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ON THE ASYMPTOTIC BEHAVIOR OF EIGENVALUES AND EIGENFUNCTIONS OF NON-SELF-ADJOINT ELLIPTIC OPERATORS

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The article concerns the study of conditions on the non-self-adjoint elliptic operator defined in the whole space $\mathbb{R}^n$, ensuring the existence and uniqueness of a constant-sign eigenfunction tending to zero at infinity. We also study the asymptotics of the corresponding eigenvalue as the coefficient in the highest-order derivative of the operator tends to zero. The result is formulated in terms connected with the variational problem for the Lagrangian on one-dimensional trajectories in the space $\mathbb{R}^n$. The explicit form of this Lagrangian is given in terms of the coefficients of the original operator.

1 INTRODUCTION

Problems pertaining to the asymptotic behavior of spectral characteristics of non-self-adjoint operators in unbounded domains arise in connection with various topics in mathematical physics, the theory of control of stochastic dynamical systems, and problems of financial mathematics. This paper is aimed at studying conditions ensuring that a non-self-adjoint elliptic operator in the entire space $\mathbb{R}^n$ admits a unique eigenfunction which is of fixed sign and tends to zero at infinity (the so-called ground state; see Conditions **A** and **B** below). We also examine asymptotics of the corresponding eigenvalue as a small parameter by the highest-order derivatives of the operator tends to zero. The result is formulated in terms of a variational problem for the Lagrangian (functional of action) on one-dimensional trajectories in $\mathbb{R}^n$. The Lagrangian is explicitly expressed through the coefficients of the given operator.

Let

$$L \equiv a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + C(x)$$

be an elliptic differential operator, $a_{ij}\xi_i\xi_j \geqslant \alpha|\xi|^2$. Here and in what follows, summation is assumed over repeated indices.

Definition 1. Condition **A** *holds for the operator* L if its coefficients satisfy the following conditions:

- $|b(x)| < C|x| + C_1$ for all $x \in \mathbb{R}^n$, where C and C_1 are constants;
- $C(x) \to -\infty$ as $|x| \to \infty$.

Definition 2. Condition **B** *holds for the operator* L if

- $(x, b(x)) > \alpha|x|^2$ for all x such that $|x| \geqslant c_1$, where $\alpha > 0$ and $c_1 > 0$ are constants;
- $C(x) < r(|x|)|x|^2$ for some function $r(s)$ which tends to zero as $s \to \infty$;
- $|b(x)| < C|x| + C_1$ for all $x \in \mathbb{R}^n$, where C and C_1 are constants.

Definition 3. A function $u(x)$ belongs to the *class* Ξ if $u(x) > 0$ and $u(x) \to 0$ as $|x| \to \infty$.

Definition 4. A function $u(x)$ belongs to the *class* Υ if $u(x) > 0$ and $u(x)\exp(\gamma|x|^2) \to 0$ as $|x| \to \infty$ for some $\gamma > 0$.

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For our next definition, we introduce an increasing sequence of domains Q_M, $Q_M \subset Q_{M+1}$, $\bigcup_{M=1}^{\infty} Q_M = \mathbb{R}^n$, and in each of these consider the spectral Dirichlet problem

$$Lu_M + \lambda_M u_M = 0 \text{ in } Q_M, \quad u_M\big|_{\partial Q_M} = 0. \tag{1}$$

According to [2], for each Q_M, there is a unique eigenvalue λ_M for which this problem admits a real-valued solution of constant sign.

Definition 5. We say that a function $u(x)$ of class Ξ is a *ground state which can be realized by a sequence of solutions in finite domains* if there is a sequence of domains Q_M with the above properties and $\lim_{M\to\infty} \lambda_M = \lambda$, $\lim_{M\to\infty} u_M(x) = u(x)$ uniformly on each compact set.

With the help of standard estimates of solutions of elliptic equations, it is easy to show that the function $u(x)$ introduced in Definition 5 is a solution of the equation $Lu + \lambda u = 0$ in $\mathbb{R}^n$.

Theorem 1. *Suppose that Condition* **A** *holds. Then there is a unique real λ for which the equation $Lu + \lambda u = 0$ admits a solution in the class Ξ. Moreover, this solution is unique (to within an arbitrary constant coefficient) both in the class Ξ and in the class of functions which decay as $|x| \to \infty$ (no condition of fixed sign is needed). Moreover, λ satisfies the inequality*

$$-C(x^*) < \lambda < -\frac{\ln \alpha(t_0)}{t_0} - C(x_*), \tag{2}$$

where x_* is a point of minimum of the function $C(x)$ on the ball Q_2; x^* is a point of maximum of the function $C(x)$ in $\mathbb{R}^n$, $Q_R = \{|x| < R\}$,

$$\alpha^{x_0}(t_0) \equiv \mathbf{P}\{\xi_{t_0}^{x_0} \in Q_1,\ \xi_s^{x_0} \in Q_2\ \forall s \in [0, t_0]\},$$

where $x_0 \in Q_1$, $\alpha(t_0) \equiv \inf_{x_0 \in Q_1} \alpha^{x_0}(t_0)$, and $\xi_s^{x_0}$ is a diffusion process with the generator

$$L \equiv a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i}$$

issuing from the point x_0; t_0 is an arbitrary positive number. The function $u(x)$ satisfies the inequality $|u(x)| < C_N(|x| + 1)^{-N}$ with an arbitrary $N > 0$, where $C_N > 0$ are constants. The ground state $u(x)$ can be realized by a sequence of solutions in finite domains.

If Condition **B** *holds, then there is a unique λ for which the equation $Lu + \lambda u = 0$ admits a solution in the class Υ, and this solution is unique to within a constant coefficient. Moreover, for this λ, the solution is unique in a wider class, namely, the class of functions $u(x)$ such that $|u(x)| \in \Upsilon$, with no condition of positivity. This solution is a ground state which can be realized by a sequence of solutions in finite domains.*

Theorem 2. *Let*

$$L_\mu \equiv \mu^2 a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \mu b_i(x)\frac{\partial}{\partial x_i} + C(x)$$

and suppose that either Condition **A** *or Condition* **B** *holds for the operator L_1. Let*

$$S(x_0, T) = \inf_{x(\cdot)} (T)^{-1} \int_0^T (a^{ij}(\dot{x}_i - b_i(x))(\dot{x}_j - b_j(x)) - C(x))\, dt,$$

where the infimum is taken over all absolutely continuous curves $x(t)$ such that $x(0) = x(T) = x_0$; a^{ij} are the elements of the matrix inverse to the matrix of coefficients a_{ij}. Let the function $u_\mu(x)$ and the real number λ_μ be such that $L_\mu u_\mu + \lambda_\mu u_\mu = 0$ and $u_\mu(x)$ is either of class Ξ or class Υ, depending on whether Condition **A** *or Condition* **B** *is satisfied (these quantities are defined uniquely in view of Theorem 1). Then there exist the limits*

$$\lim_{T\to\infty} S(x_0, T), \quad \lim_{\mu\to 0} \lambda_\mu,$$

the first limit being independent of $x_0 \in \mathbb{R}^n$. Moreover,

$$\lim_{T\to\infty} S(x_0, T) = \lim_{\mu\to 0} \lambda_\mu.$$

Remark. Direct calculations show that the differential equation

$$Lu \equiv u''(x) + xu'(x) + \left[\frac{x^2}{\sqrt{1+x^2}} + \frac{1}{(1+x^2)^{\frac{3}{2}}} - \frac{x^2}{1+x^2}\right] u = 0$$

admits the solution $u(x) = \exp(-\sqrt{1+x^2})$. Obviously, Condition **B** holds for the operator L. However, the solution $u(x)$ does not belong to the class Υ but belongs to the wider class Ξ. By Theorem 1, for some λ there is another solution $\tilde{u}(x)$ of the equation $Lu + \lambda u = 0$ in the class Υ. Thus, in the case of Condition **B**, there is no uniqueness of the ground state in the class Ξ. Moreover, it can be shown that the solution $u(x)$ is not a ground state that can be realized by solutions in finite domains.

Theorem 1 will be proved in several steps.

Step 1. Suppose that Condition **A** holds. Let us establish the existence of a real λ for which the equation $Lu + \lambda u = 0$ admits a solution of class Ξ. Consider a sequence of balls $\{Q_M\}$ of radii $M = 1, 2, \dots$ with center at the origin and the corresponding sequence of boundary-value problems

$$(L + \lambda_M) u_M(x) = 0 \ \text{ in } \ Q_M, \quad u_M(x) = 0 \ \text{ on } \ \partial Q_M. \tag{3}$$

According to the Krein–Rutman theorem on positive operators (see [2]), for each M, there is a unique real λ_M for which problem (3) has a positive solution. Let us normalize this solution so that $u_M(0) = 1$. Our aim is to show that the sequence $\{u_M\}$ contains a subsequence $\{u_{M'}\}$ uniformly convergent on each compact set $K \subset \mathbb{R}^n$, and also that the sequence $\{\lambda_{M'}\}$ is convergent, i.e., $\lim\limits_{M' \to \infty} \lambda_{M'} = \lambda$.

First, let us show that the sequence $\{\lambda_M\}$ is bounded, more precisely, that the estimate (2) from Theorem 1 holds with λ replaced by λ_M. For this purpose, in $(0, \infty) \times Q_M$ we consider a parabolic equation with the elliptic part $(L + \lambda_M)$ and the initial value $u_M(x)$ coinciding with the solution of problem (3). Then, $u_M(x)$ is a stationary solution of the parabolic equation. Let us utilize its probability representation. For each positive t, we have

$$u_M(x) = \mathbf{E}\left\{ u_M(\xi^x_{t\wedge\tau_M}) \exp\left(\int_0^{t\wedge\tau_M} (C(\xi^x_s) + \lambda_M)\, ds \right) \right\}, \tag{4}$$

where $a \wedge b \equiv \min(a, b)$, τ_M is the Markovian time at which the trajectory of the process ξ^x_s reaches the boundary of the ball Q_M, and the symbol $\mathbf{E}$ denotes the mathematical expectation. Now, let x_0 be a point of minimum of the function $u^M(x)$ on Q_1. By Ω_1 we denote the following event:

$$\Omega_1 \equiv \{\xi^{x_0}_{t_0} \in Q_1, \ \xi^{x_0}_t \in Q_2 \ \forall t \in [0, t_0]\}.$$

Since the function $u_M(x)$ is positive, the probability representation of $u_M(x)$ yields

$$u_M(x) \geqslant \mathbf{E}\left\{ \chi_{\Omega_1} u_M(\xi^{x_0}_{t_0}) \exp\left(\int_0^{t_0} (C(\xi^{x_0}_s) + \lambda_M)\, ds \right) \right\}$$

$$\geqslant u_M(x_0)\mathbf{E}\left\{ \chi_{\Omega_1} \exp\left(\int_0^{t_0} (C(\xi^{x_0}_s) + \lambda^M)\, ds \right) \right\} \geqslant u_M(x_0) \exp(t_0 C(x_*) + \lambda_M t_0) \alpha^{x_0}(t_0).$$

Therefore, $1 \geqslant \exp(t_0 C(x_*) + \lambda_M t_0) \alpha^{x_0}(t_0)$. Taking the logarithm of both sides of this inequality, we find that $-t_0 \lambda_M \geqslant t_0 C(x_*) + \ln \alpha^{x_0}(t_0)$. Hence, we obtain the upper bound in (2).

In order to establish the lower bound, assume the contrary: $-C(x_*) > \lambda_M$. Then, using the probability representation (4), we obtain $u_M(x_0) \leqslant K \exp(\beta t_0)$, where $\beta = C(x_*) + \lambda_M < 0$, $|u_M| < K$, and $K > 0$ does not depend on t_0. But then the inequality $u_M(x_0) \leqslant K \exp(\beta t_0)$ cannot be valid for all $t_0 > 0$, and this is a contradiction. Using the Harnack inequality, combined with interior estimates of derivatives of solutions of elliptic equations, and taking into account that the sequence $\{\lambda_M\}$ is bounded, we can find a subsequence $\{M'\}$ such that for each compact set K in $\mathbf{R}^n$, the sequence $\{u_{M'}\}$ is uniformly convergent on K to a function $u(x)$ defined for all $x \in \mathbb{R}^n$ and satisfying the equation $Lu + \lambda u = 0$ in $\mathbb{R}^n$, where $\lambda = \lim\limits_{M' \to \infty} \lambda_{M'}$.

Let us show that $u(x) \to 0$ as $|x| \to \infty$ and for $u(x)$ the following estimate holds:

$$|u(x)| \leqslant C_N(1+|x|)^{-N} \quad \forall N > 0.$$

Let R_0 be so large that for $|x| > R_0$ the function $C(x) + \lambda_M$ is negative for all M. We choose the constant C_N so large that $C_N|x|^{-N} \geqslant |u_{M'}|$ for $|x| = R_0$ (this choice is possible, since the sequence of functions $u_{M'}$ is uniformly bounded on each compact set $K \subset \mathbb{R}^n$ in view of the Harnack inequality). Since $u_{M'}(x) = 0$ for $|x| = M'$, we have $C_N|x|^{-N} - u_{M'} \geqslant 0$ for $|x| = R_0$ and $|x| = M'$. Let us apply the operator $L + \lambda_{M'}$ to the difference $w_{N,M'}(x) \equiv C_N|x|^{-N} - u_{M'}(x)$ and examine the sign of the function $(L + \lambda_{M'})w_{N,M'}$ in the spherical layer $\{R_0 < |x| < M'\}$. Simple calculations show that $(L + \lambda_{M'})w_{N,M'} \leqslant 0$ for large enough $R_0 > 0$, and the magnitude of R_0 depends only on $N > 0$ and the coefficients a_{ij}, b_i, and C of the operator L, but it does not depend on M'. Here, the condition $|b_i(x)| \leqslant C_1|x| + C_2$ should be used. We assume that initially R_0 has been chosen such that $(L + \lambda_{M'})w_{N,M'} \leqslant 0$ for $|x| \geqslant R_0$, and $w_{N,M'} \geqslant 0$ for $|x| = R_0$. Using the inequality $w_{N,M'} \geqslant 0$ for $|x| = M'$ and the maximum principle, we conclude that $w_{N,M'} \geqslant 0$ for $R_0 \leqslant |x| \leqslant M'$, i.e., $C_N|x|^{-N} \geqslant u_{M'}(x) \geqslant 0$ for $R_0 \leqslant |x| \leqslant M'$. Let us pass to the limit in the last inequality as $M' \to \infty$. Taking into account that $R_0 > 0$ and $C_N > 0$ is independent of M' and using the uniform convergence of $\{u^{M'}\}$ on an arbitrary compact set $K \subset \mathbb{R}^n$, we find that $C_N|x|^{-N} \geqslant u(x) \geqslant 0$ for $|x| \geqslant R_0$. Increasing (if necessary) the constant $C_N > 0$, we obtain the estimate $C_N(|x|+1)^{-N} \geqslant u(x) \geqslant 0$ for $x \in \mathbb{R}^n$.

Step 2. Let us show that every other solution $\bar{u}$ of the equation $L\bar{u} + \lambda\bar{u} = 0$ that decays as $|x| \to \infty$ and is not necessarily of fixed sign is proportional to the function $u(x)$ constructed above, i.e., $\bar{u} = \gamma u(x)$, where $\gamma \neq 0$ is a constant. First, we note that if $v(x)$ is a positive solution of the equation $Lv + \lambda v = 0$ decaying as $|x| \to \infty$, then any solution $\bar{u}$ which also decays as $|x| \to \infty$ satisfies the inequality $|\bar{u}(x)| \leqslant Kv(x)$, $K = \text{const} > 0$. Indeed, consider a ball Q_ρ of a sufficiently large radius $\rho > 0$ such that $C(x) + \lambda < 0$ for all x outside Q_ρ, and choose a constant $K > 0$ such that $Kv(x) \geqslant \bar{u}(x)$ for $|x| = \rho$. For $|x| > \rho$, the maximum principle can be applied, since $C(x) + \lambda$ is of a suitable sign. Therefore, $Kv(x) \geqslant \bar{u}(x)$ for $|x| > \rho$, but since $\bar{u}(x)$ is bounded in the ball Q_ρ, the inequality $Kv(x) \geqslant \bar{u}(x)$ holds for $x \in \mathbb{R}^n$ with a constant $K' > 0$. The inverse inequality $Kv(x) \geqslant -\bar{u}(x)$ is proved in a similar way. We finally have $Kv(x) \geqslant |\bar{u}(x)|$.

Now let $u(x)$ be the positive solution constructed above and let $u_1(x)$ be a solution that decays at infinity (it does not have to be of fixed sign). Then the function $w(x) = u_1(x) \times (u(x))^{-1}$ is defined for all $x \in \mathbb{R}^n$ and is bounded in $\mathbb{R}^n$ in view of the above estimate, and it is easy to check that it satisfies the equation $L'w(x) = 0$, where L' is a second-order elliptic operator without a potential. It would be easy to obtain an explicit expression for L', but this formula will not be used here and is, therefore, omitted. Let $\gamma = \inf\limits_{x\in\mathbb{R}^n} w(x)$. Here the infimum exists, since the function $w(x)$ is bounded. Then $L'(w(x) - \gamma) = 0$ and $\inf\limits_{x\in\mathbb{R}^n}(w(x) - \gamma) = 0$. Consider a sequence of points $\{x_n\}$ such that $\lim\limits_{n\to\infty}(w(x_n) - \gamma) = 0$. Let $x^* \in \mathbb{R}^n$ be a limit point of this sequence. Since $w(x^*) - \gamma = 0$, the maximum principle implies that $w(x) = \text{const}$ and $u_1(x) = \text{const}\, u(x)$, i.e., the desired statement is proved. If $\liminf\limits_{n\to\infty} |x_n| = \infty$, then $u_1(x)(u(x))^{-1} - \gamma = h(x)$, where $h(x_n) \to 0$ as $n \to \infty$. Consequently, $u_1(x) - \gamma u(x) = h(x)u(x)$. Let $u_1(x) - \gamma u(x) = u_2(x)$. We obviously have $u_2(x) \geqslant 0$ and $u_2(x) \to 0$ as $|x| \to \infty$. According to what has been proved above, $|u(x)| \leqslant Ku_2(x) = Kh(x)u(x)$, and, therefore, $1 \leqslant K|h(x)|$, which is a contradiction with the fact that $h(x_n) \to 0$ as $n \to \infty$. This contradiction completes the proof of the linear dependence of the functions $u_1(x)$ and $u(x)$.

Let us show that for no other constant λ' does the equation $Lu' + \lambda'u' = 0$ admit a solution in the class Ξ. Suppose the contrary. Then there exists $\lambda' \neq \lambda$ such that $Lu' = -\lambda'u'$, u' belonging to the class Ξ. Adding (if necessary) a constant to $C(x)$, we may assume without loss of generality that both $-\lambda$ and $-\lambda'$ are strictly positive. Consider the operator L^* formally conjugate to L and let $q(x) > 0$ be a solution of the equation $L^*q = -\lambda q$. The existence of such a solution is established by arguments similar to those used in the first step of the proof. These arguments show that the function $q(x)$ can be obtained as the limit (with respect to uniform convergence of each compact set K) of a sequence of solutions of the following problems: $L^*q_M = -\lambda_M q_M$ in Q_M, $q_M = 0$ on the boundary of Q_M, $q_M > 0$ in Q_M, $q_M(0) = 1$. However, we still do not have estimates characterizing the behavior of $q(x)$ at infinity. Let us obtain some integral estimates for $q_M(x)$ uniform in M (thereby, we obtain estimates for $q(x)$ for large $|x|$).

Let Q_{R_0} be a sufficiently large ball, so that $C(x) < 0$ outside Q_{R_0}, and let $M > R_0$. We have

$$(-\lambda_M)^{-1} \int\limits_{Q_M \setminus Q_{R_0}} L^* q_M \varphi \, dx = \int\limits_{Q_M \setminus Q_{R_0}} q_M \varphi \, dx \tag{5}$$

for any test function φ. We take as φ the solution of the Dirichlet problem

$$L\varphi = \bar f, \quad x \in Q_M \setminus Q_{R_0},$$
$$\varphi = 1, \quad x \in \partial Q_{R_0}, \quad \varphi = 0, \quad x \in \partial Q_M,$$

where $\bar f = \min\{0, L\psi\}$ in $Q_M \setminus Q_{R_0}$, and $\psi \equiv \frac{|x|^{-s} - M^{-s}}{R_0{}^{-s} - M^{-s}}$ for some $s > 0$. Then, by the maximum principle, the following estimate holds: $\varphi \geqslant \psi$ in $Q_M \setminus Q_{R_0}$, since the functions φ and ψ coincide on the border of the spherical layer $Q_M \setminus Q_{R_0}$, and $L(\varphi - \psi) = \min\{0, L\psi\} - L\psi \leqslant 0$. Integrating by parts and taking into account the boundary conditions for the functions φ and q_M, we obtain

$$(-\lambda_M)^{-1} \int\limits_{Q_M \setminus Q_{R_0}} L^* q_M \varphi \, dx = (-\lambda_M)^{-1} \int\limits_{\partial Q_{R_0}} \left(\frac{\partial q_M}{\partial \nu} \varphi - \frac{\partial \varphi}{\partial \nu} q_M + (b, n) \varphi q_M \right) d\sigma + (-\lambda_M)^{-1} \int\limits_{Q_M \setminus Q_{R_0}} q_M \bar f \, dx, \quad (6)$$

where $\frac{\partial}{\partial \nu}$ is the conormal derivative. The left-hand side of the last relation is positive in view of (5) and the positive sign of the functions q_M and φ. It remains bounded as $M \to \infty$, since the first term on the right-hand side of (6) is bounded uniformly in M and the second term is always negative. Thus, we have

$$\int\limits_{Q_M \setminus Q_{R_0}} q_M \psi \, dx \leqslant \int\limits_{Q_M \setminus Q_{R_0}} q_M \varphi \, dx \leqslant (-\lambda_M)^{-1} \int\limits_{Q_M \setminus Q_{R_0}} L^* q_M \varphi \, dx \leqslant C,$$

where C is independent of $M > 0$, and $s > 0$ is an arbitrary constant. Hence, using the Fatou theorem, we obtain the existence of the integrals $\int\limits_{\mathbb{R}^n} q(x)(|x| + 1)^{-s} \, dx$. We obviously have

$$\int\limits_{\mathbb{R}^n} u' q \, dx = (-\lambda')^{-1} \int\limits_{\mathbb{R}^n} L u' q \, dx = (-\lambda')^{-1} \int\limits_{\mathbb{R}^n} u' L^* q \, dx = \lambda (\lambda')^{-1} \int\limits_{\mathbb{R}^n} u' q \, dx.$$

Since the integrals in these relations are positive, we must have $\lambda = \lambda'$, which is in contradiction with our initial assumption.

Step 3. Consider the case where Condition **B** holds for the operator L. First, let us make the following observation. Suppose that a twice-differentiable function $u(x)$ is a solution of the differential equation

$$a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + C(x) u = 0.$$

Then the function

$$v(x) \equiv u(x) \exp(-\gamma |x|^2), \quad (7)$$

with a real constant γ, satisfies the differential equation

$$a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + b_i'(x) \frac{\partial v}{\partial x_i} + C'(x) v = 0,$$

with

$$b_i'(x) = b_i(x) + 4\gamma a_{ij} x_j, \quad C'(x) = C(x) + \gamma x_i b_i(x) + 2\gamma a_{ii} + 4\gamma^2 a_{ij} x_i x_j.$$

The above formulas can be verified by direct calculations.

Now suppose that Condition **B** holds for the operator L. Let us choose the constant $\gamma < 0$ such that

$$\gamma x_i b_i(x) + 4\gamma^2 a_{ij} x_i x_j < -\gamma_0 |x|^2$$

for some constant $\gamma_0 > 0$. This can be done, since Condition **B** ensures that

$$x_i b_i(x) > \alpha |x|^2.$$

It follows that Condition **A** holds for the operator L' with the coefficients a_{ij}, b_i', and C', provided that the constant $\gamma < 0$ has been chosen in a suitable manner. Let us construct a solution of the corresponding equation $L'v + \lambda v = 0$ and pass to the function $u(x) = v(x)\exp(\gamma |x|^2)$ (see (7)). We obtain a solution of the original problem in the class Υ. This solution is unique in the class Υ, since otherwise we would have a nonunique solution of the corresponding problem for the operator L' in the class Ξ.

The solutions obtained in both cases considered above are ground states which can be realized by sequences of solutions in finite domains. Indeed, for the operators subject to Condition **A** this is clear from the construction of the solution, and in the case of Condition **B**, the realizability by solutions in finite domains can be easily established with the help of the transformation (7). The proof of Theorem 1 is complete.

Let us prove Theorem 2. For this purpose we will need some auxiliary statements.

Proposition 1. *Suppose that Condition* **A** *holds. Then for all $T > 0$ there exist curves $\{x^T(t)\}$, $t \in [0, T]$, realizing the minimum in the variational problem*

$$(T)^{-1} I(x(\cdot), T) \equiv (T)^{-1} \int\limits_0^T (a^{ij}(\dot{x}_i - b_i(x))(\dot{x}_j - b_j(x)) - C(x))\, dt \to \inf, \tag{8}$$

where the infimum is taken over all absolutely continuous curves $\{x(\cdot)\}$ with arbitrary values at the end-points of the interval $[0, T]$. Moreover, there is a ball Q_R of sufficiently large radius R such that the extremal curves $\{x^T(t)\}$, $t \in [0, T]$, belong to Q_R for any T.

Proof. Suppose the contrary. Then there exists a sequence of time instants $\{T_n\}$, $T_n \to \infty$, and a sequence of points $\{Y_n\}$ on the curves $\{x^{T_n}(t)\}$ such that $|Y_n| \to \infty$ as $n \to \infty$. We further note that there exists a ball Q_ρ having a nonempty intersection with each curve $\{x^{T_n}\}$. Otherwise, there would be a sequence of balls $Q_{\rho_{n'}}$, $\rho_{n'} \to \infty$, such that the curves $\{x^{T_{n'}}\}$ lie completely outside the balls $Q_{\rho_{n'}}$, but this is in contradiction with the fact that $\{x^{T_{n'}}\}$ are extremal curves of the original variational problem. Indeed, we have $\min\limits_{x \in \mathbb{R}^n \setminus Q_{\rho_{n'}}} (-C(x)) \to \infty$ as $n' \to \infty$ and the stationary curve $x(t) \equiv 0$ being substituted into the functional yields the value $I_0 = a^{ij} b_i(0) b_j(0) - C(0)$. Therefore, for large enough n the curve $\{x^{T_n}\}$, which belongs to the exterior of the ball Q_{ρ_n}, cannot be an extremal curve of the original functional. Thus, there exists a ball Q_{ρ_0} having common points with any extremal curve $\{x^{T_n}\}$.

Further, let us take ρ_0 sufficiently large, so that

$$\min_{x \in \mathbb{R}^n \setminus Q_{\rho_{n'}}} (-C(x)) \geqslant 2I_0,$$

and denote by $[t_n^0, t_n^1]$ the time intervals on which the trajectory $\{x^{T_n}\}$ is outside the ball Q_{ρ_0}. Let us show that the sequence $t_n^1 - t_n^0$ cannot grow to infinity as $n \to \infty$. Consider two cases. If $t_n^1 \neq T_n$ and $t_n^0 \neq 0$, then $|x^{T_n}(t_n^0)| = |x^{T_n}(t_n^1)| = \rho_0$ and the piece of the trajectory $\{x^{T_n}\}$ on the interval $[t_n^0, t_n^1]$ for large n may be replaced by a curve which, moving with the unit velocity along the radius-vector, during the time ρ_0 comes from the point $\{x^{T_n}(t_n^0)\}$ to the origin, stays at the origin for the time $t_n^1 - t_n^0 - 2\rho_0$, and then the rest of the time, ρ_0 moves to the point $x^{T_n}(t_n^1)$ along the radius-vector. For instance, if $t_n^1 = T_n$, then the piece of the trajectory $\{x^{T_n}\}$ on the interval $[t_n^0, T_n]$ can be replaced by the trajectory which moves with the unit velocity from the point $\{x^{T_n}(t_n^1)\}$ to the origin, and then the rest of the time up to the instant T_n stays at the origin. In a similar way, we consider the case $t_n^0 = 0$. In all cases considered above, simple calculations show that if the length of the interval $[t_n^0, t_n^1]$ is sufficiently large, then the replacement of the piece of the trajectory leads to a smaller value of the functional on the modified trajectory, and, therefore, $\{x^{T_n}\}$ cannot be an extremal trajectory of this functional. It follows that the quantities $t_n^1 - t_n^0$ are bounded uniformly in n.

Let $\dot{x}^{T_n} - b(x^{T_n}) \equiv f^n(t)$. The quantities $\Big\{ \int\limits_{t_n^0}^{t_n^1} (f^n(t))^2\, dt \Big\}$ have to be uniformly bounded. Otherwise, one could replace the said piece of the trajectory by a more economical one (from the standpoint of the functional of the original variational problem), namely, by a uniform rectilinear motion from the point $x^{T_n}(t_n^0)$ to the point $x^{T_n}(t_n^1)$.

On the other hand, in view of Condition **A** and the known estimates for solutions of differential equations (estimates of the Gronwall–Bellmann type), we have

$$|x^{T_n}(t_n^2)| \leqslant C \exp[k(t_n^1 - t_n^0)] \Bigg(\int\limits_{t_n^0}^{t_n^1} (f^n(t))^2\, dt + |x^{T_n}(t_n^0)| \Bigg), \tag{9}$$

where t_n^2 is the time instant when the quantity $|x^{T_n}(t)|$ attains its maximal value on the interval $[t_n^0, t_n^1]$, and C and k are positive constants independent of n. If $|x^{T_n}(t_n^0)| = \rho_0$, then, according to our assumption, the left-hand side of the last inequality goes to infinity as $n \to \infty$, while all terms on the right-hand side remain bounded. This contradiction proves the statement of the lemma. If $|x^{T_n}(t_n^0)| \neq \rho_0$, then $|x^{T_n}(t_n^1)| = \rho_0$. The inequality (9) remains valid if the last term on its right-hand side is replaced by $|x^{T_n}(t_n^1)|$, which also leads to a contradiction.

Proposition 2. *Suppose that Condition* **A** *holds and let $\{x^T(t)\}$, $t \in [0,T]$, be extremal curves of the variational problem* (8) *in which the infimum is taken over all absolutely continuous curves $\{x(\cdot)\}$ such that $x(0) = x_0$ and $x(T) = y_0$, where x_0 and y_0 are arbitrary points. Then there exist compact sets K and K_1 in $\mathbb{R}^n$ such that for arbitrarily chosen $x_0, y_0 \in K$ and $T > 0$, the corresponding curves $\{x^T(t)\}$, $t \in [0,T]$, always remain within the compact set K_1.*

The proof of Proposition 2 is based on the following four lemmas.

Lemma 1. *Let $(\Delta t)_1$ be the time during which the trajectory $x^T(t)$ remains within the spherical layer $K_{R-1,R} = \{R-1 < |x| < R\}$ after crossing the sphere $\{|x| = R-1\}$ at the instant t_0. Then*

$$\int_{t_0}^{t_0+(\Delta t)_1} |\dot{x} - b(x)|^2\, dt \geqslant \max\{C(\Delta t)_1 R^2 - C'R, 0\},$$

where C and C' are positive constants independent of R and $(\Delta t)_1$.

Proof. Denote by $b_r(x)$ and $\dot{x}_r$ the projections of the corresponding vectors on the radius-vector. In view of the conditions on the vector field b, we have

$$\int_{t_0}^{t_0+(\Delta t)_1} b_r(x)\, dt \geqslant \alpha(\Delta t)_1(R-1).$$

Let us subtract from this inequality the following obvious identity:

$$\int_{t_0}^{t_0+(\Delta t)_1} \dot{x}_r\, dt = 1.$$

We get

$$\int_{t_0}^{t_0+(\Delta t)_1} (-\dot{x}_r + b_r(x))\, dt \geqslant \alpha(\Delta t)_1(R-1) - 1.$$

Using the Cauchy inequality, we find that

$$(\Delta t)_1 \int_{t_0}^{t_0+(\Delta t)_1} |-\dot{x} + b(x)|^2\, dt \geqslant \left(\int_{t_0}^{t_0+(\Delta t)_1} (-\dot{x}_r + b_r(x))dt\right)^2 \geqslant (\alpha(\Delta t)_1(R-1) - 1)^2.$$

Hence we obtain the statement of the lemma.

Lemma 2. *Let $(\Delta t)_2$ be the time during which the trajectory $x^T(t)$ remains in the spherical layer $K_{R-1,R} = \{R-1 < |x| < R\}$ after crossing the sphere $\{|x| = R\}$ at the instant t_0. Then*

$$\int_{t_0}^{t_0+(\Delta t)_2} |\dot{x} - b(x)|^2\, dt \geqslant C((\Delta t)_2 R^2 + ((\Delta t)_2)^{-1}),$$

where $C > 0$ is a constant independent of R and $(\Delta t)_2$.

Proof. In view of the restrictions on the vector field b (see Condition **B**), the following estimate holds:

$$\int\limits_{t_0}^{t_0+(\Delta t)_2} b_r(x)\,dt \geqslant \alpha(R-1)(\Delta t)_2.$$

Moreover, it is obvious that

$$\int\limits_{t_0}^{t_0+(\Delta t)_2} \dot{x}_r\,dt = -1.$$

Taking the difference of the last two relations, we get

$$\int\limits_{t_0}^{t_0+(\Delta t)_2} (b_r(x) - \dot{x}_r)\,dt \geqslant \alpha(R-1)(\Delta t)_2 + 1.$$

From the last inequality, using the Cauchy inequality, we obtain

$$(\Delta t)_2 \int\limits_{t_0}^{t_0+(\Delta t)_2} |b(x) - \dot{x}|^2\,dt \geqslant \Bigg(\int\limits_{t_0}^{t_0+(\Delta t)_2} (b_r(x) - \dot{x}_r)\,dt\Bigg)^2 \geqslant (\alpha(R-1)(\Delta t)_2 + 1)^2.$$

The last inequality for large R yields the statement of the lemma.

Lemma 3. *Let (Δt) be the time during which the extremal curve $x^T(t)$ stays outside the ball Q_R, and let $R_{\max}$ be the maximal value of $|x^T(t)|$ during this stay. Then the infinite growth of $R_{\max}$ as $T \to \infty$ implies the infinite growth of (Δt).*

Proof. Assume the contrary, namely, that (Δt) remains bounded as $R_{\max}$ goes to infinity. We have

$$-\int\limits_{t_0}^{t_0+(\Delta t)} \dot{x}_r\,dt = R_{\max} - R,$$

where t_0 is the instant at which the trajectory attains the level $R_{\max}$, $R_{\max} = |x^T(t_0)|$, and (Δt) is the time of descent to the lower level R. Moreover, according to Condition **B**, we have

$$\int\limits_{t_0}^{t_0+(\Delta t)} b_r(x)\,dt > 0.$$

Therefore,

$$\int\limits_{t_0}^{t_0+(\Delta t)} (b_r(x) - \dot{x}_r)\,dt > R_{\max} - R.$$

By the Cauchy inequality, we obtain

$$(\Delta t) \int\limits_{t_0}^{t_0+(\Delta t)} |b(x) - \dot{x}|^2\,dt \geqslant \Bigg(\int\limits_{t_0}^{t_0+(\Delta t)} (b(x) - \dot{x})\,dt\Bigg)^2 \geqslant (R_{\max} - R)^2.$$

Hence, in view of Condition **B**, for large $R_{\max}$ we have

$$\int\limits_{t_0}^{t_0+(\Delta t)} (|b(x) - \dot{x}|^2 - C(x))\,dt \geqslant (R_{\max} - R)^2(\Delta t)^{-1} - o(1)R_{\max}^2(\Delta t).$$

If (Δt) were to remain bounded, we could have replaced (for large $R_{\max}$) the curve $x^T(t)$ by the curve $y^T(t)$ representing uniform motion from the point at which the curve $x^T(t)$ enters the ball Q_R to the point at which this curve abandons the ball Q_R, so that the value of the functional on $y^T(t)$ would be smaller than on $x^T(t)$. This contradiction completes the proof of the lemma.

Lemma 4. *Let $(\Delta t) = (\Delta t)_1 + (\Delta t)_2$ be the total time during which the trajectory $x^T(t)$ stays in the spherical layer $K_{R-1,R}$. Then the value of the original functional* (8) *on the intersection of the curve $x^T(t)$ with the layer $K_{R-1,R}$ is estimated from below by $k_0R^2(\Delta t)$, where the constant k_0 is independent of R and (Δt).*

Proof. On the basis of the previous two lemmas, we have

$$\int_\Gamma |\dot{x} - b(x)|^2 \, dt \geqslant c_0 R^2((\Delta t) - c_1 R^{-1}), \tag{10}$$

where Γ is the intersection of the curve with the spherical layer mentioned in the statement of the lemma. Let us choose a constant $A > 0$ such that $2c_1 < A$, where c_1 is the constant from the preceding inequality. If $(\Delta t) < AR^{-1}$, then by Lemma 2 we have

$$\int_\Gamma |\dot{x} - b(x)|^2 \, dt \geqslant C(\Delta t)_2^{-1} \geqslant C(\Delta t)^{-1} \geqslant CA^{-1}R \geqslant CA^{-1}R(R(\Delta t)A^{-1}) \geqslant C' R^2 (\Delta t),$$

and the desired inequality is proved.

Now let $(\Delta t) \geqslant AR^{-1}$. Due to our choice of the constant $A > 0$, we have $c_1R^{-1} < 0.5AR^{-1} < 0.5(\Delta t)$, and, therefore, with the help of the inequality (10), we obtain the estimate

$$\int_\Gamma |\dot{x} - b(x)|^2 \, dt \geqslant c_0R^2((\Delta t) - \frac{1}{2}(\Delta t)) \geqslant \frac{1}{2} c_0 R^2 (\Delta t).$$

Since $C(x) \leqslant o(1)R^2$, it is obvious that

$$\int_\Gamma (|\dot{x} - b(x)|^2 - C(x)) \, dt \geqslant k_0 R^2 \Delta t.$$

The lemma is proved.

Now, we turn to the proof of Proposition 2. Let T be the time during which the trajectory stays outside the ball Q_{R_0}, and let $R_{\max}$ be the maximal radius attained by the extremal curve $x^T(t)$. We choose a radius R^* such that the curve $x^T(t)$ stays for time $T-2$ in the ball Q_{R^*}. Then,

$$\int_{\Gamma_1} (|\dot{x} - b(x)|^2 - C(x)) \, dt \geqslant k_0 {R_0}^2 (T-2)$$

due to Lemma 4. Here, Γ_1 is the part of the curve $x^T(t)$ lying in the spherical layer $K_{R_0R^*}$. Further, by Lemma 4 we have

$$\int_{\Gamma_2} (|\dot{x} - b(x)|^2 - C(x)) \, dt \geqslant k_0 2 (R^*)^2,$$

where Γ_2 is the part of the curve $x^T(t)$ lying in the spherical layer $K_{R^*R_{\max}}$. Thus,

$$\int_{\Gamma_1 \cup \Gamma_2} (|\dot{x} - b(x)|^2 - C(x)) \, dt \geqslant k_0 (2(R^*)^2 + {R_0}^2(T-2)).$$

Consider the trajectory $y^T(t)$ lying in the ball Q_{R_0} and formed by three pieces:

(1) uniform motion from the point A to the origin;

(2) staying at the origin for the time $T-2$ (stationary curve);

(3) uniform motion for the time 1 from the origin to the point B.

Here, A and B are, respectively, the points at which the extremal curve enters and leaves the ball Q_{R_0}.

Direct calculations show that the value of the functional on this curve does not exceed

$$(4{R_0}^2 + K(R_0)) + (T-2)(|b(0)| + C(0)),$$

where $K(R_0) > 0$ is a constant. Assume that as T goes to infinity, $R_{\max}$ also becomes infinitely large. Then R^* must also go to infinity, for otherwise we would obtain a contradiction with Lemma 3. But then for large enough $R_{\max}$ and R_0 such that $k_0 {R_0}^2 > |b(0)| + C(0)$, we have

$$(4{R_0}^2 + K(R_0)) < 2k_0(R^*)^2, \quad (T-2)(|b(0)| + C(0)) < k_0(T-2)(R_0)^2.$$

Thus, the original functional on the trajectory $y^T(t)$ takes a smaller value than on the trajectory $x^T(t)$, and, therefore, $x^T(t)$ cannot be an extremal trajectory. This contradiction completes the proof of Proposition 2.

The following statement gives estimates of eigenfunctions and plays an important role in the proof of Theorem 2.

Lemma 5. *Suppose that Condition* **A** *holds for the operator L and the solutions of the equation $L_\mu u_\mu + \lambda_\mu u_\mu = 0$ in the class Ξ are normalized by the condition $\int\limits_{\mathbb{R}^n} u_\mu(x)\,dx = 1$. Then, in each ball Q_ρ the following estimates hold for the functions u_μ:*

$$u_\mu(x) < c_0\mu^{-n}, \quad x \in \mathbb{R}^n; \quad u_\mu(x) > c_1 \exp(-\frac{C(\rho)}{\mu}), \quad x \in Q_\rho,$$

where c_0 and c_1 are constants independent of μ and ρ and $C(\rho)$ is a constant that depends on ρ but is independent of μ.

Proof. Note that in the variables $y = (\mu)^{-1}x$ the equation $L_\mu u_\mu + \lambda_\mu u_\mu = 0$ takes the form

$$(\tilde{L}_\mu + \lambda_\mu)u_\mu(\mu y) \equiv \left(a_{ij}\frac{\partial^2 u_\mu}{\partial y_i \partial y_j} + b_i(\mu y)\frac{\partial u_\mu}{\partial y_i} + C(\mu y)u_\mu(\mu y) + \lambda_\mu\right)u_\mu(\mu y) = 0,$$

and the small parameter μ is present only in the arguments of the coefficients which are bounded. Therefore, if $\operatorname{dist}(x', x'') < \mu$, the Harnack estimates guarantee that $c_0 \leqslant u_\mu(x')(u_\mu(x''))^{-1} \leqslant c_1$. Therefore,

$$u_\mu(x) \leqslant c \min_{z\in Q_\mu^x} u_\mu(z) \leqslant c' \min_{z\in Q_\mu^x} u_\mu(z)\mu^{-n} \int\limits_{Q_\mu^x} dz' \leqslant c' \int\limits_{\mathbb{R}^n} u_\mu(x)\,dx \mu^{-n} \leqslant c'\mu^{-n},$$

and the first inequality of this lemma is proved.

Now let $x, z \in K$ be two given points of a compact set K. Then there is a sequence of points $x_1, \dots, x_N$ such that $x_1 = x$, $x_N = z$, and $\operatorname{dist}(x_i, x_{i+1}) < \mu$, and the length N of the sequence satisfies the inequality $N < C(K)(\mu)^{-1}$, where $C(K)$ is a constant depending only on the compact set K. But then, $c_0 \leqslant u_\mu(x_i)(u_\mu(x_{i+1}))^{-1} \leqslant c_1$, and, therefore, $u_\mu(z)(u_\mu(x))^{-1} \geqslant (c_1^{-1})^{\frac{C(K)}{\mu}}$, which implies that

$$u_\mu(z) \geqslant \exp\left(-\frac{C'(K)}{\mu}\right)u_\mu(x). \tag{11}$$

Just as in the proof of Theorem 1, applying the operator $(\tilde{L}_\mu + \lambda_{\mu,M})$ to the difference

$$c_0\mu^{-n}\left(|y| - \frac{\rho}{\mu} + 1\right)^{-N} - u_{\mu,M}(\mu y)$$

in the exterior of the ball $Q_{\rho/\mu}$ and using the maximum principle, we see that $u_{\mu,M}$, together with u_μ, satisfies the following estimate for $|y| \geqslant \rho/\mu$:

$$u_\mu(\mu y) \leqslant c_0\mu^{-n}\left(|y| - \frac{\rho}{\mu} + 1\right)^{-N},$$

where $\rho = \rho(N)$ is independent of μ. In the variables x, we get

$$u_\mu(x) \leqslant c_0\mu^{N-n}\left(\frac{(|x| - \rho)}{\mu} + \mu\right)^{-N}$$

for $|x| \geqslant \rho(N)$. Taking $N = 2n + 1$ in the last inequality, it is easy to see that the following estimate holds in the ball of radius 2ρ:

$$\int\limits_{Q_{2\rho}} u_\mu(x)dx \geqslant \frac{1}{2}.$$

Therefore $\max\limits_{z \in Q_{2\rho}} u_\mu(z) > c_2 > 0$, where c_2 is independent of μ. Taking into account the estimate (11), we obtain the statement of the lemma.

Proof of Theorem 2. First, we consider the case of operators satisfying Condition **A**. Let Q_M be the ball of radius M with center at the origin. Consider the spectral problem

$$L_\mu u = \lambda u \ \text{ in } \ Q_M, \quad u\big|_{\partial Q_M} = 0, \tag{12}$$

and denote by $\lambda_{\mu,M}$ the principal eigenvalue of this problem and by $u_{\mu,M}$ the corresponding eigenfunction.

With the help of ideas used in the proof of Theorem 1 in [1], one can show that for each $M > 0$,

$$\lim_{\mu \downarrow 0} \lambda_{\mu,M} = \lim_{T \to \infty} S_M(x_0, T), \tag{13}$$

where

$$S_M(x_0, T) = \inf_{\{x(\cdot)\} \subset Q} \frac{1}{T} \int\limits_0^T \{a^{ij}(\dot{x}_i - b_i(x(t)))(\dot{x}_j - b_j(x(t))) - C(x(t))\}\, dt,$$

and the infimum is taken over all smooth curves $x(\cdot)$ that do not abandon the ball Q_M on the interval $(0, T)$.

We will need two auxiliary parabolic problems:

$$\frac{\partial}{\partial t} v_\mu = \frac{1}{\mu} L_\mu v_\mu, \quad v_\mu\big|_{t=0} = u_\mu, \tag{14}$$

where the initial value is the first eigenfunction u_μ of the operator L_μ in the entire space, and

$$\frac{\partial}{\partial t} v_{\mu,M} = \frac{1}{\mu} L_\mu v_{\mu,M}, \quad v_{\mu,M}\big|_{t=0} = u_{\mu,M}, \quad v_{\mu,M}\big|_{\partial Q_M} = 0, \tag{15}$$

where $u_{\mu,M}$ is the first eigenfunction of problem (12). We obviously have

$$v_\mu(t, x) = \exp\left(-\frac{\lambda_\mu t}{\mu}\right) u_\mu(x), \quad v_{\mu,M}(t, x) = \exp\left(-\frac{\lambda_{\mu,M} t}{\mu}\right) u_{\mu,M}(x). \tag{16}$$

Let us prove the estimate from above. For a given (small) $\delta > 0$, there is a sufficiently large T_0 such that for all $\tilde{T} > T_0$, we have $S(0, \tilde{T}) < \bar{\lambda} + \delta$. Here, for the sake of brevity, we use the notation $\bar{\lambda} = \lim\limits_{T \to \infty} S(0, T)$. With the help of the lower estimate of the principle of large deviations for the diffusion process ξ_μ with the generator

$$\mu a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i}$$

(see [4]), just as in Theorem 1 from [1], we find that for each $\gamma > 0$ there exists $\mu_0 > 0$, such that

$$v_\mu(\tilde{T}, 0) \geqslant \exp\left(-\frac{C(\rho)}{\mu}\right) \exp\left(-\frac{\tilde{T} S(0, \tilde{T}) + \gamma}{\mu}\right) \geqslant \exp\left(-\frac{C(\rho)}{\mu}\right) \exp\left(-\frac{\tilde{T}\bar{\lambda} + \tilde{T}\delta + \gamma}{\mu}\right), \quad \forall \mu < \mu_0.$$

Here, we have also used Proposition 1 and Lemma 5. Taking the logarithm of this inequality and using the estimate from Lemma 5, we get

$$\frac{C(\rho)}{\mu} - \frac{\lambda \tilde{T}}{\mu} \geqslant -\frac{C(\rho)}{\mu} - \frac{\tilde{T}\bar{\lambda} + \tilde{T}\delta + \gamma}{\mu}.$$

Dividing this by $\tilde{T}$ and taking into account that $\tilde{T}$, δ, and γ are arbitrary, we find

$$\overline{\lim_{\mu\to 0}}\,\lambda_\mu \leqslant \bar{\lambda}.$$

In order to obtain the estimate from below, we note that, due to the principle of large deviations for the process ξ_μ and the Varadhan theorem (see [3]), the following relation holds for all $\tilde{T}$:

$$\lim_{\mu\to 0}\mu\ln \mathbf{E}\exp\biggl(\frac{1}{\mu}\int\limits_0^{\tilde{T}} C(\xi_\mu^0(t))\,dt\biggr) = \tilde{T}\bar{S}(\tilde{T}),$$

where

$$\bar{S}(T) = \inf_{x(\cdot)}(\tilde{T})^{-1}\int\limits_0^T \{a^{ij}(\dot{x}_i - b_i(x(t))(\dot{x}_j - b_j(x(t)) - C(x(t))\}\,dt,$$

and the infimum is taken over all absolutely continuous curves with arbitrary end-points. Writing out the probability representation of the solution of problem (14), using Lemma 5 and the preceding equality, for each $\delta > 0$ we obtain the estimate

$$\mu\ln(v(\tilde{T},0)) = \mu\ln \mathbf{E}u_\mu(\xi_\mu^0(\tilde{T}))\exp\biggl(\frac{1}{\mu}\int\limits_0^{\tilde{T}} C(\xi_\mu^0(t))\,dt\biggr) \leqslant -C\mu\ln(\mu) + \tilde{T}S(\tilde{T}) + \delta,$$

which holds for all sufficiently small μ. Further, observe that in view of Proposition 1, the difference $(TS(T) - TS(0,T))$ is bounded uniformly in $T > 0$, and, therefore,

$$\lim_{T\to\infty} S(0,T) = \lim_{T\to\infty} S(T).$$

Taking the logarithm of the preceding inequality and dividing the result by $\tilde{T}$, we find that

$$\underline{\lim_{\mu\to 0}}\,\lambda_\mu \geqslant \bar{\lambda}.$$

This, together with the upper estimate, implies the statement of the theorem under Condition **A**.

Let us make an important observation. In view of Proposition 1, for all large enough M, we have

$$S_M(0,T) = S(0,T),$$

and, therefore, by (13)

$$\lim_{\mu\to 0}\lambda_{\mu,M} = \lim_{\mu\to 0}\lambda_\mu. \tag{17}$$

In the case where Condition **B** is satisfied, we change the unknown function by letting

$$u(x) = e^{\gamma|x|^2/(2\mu)}U(x).$$

Then the problem $L_\mu u = \lambda u$ transforms to $L_\mu^\gamma U = \lambda U$, where

$$L_\mu^\gamma = \mu^2 a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \mu(\gamma a_{ij}x_j + b_i(x))\frac{\partial}{\partial x_i} + (\gamma^2 a_{ij}x_i x_j + b_i(x)x_i + C(x)).$$

For $\gamma < 0$ sufficiently small in absolute value, Condition **B** yields

$$(\gamma a_{ij}x_j + b_i(x), x) \geqslant c|x|^2, \quad c > 0,$$
$$\gamma^2 a_{ij}x_i x_j + b_i(x)x_i + C(x) \leqslant -c|x|^2.$$

Thus, Condition **A** holds for the operator L_μ^γ and, according to what has been proved above, we have $\lim\limits_{\mu\to 0}\lambda_{\mu,M} = \lim\limits_{\mu\to 0}\lambda_\mu$. By Proposition 2, for large enough M, the quantities $S(0,T)$ and $S_M(0,T)$ coincide, and, therefore, using (13), we obtain

$$\lim_{\mu\to 0}\lambda_\mu = \lim_{\mu\to 0}\lambda_{\mu,M} = \lim_{T\to\infty} S_M(0,T) = \lim_{T\to\infty} S(0,T).$$

Theorem 2 is proved.

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