Russian Math. Surveys **57**:4 729–751

Uspekhi Mat. Nauk **57**:4 95–118
DOI 10.1070/RM2002v057n04ABEH000535

Homogenization of a random non-stationary convection-diffusion problem

M. L. Kleptsyna and A. L. Pyatnitskii [Piatnitskii]

Abstract. The homogenization problem is studied for a non-stationary convection-diffusion equation with rapidly oscillating coefficients periodic in the spatial variables and stationary random in the time. Under the assumption that the coefficients of the equation have rather good mixing properties, it is shown that, in properly chosen moving coordinates, the distribution of the solution of the original problem converges to the solution of the limit stochastic partial differential equation. The homogenized problem is well-posed and determines the limit measure uniquely.

Contents

§ 1. Introduction

We study the homogenization problem for a non-stationary convection-diffusion equation with rapidly oscillating coefficients that are random in the time and periodic in the spatial variables. We assume that the dependence of the coefficients on the time reduces to a dependence on a stationary random process $\xi_\cdot$ with values in $\mathbb{R}^d$, $d \geqslant 1$. The corresponding Cauchy problem becomes

$$\frac{\partial}{\partial t}u^\varepsilon - \frac{\partial}{\partial x_i}a_{ij}\left(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\right)\frac{\partial}{\partial x_j}u^\varepsilon - \frac{1}{\varepsilon}b_i\left(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\right)\frac{\partial}{\partial x_i}u^\varepsilon = 0, \qquad (1)$$
$$u^\varepsilon(x,0) = u_0(x),$$

and we investigate the limit behaviour of the solutions u^ε as $\varepsilon \to 0$.

This work was partially supported by INTAS under grant no. 99-559 and by RFBR under grant no. 00-01-22000.

AMS 2000 *Mathematics Subject Classification.* Primary 35B27, 35R60, 35B40; Secondary 60J60, 37A25, 60H15.

Similar problems for parabolic equations with symmetric elliptic part and with diffusion process $\xi.$ were treated earlier in [7] and [13]. In [13] it was shown that the 'classical' homogenization result holds for operators in divergence form, that is, the family of solutions of the Cauchy problem or of an initial-value boundary-value problem for the original equation converges almost surely (a.s.) as $\varepsilon \to 0$ to a solution of the corresponding Cauchy problem or initial-value boundary-value problem for the homogenized parabolic equation with constant non-random coefficients.

The picture of the limit behaviour of the solutions changes substantially in the presence of an increasing zero-order term; see [7]. In this case the limit dynamics remains random in general, and the homogenization result holds in a weaker form. Namely, the family of probability measures generated by the distributions of the solutions of the original problem in an appropriate function space is weakly convergent as $\varepsilon \to 0$ to a measure solving the limit martingale problem.

Homogenization problems for diverse elliptic and stationary parabolic operators with lower-order terms were studied in [14], [15], [24], [19], [21]. Special attention has been given in the literature to operators with incompressible convection terms; see [2], [3], [9], [10], [25].

Non-stationary parabolic equations of convection-diffusion type whose coefficients are periodic with respect to both the spatial variables and the time were treated in [11], where the homogenization result was obtained under the assumption that the oscillation in the time is 'slower' than that in the spatial variables.

The basic concepts of homogenization theory can be found, for instance, in the books [4] and [12].

As in the deterministic case, when studying the problem (1), one must take into account the convection term of order $1/\varepsilon$ in the asymptotics of u^ε. In this connection, the result on homogenization of the problem (1) is obtained below in moving coordinates $x' = x - \bar{b}t/\varepsilon$ with some constant vector $\bar{b}$. This change of variables enables us to avoid the growing velocity field (of order $1/\varepsilon$) in the effective dynamics.

We show here that the family of measures defined by the distribution laws of the solutions of the problem (1) is weakly convergent as $\varepsilon \to 0$ to a solution of the limit martingale problem in the energy function space. The diffusion arising in the limit equation is finite-dimensional and has a coefficient of the form $\Lambda\nabla u$, where $\Lambda = \{\Lambda^{ij}\}$ is a constant matrix, and the drift operator is a second-order elliptic operator with constant coefficients. The matrix Λ is of special interest because, if $\xi.$ is a diffusion process, then this matrix cannot be constructed by solving the ordinary 'local' problem on a cell with the help of the generator of the process: its construction requires a more delicate analysis.

In this paper we construct several correctors that are solutions of auxiliary partial differential equations, prove several *a priori* estimates, and then combine this technique with some ideas developed in [23] and [6]. It should be noted that some correctors constructed here depend not only on the value of the process $\xi.$ at the current time but also on the entire 'future' of the process. This differs substantially from the approaches used in [7], [13], and [6], where the diffusion nature of the correctors was used.

In § 2 we pose the problem and formulate the conditions on the coefficients and on the process. The objective of § 3 is to prove the tightness of the family of distributions for the solutions of the original problem. In § 4 we pass to the limit, construct the coefficients of the limit martingale problem, and then, using the uniqueness of the solution of the limit problem, prove the convergence of the distributions of u^ε. The last section is devoted to a special case in which $\xi.$ is a diffusion process.

§ 2. Setting of the problem

We study the asymptotic behaviour of solutions of the Cauchy problem

$$\frac{\partial}{\partial t}u^\varepsilon(x,t)-\frac{\partial}{\partial x_i}a_{ij}\left(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\right)\frac{\partial}{\partial x_j}u^\varepsilon(x,t)-\frac{1}{\varepsilon}b_i\left(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\right)\frac{\partial}{\partial x_i}u^\varepsilon(x,t)=0, \quad (2)$$
$$u^\varepsilon(x,0)=u_0(x),$$

for small $\varepsilon>0$; here $\xi.$ stands for an ergodic stationary random process defined on a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathbb{R}^d$. The symbol E always stands for the expectation. If $\xi.$ is a diffusion process, then we denote by L its infinitesimal generator,

$$L=q_{ij}(y)\frac{\partial}{\partial y_i}\,\frac{\partial}{\partial y_j}+B_i(y)\frac{\partial}{\partial y_i}\,.$$

Further conditions on the process $\xi.$ are given below in terms of strong or uniform mixing coefficients or in terms of the maximum correlation coefficient. We present here the corresponding definitions for the convenience of the reader.

Let $\mathcal{F}_{\leqslant t}$ and $\mathcal{F}_{\geqslant t}$ be the σ-algebras $\sigma(\xi_s,\ s\leqslant t)$ and $\sigma(\xi_s,\ s\geqslant t)$, respectively. The function $\alpha(s)$, $s\geqslant 0$, given by

$$\alpha(s)=\sup_{\mathcal{B}_1\in\mathcal{F}_{\leqslant 0},\,\mathcal{B}_2\in\mathcal{F}_{\geqslant s}}|\mathsf{P}(\mathcal{B}_1\cup\mathcal{B}_2)-\mathsf{P}(\mathcal{B}_1)\mathsf{P}(\mathcal{B}_2)|$$

is called the *strong mixing coefficient* of the process $\xi.$. The function $\phi(s)$, $s\geqslant 0$, given by

$$\phi(s)=\sup_{\mathcal{B}_1\in\mathcal{F}_{\leqslant 0},\,\mathcal{B}_2\in\mathcal{F}_{\geqslant s}}|\mathsf{P}(\mathcal{B}_1\mid\mathcal{B}_2)-\mathsf{P}(\mathcal{B}_1)|$$

is called the *uniform mixing coefficient* of $\xi.$. Finally, the *maximum correlation coefficient* $\rho(s)$, $s\geqslant 0$, is defined by

$$\rho(s)=\sup\frac{|\operatorname{cov}(\eta_1,\eta_2)|}{\sqrt{\mathsf{E}(\eta_1-\mathsf{E}(\eta_1))^2\mathsf{E}(\eta_2-\mathsf{E}(\eta_2))^2}}\,,$$

where the supremum is taken over all $\mathcal{F}_{\leqslant 0}$-measurable functions η_1 and all $\mathcal{F}_{\geqslant s}$-measurable functions η_2 such that $\mathsf{E}(\eta_1)^2<\infty$ and $\mathsf{E}(\eta_2)^2<\infty$, and the symbol cov stands for the covariance.

In what follows we always assume that the coefficients $a_{ij}(z,y)$ and $b_i(z,y)$ are periodic with respect to the variable z. The other assumptions are formulated below.

A1. The functions $a_{ij}(z,y)$ and $b_i(z,y)$ and their first-order derivatives with respect to z and y are uniformly bounded:

$$|a_{ij}(z,y)| + |\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \leqslant c,$$
$$|b_i(z,y)| + |\nabla_z b_i(z,y)| + |\nabla_y b_i(z,y)| \leqslant c;$$

here and below, the symbol c stands for positive constants.

A2. The matrix $a_{ij}(z,y)$ is positive definite, that is,

$$a_{ij}\eta_i\eta_j \geqslant c|\eta|^2, \quad \eta \in \mathbb{R}^n,$$

for some $c > 0$ and for any $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$.

A3. At least one of the following relations holds:

$$\int_0^\infty (\alpha(s))^{1/2}\,ds < \infty; \quad \int_0^\infty (\phi(s))^{1/2}\,ds < \infty; \quad \int_0^\infty \rho(s)\,ds < \infty.$$

For the special case in which $\xi_\cdot$ is a diffusion process, there is a natural desire to replace the above condition **A3** by some effectively verifiable sufficient condition in terms of the coefficients of the generator. However, it turns out to be much more convenient to use the generator of the time-reversed process $\zeta_s = \xi_{-s}$ rather than that of the original process ξ_s. In this connection we introduce the notation $\widetilde{L}$ for the generator of ζ_s,

$$\widetilde{L} = \widetilde{q}_{kl}(y)\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l} + \widetilde{B}_k(y)\frac{\partial}{\partial y_k},$$

and suppose that the following condition holds instead of **A3**.

A3′. The diffusion coefficients $\widetilde{q}_{kl}(y)$ and their first-order derivatives are uniformly bounded,

$$|\widetilde{q}_{kl}(y)| + |\nabla_y \widetilde{q}_{kl}(y)| \leqslant c,$$

and the operator $\widetilde{L}$ is uniformly elliptic,

$$\widetilde{q}_{kl}\eta_k\eta_l \geqslant c|\eta|^2, \qquad \eta \in \mathbb{R}^d,$$

for any $y \in \mathbb{R}^d$. The vector function $\widetilde{B}(y)$ admits the polynomial estimate

$$|\widetilde{B}(y)| + |\nabla \widetilde{B}(y)| \leqslant c(1 + |y|^\kappa)$$

for some $\kappa > 0$, and there are numbers $\mu > -1$, $R > 0$, and $c > 0$ such that

$$\frac{\widetilde{B}(y)\cdot y}{|y|} \leqslant -c|y|^\mu$$

for all y in $\{y : |y| \geqslant R\}$.

According to [20] (see also [7]), under the condition **A3′** the process $\zeta_\cdot$ (and hence the process $\xi_\cdot$ as well) admits a unique invariant measure, which has a continuous density $\rho(y)$ satisfying the equation

$$\widetilde{L}^*\rho = 0, \qquad \int_{\mathbb{R}^d} \rho(y)\,dy = 1,$$

and decaying at infinity more rapidly than any negative power of $|y|$; here the symbol $\widetilde{L}^*$ stands for the adjoint operator. Moreover, for the stationary version of the process $\zeta_\cdot$ with generator $\widetilde{L}$ the strong mixing coefficient $\alpha(s)$ decays at infinity more rapidly than any negative power of s, which implies the condition **A3**.

Further, the distribution of the stationary process $\xi_\cdot$ ($\xi_s = \zeta_{-s}$) in the space of sample paths coincides with the distribution of the stationary diffusion process with generator

$$L = (\rho(y))^{-1}\widetilde{L}^*(\rho(y)\,\cdot\,) = \widetilde{q}_{kl}(y)\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l} + \biggl((\rho(y))^{-1}\frac{\partial}{\partial y_i}[\rho(y)\widetilde{q}_{ki}(y)] - \widetilde{B}_k(y)\biggr)\frac{\partial}{\partial y_k};$$

we identify these processes in what follows.

Under the assumptions **A1**–**A3**, the problem (2) has for any initial condition $u_0 \in L^2(\mathbb{R}^n)$ and any $\varepsilon > 0$ a unique solution

$$u^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^n)) \cup C(0,T;L^2(\mathbb{R}^n))$$

almost surely. The distribution of this solution generates a Radon probability measure on the space

$$V = L^2_w(0,T;H^1(\mathbb{R}^n)) \cup C(0,T;L^2_w(\mathbb{R}^n))$$

equipped with the Borel σ-algebra; the symbol w stands here for the weak topology. This measure, defined as the distribution of u^ε in V, is denoted by Q^ε.

§ 3. Tightness

In this section we establish results on tightness of the family of measures Q^ε. Generally, this family of measures itself is not tight in V. To obtain a tight family, we introduce a moving system of coordinates $(x',t) = (x - \overline{b}/\varepsilon, t)$ with a constant vector $\overline{b}$ and show that for an appropriate choice of $\overline{b}$ the family of distributions of the functions $u^\varepsilon(x',t)$ is tight in V. This result is based on *a priori* estimates for a solution of the problem (2) and on the Prokhorov theorem (see [23], [6]).

Proposition 3.1. *The estimate*

$$\sup_{t\leqslant T}\biggl(\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\mathbb{R}^n)}\,ds\biggr) \leqslant c\|u_0\|^2_{L^2(\mathbb{R}^n)} \tag{3}$$

holds uniformly with respect to $\varepsilon > 0$. *There is a constant vector* $\overline{b}$ *such that*

$$\lim_{\nu\to 0}\sup_{\varepsilon>0}\mathsf{P}\biggl\{\sup_{|t-s|<\nu}\biggl|\biggl(u^\varepsilon(t),\varphi\biggl(\cdot+\frac{\overline{b}}{\varepsilon}t\biggr)\biggr) - \biggl(u^\varepsilon(s),\varphi\biggl(\cdot+\frac{\overline{b}}{\varepsilon}s\biggr)\biggr)\biggr| \geqslant \gamma\biggr\} = 0 \tag{4}$$

for any $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ *and any* $\gamma > 0$.

Proof. Let us consider the auxiliary problem

$$\frac{\partial}{\partial s}p_\varepsilon(z,s)+A^*p_\varepsilon(z,s)=0,\quad (z,t)\in\mathbb{T}^n\times(-\infty,(T+1)/\varepsilon^2),\qquad(5)$$
$$p_\varepsilon\big|_{s=(T+1)/\varepsilon^2}=1,$$

where
$$A^*=\frac{\partial}{\partial z_i}a^{ij}(z,\xi_s)\frac{\partial}{\partial z_j}-\frac{\partial}{\partial z_i}(b^i(z,\xi_s)\cdot).\qquad(6)$$

By the maximum principle (see [16]), the solution $p_\varepsilon(z,s)$ is strictly positive. Moreover, taking account of the structure of the equation and integrating by parts over the set $\mathbb{T}^n\times(s,(T+1)/\varepsilon^2)$, we get that

$$\int_{\mathbb{T}^n}p_\varepsilon(z,s)\,dz=1$$

for any $s\leqslant(T+1)/\varepsilon^2$. It follows from Harnack's inequality (see [22]) that

$$0<c_1\leqslant p_\varepsilon(z,s)\leqslant c_2<\infty,\qquad(7)$$

where the constants c_1 and c_2 depend neither on ε nor on the sample path of $\xi_\cdot$.

Multiplying the equation (2) by $p_\varepsilon(x/\varepsilon,s/\varepsilon^2)u^\varepsilon(x,s)$ and integrating the resulting formula over $\mathbb{R}^n\times(0,t)$, we see after simple manipulations that

$$\begin{aligned}\frac12\int_{\mathbb{R}^n}&(u^\varepsilon(x,t))^2p_\varepsilon\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)dx-\frac12\int_{\mathbb{R}^n}(u_0(x,t))^2p_\varepsilon\Big(\frac{x}{\varepsilon},0\Big)dx\\&+\int_0^t\int_{\mathbb{R}^n}a_{ij}\Big(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2}\Big)p_\varepsilon\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}u^\varepsilon(x,s)\frac{\partial}{\partial x_j}u^\varepsilon(x,s)\,dx\,ds\\&-\frac12\int_0^t\int_{\mathbb{R}^n}(u^\varepsilon)^2\Big[\frac{\partial}{\partial t}p_\varepsilon\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)+\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big(a_{ij}\Big(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2}\Big)p_\varepsilon\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\Big)\\&-\frac{\partial}{\partial x_i}\Big(b_i\Big(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2}\Big)p_\varepsilon\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\Big)\Big]\,dx\,ds=0.\end{aligned}$$

According to (5), the last integral vanishes. Therefore, by (7) we readily obtain (3).

Let us now pass to (4). The following statement is crucial here.

Lemma 1. *Let the initial condition $u_0(x)$ in (2) be of class C_0^∞. Then there is a non-random function $\kappa_1(\varepsilon)$ vanishing as $\varepsilon\downarrow0$ and such that*

$$\sup_{(x,t)\in\mathbb{R}^n\times(0,T)}\ \operatorname*{osc}_{G_x^\varepsilon}u^\varepsilon\leqslant\kappa_1(\varepsilon),\qquad(8)$$

where G_x^ε stands for the set $x+(-\varepsilon/2,\varepsilon/2)^n$.

Proof. Let us consider the auxiliary periodic Cauchy problem

$$\frac{\partial}{\partial s}v(z,s)-Av(z,s)=0,\qquad z\in\mathbb{T}^n,\quad s>s_0,$$
$$v\big|_{s=s_0}=v_0$$

with the operator

$$A = \frac{\partial}{\partial z_i} a^{ij}(z, \xi_s) \frac{\partial}{\partial z_j} + b^i(z, \xi_s) \frac{\partial}{\partial z_i}\,. \tag{9}$$

The solution $v(z, s)$ of this problem tends exponentially to a constant as $s - s_0 \to \infty$. In particular,

$$\operatorname*{osc}_{\mathbb{T}^n} v(\,\cdot\,, s_0 + \varepsilon^{-1/2}) \leqslant c' \exp(-c\varepsilon^{-1/2}) |v_0|_{L^\infty}, \tag{10}$$

where c' and c depend neither on s_0 nor on a realization of $\xi_\cdot$. Indeed, under the assumptions **A1** and **A2** the solution $v(z, s)$ satisfies the uniform Harnack inequality, which, in turn, implies the desired estimate (see, for instance, the proof of Lemma 2 below).

The operator

$$A^\varepsilon = \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\right) \frac{\partial}{\partial x_j} + \frac{1}{\varepsilon} b_i \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\right) \frac{\partial}{\partial x_i}$$

commutes with any shift of the form $S_k u(x) = u(x + \varepsilon k)$, $k \in \mathbb{Z}^n$. Therefore, by the maximum principle we get that

$$|u^\varepsilon(x + \varepsilon k, t) - u^\varepsilon(x, t)| \leqslant M\varepsilon |k|, \qquad k \in \mathbb{Z}^n, \tag{11}$$

where $M = \max_x |\nabla u_0(x)|$.

Next, let us arbitrarily choose $t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$, restrict the function $u^\varepsilon(x, t_0)$ to $G^\varepsilon_{x_0}$, and denote by $\widetilde{v}^\varepsilon_0$ the periodic extension of this restriction with period ε with respect to the coordinate directions. By (11) we have

$$|\widetilde{v}^\varepsilon_0(x) - u^\varepsilon(x, t_0)| \leqslant M\varepsilon^{1/4} \tag{12}$$

for all x such that $|x - x_0| \leqslant \varepsilon^{1/4}$.

Let $p(t, t', x, x')$ be the fundamental solution of the problem (2). According to [1], under the assumptions **A1**-**A2** the inequality

$$p(t_0, t_0 + \varepsilon^{3/2}, x, x') \leqslant c' \exp(-c|x - x'|^2 / \varepsilon^{3/2}) \tag{13}$$

holds for any x, x' such that $|x - x'| > \varepsilon^{1/4}$, where c and c' depend only on the constants in the conditions **A1**-**A2**. Integrating this inequality over the set $\{x : |x - x'| > \varepsilon^{1/4}\}$, we see that

$$\int_{|x-x'|>\varepsilon^{1/4}} p(t_0, t_0 + \varepsilon^{3/2}, x, x')\, dx \leqslant c\varepsilon. \tag{14}$$

Let $\widetilde{v}^\varepsilon(x, t)$ be the solution of the equation $\partial_t \widetilde{v}^\varepsilon - A^\varepsilon \widetilde{v}^\varepsilon = 0$ with the initial condition $\widetilde{v}^\varepsilon|_{t=t_0} = \widetilde{v}^\varepsilon_0(x)$. Then the estimate (10) gives

$$\operatorname*{osc}_{\mathbb{R}^n} \widetilde{v}^\varepsilon(\,\cdot\,, t_0 + \varepsilon^{3/2}) \leqslant c' \exp(-c\varepsilon^{-1/2}) |\widetilde{v}_0|_{L^\infty}.$$

On the other hand, it follows from (12) and (14) that

$$|u^\varepsilon(x, t_0 + \varepsilon^{3/2}) - \widetilde{v}^\varepsilon(x, t_0 + \varepsilon^{3/2})| \leqslant c\varepsilon |u_0|_{L^\infty} + cM\varepsilon^{1/4}$$

for any $x \in G^{\varepsilon}_{x_0}$. The last two inequalities imply the desired statement for any $t \geqslant \varepsilon^{3/2}$. For small t this statement is a trivial consequence of (14), which completes the proof of the lemma.

In what follows, we need a sequence of problems

$$\begin{gathered}\frac{\partial}{\partial s}p^N(z,s) + A^* p^N(z,s) = 0, \quad (z,s) \in \mathbb{T}^n \times (-\infty, N),\\ p^N(z,N) = 1,\end{gathered} \tag{15}$$

with the operator A^* given by (6). By the above arguments one can readily show that the functions p^N satisfy the estimate (7) uniformly with respect to N and s, $-\infty < s \leqslant N$.

Lemma 2. *The sequence p^N converges as $N \to \infty$ to a stationary ergodic process with values in $C(\mathbb{T}^n)$, and the realizations of this process satisfy the equation*

$$\frac{\partial}{\partial s}p + A^* p = 0, \quad \int_{\mathbb{T}^n} p(z,s)\,dz = 1, \qquad s \in (-\infty, +\infty). \tag{16}$$

Moreover, there are non-random constants $c > 0$ and $c_1 > 0$ such that

$$\max_{\substack{z\in\mathbb{T}^n\\ k\leqslant s\leqslant k+1}} |p^N(z,s) - p(z,s)| \leqslant c_1 \exp(-c(N-k)). \tag{17}$$

Proof. To prove the convergence and the estimate (17), we consider the following problem:

$$\frac{\partial}{\partial s}q + A^* q = 0, \qquad q\big|_{s=N} = q_0, \tag{18}$$

in which $q_0 \in L^2(\mathbb{T}^n)$ satisfies the equation

$$\int_{\mathbb{T}^n} q_0(z)\,dz = 0. \tag{19}$$

Let us first show that, for our purposes, it suffices to verify the exponential decay of a solution of the problem (18) as $N - s \to \infty$, that is, to obtain the estimate

$$|q(z,s)| \leqslant c_1 \|q_0\|_{L^2(\mathbb{T}^n)} \exp(-c(N-s)). \tag{20}$$

Indeed, for any $N > 0$ and $k > 0$ the difference $p^{N+k} - p^N$ is a solution of the equation

$$\frac{\partial}{\partial s}(p^{N+k} - p^N) + A^*(p^{N+k} - p^N) = 0, \qquad s < N,$$

and for any $s \leqslant N$ we have

$$\int_{\mathbb{T}^n} (p^{N+k}(z,s) - p^N(z,s))\,dz = 0, \qquad \|p^{N+k} - p^N\|_{L^\infty} \leqslant c.$$

Therefore, assuming the validity of the estimate (20), we get that

$$\|p^{N+k} - p^N\|_{C(\mathbb{T}^n\times[k,k+1])} \leqslant c_1 \exp(-c(N-k)).$$

By the Cauchy criterion, the sequence p^N converges as $N\to\infty$ to a limit continuous function, which we denote by $p(z,s)$, and this function satisfies the inequality (17). Passing to the limit in (15), we obtain (16).

To prove the estimate (20), we consider the conjugate problem

$$\begin{gathered}\frac{\partial}{\partial s}\nu(z,s) - A\nu(z,s) = 0, \quad (z,s)\in\mathbb{T}^n\times(s_0,+\infty),\\ \nu(z,s_0) = \varphi(z),\end{gathered} \tag{21}$$

where φ stands for an arbitrary $L^2(\mathbb{T}^n)$-function. Multiplying this equation by $q(z,s)$, integrating over $\mathbb{T}^n\times[s_0,N]$, and integrating by parts several times, we get that

$$\int_{\mathbb{T}^n} \nu(z,N)q_0(z)\,dz = \int_{\mathbb{T}^n} \varphi(z)q(z,s_0)\,dz. \tag{22}$$

By the Nash inequality (see [16]),

$$\max_{z\in\mathbb{T}^n} |\nu(z,s_0+1)| \leqslant c\|\varphi\|_{L^2(\mathbb{T}^n)}. \tag{23}$$

Let $\nu^+(z,s_1)$ and $\nu^-(z,s_1)$ be the positive and negative parts of $\nu(z,s_1)$, respectively:

$$\nu^+(z,s_1) = \max(\nu(z,s_1),0); \qquad \nu^-(z,s_1) = -\min(\nu(z,s_1),0).$$

Subtracting an appropriate constant if necessary, we can assume without loss of generality that

$$\max_{z\in\mathbb{T}^n}\nu(z,s_1) = -\min_{z\in\mathbb{T}^n}\nu(z,s_1),$$

and hence $\|\nu^+(\,\cdot\,,s_1)\|_{L^\infty} = \|\nu^-(\,\cdot\,,s_1)\|_{L^\infty}$. Applying Harnack's inequality to the solution to the problem

$$\begin{gathered}\frac{\partial}{\partial s}\nu^1(z,s) - A\nu^1(z,s) = 0, \quad (z,s)\in\mathbb{T}^n\times(s_1,+\infty),\\ \nu^1(z,s_1) = \nu^+(z,s_1),\end{gathered}$$

we obtain the estimate

$$\max_{z\in\mathbb{T}^n}\nu^1(z,s_1+1) \leqslant c_2 \min_{z\in\mathbb{T}^n}\nu^1(z,s_1+1)$$

with a constant c_2 depending only on the constants in the conditions **A1** and **A2**. Combining this with the obvious estimate

$$0 \leqslant \nu^1 \leqslant \|\nu^+(\,\cdot\,,s_1)\|_{L^\infty},$$

we see that

$$0 < c_3 \leqslant \nu^1(z, s_1 + 1) \leqslant \|\nu^+(\,\cdot\,, s_1)\|_{L^\infty}.$$

Similarly, if ν^2 is the solution of the problem

$$\frac{\partial}{\partial s}\nu^2(z,s) - A\nu^2(z,s) = 0, \qquad \nu^2(z,s_1) = \nu^-(z,s_1),$$

then it admits an estimate of the form

$$0 < c_3 \leqslant \nu^2(z, s_1 + 1) \leqslant \|\nu^-(\,\cdot\,, s_1)\|_{L^\infty}.$$

By the last two inequalities, it follows that there is a constant $c_4 > 0$ such that

$$\operatorname*{osc}_{\mathbb{T}^n} \nu(\,\cdot\,, s+1) \leqslant (1 - c_4) \operatorname*{osc}_{\mathbb{T}^n} \nu(\,\cdot\,, s) \tag{24}$$

for any $s > s_0 + 1$. By (23), this implies that

$$\operatorname*{osc}_{\mathbb{T}^n} \nu(\,\cdot\,, s) \leqslant c_1 \exp(-c(s - s_0))\|\varphi\|_{L^2(\mathbb{T}^n)}$$

for any $s \geqslant s_0 + 1$. Finally, it follows from (22), (19), and the last estimate that

$$\left|\int_{\mathbb{T}^n} \varphi(z) q(z, s_0)\, dz\right| = \left|\int_{\mathbb{T}^n} \nu(z, N) q_0(z)\, dz\right| \leqslant c_1 \exp(-c(N - s_0))\|q_0\|_{L^2(\mathbb{T}^n)}\|\varphi\|_{L^2(\mathbb{T}^n)}$$

for any $\varphi \in L^2(\mathbb{T}^n)$. Therefore,

$$\|q(\,\cdot\,, s_0)\|_{L^2(\mathbb{T}^n)} \leqslant c_1 \exp(-c(N - s_0))\|q_0\|_{L^2(\mathbb{T}^n)}$$

and

$$|q(z, s_0)| \leqslant c_1 \exp(-c(N - s_0))\|q_0\|_{L^2(\mathbb{T}^n)},$$

where the Nash estimate was again used in the proof of the latter estimate.

Let us consider the functions $\widetilde{p}^N(z,s) = p^{N+s}(z,s)$, where p^N is the solution of the problem (15). For any $N > 0$ the function $\widetilde{p}^N$ is defined for any $s \in \mathbb{R}$, and it is a stationary ergodic process with values in $C(\mathbb{T}^n)$. As proved above, the sequence $\widetilde{p}^N(z,s)$ converges exponentially to $p(z,s)$ as $N \to \infty$. Hence, $p(\,\cdot\,, s)$ is also a stationary ergodic process. This completes the proof of the lemma.

We now introduce the constant vector

$$\bar{b}^i = \mathsf{E}\int_{\mathbb{T}^n} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s)\right) p(z,s)\, dz, \tag{25}$$

and the following stationary random process with values in $\mathbb{R}^n$:

$$\eta^i(s) = \int_{\mathbb{T}^n} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s) - \bar{b}^i\right) p(z,s)\, dz. \tag{26}$$

All processes used in the definition of the vector $\bar{b}$ are stationary and ergodic. Thus, by Birkhoff's theorem we have

$$\bar{b}^i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{T}^n} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s)\right) p(z,s)\, dz\, ds \tag{27}$$

for almost all $\omega \in \Omega$.

Lemma 3. *The process $\eta(s)$ satisfies the functional central limit theorem (the invariance principle) with the covariance matrix*

$$\{\sigma^2\}^{ij} = \int_0^\infty \mathsf{E}\big(\eta^i(s)\eta^j(0) + \eta^i(0)\eta^j(s)\big)\, ds.$$

That is,

$$\varepsilon \int_0^{\cdot/\varepsilon^2} \eta(s)\, ds \xrightarrow[\varepsilon\to 0]{\mathcal{L}} \sigma w.$$

in $(C[0,T])^n$, where $w.$ is the standard n-dimensional Brownian motion.

Proof. Let us first prove that the inequality

$$\|\mathsf{E}\{\eta_0 \mid \mathcal{F}_{\geqslant T}\}\|_{L^2(\Omega)} \leqslant c_1\big(\exp(-cT) + \rho(T/2)\big) \tag{28}$$

holds for any $T > 0$ with some constants $c > 0$ and $c_1 > 0$. To this end, we represent the function $p(z,s)$ on the interval $0 \leqslant s \leqslant T/2$ as a sum $p(z,s) = p^1(z,s)+p^2(z,s)$, where the summands p^1 and p^2 satisfy the equation

$$\frac{\partial}{\partial s} p^i(z,s) + A^* p^i(z,s) = 0, \qquad s < T/2,$$

with the initial conditions $p^1|_{s=T/2} = 1$ and $p^2|_{s=T/2} = p(z,T/2) - 1$, respectively. Then we have $\eta(0) = \eta^1(0) + \eta^2(0)$, where

$$\eta^{m,i}(0) = \int_{\mathbb{T}^n} \bigg(\frac{\partial}{\partial z_j} a^{ij}(z,\xi_0) + b^i(z,\xi_0)\bigg) p^m(z,0)\, dz, \qquad m = 1,2.$$

It follows from the definition that $p^1(0)$ is measurable with respect to $\mathcal{F}_{\leqslant T/2}$, and hence so is $\eta^1(0)$. By the mixing condition **A3** we get that

$$\|\mathsf{E}\{\eta^1(0)|\mathcal{F}_{\geqslant T}\}\|_{L^2(\Omega)} \leqslant \rho(T/2)\|\eta^1(0)\|_{L^2(\Omega)} \leqslant C\rho(T/2). \tag{29}$$

It follows from (17) that

$$|p^2(z,0)| = |p(z,0) - p^{T/2}(z,0)| \leqslant c_1 \exp(-cT/2).$$

Hence,

$$\|\mathsf{E}\{\eta^2(0)|\mathcal{F}_{\geqslant T}\}\|_{L^2(\Omega)} \leqslant \|\eta^2(0)\|_{L^2(\Omega)} \leqslant C\exp(-cT/2),$$

and this inequality, together with (29), gives the desired estimate (28).

According to [18], Chapter 9, under the assumption **A3** the inequality (28) ensures the functional central limit theorem (CLT) for the process $\eta(-s)$, which in turn implies the functional CLT for $\eta(s)$. Indeed, by the Prokhorov theorem (see [5]), the family $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(s)\, ds\right\}$ is tight in $(C([0,T])^n$ if the family $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(-s)\, ds\right\}$ is tight and the process is stationary. The convergence of the finite-dimensional distributions of the process $\left\{\varepsilon\int_0^{t/\varepsilon^2}\eta(s)\,ds\right\}$ is obviously equivalent to the convergence of the corresponding finite-dimensional distributions of $\left\{\varepsilon\int_0^{t/\varepsilon^2}\eta(-s)\,ds\right\}$.

Remark. We have proved the assertion of Lemma 3 under the assumption that the last inequality in **A3** holds. The other two cases (of strong or uniform mixing) can be treated similarly.

Let us consider another auxiliary problem,

$$\frac{\partial}{\partial s}\psi^i(z,s)+A^*\psi^i(z,s)=-\frac{\partial}{\partial z_j}(a^{ij}(z,\xi_s)p(z,s))-a^{ij}(z,\xi_s)\frac{\partial}{\partial z_j}p(z,s)$$
$$+b^i(z,\xi_s)p(z,s)-\bar{b}^i p(z,s)-\eta^i(s)p(z,s),\quad (z,s)\in\mathbb{T}^n\times(-\infty,+\infty),\tag{30}$$

where A^* is given by (6).

Lemma 4. *The problem* (30) *has a stationary ergodic solution. Under the normalization condition*

$$\int_{\mathbb{T}^n}\psi(z,s)\,dz=0$$

this solution is unique.

Proof. For brevity, we denote the right-hand side of (30) by $F(z,s)$. Let us consider the following sequence of Cauchy problems:

$$\frac{\partial}{\partial s}\psi^N(z,s)+A^*\psi^N(z,s)=\mathbf{1}_{\{N-1<s\leqslant N\}}F(z,s),\quad (z,s)\in\mathbb{T}^n\times(-\infty,N),$$
$$\psi^N\big|_{s=N}=0,$$

where $\mathbf{1}_{\{N-1<s\leqslant N\}}$ stands for the indicator function of the interval $(N-1,N)$. Taking into account the equality $\int_{\mathbb{T}^n}F(z,s)dz\equiv 0$, we can readily see that

$$\int_{\mathbb{T}^n}\psi^N(z,s)\,dz=0$$

for any s, where we assume for convenience that $\psi^N=0$ for $s>N$. By (3.20) we obtain the estimate

$$\|\psi^N\|_{L^\infty(\mathbb{T}^n\times(k,k+1))}\leqslant c_1\exp(-c(N-k))$$

for all $k\leqslant N$. Summing the functions over N, $-\infty<N<+\infty$, we obtain the stationary solution $\psi=\sum_{N=-\infty}^{+\infty}\psi^N$ of the problem (30). Moreover,

$$\|\psi\|_{L^\infty(\mathbb{T}^n\times(-\infty,+\infty))}\leqslant C\tag{31}$$

for some non-random constant C. The uniqueness and ergodicity can now be proved in the same way as in Lemma 2.

To complete the proof of the tightness of the family of distributions of $\{u^\varepsilon\}$, we consider the expression

$$(\widetilde{u}^\varepsilon, \widetilde{p}^\varepsilon \varphi) + \varepsilon(\widetilde{u}^\varepsilon, \widetilde{\psi}^\varepsilon \nabla_x \varphi)$$

for an arbitrary test function $\varphi \in C_0^\infty(\mathbb{R}^n)$, where

$$\begin{aligned}
\widetilde{u}^\varepsilon(x,t) &= u^\varepsilon\biggl(x - \frac{1}{\varepsilon}\bar{b}t - \frac{1}{\varepsilon}\int_0^t \eta\Bigl(\frac{s}{\varepsilon^2}\Bigr)\,ds, t\biggr),\\
\widetilde{p}^\varepsilon(x,t) &= p\biggl(\frac{x}{\varepsilon} - \varepsilon^{-2}\bar{b}t - \varepsilon^{-2}\int_0^t \eta\Bigl(\frac{s}{\varepsilon^2}\Bigr)ds, \frac{t}{\varepsilon^2}\biggr),\\
\widetilde{\psi}^\varepsilon(x,t) &= \psi\biggl(\frac{x}{\varepsilon} - \varepsilon^{-2}\bar{b}t - \varepsilon^{-2}\int_0^t \eta\Bigl(\frac{s}{\varepsilon^2}\Bigr)ds, \frac{t}{\varepsilon^2}\biggr).
\end{aligned}$$

We also use the notation

$$\widetilde{\varphi} = \varphi\biggl(x + \frac{1}{\varepsilon}\bar{b}t + \frac{1}{\varepsilon}\int_0^t \eta\Bigl(\frac{s}{\varepsilon^2}\Bigr)ds\biggr) \tag{32}$$

and write

$$\nabla_x^\varepsilon r(x) = \nabla_z r(z)\big|_{z=x/\varepsilon}, \qquad \partial_t^\varepsilon r(t) = \frac{\partial}{\partial s} r(s)\Big|_{s=t/\varepsilon^2}$$

for an arbitrary function r. Using the equations (16) and (30), integrating by parts, and making simple manipulations, we get that

$$\begin{aligned}
\frac{d}{dt}\bigl[(\widetilde{u}^\varepsilon, \widetilde{p}^\varepsilon\varphi) + \varepsilon(\widetilde{u}^\varepsilon, \widetilde{\psi}^\varepsilon \nabla_x\varphi)\bigr] &= \frac{d}{dt}\bigl[(u^\varepsilon, p^\varepsilon\widetilde{\varphi}) + \varepsilon(u^\varepsilon, \psi^\varepsilon\nabla_x\widetilde{\varphi})\bigr]\\
= (A^\varepsilon u^\varepsilon, p^\varepsilon\widetilde{\varphi}) &+ \Bigl(u^\varepsilon, \widetilde{\varphi}\frac{\partial}{\partial t}p^\varepsilon\Bigr) + \frac{1}{\varepsilon}(\bar{b} + \eta(t/\varepsilon^2))\cdot(u^\varepsilon, p^\varepsilon\nabla_x\widetilde{\varphi})\\
&+ \varepsilon(A^\varepsilon u^\varepsilon, \psi^\varepsilon\nabla_x\widetilde{\varphi}) + \varepsilon\Bigl(u^\varepsilon, \nabla_x\widetilde{\varphi}\frac{\partial}{\partial t}\psi^\varepsilon\Bigr) + (\bar{b} + \eta(t/\varepsilon^2))\cdot(u^\varepsilon, \psi^\varepsilon\nabla_x\nabla_x\widetilde{\varphi})\\
= (u^\varepsilon, p^\varepsilon a^\varepsilon\nabla_x\nabla_x\widetilde{\varphi}) &+ \bigl(u^\varepsilon\nabla_x\nabla_x\widetilde{\varphi}, [\nabla_x^\varepsilon(a\psi) + a^\varepsilon\nabla_x^\varepsilon\psi - b^\varepsilon\psi^\varepsilon + \bar{b}\psi^\varepsilon + \eta^\varepsilon\psi^\varepsilon]\bigr)\\
&+ \varepsilon(u^\varepsilon\psi^\varepsilon, a^\varepsilon\nabla_x\nabla_x\nabla_x\widetilde{\varphi}).
\end{aligned} \tag{33}$$

By (3), this implies the inequality

$$|(\widetilde{u}^\varepsilon(t), \widetilde{p}^\varepsilon(t)\varphi) - (\widetilde{u}^\varepsilon(s), \widetilde{p}^\varepsilon(s)\varphi)| \leqslant c|t-s|\,\|\varphi\|_{C^3}. \tag{34}$$

Approximating the initial condition $u_0 \in L^2(\mathbb{R}^n)$ in the problem (1) by a sequence $u_0^N \in C_0^\infty(\mathbb{R}^n)$ if necessary and using (3), we can always assume that $u_0 \in C_0^\infty(\mathbb{R}^n)$. Then by (34) and Lemma 1, we get that

$$|(\widetilde{u}^\varepsilon(t), \varphi) - (\widetilde{u}^\varepsilon(s), \varphi)| \leqslant c|t-s|\,\|\varphi\|_{C^3} + c\kappa_1(\varepsilon)\|\varphi\|_{C^1}. \tag{35}$$

Writing

$$\widehat{u}^\varepsilon(x,t) = u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t), \tag{36}$$

we arrive at the relation

$$
\begin{aligned}
&|(\widehat{u}^\varepsilon(t),\varphi)-(\widehat{u}^\varepsilon(s),\varphi)|\\
&\quad=\biggl|\biggl(\widetilde{u}^\varepsilon(t),\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^t\eta(\tau/\varepsilon^2)\,d\tau\biggr)\biggr)-\biggl(\widetilde{u}^\varepsilon(s),\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^s\eta(\tau/\varepsilon^2)\,d\tau\biggr)\biggr)\biggr|\\
&\quad\leqslant\biggl|\biggl(\widetilde{u}^\varepsilon(t),\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^s\eta(\tau/\varepsilon^2)\,d\tau\biggr)\biggr)-\biggl(\widetilde{u}^\varepsilon(s),\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^s\eta(\tau/\varepsilon^2)\,d\tau\biggr)\biggr)\biggr|\\
&\qquad+\biggl|\biggl(\widetilde{u}^\varepsilon(t),\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^t\eta(\tau/\varepsilon^2)\,d\tau\biggr)-\varphi\biggl(\cdot-\frac{1}{\varepsilon}\int_0^s\eta(\tau/\varepsilon^2)\,d\tau\biggr)\biggr)\biggr|\\
&\quad\leqslant c|t-s|\,\|\varphi\|_{C^3}+c\kappa(\varepsilon)\|\varphi\|_{C^1}+c\|\varphi\|_{C^1}\biggl|\frac{1}{\varepsilon}\int_s^t\eta(\tau/\varepsilon^2)\,d\tau\biggr|.
\end{aligned}\tag{37}
$$

For any function $\varphi\in C_0^\infty(\mathbb{R}^n)$ the first two terms on the right-hand side of (37) vanish as $\varepsilon\to0$ and $|t-s|\to0$, uniformly with respect to t, s, and $\omega\in\Omega$. By Lemma 3, the integral $\dfrac{1}{\varepsilon}\displaystyle\int_0^t\eta(\tau/\varepsilon^2)\,d\tau$ satisfies the functional central limit theorem in $(C[0,T])^n$ on any finite interval $[0,T]$. Therefore, applying the Prokhorov theorem twice, we see that the family of distributions of $(\widetilde{u}^\varepsilon,\varphi)$ is tight in $C(0,T)$.

§ 4. Passage to the limit

The objective of this section is to show that the family $\{\widetilde{u}^\varepsilon\}$ of functions converges almost surely to a solution of the Cauchy problem for a limit deterministic parabolic equation with constant coefficients, and to obtain the main results on convergence in distribution for the family of solutions of the problem (1). We first prove convergence in the space V introduced in § 2. Then we prove the same convergence in a stronger topology.

We have already proved that the family $\{\widetilde{u}^\varepsilon\}$ is tight. Therefore, to find the limit distribution, it suffices to pass to the limit in the expressions of the form $(\widetilde{u}^\varepsilon,\varphi)$ for an arbitrary function $\varphi\in C_0^\infty$.

By (33) and by Lemma 1 we get that

$$
\begin{aligned}
&(\widetilde{u}^\varepsilon(t),\varphi)-(u_0,\varphi)\\
&\quad=(\widetilde{u}^\varepsilon(t),p^\varepsilon(t)\varphi)+\varepsilon(\widetilde{u}^\varepsilon(t),\psi^\varepsilon(t)\nabla_x\varphi)-(u_0,p^\varepsilon(0)\varphi)-\varepsilon(u_0,\psi^\varepsilon(0)\nabla_x\varphi)+O(\varepsilon)\\
&\quad=\int_0^t(u^\varepsilon(s),a^\varepsilon p^\varepsilon(s)\nabla_x\nabla_x\widetilde{\varphi})\,ds+\int_0^t(u^\varepsilon(s),a^\varepsilon\nabla_x^\varepsilon\psi^\varepsilon(s)\nabla_x\nabla_x\widetilde{\varphi})\,ds\\
&\qquad+\int_0^t(u^\varepsilon(s),\nabla_x^\varepsilon(a^\varepsilon\psi^\varepsilon(s))\nabla_x\nabla_x\widetilde{\varphi})\,ds-\int_0^t(u^\varepsilon(s),b^\varepsilon\psi^\varepsilon(s)\nabla_x\nabla_x\widetilde{\varphi})\,ds\\
&\qquad+\int_0^t(u^\varepsilon(s),\bar{b}\psi^\varepsilon(s)\nabla_x\nabla_x\widetilde{\varphi})\,ds+\int_0^t(u^\varepsilon(s),\eta^\varepsilon(s)\psi^\varepsilon(s)\nabla_x\nabla_x\widetilde{\varphi})\,ds+O(\varepsilon).
\end{aligned}\tag{38}
$$

The following lemma will help us to pass to the limit in this expression.

Lemma 5. *Let $\zeta(z,s)$ be a stationary ergodic process with values in $L^2(\mathbb{T}^n)$ (or $C(\mathbb{T}^n)$) and let*

$$\|\zeta\|_{L^2(\mathbb{T}^n\times(0,1))}\leqslant C$$

for a non-random constant C. Then the relation

$$\lim_{\varepsilon\to 0}\sup_{t\leqslant T}\left|\int_0^t (u^\varepsilon(s),\zeta^\varepsilon(s)\widetilde{\varphi})\,ds - \overline{\langle\zeta\rangle}\int_0^t (u^\varepsilon(s),\widetilde{\varphi})\,ds\right| = 0 \tag{39}$$

holds almost surely for any C_0^∞-function φ, where

$$\overline{\langle\zeta\rangle} = \mathsf{E}\int_{\mathbb{T}^n}\zeta(z,s)\,dz,$$

and $\widetilde{\varphi}$ is defined in (32).

Proof. We can assume without loss of generality that $\overline{\langle\zeta\rangle} = 0$, and we write $\mu(s) = \int_{\mathbb{T}^n}\zeta(z,s)dz$. By Lemma 1,

$$\sup_{t\leqslant T}\left|\int_0^t \left[(u^\varepsilon(\tau),\zeta^\varepsilon(\tau)\widetilde{\varphi}) - \mu\left(\frac{\tau}{\varepsilon^2}\right)(u^\varepsilon(\tau),\widetilde{\varphi})\right]d\tau\right| \leqslant C\kappa_1(\varepsilon)$$

with a non-random constant C. Thus, it suffices to show that

$$\lim_{\varepsilon\to 0}\sup_{t\leqslant T}\left|\int_0^t \mu\left(\frac{\tau}{\varepsilon^2}\right)(u^\varepsilon(\tau),\widetilde{\varphi})\,d\tau\right| = 0. \tag{40}$$

By the Arzelà theorem, it follows from (35) that for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ there is a compact subset $K \subset C[0,T]$ for which $(\widetilde{u}^\varepsilon(\cdot),\varphi) \in K$ almost surely for any $\varepsilon > 0$. The rest of the proof is standard. We construct a finite δ-net consisting of step functions and use Birkhoff's theorem to prove the limit relation

$$\lim_{\varepsilon\to 0}\left|\int_0^t \mu\left(\frac{\tau}{\varepsilon^2}\right)(u^\varepsilon(\tau),\widetilde{\varphi})\,d\tau\right| = 0$$

for any $t \leqslant T$. This, together with the simple estimate

$$\left|\int_s^t \mu\left(\frac{\tau}{\varepsilon^2}\right)(u^\varepsilon(\tau),\widetilde{\varphi})\,d\tau\right| \leqslant c\left\|\mu\left(\frac{\cdot}{\varepsilon^2}\right)\right\|_{L^2(0,T)}|t-s|^{1/2},$$

establishes the relation (40), which completes the proof.

We denote by $\bar{a}^{ij}$ the 'homogenized' matrix given by

$$\bar{a}^{ij} = \mathsf{E}\int_{\mathbb{T}^n}\left[a^{ij}(z,\xi_s)p(z,s) + a^{ik}(z,\xi_s)\frac{\partial}{\partial z_k}\psi^j(z,s) - b^i(z,\xi_s)\psi^j(z,s)\right]dz. \tag{41}$$

This definition is natural since the terms on the right-hand side of (38) which do not enter the definition of $\bar{a}^{ij}$ satisfy the equation

$$\int_{\mathbb{T}^n}\left(\frac{\partial}{\partial z_k}\big(a^{ik}(z,\xi_s)\psi^j(z,s)\big) + \bar{b}^i\psi^j(z,s) + \eta^i(s)\psi^j(z,s)\right)dz = 0.$$

By Lemma 5, (38), and the last equation, we get that

$$\lim_{\varepsilon\to 0}\sup_{t\leqslant T}\left|(\widetilde{u}^\varepsilon(t),\varphi)-(u_0,\varphi)-\int_0^t(\widetilde{u}^\varepsilon(s),\bar{a}\nabla_x\nabla_x\varphi)\,ds\right|=0$$

almost surely for any $\varphi\in C_0^\infty$. Thus, any limit point of $\{\widetilde{u}^\varepsilon\}$ in V satisfies the equation

$$\frac{\partial}{\partial t}u^0-\bar{A}u^0=0,\qquad u^0\big|_{t=0}=u_0, \tag{42}$$

for any typical realization of ξ. and for $\bar{A}=\bar{a}^{ij}\dfrac{\partial}{\partial x_i}\dfrac{\partial}{\partial x_j}$. A solution of the problem (42) is unique, and hence the family $\{\widetilde{u}^\varepsilon\}$ converges almost surely in V to the function u^0 as $\varepsilon\to 0$.

In fact, the above convergence result can be improved. Let us show that there is a stationary ergodic process $\chi(z,s)$ with values in $C(\mathbb{T}^n)$ for which the function

$$\begin{aligned}v^\varepsilon(x,t)\equiv u^\varepsilon(x,t)&-u^0\bigg(x+\bar{b}\frac{t}{\varepsilon}+\frac{1}{\varepsilon}\int_0^t\eta\Big(\frac{\tau}{\varepsilon^2}\Big)d\tau,t\bigg)\\&-\varepsilon\nabla_x u^0\bigg(x+\bar{b}\frac{t}{\varepsilon}+\frac{1}{\varepsilon}\int_0^t\eta\Big(\frac{\tau}{\varepsilon^2}\Big)d\tau,t\bigg)\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\end{aligned} \tag{43}$$

converges almost surely to 0 with respect to the norm of the space $L^\infty(\mathbb{R}^n\times[0,T])\cap L^2(0,T;H^1(\mathbb{R}^n))$. To this end, we substitute the expression (43) into the original equation, collect the terms with like powers of ε, and equate the resulting expressions to zero. The first equation in this chain is generated by the terms of order ε^{-1}, and it is

$$\begin{aligned}&\Big(\frac{\partial}{\partial s}\chi(z,s)-A\chi(z,s)\Big)\Big|_{z=\frac{x}{\varepsilon},\,s=\frac{t}{\varepsilon^2}}\nabla_x u^0_\varepsilon(x,t)\\&\qquad=\big[\nabla_z a(z,\xi_s)+b(z,\xi_s)-\bar{b}-\eta(s)\big]\Big|_{z=\frac{x}{\varepsilon},\,s=\frac{t}{\varepsilon^2}}\nabla_x u^0_\varepsilon(x,t);\end{aligned}$$

here and henceforth, we use the notation

$$u^0_\varepsilon(x,t)=u^0\bigg(x+\bar{b}\frac{t}{\varepsilon}+\frac{1}{\varepsilon}\int_0^t\eta\Big(\frac{\tau}{\varepsilon^2}\Big)d\tau,t\bigg).$$

Hence, we want to choose the function $\chi(z,s)$ as a solution of the equation

$$\frac{\partial}{\partial s}\chi^j(z,s)-A\chi^j(z,s)=\nabla_{z_i}a^{ij}(z,\xi_s)+b^j(z,\xi_s)-\bar{b}^j-\eta^j(s). \tag{44}$$

Let us study this equation in more detail. Multiplying it by $p(z,s)$, integrating over the cylinder $\mathbb{T}^n \times [s_1, s_2]$, and recalling the definitions of $p(z,s)$, $\bar{b}$, and $\eta(s)$, we can readily see that

$$\int_{\mathbb{T}^n} \chi(z,s)p(z,s)\,dz = \text{const}.$$

Further, arguing as in the proof of Lemma 4, one can show that the equation (44) has a stationary solution $\chi(z,s)$ which is unique up to an additive constant. Moreover, $\chi(\cdot,s)$ is an ergodic process with values in $C(\mathbb{T}^n)$. For definiteness, we set $\displaystyle\int_{\mathbb{T}^n} \chi(z,s)p(z,s)\,dz = 0$.

Under the above choice of χ we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - A^\varepsilon\Big)v^\varepsilon = \Big(-\frac{\partial}{\partial t} + \breve{a}^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big)u^0 + \Big(a^\varepsilon + \nabla_x^\varepsilon(a^\varepsilon\chi^\varepsilon) + a^\varepsilon\nabla_x^\varepsilon\chi^\varepsilon \\ + b^\varepsilon\chi^\varepsilon - \bar{b}\chi^\varepsilon - \eta\Big(\frac{t}{\varepsilon^2}\Big)\chi^\varepsilon - \breve{a}\Big)\nabla_x\nabla_x u^0 + O(\varepsilon),\end{aligned} \tag{45}$$

where $O(\varepsilon)$ tends to zero in the $L^\infty((0,T);H^{-1}(\mathbb{R}^n)$-norm uniformly with respect to $\omega \in \Omega$, and

$$\begin{aligned}\breve{a}^{ij} = \mathsf{E}\int_{\mathbb{T}^n} \big(a^{ij}(z,\xi_s) + \nabla_{z_k}(a^{ik}(z,\xi_s)\chi^j(z,s)) \\ + a^{ik}(z,\xi_s)\nabla_{z_k}\chi^j(z,s) + b^i(z,\xi_s)\chi^j(z,s)\big)p(z,s)\,dz.\end{aligned}$$

For brevity, let us use the notation

$$\begin{aligned}\breve{a}^\varepsilon_{\text{aux}}(x,t) = \{\breve{a}^{\varepsilon,ij}_{\text{aux}}(x,t)\} = a\Big(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2}\Big) + \nabla_x^\varepsilon\Big(a\Big(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2}\Big)\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\Big) \\ + a\Big(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2}\Big)\nabla_x^\varepsilon\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) \\ + b\Big(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2}\Big)\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) - \Big(\bar{b} + \eta\Big(\frac{t}{\varepsilon^2}\Big)\Big)\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\end{aligned}$$

and

$$\begin{aligned}\langle \breve{a}_{\text{aux}}\rangle(s) = \int_{\mathbb{T}^n} \big\{a^{ij}(z,\xi_s) + \nabla_{z_k}\big(a^{ik}(z,\xi_s)\chi^j(z,s)\big) \\ + a^{ik}(z,\xi_s)\nabla_{z_k}\chi^j(z,s) + b^i(z,\xi_s)\chi^j(z,s)\big\}p(z,s)\,dz.\end{aligned}$$

We shall see below that $\breve{a} = \bar{a}$. Thus, the first term on the right-hand side of (45) vanishes. To obtain the energy estimate, we multiply the equation (45) by $p^\varepsilon v^\varepsilon$

and integrate the resulting formula over $\mathbb{R}^d \times (0,T)$. After simple manipulations, this gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} (v^\varepsilon(x,t))^2 p\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)dx + \int_0^t\int_{\mathbb{R}^n} p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)a^{ij}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_i}v^\varepsilon(x,s)\frac{\partial}{\partial x_j}v^\varepsilon(x,s)\,dx\,ds\\
&\quad = \varepsilon^2\int_{\mathbb{R}^n} p\Big(\frac{x}{\varepsilon},0\Big)\Big(\chi^i\Big(\frac{x}{\varepsilon},0\Big)\frac{\partial}{\partial x_i}u_0(x)\Big)^2dx\\
&\qquad + \int_0^t\int_{\mathbb{R}^n}[\check a^{\varepsilon,ij}_{\rm aux}(x,s)-\check a^{ij}]p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\,dx\,ds\\
&\quad = \varepsilon^2\int_{\mathbb{R}^n} p\Big(\frac{x}{\varepsilon},0\Big)\Big(\chi^i\Big(\frac{x}{\varepsilon},0\Big)\frac{\partial}{\partial x_i}u_0(x)\Big)^2dx\\
&\qquad + \int_0^t\int_{\mathbb{R}^n}\Big[\check a^{\varepsilon,ij}_{\rm aux}(x,s)-\langle\check a_{\rm aux}\rangle^{ij}\Big(\frac{s}{\varepsilon^2}\Big)\Big]p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\,dx\,ds\\
&\qquad + \int_0^t\int_{\mathbb{R}^n}\Big[\langle\check a_{\rm aux}\rangle^{ij}\Big(\frac{s}{\varepsilon^2}\Big)-\check a^{ij}\Big]p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\,dx\,ds.
\end{aligned}\tag{46}$$

The second integral on the right-hand side admits the estimate

$$\begin{aligned}
&\bigg|\int_0^t\int_{\mathbb{R}^n}\Big[\check a^{\varepsilon,ij}_{\rm aux}(x,s)-\langle\check a_{\rm aux}\rangle^{ij}\Big(\frac{s}{\varepsilon^2}\Big)\Big]p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\,dx\,ds\bigg|\\
&\quad = \varepsilon\bigg|\int_0^t\int_{\mathbb{R}^n}\Psi^{ij,k}\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_k}\Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\Big)\,dx\,ds\bigg|\leqslant c\varepsilon
\end{aligned}$$

with non-random constant c; here the functions $\Psi^{ij,k}(z,s)$ are chosen to satisfy the equation

$$\operatorname{div}\Psi^{ij}(z,s) = \big(\check a^{\varepsilon,ij}_{\rm aux}(z,s)-\langle\check a_{\rm aux}\rangle^{ij}\big)p(z,s).$$

Let us estimate the last term in (46). It follows from the definition of $v^\varepsilon(x,t)$ that

$$v^\varepsilon(x,t)=u^\varepsilon(x,t)-u^0\Big(x-\bar b\frac{t}{\varepsilon}-\frac{1}{\varepsilon}\int_0^t\eta\Big(\frac{\tau}{\varepsilon^2}\Big)d\tau,t\Big)+O(\varepsilon)=u^\varepsilon(x,t)-u^0_\varepsilon(x,t)+O(\varepsilon)$$

uniformly with respect to x, t, and ω. Further, by (35), the family

$$\big(u^\varepsilon(t)-u^0_\varepsilon(t),\nabla_x\nabla_x u^0_\varepsilon(t)\big)=\big(\widetilde u^\varepsilon(t)-u^0(t),\nabla_x\nabla_x u^0(t)\big)$$

is compact in $(C[0,T])^{n^2}$. Therefore, it follows from Birkhoff's theorem that the limit relation

$$\lim_{\varepsilon\to0}\sup_{t\leqslant T}\bigg|\int_0^t\int_{\mathbb{R}^n}\Big[\langle\check a_{\rm aux}\rangle^{ij}\Big(\frac{s}{\varepsilon^2}\Big)-\check a^{ij}\Big]p\Big(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}\Big)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\varepsilon(x,s)v^\varepsilon(x,s)\,dx\,ds\bigg|=0$$

holds almost surely. We arrive at the following lemma.

Lemma 6. *The difference $(u^\varepsilon - u^0_\varepsilon)$, and hence the difference $(\widetilde u^\varepsilon - u^0)$ as well, converges to zero almost surely with respect to the $L^\infty(0,T;L^2(\mathbb{R}^n))$-norm as $\varepsilon\to0$.*

Let us now pass to the main results of the paper. We denote by $\widehat{Q}^\varepsilon$ the distribution of the function $\widehat{u}^\varepsilon(x,t) = u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t)$ in V, where the vector $\bar{b}$ is given by (25). We also recall that the matrix $\{\bar{a}^{ij}\}$ was introduced in (41).

Theorem 1. *Let $\{u^\varepsilon\}$ be the family of solutions of the problem* (1). *Suppose that $u_0 \in L^2(\mathbb{R}^n)$. Then the measures $\widehat{Q}^\varepsilon$ converge weakly in V as $\varepsilon \to 0$ to the unique solution of the stochastic partial differential equation*

$$du(t) = \left(\bar{a}^{ij} + \frac{1}{2}(\Lambda^2)^{ij}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u(t)\,dt + \Lambda\nabla_x u(t)\,dw_t, \qquad u\big|_{t=0} = u_0, \tag{47}$$

where the matrix Λ is given by the formula

$$(\Lambda^2)^{ij} = \int_0^\infty \mathsf{E}\big(\eta^i(0)\eta^j(s) + \eta^j(0)\eta^i(s)\big)\,ds$$

and w_t stands for the standard n-dimensional Wiener process.

Proof. One can readily see by Itô's formula that the function $u^0(x - \Lambda w_t, t)$ is a solution of the problem (47). According to [8], this problem is well-posed and has a unique solution.

Let us transform the function $u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t)$ as follows:

$$u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) = u^0\left(x + \frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right) + u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) - u^0\left(x + \frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right).$$

It follows from Lemma 6 that the $L^\infty(0,T;L^2(\mathbb{R}^n))$-norm of the difference between the second and third terms on the right-hand side vanishes almost surely as $\varepsilon \to 0$. Thus, for any $\varphi \in C_0^\infty$ the families

$$\big(u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t), \varphi\big) \quad \text{and} \quad \left(u^0\left(x + \frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right), \varphi\right)$$

converge in distribution in $C(0,T)$ to the same limit distribution.

The map $F_\varphi\colon (C(0,T))^n \to C(0,T)$ defined by

$$F_\varphi(\theta(\,\cdot\,)) = \big(u^0(\,\cdot + \theta(t), t), \varphi\big)$$

is continuous, and since the functions

$$\left\{\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds\right\}$$

converge in distribution to Λw_t in $(C(0,T))^n$, it follows that the functions

$$\left(u^0\left(x+\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right), \varphi\right)$$

converge in distribution to $(u^0(x+\Lambda w_t, t), \varphi)$, which proves the theorem.

The topology of V is very weak. In fact, the same convergence holds in a stronger topology.

Theorem 2. *The family* $\widehat{u}^\varepsilon$ *converges in distribution in the function space* $V_1 = L^\infty(0,T;L^2(\mathbb{R}^n))$ *endowed with the topology of convergence in norm* (*the strong topology*).

Proof. As already proved, the difference

$$u^\varepsilon(x-\varepsilon^{-1}\bar{b}t, t) - u^0\left(x+\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right)$$

converges to zero almost surely in V^1, that is, with respect to the $L^\infty(0,T;L^2(\mathbb{R}^n))$-norm. By the Prokhorov theorem and Lemma 3, for any $\delta > 0$ there is a compact subset $K \subset (C(0,T))^n$ such that

$$\sup_{\varepsilon>0} \mathsf{P}\left\{\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds \notin K\right\} < \delta.$$

The map $\Phi\colon (C(0,T))^n \longrightarrow V^1$ defined by

$$\Phi(\theta(\,\cdot\,)) = u^0(x+\theta(t), t),$$

is continuous. Therefore, for any $\delta > 0$ there is a compact subset K^1 of V^1 such that

$$\sup_{\varepsilon>0} \mathsf{P}\left\{u^0\left(x+\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right) \notin K^1\right\} < \delta.$$

By Lemma 6, this implies that the families

$$\left\{u^\varepsilon(x-\varepsilon^{-1}\bar{b}t, t)\right\} \quad \text{and} \quad \left\{u^0\left(x+\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right)\right\}$$

have the same limits in distribution in V^1. The convergence in distribution of the expressions $u^0\left(x+\frac{1}{\varepsilon}\int_0^t \eta\left(\frac{s}{\varepsilon^2}\right)ds, t\right)$ to $u^0(x+\Lambda w_t, t)$ follows from Lemma 3 and the continuity of the above map Φ.

§ 5. Operators with diffusion coefficients

In this section we assume that the process $\xi.$ in the definition of the coefficients of the equation (1) is a diffusion process. In this special case, the coefficients of the effective equation (47) can be found in terms of solutions of auxiliary deterministic partial differential equations. Moreover, diverse sufficient conditions ensuring the mixing property **A3** of the process $\xi.$ can be formulated in terms of the coefficients of the generator of $\xi.$.

Let us recall the notation $\zeta_s = \xi_{-s}$ and consider the diffusion process $(\widetilde{X}_s, \zeta_s)$ with values in $\mathbb{T}^n \times \mathbb{R}^d$ that corresponds to the operator

$$A + \widetilde{L} = \frac{\partial}{\partial z_i} a_{ij}(z,y) \frac{\partial}{\partial z_j} + b_i(z,y) \frac{\partial}{\partial z_i} + \widetilde{q}_{kl}(y) \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_l} + \widetilde{B}_k(y) \frac{\partial}{\partial y_k} .$$

According to [20] and [7], under the assumptions **A1**, **A2**, and **A3′** the process $(\widetilde{X}_s, \zeta_s)$ has a unique invariant measure, and the density of this measure satisfies the equation

$$(A^* + \widetilde{L}^*)\widetilde{\rho}(z,y) = 0, \qquad \int_{\mathbb{T}^n} \int_{\mathbb{R}^d} \widetilde{\rho}(z,y)\, dy\, dz = 1. \tag{48}$$

Moreover, $\widetilde{\rho}(z,y)$ decays as $|y| \to \infty$ more rapidly than any negative power of $|y|$. Further, if $f(z,y)$ is a function of polynomial growth with respect to y, then the equation

$$(A + \widetilde{L}\,)\widetilde{\chi}(z,y) = f(z,y)$$

is solvable if and only if

$$\int_{\mathbb{T}^n} \int_{\mathbb{R}^d} f(z,y) \widetilde{\rho}(z,y)\, dy\, dz = 0; \tag{49}$$

any corresponding solution is also of polynomial growth in y.

In what follows, we always suppose that $(\widetilde{X}_s, \zeta_s)$ is a stationary process with density $\widetilde{\rho}(z,y)$.

Let $p(z,s)$ be the conditional density of X_s given ξ_τ, $\tau \in [s, +\infty)$. Then $p(z,s)$ satisfies the equation (16) almost surely (see [17]), and the process η defined in (26) admits the representation

$$\eta(s) = \mathsf{E}\big\{ \mathrm{div}[a(X_s, \xi_s)] + b^i(X_s, \xi_s) - \bar{b}^i \mid \sigma(\xi_\tau, \tau \geqslant s) \big\}.$$

We note that the condition (49) is equivalent to $\mathsf{E} f(\widetilde{X}_s, \zeta_s) = 0$. In particular, the solvability condition holds for the function $f^b(z,y) = (\mathrm{div}_z[a(z,y)] + b^i(z,y) - \bar{b}^i)$. We denote the corresponding solution by $\chi_b(z,y)$. Applying Itô's formula to the expression $\chi_b(\widetilde{X}_s, \zeta_s)$, we get after simple manipulations that

$$\int_0^t f^b(\widetilde{X}_s, \zeta_s)\, ds = \chi_b(\widetilde{X}_t, \zeta_t) - \chi_b(\widetilde{X}_0, \zeta_0) + \int_0^t \sigma(\widetilde{X}_s, \zeta_s) \nabla_z \chi_b(\widetilde{X}_s, \zeta_s)\, dw_s^1 + \int_0^t \Sigma(\zeta_s) \nabla_y \chi_b(\widetilde{X}_s, \zeta_s)\, dw_s^2,$$

where w^1 and w^2 are independent Wiener processes of dimensions n and d, respectively, and

$$\sigma(z,y) = \sqrt{\{a_{ij}(z,y)\}}, \qquad \Sigma(y) = \sqrt{\{q_{kl}(y)\}}.$$

Passing in the preceding formula to the conditional expectations with respect to the σ-algebra $\widetilde{\mathcal{F}}_{-\infty,t} = \sigma\{\zeta_\tau, -\infty < \tau \leqslant t\} = \mathcal{F}_{-t,+\infty}$ and taking the independence of $\{\zeta.\}$ and w^1 into account, we get that

$$\begin{aligned}\frac{1}{\sqrt{t}}\int_0^t \eta_{-s}\,ds &= \frac{1}{\sqrt{t}}\mathsf{E}\{\chi_b(\widetilde{X}_t,\zeta_t)\mid\widetilde{\mathcal{F}}_{-\infty,t}\}\\ &\quad+\frac{1}{\sqrt{t}}\mathsf{E}\biggl\{\int_0^t \Sigma(\zeta_s)\nabla_y\chi_b(\widetilde{X}_s,\zeta_s)\,dw_s^2\biggm|\widetilde{\mathcal{F}}_{-\infty,t}\biggr\}\\ &=\frac{1}{\sqrt{t}}\mathsf{E}\{\chi_b(\widetilde{X}_t,\zeta_t)\mid\widetilde{\mathcal{F}}_{-\infty,t}\}\\ &\quad+\frac{1}{\sqrt{t}}\int_0^t \mathsf{E}\{\Sigma(\zeta_s)\nabla_y\chi_b(\widetilde{X}_s,\zeta_s)\mid\widetilde{\mathcal{F}}_{-\infty,s}\}\,dw_s^2,\end{aligned}$$

where the process η_s is defined in (26). The first term on the right-hand side vanishes as $t\to\infty$. Calculating the quadratic characteristic of the Itô integral in the second term and applying Birkhoff's theorem, we see that

$$\begin{aligned}&\frac{1}{t}\int_0^t q(\zeta_s)\mathsf{E}\{\nabla_y\chi_b^i(\widetilde{X}_s,\zeta_s)\mid\widetilde{\mathcal{F}}_{-\infty,s}\}\mathsf{E}\{\nabla_y\chi_b^j(\widetilde{X}_s,\zeta_s)\mid\widetilde{\mathcal{F}}_{-\infty,s}\}\,ds\\ &\quad\to\mathsf{E}\Bigl(q(\zeta_0)\mathsf{E}\{\nabla_y\chi_b^i(\widetilde{X}_0,\zeta_0)\mid\widetilde{\mathcal{F}}_{-\infty,0}\}\mathsf{E}\{\nabla_y\chi_b^j(\widetilde{X}_0,\zeta_0)\mid\widetilde{\mathcal{F}}_{-\infty,0}\}\Bigr)\\ &\quad=\mathsf{E}\Bigl(q(\xi_0)\mathsf{E}\{\nabla_y\chi_b^i(\widetilde{X}_0,\xi_0)\mid\mathcal{F}_{0,+\infty}\}\mathsf{E}\{\nabla_y\chi_b^j(\widetilde{X}_0,\xi_0)\mid\mathcal{F}_{0,+\infty}\}\Bigr)=(\Lambda^2)^{ij}.\end{aligned}$$

Finally, by Theorem 9.1 and 9.2 in [18] we have the convergence

$$\frac{1}{\sqrt{\tau}}\int_0^{t\tau}\eta_{-s}\,ds\xrightarrow[\tau\to\infty]{\mathcal{L}}\Lambda W_t,$$

where W_t is the standard n-dimensional Wiener process.

Bibliography

[1] D. G. Aronson, "Bounds for the fundamental solutions of a parabolic equation", *Bull. Amer. Math. Soc.* **73** (1967), 890–896.

[2] M. Avellaneda and A. J. Majda, "Mathematical models with exact renormalization for turbulent transport", *Comm. Math. Phys.* **131** (1990), 381–429.

[3] M. Avellaneda and A. J. Majda, "Simple examples with features of renormalization for turbulent transport", *Philos. Trans. Roy. Soc. London Ser.* A **346** (1994), 205–233.

[4] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam 1978.

[5] P. Billingsley, *Convergence of probability measures*, Wiley, New York 1968; Russian transl., Nauka, Moscow 1977.

[6] R. Bouc and E. Pardoux, "Asymptotic analysis of p.d.e.s with wide-band noise disturbances, and expansion of the moments", *Stochastic Anal. Appl.* **2** (1984), 369–422.

[7] F. Campillo, M. Kleptsyna, and A. Piatnitskii, "Homogenization of random parabolic operator with large potential", *Stochastic Process. Appl.* **93** (2001), 57–85.

[8] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge Univ. Press, Cambridge 1992.

[9] A. Fannjiang and G. Papanicolaou, "Convection enhanced diffusion for periodic flows", *SIAM J. Appl. Math.* **54** (1994), 333–408.

[10] A. Fannjiang and G. Papanicolaou, "Diffusion in turbulence", *Probab. Theory Related Fields* **105** (1996), 279–334.

[11] J. Garnier, "Homogenization in a periodic and time-dependent potential", *SIAM J. Appl. Math.* **57** (1997), 95–111.

[12] V. Jikov, S. Kozlov, and O. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin 1994.

[13] M. Kleptsyna and A. Piatnitskii, "Homogenization of random parabolic operators", Homogenization and Applications to Material Sciences (Proc. Internat. Conf., Nice, 1995), Gakkōtosho, Tokyo, 1997, pp. 241–255.

[14] S. M. Kozlov, "Reducibility of quasiperiodic differential operators and averaging", *Trudy Moskov. Mat. Obshch.* **46** (1983), 99–123; English transl., *Trans. Moscow Math. Soc.* **1984**, no. 2, 101–126.

[15] S. M. Kozlov and A. L. Piatnitskii, "Averaging on a background of vanishing viscosity", *Mat. Sb.* **181** (1990), 813–832; English transl., *Math. USSR-Sbornik* **70** (1991), 241–261.

[16] O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, Nauka, Moscow 1967; English transl., Amer. Math. Soc., Providence, RI 1968.

[17] R. Sh. Liptser and A. N. Shiryaev, *Statistics of random processes*, Nauka, Moscow 1974; English transl., Springer-Verlag, New York–Berlin 1977.

[18] R. Sh. Liptser and A. N. Shiryaev, *Theory of martingales*, Nauka, Moscow 1986; English transl., Kluwer, Dordrecht 1989.

[19] E. Pardoux, "Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: a probabilistic approach", *J. Funct. Anal.* **167** (1999), 498–520.

[20] E. Pardoux and A. Yu. Veretennikov, "On the Poisson equation and diffusion approximation. I", *Ann. Probab.* **29** (2001), 1061–1085.

[21] A. L. Pyatnitskii [Piatnitskii], "Averaging of a singularly perturbed equation with rapidly oscillating coefficients in a layer", *Mat. Sb.* **121** (1983), 18–39; English transl., *Math. USSR-Sbornik* **121** (1984), 19–40.

[22] N. S. Trudinger, "Pointwise estimates and quasilinear parabolic equations", *Comm. Pure Appl. Math.* **21** (1968), 205–226.

[23] M. Viot, *Solutions faibles d'équations dérivées stochastiques non linéaires*, Thèse, Université Paris VI, Paris 1976.

[24] V. V. Zhikov, "Remarks on a problem of residual diffusion", *Uspekhi Mat. Nauk* **44**:6 (1989), 155–156; English transl., *Russian Math Surveys* **44**:6 (1989), 194-195.

[25] V. V. Zhikov, "Diffusion in an incompressible random flow", *Funktsional. Anal. i Prilozhen.* **31**:3 (1997), 10–22; English transl., *Funct. Anal. Appl.* **31**:3 (1997), 1–11.

Institute for Information Transmission Problems,
Russian Academy of Sciences,
Narvik Institute of Technology, Norway;
Lebedev Physical Institute,
Russian Academy of Sciences
E-mail address: Masha.Kleptsyna@id.ru, andrey@sci.lebedev.ru

Received 5/APR/02

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX