{2d_\varepsilon} + o(1), \quad \varepsilon \to 0. \tag{7.6}$$

Now let K^z_h be an open square with length h $(0 < \varepsilon \ll h < 1)$ centred at $z \in \Omega$.

First we check assumption 4.1. Since $\operatorname{meas}(K^z_h \cap \Omega^\varepsilon_{\mathrm{f}}) \sim 2d_\varepsilon\varepsilon^{-1}h^2$, it is clear that assumption 4.1 is satisfied and

$$\rho(x) = 1. \tag{7.7}$$

The fact that the family of domains $\{\Omega^\varepsilon_{\mathrm{f}}\}$ satisfies assumption 4.2 (the SC condition) is known from [27] (see also [20]).

Assumption 4.3 was considered in [3, 23] in a more general situation. Applying the results of [23] we get

$$A(x,\boldsymbol{q}) = \tfrac{1}{2}\alpha_{\mathrm{f}}(|q_1|^p + |q_2|^p). \tag{7.8}$$

It remains to check assumption 4.4. Denote by $\mathcal{M}$ the unit square in the space $\mathbb{R}^2$, $\mathcal{M} = \{x \in \mathbb{R}^2 : |x_i| < \frac{1}{2}\}$. Consider the following boundary-value problem:

$$\left.\begin{aligned} \Delta_p W^\varepsilon + \beta^\varepsilon|W^\varepsilon|^{p-2}W^\varepsilon &= 0 \quad \text{in } \mathcal{M}, \\ W^\varepsilon &= 1 \quad \text{on } \partial\mathcal{M}, \end{aligned}\right\} \tag{7.9}$$

where Δ_p denotes the p-Laplacian and

$$\beta^\varepsilon = \frac{g}{\alpha_m}\frac{(\varepsilon - d_\varepsilon)^p}{\varepsilon^\theta}. \tag{7.10}$$

The functional $b^{\varepsilon,h}(z)$ in our case has the form:

$$b^{\varepsilon,h}(z) = \inf_{w^\varepsilon}\mu^\varepsilon\int_{K^z_h}\{a^\varepsilon(x)|\nabla w^\varepsilon|^p + g\mathbf{1}^\varepsilon_m|w^\varepsilon|^p + h^{-p-\gamma}\mathbf{1}^\varepsilon_{\mathrm{f}}|w^\varepsilon - 1|^p\}\,\mathrm{d}x, \tag{7.11}$$

where a^ε is defined in (7.2) and the infimum is taken over $w^\varepsilon \in W^{1,p}(K^z_h)$. We seek for a function w^ε minimizing (7.11) in the form

$$w^\varepsilon(x) = \vartheta^\varepsilon(x) + \zeta^\varepsilon(x), \tag{7.12}$$

where

$$\vartheta^\varepsilon(x) = \begin{cases} W^\varepsilon\left(\dfrac{x - x^{i,\varepsilon}}{\varepsilon - d_\varepsilon}\right) & \text{in } \mathcal{M}_i^\varepsilon \cap K_h^z, \\ 1 & \text{in } \Omega_{\mathrm{f}}^\varepsilon \cap K_h^z. \end{cases} \tag{7.13}$$

Then

$$b^{\varepsilon,h}(z) = \mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x)|\nabla\vartheta^\varepsilon + \nabla\zeta^\varepsilon|^p + g\mathbf{1}_m^\varepsilon|\vartheta^\varepsilon + \zeta^\varepsilon|^p + h^{-p-\gamma}\mathbf{1}_{\mathrm{f}}^\varepsilon|\vartheta^\varepsilon + \zeta^\varepsilon - 1|^p\}\,\mathrm{d}x. \tag{7.14}$$

We will prove that the function ζ^ε gives a vanishing contribution (as $\varepsilon \to 0$ and $h \to 0$) in (7.11). Since the function $w^\varepsilon = \vartheta^\varepsilon + \zeta^\varepsilon$ minimizes the functional (7.11), and $\vartheta^\varepsilon = 1$ in $\Omega_{\mathrm{f}}^\varepsilon$, we have

$$b^{\varepsilon,h}(z) \leqslant \mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x)|\nabla\vartheta^\varepsilon|^p + g\mathbf{1}_m^\varepsilon|\vartheta^\varepsilon|^p\}\,\mathrm{d}x \equiv \Theta^{\varepsilon,h}(z). \tag{7.15}$$

Now let us estimate the functional (7.11) from below. To this end we make use of

$$|\xi_1 + \xi_2|^p \geqslant |\xi_1|^p + \delta_p|\xi_2|^p + p|\xi_1|^{p-2}(\xi_1, \xi_2), \tag{7.16}$$

where ξ_1 and ξ_2 are arbitrary vectors from the space $\mathbb{R}^n$, $n = 2, 3$, $0 < \delta_p \leqslant 1$ ($\delta_p = 1$ when $p = 2$). We have

$$\begin{aligned} b^{\varepsilon,h}(z) \geqslant \Theta^{\varepsilon,h}(z) &+ \delta_p\mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x)|\nabla\zeta^\varepsilon|^p + g\mathbf{1}_m^\varepsilon|\zeta^\varepsilon|^p + h^{-p-\gamma}\mathbf{1}_{\mathrm{f}}^\varepsilon|\zeta^\varepsilon|^m\}\,\mathrm{d}x \\ &+ p\mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x)|\nabla\vartheta^\varepsilon|^{p-2}(\nabla\vartheta^\varepsilon, \nabla\zeta^\varepsilon) + g\mathbf{1}_m^\varepsilon\vartheta^\varepsilon|\nabla\vartheta^\varepsilon|^{p-2}\zeta^\varepsilon\}\,\mathrm{d}x. \end{aligned} \tag{7.17}$$

It now follows from (7.9), (7.15) and (7.17) that

$$\begin{aligned} \Upsilon^{\varepsilon,h}[\zeta^\varepsilon] &\equiv \mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x)|\nabla\zeta^\varepsilon|^p + g\mathbf{1}_m^\varepsilon|\zeta^\varepsilon|^p + h^{-p-\gamma}\mathbf{1}_{\mathrm{f}}^\varepsilon|\zeta^\varepsilon|^p\}\,\mathrm{d}x \\ &\leqslant \frac{\alpha_m p\varepsilon^\theta\mu^\varepsilon}{\delta_p} \int_{K_h^z\cap\partial\Omega_{\mathrm{m}}^\varepsilon} \left|\frac{\partial\vartheta^\varepsilon}{\partial\nu}\right| |\nabla\vartheta^\varepsilon|^{p-2}|\zeta^\varepsilon|\,\mathrm{d}\sigma. \end{aligned} \tag{7.18}$$

It is easy to see that, for any $v \in W^{1,p}(\Omega)$, the following inequality holds:

$$\int_{K_h^z\cap\partial\Omega_{\mathrm{m}}^\varepsilon} |v|^p d\sigma \leqslant C\left\{\frac{1}{d_\varepsilon}\int_{K_h^z\cap\Omega_{\mathrm{f}}^\varepsilon} |v|^p\,\mathrm{d}x + d_\varepsilon^{p-1}\int_{K_h^z\cap\Omega_{\mathrm{f}}^\varepsilon} |\nabla v|^p\,\mathrm{d}x\right\}, \tag{7.19}$$

where C is a constant independent of ε. Then, from (7.18), (7.19) and Hölder's inequality, we obtain

$$\begin{aligned} \Upsilon^{\varepsilon,h}[\zeta^\varepsilon] \leqslant C\mu^\varepsilon\varepsilon^\theta &\left(\sum_i \int_{K_h^z\cap\partial\mathcal{M}_i^\varepsilon} \left\{\left|\frac{\partial\vartheta^\varepsilon}{\partial\nu}\right| |\nabla\vartheta^\varepsilon|^{p-2}\right\}^{p/p-1} \mathrm{d}\sigma\right)^{(p-1)/p} \\ &\times \left(\frac{1}{d_\varepsilon}\int_{K_h^z\cap\Omega_{\mathrm{f}}^\varepsilon} |\zeta^\varepsilon|^p\,\mathrm{d}x + d_\varepsilon^{p-1}\int_{K_h^z\cap\Omega_{\mathrm{f}}^\varepsilon} |\nabla\zeta^\varepsilon|^p\,\mathrm{d}x\right)^{1/p}. \end{aligned} \tag{7.20}$$

The estimate of the first factor on the right-hand side relies on the following lemma.

LEMMA 7.2. *Let ϑ^ε be defined by (7.13), where W^ε is the solution of problem (7.9). We then have*

$$|\nabla \vartheta^\varepsilon| + \left|\frac{\partial \vartheta^\varepsilon}{\partial \nu}\right| \leqslant C\varepsilon^{-\theta/p} \quad \text{on } \partial\mathcal{M}_i^\varepsilon. \tag{7.21}$$

Proof of lemma 7.2. For simplicity of notation, we assume that $\mathcal{M}_i^\varepsilon = \{x \in \mathbb{R}^2 : 0 < x_k < (\varepsilon - d_\varepsilon)\}$. It follows from (7.9) that ϑ^ε satisfies

$$\left.\begin{aligned} \Delta_p \vartheta^\varepsilon - \tilde{\beta}^\varepsilon |\vartheta^\varepsilon|^{p-2}\vartheta^\varepsilon &= 0 \quad \text{in } \mathcal{M}_i^\varepsilon, \\ \vartheta^\varepsilon &= 1 \quad \text{on } \partial\mathcal{M}_i^\varepsilon, \end{aligned}\right\} \tag{7.22}$$

where

$$\tilde{\beta}^\varepsilon = \frac{g}{\alpha_m \varepsilon^\theta}.$$

Consider the function

$$v^\varepsilon = \exp\left\{-\left(\frac{\tilde{\beta}^\varepsilon}{p-1}\right)^{1/p} x_1\right\}. \tag{7.23}$$

It is clear that $v^\varepsilon(x)$ satisfies (7.22) and that $v^\varepsilon(x) = 1$ on the face $\{x_1 = 0\}$.

Then, according to the comparison principle for quasilinear elliptic equations (see, for example, [16]), the function $(\vartheta^\varepsilon - v^\varepsilon)(x)$ attains its positive maximum (or negative minimum) on $\partial\mathcal{M}_i^\varepsilon$. Since the function $(\vartheta^\varepsilon - v^\varepsilon)(x) = 0$ on the face $\{x_1 = 0\}$ and it is positive on the other faces of the cube $\mathcal{M}_i^\varepsilon$, we have

$$\vartheta^\varepsilon - v^\varepsilon \geqslant 0, \quad x \in \overline{\mathcal{M}}_i^{(\varepsilon)}. \tag{7.24}$$

On the other hand, $\vartheta^\varepsilon \leqslant 1$ in $\overline{\mathcal{M}}_i^\varepsilon$. It then follows from (7.24) that

$$1 - v^\varepsilon \geqslant 1 - \vartheta^\varepsilon \geqslant 0 \quad \text{in } \overline{\mathcal{M}}_i^\varepsilon \tag{7.25}$$

and, since $v^\varepsilon(0, x_2) = \vartheta^\varepsilon(0, x_2) = 1$, we get

$$\frac{v^\varepsilon(0, x_2) - v^\varepsilon(\delta, x_2)}{\delta} \geqslant \frac{\vartheta^\varepsilon(0, x_2) - \vartheta^\varepsilon(\delta, x_2)}{\delta} \geqslant 0, \quad \delta > 0. \tag{7.26}$$

Passing to the limit in (7.26) as $\delta \to 0$, we obtain

$$0 \leqslant \frac{\partial \vartheta^\varepsilon}{\partial \nu} \leqslant \frac{\partial v^\varepsilon}{\partial \nu} \quad \text{on } \{x_1 = 0\},$$

with

$$\left.\frac{\partial v^\varepsilon}{\partial \nu}\right|_{x_1=0} = \frac{g}{\alpha_m(p-1)}\varepsilon^{-\theta/p}.$$

Moreover, since

$$\frac{\partial \vartheta^\varepsilon}{\partial x_2} = 0 \quad \text{on } \{x_1 = 0\},$$

we have

$$|\nabla \vartheta^\varepsilon||_{x_1=0} \leqslant \frac{g}{\alpha_m(p-1)}\varepsilon^{-\theta/p}.$$

Clearly, the other faces of $\mathcal{M}_i^\varepsilon$ can be treated in the same way. Lemma 7.2 is proved. □

It now follows from (7.21) that

$$\sum_i \int_{K_h^z \cap \partial \mathcal{M}_i^\varepsilon} \left\{ \left| \frac{\partial \vartheta^\varepsilon}{\partial \nu} \right| |\nabla \vartheta^\varepsilon|^{p-2} \right\}^{p/p-1} \mathrm{d}\sigma \leqslant C \frac{h^2}{\varepsilon^2} \varepsilon [(\varepsilon^{-\theta/p})^{p-1}]^{p/(p-1)} = C \frac{h^2}{\varepsilon^{\theta+1}}. \tag{7.27}$$

Then it is easy to show that

$$\Upsilon^{\varepsilon,h}[\zeta^\varepsilon] \leqslant C h^{3+(\gamma-2)/p} (\Upsilon^{\varepsilon,h}[\zeta^\varepsilon])^{1/p}$$

for ε sufficiently small. Therefore, the function ζ^ε gives a vanishing contribution in the functional $b^{\varepsilon,h}(z)$, namely

$$\limsup_{\varepsilon \to 0} \Upsilon^{\varepsilon,h}[\zeta^\varepsilon] = o(h^2) \tag{7.28}$$

as $h \to 0$. This yields

$$b^{\varepsilon,h}(z) = \mu^\varepsilon \int_{K_h^z} \{a^\varepsilon(x) |\nabla \vartheta^\varepsilon|^p + g \mathbf{1}_m^\varepsilon |\vartheta^\varepsilon|^p\} \, \mathrm{d}x + o(h^2) \tag{7.29}$$

as $h \to 0$ for sufficiently small ε. Thus, by (7.13) and (7.29) we obtain

$$\begin{aligned} \lim_{h \to 0} h^{-2} \limsup_{\varepsilon \to 0} b^{\varepsilon,h}(z) &= \lim_{h \to 0} h^{-2} \liminf_{\varepsilon \to 0} b^{\varepsilon,h}(z) \\ &= \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} W^\varepsilon(x) |W^\varepsilon(x)|^{p-2} \, \mathrm{d}x, \end{aligned} \tag{7.30}$$

provided that the last limit exists.

Now it remains to obtain an asymptotic formula for the integral in (7.30). Let

$$U^\varepsilon(x) = \sum_{j=1}^{2} \{V_{\varepsilon j}^+(x) + V_{\varepsilon j}^-(x)\} \tag{7.31}$$

with

$$V_{\varepsilon j}^{\pm}(x) = \exp\left\{ \pm \left(\frac{\beta^\varepsilon}{p-1} \right)^{1/p} (x_j \mp \tfrac{1}{2}) \right\}, \quad j = 1, 2, \tag{7.32}$$

and β^ε defined in (7.10). Following the arguments from [25, lemma 7.2], we can show that

$$\begin{aligned} \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} W^\varepsilon(x) |W^\varepsilon(x)|^{p-2} \, \mathrm{d}x &= \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} (U^\varepsilon(x))^{p-1} \, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} 4 \mu^\varepsilon g \int_{\mathcal{M}} (V_{\varepsilon 1}^+(x))^{p-1} \, \mathrm{d}x. \end{aligned} \tag{7.33}$$

After straightforward computation we have

$$4 \mu^\varepsilon g \int_{\mathcal{M}} (V_{\varepsilon 1}^+(x))^{p-1} \, \mathrm{d}x = 4 \frac{\varepsilon}{2 d \varepsilon^{\theta/p}} \frac{\varepsilon^{\theta/p}}{\varepsilon} (\alpha_m)^{1/p} \left(\frac{g}{p-1} \right)^{(p-1)/p} + o(1), \quad \varepsilon \to 0.$$

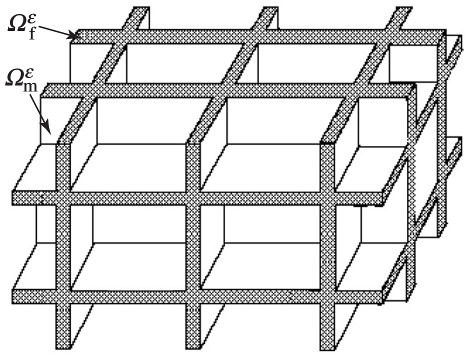

Figure 1. A three-dimensional example of the microstructure of the domain Ω.

Thus,

$$b(z) = b = \lim_{\varepsilon \to 0} \mu^{\varepsilon} g \int_{\mathcal{M}} W^{\varepsilon}(x)|W^{\varepsilon}(x)|^{p-2}\,\mathrm{d}x = \frac{2(\alpha_m)^{1/p}}{d}\left(\frac{g}{p-1}\right)^{(p-1)/p}. \tag{7.34}$$

Moreover, since the solution u of (7.4) is a smooth function in Ω, it follows from remark 3.2 that

$$\lim_{\varepsilon \to 0} \frac{1}{\operatorname{meas} \Omega_{\mathrm{f}}^{\varepsilon}} \|u^{\varepsilon} - u\|^{p}_{\Omega_{\mathrm{f}}^{\varepsilon}} = 0.$$

Thus, theorem 7.1 is proved.

REMARK 7.3. Let us notice that if the thickness $d_{\varepsilon} \gg \varepsilon^{\theta/p}$, then $\{u^{\varepsilon}\}$, the sequence of solutions of problem (7.1), $D^{p}_{\Omega_{\mathrm{f}}^{\varepsilon}}$-converges to u the solution of the following problem:

$$\int_{\Omega} \{\tfrac{1}{2}\alpha_{\mathrm{f}}(|u_{x_1}|^p + |u_{x_2}|^p) + g|u|^p - pS(x)u\}\,\mathrm{d}x \to \inf, \quad u \in W^{1,p}(\Omega). \tag{7.35}$$

This case corresponds to a model where the process is governed only by the fissures system.

7.2. Three-dimensional periodic example

Let $\Omega = \Omega_{\mathrm{f}}^{\varepsilon} \cup \overline{\Omega}_{\mathrm{m}}^{\varepsilon}$ be a bounded domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$. Following [13], we assume that the fissures system $\Omega_{\mathrm{f}}^{\varepsilon}$, i.e. the highly permeable material, is distributed in thin orthogonal layers of thickness $d_{\varepsilon} = d\varepsilon^{\theta/p}$, ($d > 0$, $\theta > p \geqslant 2$) and the matrix part $\Omega_{\mathrm{m}}^{\varepsilon}$ is made of low-permeability cubic porous blocks $\mathcal{M}_i^{\varepsilon}$ centred at $x^{i,\varepsilon} \in \Omega$. The centres $x^{i,\varepsilon}$ are periodically distributed in Ω, with period ε (see figure 1).

Consider the variational problem (7.1), (7.2). The homogenization result for this example is as follows.

THEOREM 7.4. *Let $\{u^{\varepsilon}\}$ be the sequence of solutions of problem (7.1), (7.2). Then $\{u^{\varepsilon}\}$ $D^{p}_{\Omega_{\mathrm{f}}^{\varepsilon}}$-converges to the solution u of the problem:*

$$\int_{\Omega} \{\mathsf{A}(\nabla u) + B|u|^p - pS(x)u\}\,\mathrm{d}x \to \inf, \quad u \in W^{1,p}(\Omega),$$

where

$$\mathsf{A}(\nabla u) = \tfrac{1}{3}\alpha_{\mathrm{f}}(|(u_{x_1})^2 + (u_{x_2})^2|^{p/2} + |(u_{x_1})^2 + (u_{x_3})^2|^{p/2} + |(u_{x_2})^2 + (u_{x_3})^2|^{p/2})$$

and B is given by (7.5).

The proof of theorem 7.4 is similar to that of theorem 7.1.

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