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HOMOGENIZATION OF VARIATIONAL FUNCTIONALS WITH NONSTANDARD GROWTH IN PERFORATED DOMAINS

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Abstract. The aim of the paper is to study the asymptotic behavior of solutions to a Neumann boundary value problem for a nonlinear elliptic equation with nonstandard growth condition of the form

$$-\mathrm{div}\left(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\,\nabla u^\varepsilon\right)+|u^\varepsilon|^{p_\varepsilon(x)-2}\,u^\varepsilon=f(x)$$

in a perforated domain Ω^ε, ε being a small parameter that characterizes the microscopic length scale of the microstructure. Under the assumption that the functions $p_\varepsilon(x)$ converge uniformly to a limit function $p_0(x)$ and that p_0 satisfy certain logarithmic uniform continuity condition, it is shown that u^ε converges, as $\varepsilon\to 0$, to a solution of homogenized equation whose coefficients are calculated in terms of local energy characteristics of the domain Ω^ε. This result is then illustrated with periodic and locally periodic examples.

1. **Introduction.** In recent years, increasing attention has been paid to the study of the so called differential equations and variational problems with nonstandard $p(x)$-growth motivated by their applications to the mathematical modeling in continuum mechanics. Such equations arise, for example, from the modeling of non-Newtonian fluids with thermo-convective effects (see, e.g., [9, 10]), the modeling of electro-rheological fluids (see, e.g., [28, 29]), the thermistor problem (see, e.g., [37]), the problem of image recovery (see, e.g., [16]), and the motion of a compressible fluid in a heterogeneous anisotropic porous medium obeying to the nonlinear Darcy law (see, e.g., [11, 14]). There is an extensive literature on this subject. We will not attempt a review of the literature here, but merely mention a few references,

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see for instance [3, 4, 12, 16, 19, 22, 30, 32], and references therein. Recently, there also appeared a research group on variable exponent Lebesgue and Sobolev spaces; we refer to their web page

http://www.math.helsinki.fi/analysis/varsobgroup/.

This paper is aimed at homogenization of the Neumann problem for the following equation with nonstandard growth:

$$-\operatorname{div}\left(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\,\nabla u^\varepsilon\right)+|u^\varepsilon|^{p_\varepsilon(x)-2}\,u^\varepsilon=f(x)\quad\text{in }\Omega^\varepsilon,\tag{1}$$

where ε is a small positive parameter, $\Omega^\varepsilon=\Omega\setminus\mathcal{F}^\varepsilon$ is a perforated domain in $\mathbb{R}^n$ $(n\geqslant 2)$, Ω is a bounded Lipschitz domain, and p_ε is a continuous positive function in Ω which converges to a function p_0 and satisfies some conditions which will be specified in Section 3; f is a given function. Equations of such type are called $p_\varepsilon(x)$–Laplacian equations with nonstandard growth conditions.

Note that the Neumann boundary value problem in perforated domains for linear and nonlinear equations with standard growth, i.e., $p_\varepsilon(x)=p=\text{Const}$ in Ω^ε, was considered earlier by many authors (see, e.g., [17, 26, 38] and the bibliography therein). Homogenization problems for functionals with periodic and locally periodic rapidly oscillating Lagrangians of p-growth with a constant p are well studied now, see for instance [15, 18] and the bibliography therein.

The works [33, 34, 35, 36, 24] (see also [38]) focus on the variational functionals with non-standard growth conditions. In particular, the homogenization and Γ-convergence problems for Lagrangians with variable rapidly oscillating exponents $p(x)$ were considered in [34, 35]. It was shown that the energy minimums and the homogenized Lagrangians in the spaces $W^{1,r}$ might depend on the value of r (so called Lavrentiev phenomenon). For example, such a behaviour can be observed for the Lagrangian $|\nabla u|^{p(x/\varepsilon)}$ with a periodic "chess-board" exponent $p(y)$ and a small parameter $\varepsilon>0$.

Another interesting example of Lagrangian with rapidly oscillating exponent was considered in [24]. Namely, for the functional

$$\mathcal{J}_\varepsilon[u]=\int|\nabla u|^{p(x/\varepsilon)}dx$$

with a smooth periodic exponent $p(y)$ such that $p(x)>1$, it was shown that the limit functional is bounded on Sobolev-Orlicz space of functions with gradient in a L^α log-space where α is the fiber percolation level of $p(x)$. Variational functionals with non-standard growth conditions have also been considered in [15]. Chapter 21 of this book focuses on the Γ-convergernce of such functionals in L^p spaces. The Γ–convergence of the variational functionals for Lagrangians of p_ε growth with rapidly oscillating coefficients in variable Sobolev spaces $W^{1,p_\varepsilon(\cdot)}$ was studied in [8].

Let us also mention that the homogenization of the Dirichlet variational problem corresponding to the nonlinear equation (1) was studied in [5, 7]. To our knowledge, the homogenization problems in perforated domain for $p_\varepsilon(x)$–Laplacian with nonstandard growth conditions have not been studied earlier. In this paper we deal with the Neumann variational problem for the nonlinear equation (1). Here, problem (1) is considered in the framework of Sobolev spaces with variable exponents which will be briefly described in the following Section. For a more general discussion on the variational homogenization methods used in this paper, we refer to [15, 18, 26].

Following the approach developed in [26], instead of a classical periodicity assumption on the structure of the perforated domain Ω^ε, we impose certain conditions on the so–called local energy characteristics associated with the equation (1). It will be shown that the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε is described by the Neumann problem for the following nonlinear equation:

$$-\partial_{x_i} a_i(x, \nabla u) + a_0(x)|u|^{p_0(x)-2}u = \rho(x)f(x) \quad \text{in } \Omega, \tag{2}$$

where the functions a_i $(i = 1, 2, .., n)$, a_0, and ρ are defined in terms of the above mentioned local characteristics.

The proof of the main result is based on the application of the notion of Γ–convergence and the variational homogenization technique which is nowadays widely used in the homogenization theory (see, e.g., [15, 26, 38] and references therein).

The outline of the rest of the paper is as follows. In Section 2, for the sake of completeness, we recall the definition and main results on the Lebesgue and Sobolev spaces with variable exponents which will be used in the sequel. In Section 3 all necessary mathematical notation is defined, the microscopic problem is formulated, the general assumptions are stated, and the main result is formulated. The proof of the convergence result is carried out in Section 5; it relies on auxiliary results from Section 4. Two examples of periodic and locally periodic structures are considered in Section 6.

Notational convention. In what follows C, C_1, C_2, etc. are generic constants independent of ε.

2. **Sobolev spaces with variable exponents.** For the reader's convenience, we recall some basic facts concerning Sobolev spaces with variable exponents, see for instance [13] or [20] and the bibliography herein.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geqslant 2)$ and the exponent function $p(x)$ satisfies the following conditions:

$$1 < p^- = \inf_\Omega p(x) \leqslant p(x) \leqslant \sup_\Omega p(x) = p^+ < +\infty. \tag{3}$$

For all $x, y \in \Omega$,

$$|p(x) - p(y)| \leqslant \omega(|x - y|) \quad \text{with} \quad \overline{\lim_{\tau\to 0}}\, \omega(\tau) \ln\left(\frac{1}{\tau}\right) \leqslant C, \tag{4}$$

where C is a constant.

1. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions f in Ω such that

$$A_{p(\cdot),\Omega}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < +\infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \; A_{p(\cdot),\Omega}\left(\frac{f}{\lambda}\right) \leqslant 1\right\} \tag{5}$$

becomes a Banach space.

2. The following inequalities hold:

$$\begin{cases} \min\left(\|f\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right) \leqslant A_{p(\cdot),\Omega}(f) \leqslant \max\left(\|f\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right), \\ \min\left(A^{\frac{1}{p^-}}_{p(\cdot),\Omega}(f), A^{\frac{1}{p^+}}_{p(\cdot),\Omega}(f)\right) \leqslant \|f\|_{L^{p(\cdot)}(\Omega)} \leqslant \max\left(A^{\frac{1}{p^-}}_{p(\cdot),\Omega}(f), A^{\frac{1}{p^+}}_{p(\cdot),\Omega}(f)\right). \end{cases} \tag{6}$$

3. Let $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{q(\cdot)}(\Omega)$ with

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad 1 < p^- \leqslant p(x) \leqslant p^+ < +\infty, \quad 1 < q^- \leqslant q(x) \leqslant q^+ < +\infty.$$

Then the Hölder's inequality holds

$$\int_\Omega |f\, g|\, dx \leqslant 2\, \|f\|_{L^{p(\cdot)}(\Omega)}\, \|g\|_{L^{q(\cdot)}(\Omega)}\,. \tag{7}$$

4. According to (7), for every $1 \leqslant q = const < p^- \leqslant p(x) < +\infty$

$$\|f\|_{L^q(\Omega)} \leqslant C\ \|f\|_{L^{p(\cdot)}(\Omega)} \quad \text{with the constant} \quad C = 2\, \|1\|_{L^{\frac{p(\cdot)}{p(\cdot)-q}}(\Omega)}\,. \tag{8}$$

It is straightforward to check that for domains Ω such that $\operatorname{meas}\Omega < +\infty$,

$$\|1\|_{L^{p(\cdot)}(\Omega)} \leqslant 2\, \max\left\{[\operatorname{meas}\Omega]^{2/p^-}, [\operatorname{meas}\Omega]^{1/2p^+}\right\}. \tag{9}$$

5. The space $W^{1,\,p(\cdot)}(\Omega)$, $p(\cdot) \in [p^-, p^+] \subset]1, +\infty[$, is defined by

$$W^{1,\,p(\cdot)}(\Omega) = \left\{f \in L^{p(\cdot)}(\Omega) : |\nabla f| \in L^{p(\cdot)}(\Omega)\right\}.$$

If condition (4) is fulfilled, $W_0^{1,p(\cdot)}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p(\cdot)}(\Omega)$. If the boundary of Ω is Lipschitz–continuous and $p(x)$ satisfies (4), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$. The norm in the space $W_0^{1,\,p(\cdot)}$ is defined by

$$\|u\|_{W_0^{1,\,p(\cdot)}} = \sum_i \|D_i u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$

If the boundary of Ω is Lipschitz and $p \in C^0(\Omega)$, then the norm $\|\cdot\|_{W_0^{1,\,p(\cdot)}(\Omega)}$ is equivalent to the norm

$$\widetilde{\|u\|}_{W_0^{1,\,p(x)}(\Omega)} = \sum_i \|D_i u\|_{L^{p(\cdot)}(\Omega)}. \tag{10}$$

6. If $p \in C(\overline{\Omega})$, then $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive.
7. If $p,\, q \in C(\overline{\Omega})$,

$$p_*(x) = \begin{cases} \dfrac{p(x)n}{n - p(x)} & \text{if } p(x) < n, \\ +\infty & \text{if } p(x) > n, \end{cases} \quad \text{and} \quad 1 < q(x) \leqslant \sup_\Omega q(x) < \inf_\Omega p_*(x),$$

then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

8. Friedrichs' inequality is valid in the following form: if $p(x)$ satisfies conditions (3)–(4), then there exists a constant $C > 0$ such that for every $f \in W_0^{1,p(\cdot)}(\Omega)$
$$\|f\|_{L^{p(\cdot)}(\Omega)} \leqslant C \, \|\nabla f\|_{L^{p(\cdot)}(\Omega)} \, . \tag{11}$$

3. **Statement of the problem and main results.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geqslant 2)$. Let $\{\mathcal{F}^\varepsilon\}_{(\varepsilon>0)}$ be a family of open subsets in Ω; in the sequel ε is a small positive parameter characterizing the microscopic length scale. We assume that:

(F1) the set $\mathcal{F}^\varepsilon$ consists of N_ε $(N_\varepsilon \to +\infty$ as $\varepsilon \to 0)$ small isolated components such that their diameters go to zero as $\varepsilon \to 0$;

(F2) the set $\mathcal{F}^\varepsilon$ is distributed in an asymptotically regular way in Ω, i.e., for any ball $\mathcal{B}(y,r)$ of radius r centered at $y \in \Omega$ and sufficiently small $\varepsilon > 0$ $(\varepsilon \leqslant \varepsilon_0(r))$, we have that $\mathcal{B}(y,r) \cap \mathcal{F}^\varepsilon \neq \emptyset$ and $\mathcal{B}(y,r) \cap (\Omega \setminus \mathcal{F}^\varepsilon) \neq \emptyset$.

We set
$$\Omega^\varepsilon = \Omega \setminus \overline{\mathcal{F}^\varepsilon}. \tag{12}$$

A sequence of functions $\{p_\varepsilon\}_{(\varepsilon>0)}$ is said to belong to the class $\mathfrak{L}^\varepsilon_{p_0(\cdot)}$ if this sequence possesses the following properties:

(A1) *for any $\varepsilon > 0$, p_ε is bounded in the following sense:*
$$1 < \mathsf{p}^- \leqslant p_\varepsilon^- \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^+ \leqslant \mathsf{p}^+ < +\infty \quad \text{in } \overline{\Omega}; \tag{13}$$

(A2) *for any $x, y \in \Omega$ and any $\varepsilon > 0$, the function p_ε satisfies the* local log–Hölder continuity property, *i.e.,*
$$|p_\varepsilon(x) - p_\varepsilon(y)| \leqslant \omega_\varepsilon(|x-y|) \quad \text{with} \quad \overline{\lim_{\tau\to 0}}\, \omega_\varepsilon(\tau) \ln\left(\frac{1}{\tau}\right) \leqslant C_\varepsilon. \tag{14}$$

(A3) *the function p_ε converges uniformly in Ω to a function p_0, i.e.,*
$$\lim_{\varepsilon\to 0} \|p_\varepsilon - p_0\|_{C(\overline{\Omega})} = 0, \tag{15}$$
where the limit function p_0 is assumed to satisfy (4);

Notice that the sequence $p_\varepsilon = p_0$ for any $\varepsilon > 0$ belongs to the family $\mathfrak{L}^\varepsilon_{p_0(\cdot)}$.

On the space $L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ we define the functional $J^\varepsilon : L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon) \longrightarrow \mathbb{R} \cup \{+\infty\}$:
$$J^\varepsilon[u] = \begin{cases} \displaystyle\int_{\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} |\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)} |u|^{p_\varepsilon(x)} - f(x)\, u \right\} dx, & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon); \\ +\infty, \quad \text{otherwise}, \end{cases} \tag{16}$$
where $f \in C(\overline{\Omega})$.

We study the asymptotic behavior of J^ε and their minimizers as $\varepsilon \to 0$. The classical periodicity assumption is here substituted by an abstract one covering a variety of concrete behaviors such as the periodicity, the almost periodicity, and many more besides. For this, we assume that $\Omega^\varepsilon \subset \Omega$ is a disperse medium, i.e., the following assumptions hold:

(C1) *the local concentration of the set Ω^ε has a positive continuous limit, that is the indicator of Ω^ε converges weakly in $L^2(\Omega)$ to a continuous positive limit. This implies that there exists a continuous positive function $\rho = \rho(x)$ such that*
$$\lim_{h\to 0} \lim_{\varepsilon\to 0} h^{-n} \text{meas}\,(K_h^x \cap \Omega^\varepsilon) = \rho(x)$$

for any open cube K_h^x centered at $x \in \Omega$ with lengths equal to $h > 0$;

(C2) *for any $q \in [\mathsf{p}^-, \mathsf{p}^+]$ there exists a family of extension operators $\mathsf{P}_q^\varepsilon : W^{1,q}(\Omega^\varepsilon) \to W^{1,q}(\Omega)$ such that*

<u>global:</u> *for any $u^\varepsilon \in W^{1,q}(\Omega^\varepsilon)$,*

$$\|\mathsf{u}^\varepsilon\|_{L^q(\Omega)} \leqslant C \|u^\varepsilon\|_{L^q(\Omega^\varepsilon)}, \qquad \|\nabla \mathsf{u}^\varepsilon\|_{L^q(\Omega)} \leqslant C \|\nabla u^\varepsilon\|_{L^q(\Omega^\varepsilon)} \tag{17}$$

uniformly in $\varepsilon > 0$, where $\mathsf{u}^\varepsilon = \mathsf{P}_q^\varepsilon u^\varepsilon$ and $\mathsf{u}^\varepsilon = u^\varepsilon$ in Ω^ε.

<u>local:</u> *for any $h > 0$ there is $\varepsilon_0(h) > 0$ such that for all $\varepsilon < \varepsilon_0(h)$, $z \in \Omega$ and any function $u \in W^{1,q}((z + [-2h, 2h]^n) \cap \Omega^\varepsilon)$ the estimates hold*

$$\|\mathsf{u}^\varepsilon\|_{L^q((z+[-h,h]^n)\cap\Omega)} \leqslant C \|u^\varepsilon\|_{L^q((z+[-2h,2h]^n)\cap\Omega^\varepsilon)}, \tag{18}$$

$$\|\nabla\mathsf{u}^\varepsilon\|_{L^q((z+[-h,h]^n)\cap\Omega)} \leqslant C \|\nabla u^\varepsilon\|_{L^q((z+[-2h,2h]^n)\cap\Omega^\varepsilon)}. \tag{19}$$

Remark 1. Notice that in condition **(C2)** we require the existence of extension operators only in usual Sobolev spaces $W^{1,q}$ with constant q. In this case the extension condition is well studied in the existing mathematical literature (see, e.g., [1, 6, 17, 26, 27]). For instance, it holds for a wide class of disperse media (see, for instance, [26]).

One more condition is imposed on the so called local characteristic of the set $\mathcal{F}^\varepsilon$ associated to the functional (16). In order to formulate this condition we denote by K_h^z an open cube centered at $z \in \Omega$ with edge length h $(0 < \varepsilon \ll h \ll 1)$, and introduce the functional:

$$c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z, \mathbf{b}) = \inf_{v^\varepsilon} \int\limits_{K_h^z \cap \Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} |\nabla v^\varepsilon|^{p_\varepsilon(\cdot)} + h^{-p_\varepsilon(x)-\gamma} |v^\varepsilon - (x - z, \mathbf{b})|^{p_\varepsilon(x)} \right\} dx, \tag{20}$$

where $\gamma > 0$, $\mathbf{b} \in \mathbb{R}^n$, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega^\varepsilon)$. Here $(\cdot\,, \cdot)$ stands for the scalar product in $\mathbb{R}^n$. We assume that:

(C3) *there is a continuous, with respect to $x \in \overline{\Omega}$, function $T(x, \mathbf{b})$ and $\gamma = \gamma_0$ $(0 < \gamma_0 < \mathsf{p}^-)$ such that, for any $\{p_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{L}_{p_0(\cdot)}^\varepsilon$, any $x \in \Omega$ and any $\mathbf{b} \in \mathbb{R}^n$,*

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} h^{-n} c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(x, \mathbf{b}) = \lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}} h^{-n} c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(x, \mathbf{b}) = T(x, \mathbf{b}). \tag{21}$$

Remark 2. Condition **(C3)** is always fulfilled for periodic and locally periodic structures.

Remark 3. It is crucial in condition **(C3)** that the limit function $T(x, \mathbf{b})$ does not depend on the particular choice of the sequence $p_\varepsilon \to p_0$. Notice that this is always the case for periodic and locally periodic perforated media. These media will be considered in detail in the last section of the paper.

We define the strong convergence in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ in the following way.

Definition 3.1 (*Strong convergence in $L^{p_0(\cdot)}(\Omega^\varepsilon)$*)**.** The sequence $\{\omega^\varepsilon\} \subset L^{p_0(\cdot)}(\Omega^\varepsilon)$ is said to converge strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ to a function $\omega \in L^{p_0(\cdot)}(\Omega)$ if

$$\lim_{\varepsilon\to 0} \|\omega^\varepsilon - \omega\|_{L^{p_0(\cdot)}(\Omega^\varepsilon)} = 0.$$

We also recall the definition of the Γ–convergence (see, e.g., [15, 18]). In our case it reads.

Definition 3.2 ($\Gamma_{p_0(\cdot)}$*–convergence*)**.** The functionals $I^\varepsilon : L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon) \longrightarrow \mathbb{R}\cup\{+\infty\}$ are said to $\Gamma_{p_0(\cdot)}$–converge to a functional $I : L^{p_0(\cdot)}(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ if
(a) ("lim inf"– inequality) for any $u \in L^{p_0(\cdot)}(\Omega)$ and any sequence $\{u^\varepsilon\} \subset L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ which converges to the function u strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ we have:

$$\varliminf_{\varepsilon\to 0} I^\varepsilon[u^\varepsilon] \geqslant I[u];$$

(b) ("lim sup"– inequality) for any $u \in L^{p_0(\cdot)}(\Omega)$ there is a sequence $\{w^\varepsilon\} \subset L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ which converges to the function u strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ such that

$$\varlimsup_{\varepsilon\to 0} I^\varepsilon[w^\varepsilon] \leqslant I[u].$$

Now we are in position to formulate the first convergence result of the paper.

Theorem 3.3. *Assume that* $\{p_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{L}^\varepsilon_{p_0(\cdot)}$*, and let conditions* **(C1)–(C3)** *be satisfied. Then the functionals* J^ε *defined in (16),* $\Gamma_{p_0(\cdot)}$*–converge to the functional* $J_{\rm hom} : L^{p_0(\cdot)}(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ *given by*

$$J_{\rm hom}[u] = \begin{cases} \displaystyle\int_\Omega \left\{ T(x,\nabla u) + \frac{\rho(x)}{p_0(x)}|u|^{p_0(x)} - \rho(x)\, f(x)\, u \right\} dx, & \text{if } u \in W^{1,p_0(\cdot)}(\Omega); \\ +\infty, & \text{otherwise.} \end{cases} \tag{22}$$

Now let us formulate the convergence result for the minimizers of the functionals J^ε. Consider the variational problem:

$$J^\varepsilon[u^\varepsilon] \longrightarrow \min, \quad u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon). \tag{23}$$

According to [2, 13, 14, 21], for each $\varepsilon > 0$, problem (23) has a unique solution $u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$.

The following convergence result holds.

Theorem 3.4. *Under the assumptions of Theorem 3.3, the solution* u^ε *of the variational problem (23) converges strongly in* $L^{p_0(\cdot)}(\Omega^\varepsilon)$ *to a solution of the problem:*

$$J_{\rm hom}[u] \longrightarrow \min, \quad u \in W^{1,p_0(\cdot)}(\Omega). \tag{24}$$

4. **Properties of the homogenized problem.** In this Section we deal with the properties of the homogenized problem (24). First, we will describe the properties of the function $T(x,\mathbf{b})$ defined in (21). Then using this result we will show the continuity of the homogenized functional $J_{\rm hom}$ in the space $W^{1,p_0(\cdot)}(\Omega)$. Finally, we will prove that the variational problem (24) has a unique solution $u \in W^{1,p_0(\cdot)}(\Omega)$.

Properties of the function $T(x,\mathbf{b})$ are given by the following lemma.

Lemma 4.1. *Under the assumptions of Theorem 3.4 the function* $T(x,\mathbf{b})$ *has the following properties:*

(i) *it is convex with respect to the variable* $\mathbf{b}$*, i.e.,*

$$T(x,\mathbf{b}_\tau) \leqslant \tau T(x,\mathbf{b}_1) + (1-\tau)T(x,\mathbf{b}_2), \tag{25}$$

for any $\tau \in [0,1]$*, where* $\mathbf{b}_\tau = \tau\mathbf{b}_1 + (1-\tau)\mathbf{b}_2$*;*
(ii) *it satisfies the bound:*

$$|T(x,\mathbf{b})| \leqslant C\ |\mathbf{b}|^{p_0(x)}\,; \tag{26}$$

(iii) *it is locally Lipschitz in the following sense:*

$$|T(x,\mathbf{b}_1)-T(x,\mathbf{b}_2)| \leqslant C\left(1+|\mathbf{b}_1|+|\mathbf{b}_2|\right)^{p_0(x)-1}|\mathbf{b}_1-\mathbf{b}_2|. \tag{27}$$

Proof of Lemma 4.1. We begin by proving the statement (i) of Lemma 4.1. Let v_1^ε, v_2^ε, and $v_{1,2}^\varepsilon$ be minimizers of the functional in (20) with $\mathbf{b}=\mathbf{b}_1$, $\mathbf{b}=\mathbf{b}_2$, and $\mathbf{b}_\tau=\tau\mathbf{b}_1+(1-\tau)\mathbf{b}_2$, respectively. Then we have

$$\begin{aligned} c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_\tau) &= \int\limits_{K_h^z\cap\Omega^\varepsilon}\left\{\frac{1}{p_\varepsilon(x)}|\nabla v^\varepsilon_{1,2}|^{p_\varepsilon(x)}+h^{-\gamma-p_\varepsilon(x)}|v^\varepsilon_{1,2}-(x-z,\mathbf{b}_\tau)|^{p_\varepsilon(x)}\right\}dx \\ &\leqslant \int\limits_{K_h^z\cap\Omega^\varepsilon}\left\{\frac{1}{p_\varepsilon(x)}|\nabla v^\varepsilon_{\tau}|^{p_\varepsilon(x)}+h^{-\gamma-p_\varepsilon(x)}|v^\varepsilon_{\tau}-(x-z,\mathbf{b}_\tau)|^{p_\varepsilon(x)}\right\}dx, \end{aligned} \tag{28}$$

where $v^\varepsilon_\tau=\tau v^\varepsilon_1+(1-\tau)v^\varepsilon_2$. Then from (28) we get:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_\tau)\leqslant \tau c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1)+(1-\tau)c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_2). \tag{29}$$

Now the statement (i) of Lemma 4.1 immediately follows from (29) and the condition **(C3)**.

We turn to the statement (ii) of Lemma 4.1. Let v^ε be the minimizer of the functional in (20). Taking $w_{\mathbf{b}}=(x-z,\mathbf{b})$ as a test function in (20) we get:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})\leqslant \int\limits_{K_h^z\cap\Omega^\varepsilon}\frac{1}{p_\varepsilon(x)}|\nabla w_{\mathbf{b}}|^{p_\varepsilon(x)}\,dx.$$

This inequality and (13) immediately imply that

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})\leqslant \frac{1}{\mathsf{p}^-}\int\limits_{K_h^z\cap\Omega^\varepsilon}|\mathbf{b}|^{p_\varepsilon(x)}\,dx. \tag{30}$$

Then we have:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})\leqslant \frac{1}{\mathsf{p}^-}\int\limits_{K_h^z\cap\Omega^\varepsilon}|\mathbf{b}|^{p_0(x)}\,dx+\frac{1}{\mathsf{p}^-}\int\limits_{K_h^z\cap\Omega^\varepsilon}\left\{|\mathbf{b}|^{p_\varepsilon(x)}-|\mathbf{b}|^{p_0(x)}\right\}dx.$$

Now using the assumption **(A3)** we obtain

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})\leqslant \frac{1}{\mathsf{p}^-}\int\limits_{K_h^z\cap\Omega^\varepsilon}|\mathbf{b}|^{p_0(x)}\,dx+o(1)\quad\text{as }\varepsilon\to 0. \tag{31}$$

This inequality and the assumption **(A2)** imply that, for ε sufficiently small,

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})\leqslant C\,h^n|\mathbf{b}|^{p_0(z)}+o(h^n)\quad\text{as }h\to 0. \tag{32}$$

Now the statement (ii) of Lemma 4.1 immediately follows from (32) and the condition **(C3)**.

It remains to prove the statement (iii) of the lemma. Let τ be defined by

$$\tau=\frac{|\mathbf{b}_1-\mathbf{b}_2|}{1+|\mathbf{b}_1|+|\mathbf{b}_2|}. \tag{33}$$

Consider the functional $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1)$. It can be represented as follows:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1)=c^{\varepsilon,h}_{p_\varepsilon(\cdot)}\left(z,(1-\tau)\mathbf{b}_2+\tau\left(\mathbf{b}_2+\tau^{-1}(\mathbf{b}_1-\mathbf{b}_2)\right)\right).$$

Then it follows from (29) that

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1) \leqslant (1-\tau)\, c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_2) + \tau\, c^{\varepsilon,h}_{p_\varepsilon(\cdot)}\left(z,\mathbf{b}_2 + \tau^{-1}(\mathbf{b}_1-\mathbf{b}_2)\right). \tag{34}$$

We apply the inequality (32) to estimate the second term of the right hand side of (34). For ε sufficiently small and $h \to 0$, from (33), we have:

$$\tau\, c^{\varepsilon,h}_{p_\varepsilon(\cdot)}\left(z,\mathbf{b}_2 + \tau^{-1}(\mathbf{b}_1-\mathbf{b}_2)\right) \leqslant C\, h^n \frac{|\mathbf{b}_1-\mathbf{b}_2|}{1+|\mathbf{b}_1|+|\mathbf{b}_2|} \left|\mathbf{b}_2 + \tau^{-1}(\mathbf{b}_1-\mathbf{b}_2)\right|^{p_0(x)}$$

$$+\, o(h^n) \leqslant 2^n\, C\, h^n\, (1+|\mathbf{b}_1|+|\mathbf{b}_2|)^{p_0(x)-1}\, |\mathbf{b}_1-\mathbf{b}_2| + o(h^n) \quad \text{as, } h\to 0. \tag{35}$$

Then from (33)–(35), for ε sufficiently small and $h \to 0$, we obtain:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1) - c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_2) \leqslant C_1\, h^n\, (1+|\mathbf{b}_1|+|\mathbf{b}_2|)^{p_0(x)-1}\, |\mathbf{b}_1-\mathbf{b}_2| + o(h^n), \tag{36}$$

as $h \to 0$. In the same way, for ε sufficiently small and $h \to 0$,

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_1) - c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}_2) \geqslant -C_1\, h^n\, (1+|\mathbf{b}_1|+|\mathbf{b}_2|)^{p_0(x)-1}\, |\mathbf{b}_1-\mathbf{b}_2| + o(h^n). \tag{37}$$

The statement (iii) of Lemma 4.1 follows from (36)–(37) and the condition **(C3)**. This completes the proof of Lemma 4.1. □

Our next aim is to show that the homogenized functional J_{hom} is continuous in the space $W^{1,p_0(\cdot)}(\Omega)$.

Lemma 4.2. *Under the assumptions of Theorem 3.4 the functional* J_{hom} *satisfies the inequality:*

$$|J_{\text{hom}}[u] - J_{\text{hom}}[v]| \leqslant L\, \|u-v\|_{W^{1,p_0(\cdot)}(\Omega)}\ , \tag{38}$$

where $L = L\left(\operatorname{meas}\Omega, p_0^{\pm}, \|u\|_{W^{1,p_0(\cdot)}(\Omega)}, \|v\|_{W^{1,p_0(\cdot)}(\Omega)}\right)$.

Proof of Lemma 4.2. From the definition of the homogenized functional J_{hom} and regularity properties of functions p_0, ρ, f, we get:

$$|J_{\text{hom}}[u] - J_{\text{hom}}[v]|$$

$$\leqslant C \int_\Omega \left\{ |T(x,\nabla u) - T(x,\nabla v)| + \left| |u|^{p_0(x)} - |v|^{p_0(x)} \right| + |u-v| \right\} dx. \tag{39}$$

Let us estimate the right hand side of (39). For the first term, by (27), we have:

$$\int_\Omega |T(x,\nabla u) - T(x,\nabla v)|\, dx \leqslant C \int_\Omega (1+|\nabla u|+|\nabla v|)^{p_0(x)-1}\, |\nabla u - \nabla v|\, dx. \tag{40}$$

To estimate the integral on the right hand side of (40), we apply Hölder's inequality (7) and inequalities (6). Then we obtain:

$$\int_\Omega (1+|\nabla u|+|\nabla v|)^{p_0(x)-1}\, |\nabla u - \nabla v|\, dx \leqslant C\, \Upsilon_1\, \|\nabla u - \nabla v\|_{L^{p_0(\cdot)}(\Omega)}, \tag{41}$$

where $\Upsilon_1 = \max\left\{ A^{1/q_0^-}_{p_0(\cdot),\Omega}(1+|\nabla u|+|\nabla v|),\, A^{1/q_0^+}_{p_0(\cdot),\Omega}(1+|\nabla u|+|\nabla v|) \right\}$ and

$$\frac{1}{p_0(\cdot)} + \frac{1}{q_0(\cdot)} = 1 \quad \text{with} \quad 1 < q_0^- \leqslant q_0(x) \leqslant q_0^+. \tag{42}$$

In a similar way one can estimate the second and the third terms on the right hand side of (39). Finally, this yields the desired inequality (38). Lemma 4.2 is proved. □

Lemma 4.3. *Under the assumptions of Theorem 3.4 there exists a unique solution $u \in W^{1,p_0(\cdot)}(\Omega)$ of the variational problem (24).*

Proof of Lemma 4.3. The existence of the minimizer to the functional (16) is a consequence of the proof of Theorem 3.4 presented in Section 5.3. The uniqueness of the solution of the homogenized problem (24) immediately follows from the strict convexity of the homogenized functional $J_{\rm hom}$. □

5. **Proof of main results.** The proof of main results is based on the Γ–convergence and variational homogenization techniques (see for instance [26]). The proof of Theorem 3.3 is given below in Sections 5.1 and 5.2. First, we show that the Γ-limit functional takes on finite values only for $u \in W^{1,p_0(\cdot)}(\Omega)$. Then we obtain the *"lim inf"–inequality* and the *"lim sup"–inequality*.

The assertion of Theorem 3.4 is then a consequence of Theorem 3.3. It is shown in Section 5.3.

The following statement characterizes the domain of the Γ-limit functional.

Lemma 5.1. *Let a family $\{u^\varepsilon\}$, $u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$, be such that*

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] < \infty.$$

Then there is a function $u^0 \in W^{1,p_0(\cdot)}(\Omega)$ such that along a subsequence

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u^0\|_{L^{p_0(\cdot)}(\Omega^\varepsilon)} = 0. \tag{43}$$

Proof of Lemma 5.1. Considering the coercive properties of the functional J^ε it is easy to see that, for some subsequence ε_k, the uniform bound holds

$$\|u^{\varepsilon_k}\|_{W^{1,p_{\varepsilon_k}(\cdot)}(\Omega^{\varepsilon_k})} \leqslant C.$$

Exploiting the extension condition **(C2)**, the continuity of $p_0(x)$ and the fact that p^ε converges to p_0 uniformly in Ω, we conclude that for any $\delta > 0$ there exist $h = h(\delta) > 0$ and piece-wise constant function $\hat p_\delta$ such that for all sufficiently small $\varepsilon > 0$:
(i) $\hat p_\delta(x) > p_0(x) - \delta$ in Ω;
(ii) $\hat p_\delta(x) = const$ on each cube of the form $2hj + [-h,h]^n$, $j \in \mathbb{Z}^n$;
(iii) $\|u^{\varepsilon_k}\|_{W^{1,\hat p_\delta(\cdot)}(\Omega)} \leqslant C$.

By the Sobolev embedding theorem, the sequence $\{u^{\varepsilon_k}\}$ is compact in $L^{p_0(\cdot)}(\Omega)$. Thus, there is $u_0 \in L^{p_0(\cdot)}(\Omega)$ such that (43) holds, after probably taking another subsequence.

We assert that $u_0 \in W^{1,p_0(\cdot)}(\Omega)$. Indeed, it follows from the uniform bound (iii) that $\|\nabla u^0\|_{L^{\hat p_\delta(\cdot)}(\Omega)} \leqslant C$ with a constant C being independent of δ. Passing to the limit as $\delta \to 0$ and applying the Fatou theorem, we arrive at the desired statement. □

5.1. **Proof of the "lim inf"–inequality.** The proof of the *"lim inf"–inequality* is done in two main steps. At the first step we introduce an auxiliary functional $\widetilde J^\varepsilon$ and prove the "lim inf"–inequality for this functional. On the second step, using the condition **(C3)**, we obtain this inequality for the functional J^ε.

Step 1. An auxiliary inequality. Let $\pi_\varepsilon(x) = \min\{p_\varepsilon(x), p_0(x)\}$. It is clear that $\{\pi_\varepsilon\} \subset \mathfrak{L}^\varepsilon_{p_0(\cdot)}$.

On the space $L^{\pi_\varepsilon(\cdot)}(\Omega^\varepsilon)$ we define the functional $\widetilde{J}^\varepsilon : L^{\pi_\varepsilon(\cdot)}(\Omega^\varepsilon) \longrightarrow \mathbb{R}\cup\{+\infty\}$ by setting:

$$\widetilde{J}^\varepsilon[u] = \begin{cases} \int\limits_{\Omega^\varepsilon} \left\{ \dfrac{1}{p_\varepsilon(x)}|\nabla u|^{\pi_\varepsilon(x)} + \dfrac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)\,u \right\} dx, & \text{if } u \in W^{1,\pi_\varepsilon(\cdot)}(\Omega^\varepsilon); \\ +\infty, \quad \text{otherwise.} \end{cases} \tag{44}$$

Notice that the functional $\widetilde{J}^\varepsilon$ is continuous in $W^{1,\pi_\varepsilon(\cdot)}(\Omega^\varepsilon)$. More precisely, the following inequality holds:

$$|\widetilde{J}^\varepsilon[u] - \widetilde{J}^\varepsilon[v]| \leqslant C\,\Upsilon_2\,\|u-v\|_{W^{1,\pi_\varepsilon(\cdot)}(\Omega^\varepsilon)}, \tag{45}$$

where

$$\Upsilon_2 = \max\left\{ A^{1/q_0^-}_{p_0(\cdot),\Omega^\varepsilon}(1+|u|+|\nabla u|+|v|+|\nabla v|)\,,\, A^{1/q_0^+}_{p_0(\cdot),\Omega^\varepsilon}(1+|u|+|\nabla u|+|v|+|\nabla v|) \right\},$$

the exponent $q_0 = q_0(x)$ and the value q_0^- are defined in (42). Notice also, that the statement of Lemma 5.1 remains valid for the functional $\widetilde{J}^\varepsilon$.

Now let u be an arbitrary $C^\infty(\overline{\Omega})$ function and $\{u^\varepsilon\}$ be a sequence which converges to the function u strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ and such that $\widetilde{J}^\varepsilon[u^\varepsilon] \leqslant C$. We will show that

$$\varliminf_{\varepsilon\to 0} \widetilde{J}^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]. \tag{46}$$

Let $\{x^\alpha\}$ be a set of points in the domain Ω that form an h–periodic space lattice. Let us cover the domain Ω by the cubes K_h^α with nonintersecting interiors and introduce the notation:

$$\Omega_h = \{\cup_\alpha K_h^\alpha; \quad K_h^\alpha \subset \Omega\}; \quad \widetilde{\Omega}_h = \Omega \setminus \Omega_h; \quad \Omega_h^\varepsilon = \Omega^\varepsilon \cap \Omega_h; \quad \widetilde{\Omega}_h^\varepsilon = \Omega^\varepsilon \cap \widetilde{\Omega}_h.$$

Moreover,

$$\operatorname{meas}\widetilde{\Omega}_h = O(h) \quad \text{as } h \to 0. \tag{47}$$

Consider now $\widetilde{J}^\varepsilon\,[u^\varepsilon]$. It is clear that

$$\widetilde{J}^\varepsilon\,[u^\varepsilon] = \int\limits_{\Omega_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon}(x,u^\varepsilon,\nabla u^\varepsilon)\,dx + \int\limits_{\widetilde{\Omega}_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon}(x,u^\varepsilon,\nabla u^\varepsilon)\,dx, \tag{48}$$

where

$$\mathsf{F}_{\pi_\varepsilon}(x,u,\nabla u) = \frac{1}{p_\varepsilon(x)}|\nabla u|^{\pi_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)\,u. \tag{49}$$

Consider, first, the second term on the right hand side of (48). It follows from the strong convergence of the sequence $\{u^\varepsilon\}$ to $u \in C^\infty(\overline{\Omega})$ in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ and (47) that

$$\varliminf_{h\to 0}\varliminf_{\varepsilon\to 0} \int\limits_{\widetilde{\Omega}_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon}(x,u^\varepsilon,\nabla u^\varepsilon)\,dx \geqslant 0. \tag{50}$$

Consider now the first term on the right hand side of (48). We have:

$$\begin{aligned} &\int\limits_{\Omega_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon}(x,u^\varepsilon,\nabla u^\varepsilon)\,dx \\ &\quad = \sum_{K_h^\alpha\subset\Omega}\ \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{\pi_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u^\varepsilon|^{p_\varepsilon(x)} - f(x)\,u^\varepsilon \right\} dx. \end{aligned} \tag{51}$$

For any α such that $K_h^\alpha \subset \Omega$, we set

$$v^\varepsilon(x) = u^\varepsilon(x) - u(x^\alpha) \tag{52}$$

and consider the first term on the right hand side of (51). Bearing in mind (20), as $\varepsilon \to 0$, we have:

$$\begin{aligned}\int\limits_{K_h^\alpha\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{\pi_\varepsilon(x)}\,dx &= \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} \frac{1}{\pi_\varepsilon(x)}|\nabla v^\varepsilon|^{\pi_\varepsilon(x)}\,dx + o(1)\\ = \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} &\left\{\frac{1}{\pi_\varepsilon(x)}|\nabla v^\varepsilon|^{\pi_\varepsilon(x)} + h^{-\gamma-\pi_\varepsilon(x)}|v^\varepsilon - (x-z,\mathbf{b})|^{\pi_\varepsilon(x)}\right\}dx \\ -h^{-\gamma}&\int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}|v^\varepsilon - (x-x^\alpha,\mathbf{b})|^{\pi_\varepsilon(x)}\,dx + o(1).\end{aligned} \tag{53}$$

Let us estimate the second integral on the right hand side of (53). It follows from the regularity of the function u and assumptions **(A1)**, **(A3)** that, for any $\mathbf{b} \in \mathbb{R}^n$ and any $\varepsilon > 0$, we have:

$$\begin{aligned}&\int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}|v^\varepsilon - (x-x^\alpha,\mathbf{b})|^{\pi_\varepsilon(x)}\,dx\\ &= \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}\,\big|(u^\varepsilon-u)(x) + \{u(x)-u(x^\alpha)-(x-x^\alpha,\nabla u(x^\alpha))\}\\ &\qquad + (x-x^\alpha,\nabla u(x^\alpha)-\mathbf{b})\big|^{\pi_\varepsilon(x)}\,dx.\end{aligned}$$

It is easy to see that, for $h \to 0$,

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}\,|u(x)-u(x^\alpha)-(x-x^\alpha,\nabla u(x^\alpha))|^{\pi_\varepsilon(x)}\,dx = O\left(h^{n+\mathsf{p}^-}\right). \tag{54}$$

Now it follows from (54) that

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}|v^\varepsilon - (x-x^\alpha,\mathbf{b})|^{\pi_\varepsilon(x)}\,dx\\ &\leqslant C\,\overline{\lim_{\varepsilon\to 0}}\left\{\int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}\,|u^\varepsilon(x)-u(x)|^{\pi_\varepsilon(x)}\,dx\right.\\ &\left. + \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\pi_\varepsilon(x)}\,|(x-x^\alpha,\nabla u(x^\alpha)-\mathbf{b})|^{\pi_\varepsilon(x)}\,dx\right\} + O\left(h^{n+\mathsf{p}^-}\right),\ \text{as } h\to 0.\end{aligned} \tag{55}$$

We set $\mathbf{b} = \mathbf{b}_\alpha = \nabla u(x^\alpha)$. Then it follows from the strong convergence of the sequence $\{u^\varepsilon\}$ to u in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ and (55) that

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-\gamma-\pi_\varepsilon(x)}|v^\varepsilon - (x-z,\nabla u(x^\alpha))|^{p_\varepsilon(x)}\,dx = O\left(h^{n+\mathsf{p}^- -\gamma}\right), \tag{56}$$

as $h \to 0$. Finally, from the definition (20) and relations (53), (56) we get:

$$\underline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{\pi_\varepsilon(x)}\,dx \geqslant \underline{\lim_{\varepsilon\to 0}}\, c^{\varepsilon,h}_{\pi_\varepsilon(\cdot)}(x^\alpha,\nabla u(x^\alpha)) - O\left(h^{n+\mathsf{p}^- -\gamma}\right). \tag{57}$$

Therefore, by (57), the first term on the right hand side of (48) can be estimated as follows:

$$\varliminf_{\varepsilon\to 0}\int\limits_{\Omega_h^\varepsilon}\mathsf{F}_{\pi_\varepsilon}(x,u^\varepsilon,\nabla u^\varepsilon)\,dx \geqslant \varliminf_{\varepsilon\to 0}\sum_{K_h^\alpha\subset\Omega} c^{\varepsilon,h}_{\pi_\varepsilon(\cdot)}(x^\alpha,\nabla u(x^\alpha))$$

$$+\varliminf_{\varepsilon\to 0}\int\limits_{\Omega_h^\varepsilon}\left\{\frac{1}{p_\varepsilon}|u^\varepsilon|^{p_\varepsilon}-f\,u^\varepsilon\right\}dx - O\left(h^{\mathsf{p}^- -\gamma}\right),\quad \text{as } h\to 0 \quad (0<\gamma<\mathsf{p}^-). \tag{58}$$

Finally, from (50), (58) we have:

$$\varliminf_{\varepsilon\to 0}\widetilde{J}^\varepsilon[u^\varepsilon] \geqslant \varliminf_{\varepsilon\to 0}\sum_{K_h^\alpha\subset\Omega} c^{\varepsilon,h}_{\pi_\varepsilon(\cdot)}(x^\alpha,\nabla u(x^\alpha))$$

$$+\varliminf_{\varepsilon\to 0}\int\limits_{\Omega_h^\varepsilon}\left\{\frac{1}{p_\varepsilon(x)}|u^\varepsilon|^{p_\varepsilon(x)}-f(x)\,u^\varepsilon\right\}dx - O\left(h^{\mathsf{p}^- -\gamma}\right),\quad \text{as } h\to 0. \tag{59}$$

We pass to the limit in the inequality (59) first as $\varepsilon\to 0$ and then as $h\to 0$. Taking into account the strong convergence of the sequence $\{u^\varepsilon\}$ to u in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$, the regularity of the function f, the properties of the function p_ε, and conditions **(C1)**, **(C3)** we obtain the desired inequality (46).

By the definition of $\pi_\varepsilon(x)$ we have $\pi_\varepsilon(x)\leqslant p_0(x)$ in Ω. Therefore, the family $\{\widetilde{J}^\varepsilon\}$ is uniformly in ε continuous in $W^{1,p_0(\cdot)}(\Omega^\varepsilon)$ topology. In addition, by Lemma 4.2 the functional J_{hom} is continuous in $W^{1,p_0(\cdot)}(\Omega)$ topology. Then the fact that inequality (46) holds for any $u\in C^\infty(\overline{\Omega})$ implies that (46) holds for all $u\in W^{1,p_0(\cdot)}(\Omega)$. This completes the proof of the "lim inf"–inequality for the functional $\widetilde{J}^\varepsilon$.

Step 2. "Lim inf"–inequality for the initial functional. Let u be an arbitrary function from $L^{p_0(\cdot)}(\Omega)$ and $\{u^\varepsilon\}$ be a sequence which converges to the function u strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ and such that $J^\varepsilon[u^\varepsilon]\leqslant C$. First we remark that one can prove the inequality

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon]\geqslant J_{\text{hom}}[u]\quad \forall u\in C^\infty(\Omega) \tag{60}$$

in the same way as the inequality (46).

Notice that in contrast with $\widetilde{J}^\varepsilon$, the functional J^ε is not continuous in $W^{1,p_0(\cdot)}$ topology. Therefore, the fact that (60) holds for any C^∞–function does not imply this inequality for any $u\in W^{1,p_0(\cdot)}(\Omega)$. To prove (60) for any $u\in W^{1,p_0(\cdot)}(\Omega)$ we use another technique based on the assumption **(C3)**. Namely, let $u\in W^{1,p_0(\cdot)}(\Omega)\setminus C^\infty(\Omega)$. Consider the value

$$I^\varepsilon[u^\varepsilon]=\int\limits_{\Omega^\varepsilon}\frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{p_\varepsilon(x)}\,dx-\int\limits_{\Omega^\varepsilon}\frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{\pi_\varepsilon(x)}\,dx$$

$$=\int\limits_{\Omega^\varepsilon}|\nabla u^\varepsilon|^{\pi_\varepsilon(x)}\left\{\frac{1}{p_\varepsilon(x)}|\nabla u^\varepsilon|^{p_\varepsilon(x)-\pi_\varepsilon(x)}-\frac{1}{p_\varepsilon(x)}\right\}dx. \tag{61}$$

It is easy to see that for all $x\in\Omega^\varepsilon$

$$\max_{0<B<1}\left(-B^{\pi_\varepsilon(x)}\left(\frac{1}{p_\varepsilon(x)}B^{p_\varepsilon(x)-\pi_\varepsilon(x)}-\frac{1}{p_\varepsilon(x)}\right)\right)\leqslant C|p_\varepsilon(x)-p_0(x)|, \tag{62}$$

with a constant C that does not depend on x and ε. Then

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] &\geqslant \underline{\lim}_{\varepsilon\to 0} I^\varepsilon[u^\varepsilon] + \underline{\lim}_{\varepsilon\to 0} \widetilde{J}^\varepsilon[u^\varepsilon] \\ \geqslant \underline{\lim}_{\varepsilon\to 0} \int\limits_{\{|\nabla u^\varepsilon|<1\}\cap\Omega^\varepsilon} |\nabla u^\varepsilon|^{\pi_\varepsilon(x)} &\left\{ \frac{1}{p_\varepsilon(x)} |\nabla u^\varepsilon|^{p_\varepsilon(x)-\pi_\varepsilon(x)} - \frac{1}{p_\varepsilon(x)} \right\} dx + J_{\rm hom}[u].\end{aligned} \tag{63}$$

Now it follows from (62), (63) that

$$\underline{\lim}_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] \geqslant J_{\rm hom}[u] \quad \forall u \in W^{1,p_0(\cdot)}(\Omega) \setminus C^\infty(\Omega). \tag{64}$$

Inequalities (60), (64) mean that if u is an arbitrary function from $W^{1,p_0(\cdot)}(\Omega)$ and $\{u^\varepsilon\}$ is a sequence converging to the function u strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ then

$$\underline{\lim}_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] \geqslant J_{\rm hom}[u] \tag{65}$$

and the "lim inf"–inequality is proved.

5.2. **Proof of the "lim sup"– inequality. Step 1. Upper bound.** Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h' = h - h^{1+\gamma/\mathsf{p}^+}$ ($\varepsilon \ll h \ll 1$, $0<\gamma<\mathsf{p}^-$). Let us cover the domain Ω by cubes K_h^α of length $h>0$ centered at points x^α. We associate with this covering a partition of unity $\{\varphi_\alpha\}: 0\leqslant \varphi_\alpha(x)\leqslant 1$; $\varphi_\alpha(x)=0$ for $x\notin K_h^\alpha$; $\varphi_\alpha(x)=1$ for $x\in K_h^\alpha\setminus \cup_{\beta\neq\alpha}K_h^\beta$; $\sum_\alpha \varphi_\alpha(x)=1$ for $x\in\Omega$; $|\nabla\varphi_\alpha(x)|\leqslant Ch^{-1-\gamma/\mathsf{p}^+}$.

Let now $v_\alpha^\varepsilon = v_\alpha^\varepsilon(x)$ be a function minimizing functional (20) with $\mathbf{b}=\mathbf{b}_\alpha$ and $z=x^\alpha$, where $\mathbf{b}_\alpha$ is a constant vector which will be specified later on. It follows from conditions **(A1)** and **(C3)** that, as $h\to 0$,

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} |\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)}\,dx = O\left(h^n\right);\\ \overline{\lim_{\varepsilon\to 0}} \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} h^{-p_\varepsilon(x)}|v_\alpha^\varepsilon - (x-z,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\,dx = O\left(h^{n+\gamma}\right).\end{aligned} \tag{66}$$

Denote by $K_{h'}^\alpha$ the cube of length h' centered at the point x^α, and by Π_h^α the set $K_h^\alpha \setminus K_{h'}^\alpha$. By (66) and condition **(A1)** we have:

$$\begin{aligned}&\int\limits_{\Pi_h^\alpha\cap\Omega^\varepsilon} \left\{|\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}|v_\alpha^\varepsilon-(x-z,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\right\}dx \\ &= \int\limits_{K_h^\alpha\cap\Omega^\varepsilon} \left\{|\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}|v_\alpha^\varepsilon-(x-z,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\right\}dx \\ &- \int\limits_{K_{h'}^\alpha\cap\Omega^\varepsilon} \left\{|\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)} + (h')^{-p_\varepsilon(x)-\gamma}|v_\alpha^\varepsilon-(x-z,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\right\}dx + o(h^n),\end{aligned} \tag{67}$$

as $h\to 0$. Then from the definition of $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\mathbf{b})$ (see (20)) it follows that, for sufficiently small ε, the bound holds

$$\begin{aligned}&\int\limits_{\Pi_h^\alpha\cap\Omega^\varepsilon} \left\{|\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}|v_\alpha^\varepsilon-(x-z,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\right\}dx \\ &\leqslant c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\mathbf{b}_\alpha) - c_{p_\varepsilon(\cdot)}^{\varepsilon,h'}(z,\mathbf{b}_\alpha) + o(h^n), \text{ as } h\to 0.\end{aligned}$$

Considering now condition **(C3)**, we obtain that, as $h \to 0$,

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\Pi_h^\alpha \cap \Omega^\varepsilon} |\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)}\, dx = o\left(h^n\right); \qquad (68)$$

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\Pi_h^\alpha \cap \Omega^\varepsilon} h^{-p_\varepsilon(x)} |v_\alpha^\varepsilon - (x-z, \mathbf{b}_\alpha)|^{p_\varepsilon(x)}\, dx = o\left(h^{n+\gamma}\right).$$

Let u be a smooth function in Ω. In the domain Ω^ε we define

$$w_h^\varepsilon(x) = \sum_\alpha \left\{u(x) + v_\alpha^\varepsilon(x) - (x - x^\alpha, \mathbf{b}_\alpha)\right\} \varphi_\alpha(x). \qquad (69)$$

It is clear that $w_h^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ and that

$$J^\varepsilon[w_h^\varepsilon] \leqslant \sum_\alpha \int\limits_{K_{h'}^\alpha \cap \Omega^\varepsilon} \mathsf{F}_\varepsilon(x, w_h^\varepsilon, \nabla w_h^\varepsilon)\, dx + \sum_{\alpha,\beta} \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} |\mathsf{F}_\varepsilon(x, w_h^\varepsilon, \nabla w_h^\varepsilon)|\; dx, \quad (70)$$

where the function F_ε is defined by

$$\mathsf{F}_\varepsilon(x, u, \nabla u) = \frac{1}{p_\varepsilon(x)} |\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)} |u|^{p_\varepsilon(x)} - f(x)\, u. \qquad (71)$$

First, we consider the second sum on the right hand side of (70). Considering the properties of the partition of unity $\{\varphi_\alpha\}$, it is not difficult to check that for any α and β the number of terms which are nontrivial in $K_h^\alpha \cap K_h^\beta$ is finite and does not depend on ε. Then in order to estimate the second term on the right hand side of (70) it is sufficient to estimate the following integral:

$$\begin{aligned}\mathbf{j}^\varepsilon[w_h^\varepsilon] = \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} \Bigg\{ &\frac{1}{p_\varepsilon(x)} \left|\nabla\left(u + \left[v_\alpha^\varepsilon - (x - x^\alpha, \mathbf{b}_\alpha)\right]\varphi_\alpha\right)\right|^{p_\varepsilon(x)} \\ &+ \frac{1}{p_\varepsilon(x)} \left|u + \left[v_\alpha^\varepsilon - (x - x^\alpha, \mathbf{b}_\alpha)\right]\varphi_\alpha\right|^{p_\varepsilon(x)} \\ - f(x)\,\left(u + \left[v_\alpha^\varepsilon - (x - x^\alpha, \mathbf{b}_\alpha)\right]\varphi_\alpha\right) &\Bigg\}\, dx \equiv \mathbf{j}_1^\varepsilon[w_h^\varepsilon] + \mathbf{j}_2^\varepsilon[w_h^\varepsilon] + \mathbf{j}_3^\varepsilon[w_h^\varepsilon]. \end{aligned} \qquad (72)$$

For the first term on the right hand side of (72) we have:

$$\begin{aligned}\mathbf{j}_1^\varepsilon[w_h^\varepsilon] &= \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)} \left|\nabla\left(u + \left[v_\alpha^\varepsilon - (x - x^\alpha, \mathbf{b}_\alpha)\right]\varphi_\alpha\right)\right|^{p_\varepsilon(x)}\, dx \\ &\leqslant C_1 \Bigg\{ \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} |\nabla u|^{p_\varepsilon(x)}\, dx + \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)} |\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)}\, dx \\ &+ \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} |\nabla(x - x^\alpha, \mathbf{b}_\alpha)|^{p_\varepsilon(x)}\, dx + \int\limits_{(K_h^\alpha \cap K_h^\beta)\cap\Omega^\varepsilon} \left|\left[v_\alpha^\varepsilon - (x - x^\alpha, \mathbf{b}_\alpha)\right]\nabla\varphi_\alpha\right|^{p_\varepsilon(x)}\, dx \Bigg\}. \end{aligned} \qquad (73)$$

By the condition **(A1)**, taking into account the relation $\operatorname{meas}(K_h^\alpha \cap K_h^\beta) = o(h^n)$ as $h \to 0$, and the fact that u is a smooth function in Ω and $\mathbf{b}_\alpha$ is a constant vector, we obtain

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} \Big\{|\nabla u|^{p_\varepsilon(x)} + |\nabla(x-x^\alpha,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\Big\}\,dx = o(h^n), \quad \text{as } h\to 0. \tag{74}$$

It also follows from (68) that

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)}\,|\nabla v_\alpha^\varepsilon|^{p_\varepsilon(x)}\,dx = o(h^n), \quad \text{as } h\to 0. \tag{75}$$

Due to the properties of $\{\varphi_\alpha\}$, for the last term on the right hand side of (73), we have:

$$\int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} |[v_\alpha^\varepsilon-(x-x^\alpha,\mathbf{b}_\alpha)]\,\nabla\varphi_\alpha|^{p_\varepsilon(x)}\,dx$$

$$\leqslant C \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} h^{-p_\varepsilon(x)}\cdot h^{-\gamma(p_\varepsilon(x)/\mathsf{p}^+)}\,|v_\alpha^\varepsilon-(x-x^\alpha,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\,dx.$$

It is clear that $p_\varepsilon(x)/\mathsf{p}^+ \leqslant 1$. Then $h^{-\gamma(p_\varepsilon(x)/\mathsf{p}^+)} \leqslant h^{-\gamma}$ and

$$\int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} h^{-p_\varepsilon(x)}\cdot h^{-\gamma(p_\varepsilon(x)/\mathsf{p}^+)}\,|v_\alpha^\varepsilon-(x-x^\alpha,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\,dx$$

$$\leqslant h^{-\gamma} \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} h^{-p_\varepsilon(x)}\,|v_\alpha^\varepsilon-(x-x^\alpha,\mathbf{b}_\alpha)|^{p_\varepsilon(x)}\,dx.$$

Therefore, from the second estimate in (68) we deduce that

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} |[v_\alpha^\varepsilon-(x-x^\alpha,\mathbf{b}_\alpha)]\,\nabla\varphi_\alpha|^{p_\varepsilon(x)}\,dx = o(h^n), \quad \text{as} \quad h\to 0. \tag{76}$$

Finally, (74)–(76) yield:

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathbf{j}_1^\varepsilon[w_h^\varepsilon] = 0. \tag{77}$$

In a similar way we can show that $\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathbf{j}_2^\varepsilon[w_h^\varepsilon] = 0$, and $\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathbf{j}_3^\varepsilon[w_h^\varepsilon] = 0$. This implies that the contribution of the second term on the right hand side of (70) is asymptotically negligible, that is

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\sum_{\alpha,\beta}\ \int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon} |\mathsf{F}_\varepsilon(x,w_h^\varepsilon,\nabla w_h^\varepsilon)|\,dx = 0. \tag{78}$$

Consider now the first term on the right hand side of (70). We set $\mathbf{b}_\alpha = \nabla u(x^\alpha)$. It follows from the definition of the function w_h^ε that, for any α,

$$w_h^\varepsilon(x) = u(x) + v_\alpha^\varepsilon(x) - (x-x^\alpha,\nabla u(x^\alpha)) \quad \text{in } K_{h'}^\alpha\cap\Omega^\varepsilon \tag{79}$$

and

$$\nabla w_h^\varepsilon = (\nabla u(x)-\nabla u(x^\alpha)) + \nabla v_\alpha^\varepsilon(x) \quad \text{in } K_{h'}^\alpha\cap\Omega^\varepsilon. \tag{80}$$

Then we have that

$$\int\limits_{K_{h'}^\alpha\cap\Omega^\varepsilon} \mathsf{F}_\varepsilon(x,w_h^\varepsilon,\nabla w_h^\varepsilon)\,dx = \int\limits_{K_{h'}^\alpha\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon(x)}|\nabla w_h^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{p_0(x)}|u|^{p_0(x)} - f\,u\right\} dx$$

$$+\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}\frac{1}{p_\varepsilon(x)}\left\{|w^\varepsilon_h|^{p_\varepsilon(x)}-|u|^{p_\varepsilon(x)}\right\}dx+\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}\left\{\frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)}-\frac{1}{p_0(x)}|u|^{p_0(x)}\right\}dx$$
$$-\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}f(x)\left\{v^\varepsilon_\alpha(x)-(x-x^\alpha,\nabla u(x^\alpha))\right\}dx.$$

In order to estimate the second term on the right–hand side of this relation we apply the inequality:

$$\left|(\xi+\eta)^{p_\varepsilon(\cdot)}-\xi^{p_\varepsilon(\cdot)}\right|\leqslant A\,\eta\left(1+\xi^{p_\varepsilon(\cdot)-1}+\eta^{p_\varepsilon(\cdot)-1}\right),\tag{81}$$

where $\xi,\eta\geqslant 0$ and $A=A(\mathsf{p}^{(-)},\mathsf{p}^{(+)})$ is a constant. Then from condition **(A3)**, (66) with $\mathbf{b}_\alpha=\nabla u(x^\alpha)$, (79), (81), and the regularity of u,f,p_ε,p_0, for sufficiently small ε and $h\to 0$, we get:

$$\begin{aligned}&\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}\mathsf{F}_\varepsilon(x,w^\varepsilon_h,\nabla w^\varepsilon_h)\,dx\\&=\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}\left\{\frac{1}{p_\varepsilon(x)}|\nabla w^\varepsilon_h|^{p_\varepsilon(x)}+\frac{1}{p_0(x)}|u|^{p_0(x)}-f\,u\right\}dx+o(h^n).\end{aligned}\tag{82}$$

Now it follows from (80), (82), the regularity properties of functions u,f, and the definition (20) that, for sufficiently small ε and $h\to 0$,

$$\begin{aligned}\int\limits_{K^{\alpha}_{h'}\cap\Omega^{\varepsilon}}\mathsf{F}_\varepsilon(x,w^\varepsilon_h,\nabla w^\varepsilon_h)\,dx\leqslant h^n\,\frac{c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x^\alpha,\nabla u(x^\alpha))}{h^n}\\+h^n\left\{\frac{1}{p_0(x^\alpha)}|u(x^\alpha)|^{p_0(x^\alpha)}-f(x^\alpha)\,u(x^\alpha)\right\}\frac{\text{meas}\,(K^\alpha_h\cap\Omega^\varepsilon)}{h^n}+o(h^n).\end{aligned}\tag{83}$$

Now we take the union in (83) over all cubes and pass to the limit first as $\varepsilon\to 0$ and then as $h\to 0$. Taking into account (78) and the condition **(C3)**, we obtain that for any smooth function u

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,J^\varepsilon[w^\varepsilon_h]\leqslant J_{\text{hom}}[u].\tag{84}$$

This inequality also holds true for any $u\in W^{1,p_0(\cdot)}(\Omega)$. This fact immediately follows from density arguments and the continuity of the homogenized functional in $W^{1,p_0(\cdot)}(\Omega)$ (cf. Lemma 4.2).

Step 2. Construction of the recovery sequence. Consider the sequence $\{w^\varepsilon_h\}$ defined by (69). Let $\hat\varepsilon(h)$ be a decreasing function such that $\lim\limits_{h\to 0}\hat\varepsilon(h)=0$. We set

$$h(\varepsilon)=\frac{1}{j}\quad\text{for}\quad\hat\varepsilon\left(\frac{1}{j+1}\right)\leqslant\varepsilon\leqslant\hat\varepsilon\left(\frac{1}{j}\right),\ j=1,2,...$$

and

$$w^\varepsilon=w^\varepsilon_h\Big|_{h=h(\varepsilon)}.$$

It is clear that the sequence w^ε converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ to the function $u=u(x)$ and satisfies the inequality:

$$\overline{\lim_{\varepsilon\to 0}}\,J^\varepsilon[w^\varepsilon]\leqslant J_{\text{hom}}[u].\tag{85}$$

This completes the proof of the "lim sup"–inequality and of Theorem 3.3.

5.3. **Proof of Theorem 3.4.** Let u^ε be a solution of the variational problem (23). Then from (6), (7), (23), and the properties of the functions f, p_ε we have:

$$\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)} \leqslant C \quad \text{and} \quad A_{p_\varepsilon(\cdot),\Omega^\varepsilon}(u^\varepsilon) + A_{p_\varepsilon(\cdot),\Omega^\varepsilon}(\nabla u^\varepsilon) \leqslant C. \tag{86}$$

It follows from the properties **(F1)** and **(F2)** of the set $\mathcal{F}^\varepsilon$ (see the beginning of Section 3) and the continuity of the function p_0 that we can cover the domain Ω by the finite number of subdomains Θ_α $(\alpha = 1, 2, .., M)$ with nonintersecting interiors such that, for any α, $\partial\Theta_\alpha \cap \mathcal{F}^\varepsilon = \emptyset$ and

$$\mathsf{p}_\alpha^\star = \frac{n\mathsf{p}_\alpha^-}{n - \mathsf{p}_\alpha^-} > \max_{\overline{\Theta}_\alpha} p_0(x) \equiv \mathsf{p}_\alpha^+ \quad \text{with } \mathsf{p}_\alpha^- = \min_{\overline{\Theta}_\alpha} p_0(x). \tag{87}$$

Since the number of the subdomains Θ_α is finite, then inequalities (86) imply that

$$\|u^\varepsilon\|_{W^{1,\mathsf{p}^\alpha_{min}}(\Omega^\varepsilon)} \leqslant C, \tag{88}$$

where $\mathsf{p}^\alpha_{min} = \min_\alpha \mathsf{p}_\alpha^-$. Now it follows from condition **(C3)** that

$$\|u^\varepsilon\|_{W^{1,\mathsf{p}^\alpha_{min}}(\Omega)} \leqslant C \tag{89}$$

and inequalities (87), (89) imply that the family $\{u^\varepsilon\}$ is a compact set in the space $L^{p_0(\cdot)}(\Omega)$. Hence, one can extract a subsequence $\{\mathsf{u}^\varepsilon,\ \varepsilon = \varepsilon_k \to 0\}$ that converges strongly in $L^{p_0(\cdot)}(\Omega)$ to a function $u \in W^{1,p_0(\cdot)}(\Omega)$. Let us show that $u = u(x)$ is a solution of the variational problem (24).

First, it is clear that since u^ε is the solution of the variational problem (23), then

$$J^\varepsilon[u^\varepsilon] \leqslant J^\varepsilon[w_h^\varepsilon],$$

where the function w_h^ε is given by (69). Now the "lim sup"–inequality (84) immediately implies that, for any $w \in W^{1,p_0(\cdot)}(\Omega)$,

$$\overline{\lim_{\varepsilon=\varepsilon_k\to 0}}\, J^\varepsilon[u^\varepsilon] \leqslant J_{\text{hom}}[w]. \tag{90}$$

On the other hand, from the "lim inf"–inequality, we have:

$$\underline{\lim_{\varepsilon=\varepsilon_k\to 0}}\, J^\varepsilon[u^\varepsilon] \geqslant J_{\text{hom}}[u]. \tag{91}$$

Now inequalities (90) and (91) imply that if a subsequence of solutions of problem (23) converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ to a function $u = u(x)$, then, for any $w \in W^{1,p_0(\cdot)}(\Omega)$,

$$J_{\text{hom}}[u] \leqslant J_{\text{hom}}[w] \tag{92}$$

and u is the solution of (24). Since this problem has a unique solution, then the whole sequence of solutions of problem (23) converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$ to the function u.

This completes the proof of Theorem 3.4.

6. **Periodic and locally periodic examples.** Theorems 3.3 and 3.4 of Section 3 provide sufficient conditions for the existence of the Γ–limit functional (22) and for the convergence of minimizers of the variational problem (23) to the minimizer of the homogenized variational problem (24). It is important to show that the class of functions which satisfy the conditions of these theorems is not empty. The goal of this Section is to prove that for periodic and locally periodic media all conditions of the above mentioned theorems are satisfied and to compute the coefficients of the homogenized functional (22) in terms of solutions of auxiliary cell problems.

In fact, we will prove that conditions **(C1), (C3)** are always satisfied in the periodic case if the boundary of inclusions is regular enough (see Proposition 1), and that the extension condition **(C2)** can also be replaced with the assumption on the regularity of the inclusions geometry (see the beginning of Appendix).

6.1. **A periodic example.** Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$ with sufficiently smooth boundary. We assume that, in the standard periodic cell $Y = (-1/2, 1/2)^n$, there is an obstacle $F \subset Y$ being an open set with a sufficiently smooth boundary ∂F such that $\bar{F} \subset Y$. We assume that this geometry is repeated periodically in the whole $\mathbb{R}^n$. The geometric structure within the domain Ω is then obtained by intersecting the ε–multiple of this geometry with Ω, ε being a small positive parameter. Let $\{x^{\mathbf{k},\varepsilon}\}$ be an ε–periodic grid in $\mathbb{R}^n$: $x^{\mathbf{k},\varepsilon} = \varepsilon\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^n$. Then we define $\mathcal{F}^\varepsilon$ as the union of sets $\mathcal{F}^\varepsilon_{\mathbf{k}} \subset K^{\mathbf{k}}_\varepsilon$ obtained from εF by translations with vectors $\varepsilon\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^n$, i.e.,

$$\mathcal{F}^\varepsilon_{\mathbf{k}} = \varepsilon\mathbf{k} + \varepsilon F, \quad \mathcal{F}^\varepsilon = \bigcup_{\mathbf{k}} (\mathcal{F}^\varepsilon_{\mathbf{k}} \cap \Omega), \quad \text{and} \quad \Omega^\varepsilon_B = \Omega \setminus \overline{\mathcal{F}^\varepsilon}, \tag{93}$$

and $K^{\mathbf{k}}_\varepsilon = \varepsilon\mathbf{k} + \varepsilon Y$.

Notice that the geometry of the inclusions having a nontrivial intersection with the domain boundary, might be rather complicated. In particular, the extension condition **(C2)** might be violated for these inclusions. To avoid these technical difficulties we will often assume below that the domain Ω is not perforated in a small neighbourhood of its boundary $\partial\Omega$.

Denote by $\mathcal{K}_\varepsilon$ the union of $\mathbf{k} \in \mathbb{Z}^n$ such that $K^{\mathbf{k}}_\varepsilon \subset \Omega$, and set

$$\tilde{\Omega}^\varepsilon = \bigcup_{\mathbf{k}\in\mathcal{K}_\varepsilon} (K^{\mathbf{k}}_\varepsilon \cap \mathcal{F}^\varepsilon_{\mathbf{k}}), \qquad \Omega^\varepsilon = \Omega \setminus \bigcup_{\mathbf{k}\in\mathcal{K}_\varepsilon} \mathcal{F}^\varepsilon_{\mathbf{k}}. \tag{94}$$

Let a family of continuous functions $\{p_\varepsilon\}_{(\varepsilon>0)}$ and a function p_0 satisfy conditions **(A1)–(A3)** from Section 3.

On the space $L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ we define the functional $J^\varepsilon : L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon) \longrightarrow \mathbb{R} \cup \{+\infty\}$:

$$J^\varepsilon[u] = \begin{cases} \displaystyle\int\limits_{\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} \left(|\nabla u|^{p_\varepsilon(x)} + |u|^{p_\varepsilon(x)} \right) - f(x)u \right\} dx, & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon); \\ +\infty, \quad \text{otherwise,} & \end{cases} \tag{95}$$

where $f \in C(\overline{\Omega})$, and on the space $L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon_B)$ - the functional

$$J^\varepsilon_B[u] = \begin{cases} \displaystyle\int\limits_{\Omega^\varepsilon_B} \left\{ \frac{1}{p_\varepsilon(x)} \left(|\nabla u|^{p_\varepsilon(x)} + |u|^{p_\varepsilon(x)} \right) - f(x)u \right\} dx, & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon_B); \\ +\infty, \quad \text{otherwise,} & \end{cases} \tag{96}$$

We study the asymptotic behavior of the functional J^ε and its minimizer as $\varepsilon \to 0$. To formulate the main result of this section we will introduce some notation. We denote by $U^{\mathbf{b}} = U^{\mathbf{b}}(p, y)$ a minimizer of the following variational problem:

$$\int\limits_{Y^\star} \frac{1}{p} \left| \nabla_y U^{\mathbf{b}} - \mathbf{b} \right|^p \, dy \longrightarrow \min, \quad u \in W^{1,p}_{\text{per}}(Y^\star). \tag{97}$$

where $Y^\star = Y \setminus \overline{F}$, and $\mathbf{b} = (b_1, b_2, ..., b_n)$ is a vector in $\mathbb{R}^n$, $p > 1$ is a parameter.

If $p \geqslant 2$ then the solution $U^{\mathbf{b}}$ coincides with a unique solution in $W^{1,p}_{\mathrm{per}}(Y^\star)$ of the following cell problem:

$$\begin{cases} \mathrm{div}_y \left(\left|\nabla_y U^{\mathbf{b}}\right|^{p-2} \nabla_y U^{\mathbf{b}} \right) = 0 \quad \text{in } Y^\star; \\ \left(\left|\nabla_y U^{\mathbf{b}}\right|^{p-2} \nabla_y U^{\mathbf{b}} - \mathbf{b}, \vec{\nu} \right) = 0 \quad \text{on } \partial F; \\ y \to U^{\mathbf{b}}(y) \quad Y-\text{periodic}, \end{cases} \tag{98}$$

here $\vec{\nu}$ is the outward normal to ∂F.

The following result holds.

Theorem 6.1. *The sequence of functionals* $\{J^\varepsilon\}_{(\varepsilon>0)}$ *defined in (95),* $\Gamma_{p_0(\cdot)}$*– converges to the functional* $J_{\mathrm{hom}} : L^{p_0(\cdot)}(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ *given by:*

$$J_{\mathrm{hom}}[u] = \begin{cases} \int\limits_\Omega \left\{ T(x,\nabla u) + \dfrac{\rho}{p_0(x)}|u|^{p_0(x)} - \rho f(x)\, u \right\} dx, & \text{if } u \in W^{1,p_0(\cdot)}(\Omega); \\ +\infty, & \text{otherwise,} \end{cases} \tag{99}$$

where

$$\rho = \mathrm{meas}\, Y^\star \quad \text{and} \quad T(x,\mathbf{b}) = \int\limits_{Y^\star} \frac{1}{p_0(x)} \left|\nabla_y U^{\mathbf{b}}(p_0(x), y) - \mathbf{b}\right|^{p_0(x)} dy. \tag{100}$$

Moreover, a minimizer u^ε *of the functional (95) converges strongly in the space* $L^{p_0(\cdot)}(\Omega^\varepsilon)$ *to* u *the minimizer of the homogenized functional (99).*

The sequence $\{J^\varepsilon_B\}_{(\varepsilon>0)}$ *defined in (96), also* $\Gamma_{p_0(\cdot)}$*–converges to the functional* J_{hom}.

Remark 4. In general, the existence of a minimizer of the functional (96) is a complicated problem because the geometry of Ω^ε_B might be rather complex.

6.1.1. *Proof of Theorem 6.1.* Theorem 6.1 can be proved in two different ways. One of them is to check that under the assumptions of Theorem 6.1 conditions **(C1)** – **(C3)** are satisfied and that the characteristics introduced in conditions **(C1)** and **(C3)** coincide with those defined in (100). In order to make the results of Theorems 3.3 and 6.1 compatible, we will prove in this section that the mentioned characteristics do coincide.

On the other hand, in the periodic case the direct Γ-convergence techniques apply. This allows us to simplify the proof and to obtain formula (100) by means of Γ-convergence approach used in periodic homogenization. In this connection, we will provide below the proof of "liminf" inequality for the stated in Theorem 6.1 Γ-convergence. Since the proof of "limsup" inequality and of the convergence of minimizers is standard, it will be omitted.

Let us show that conditions **(C1)**–**(C3)** are satisfied in the periodic case under consideration. The following result holds.

Proposition 1. *Let* Ω^ε *be a perforated domain defined in (94). Then conditions* **(C1)**–**(C3)** *of Theorem 3.4 are fulfilled and*

$$\rho = \mathrm{meas}\, Y^\star \quad \text{and} \quad T(x,\mathbf{b}) = \int\limits_{Y^\star} \frac{1}{p_0(x)} \left|\nabla_y U^{\mathbf{b}}(p_0(x), y) - \mathbf{b}\right|^{p_0(x)} dy. \tag{101}$$

The proof of Proposition 1 will be presented in Appendix. Due to this proposition, Theorem 6.1 it a consequence of Theorems 3.3 and 3.4.

We proceed with the direct proof of Theorem 6.1.

$\Gamma_{p_0(\cdot)}$**–convergence of the family** $\{J^\varepsilon\}_{(\varepsilon>0)}$**.** First we justify the "lim inf"–inequality for the functional J^ε defined in (95). To this end we cover the domain Ω with the cubes $K_h^\alpha = x^\alpha + [-h/2, h/2]^n$ centered at the points $\{x^\alpha\}$, where $\{x^\alpha\}$ is a set of points in Ω that form a space lattice with a period h in each coordinate directions, we assume that $0 < \varepsilon \ll h \ll 1$. For any α we define the functional $J^\varepsilon_{\alpha,h} : L^{p_\varepsilon(\cdot)}(K_h^\alpha \cap \Omega^\varepsilon) \longrightarrow \mathbb{R} \cup \{+\infty\}$:

$$J^\varepsilon_{\alpha,h}[u] = \begin{cases} \displaystyle\int\limits_{K_h^\alpha \cap \Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} \Big(|\nabla u|^{p_\varepsilon(x)} + |u|^{p_\varepsilon(x)} \Big) - f u \right\} dx, & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(K_h^\alpha \cap \Omega^\varepsilon); \\ +\infty, & \text{otherwise.} \end{cases} \tag{102}$$

Now we introduce the values $p_{\alpha,h}$ such that, for ε sufficiently small,

$$\begin{gathered} p_{\alpha,h} \leqslant \min_{\overline{K_h^\alpha}} \{p_0(x), p_\varepsilon(x)\}, \qquad \max_{\overline{K_h^\alpha}} |p_\varepsilon(x) - p_{\alpha,h}| = o(1), \\ \max_{\overline{K_h^\alpha}} |p_0(x) - p_{\alpha,h}| = o(1), \quad \text{as } h \to 0; \end{gathered} \tag{103}$$

such a choice of $p_{\alpha,h}$ is possible due to the continuity of p_0 and the uniform convergence of p_ε to p_0. We define the functional $\widetilde{J}^\varepsilon_{\alpha,h} : L^{p_\varepsilon(\cdot)}(K_h^\alpha \cap \Omega^\varepsilon) \longrightarrow \mathbb{R} \cup \{+\infty\}$:

$$\widetilde{J}^\varepsilon_{\alpha,h}[u] = \begin{cases} \displaystyle\int\limits_{K_h^\alpha \cap \Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} \Big(|\nabla u|^{p_{\alpha,h}} + |u|^{p_\varepsilon(x)} \Big) - f u \right\} dx, & \text{if } u \in W^{1,p_{\alpha,h}}(K_h^\alpha \cap \Omega^\varepsilon); \\ +\infty, & \text{otherwise.} \end{cases} \tag{104}$$

By arguments similar to those used in the proof of the inequality (64) from Section 5.1 we obtain that, for any $u \in W^{1,p_0(\cdot)}(\Omega)$ and any sequence $\{u^\varepsilon\}$ which converges to the function u strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$,

$$\varliminf_{\varepsilon\to 0} J^\varepsilon_{\alpha,h}[u^\varepsilon] \geqslant \varliminf_{\varepsilon\to 0} \widetilde{J}^\varepsilon_{\alpha,h}[u^\varepsilon]. \tag{105}$$

Since the exponent $p_{\alpha,h}$ in the definition of the functional $\widetilde{J}^\varepsilon_{\alpha,h}$ does not depend on the space variable, the Γ–convergence result for $\widetilde{J}^\varepsilon_{\alpha,h}$ is well known (see, e.g., [15]). In particular, the "lim inf"–inequality for $\widetilde{J}^\varepsilon_{\alpha,h}$ reads:

$$\varliminf_{\varepsilon\to 0} \widetilde{J}^\varepsilon_{\alpha,h}[u^\varepsilon] \geqslant \int\limits_{K_h^\alpha} \left\{ T_{\alpha,h}(\nabla u) + \frac{\rho}{p_{\alpha,h}} |u|^{p_0(x)} - \rho\, f\, u \right\} dx \equiv J_{\alpha,h}[u] \tag{106}$$

for any $\{u^\varepsilon\}$ which converges to u in $L^{p_0(x)}(K_h^\alpha)$; here

$$\rho = \operatorname{meas} Y^\star \quad \text{and} \quad T_{\alpha,h}(\mathbf{b}) = \frac{1}{p_{\alpha,h}} \int\limits_{Y^\star} \left| \nabla_y U^{\mathbf{b}}(p_{\alpha,h}, y) - \mathbf{b} \right|^{p_{\alpha,h}} dy, \tag{107}$$

Now from (105), (106), for any $u \in W^{1,p_0(\cdot)}(\Omega)$ and any sequence $\{u^\varepsilon\}$ which converges to the function u strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$, we get:

$$\varliminf_{\varepsilon\to 0} J^\varepsilon_{\alpha,h}[u^\varepsilon] \geqslant J_{\alpha,h}[u]. \tag{108}$$

Denote

$$\mathcal{T}(p,\mathbf{b}) = \int\limits_{Y^\star} \frac{1}{p} \left|\nabla_y U^{\mathbf{b}}(p,y) - \mathbf{b}\right|^p \, dy.$$

Then $T(x,\mathbf{b}) = \mathcal{T}(p_0(x),\mathbf{b})$. Our analysis will also rely on the following statement.

Lemma 6.2. *The function $\mathcal{T}(p,\mathbf{b})$ is continuous in p, i.e.,*

$$|\mathcal{T}(p+\delta,\mathbf{b}) - \mathcal{T}(p,\mathbf{b})| = o(1) \quad \text{as } \delta \to 0. \tag{109}$$

The proof of the lemma relies on the properties of solution $U^{\mathbf{b}}$ of the cell problem (97), (98) (see Lemma 6.4 in Appendix).

Lemma 6.2 implies that, for $x \in K_h^\alpha$,

$$T_{\alpha,h}(\mathbf{b}) = T(x,\mathbf{b}) + o(1) \quad \text{as } h \to 0. \tag{110}$$

Then from the inequality (108) and the relation (110) we obtain:

$$\varliminf_{\varepsilon\to 0} J^\varepsilon_{\alpha,h}[u^\varepsilon] \geqslant \int\limits_{K_h^\alpha} \left\{ T(x,\nabla u) + \frac{\rho}{p_0(x)}|u|^{p_0(x)} - \rho f\, u \right\} dx \, + o(h^n), \quad \text{as } h \to 0. \tag{111}$$

Summing up (111) over α leads to the desired "lim inf"–inequality for the functional J^ε, i.e., for any $u \in W^{1,p_0(\cdot)}(\Omega)$ and any sequence $\{u^\varepsilon\}$ which converges to u strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon)$,

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] \geqslant J_{\text{hom}}[u], \tag{112}$$

where the functional J_{hom} is given in (99). This completes the proof of the "lim inf"–inequality.

The proof of the "lim sup"–inequality is standard, we should just use the continuity of $\mathcal{T}(p,\mathbf{b})$ stated in Lemma 6.2.

The convergence of minimizers is a consequence of the Γ-convergence result and the extension properties of u^ε. This completes the proof of Theorem 6.1.

6.2. **A locally periodic example.** In this section we generalize the result obtained in Section 6.1 to the case of locally periodic perforation.

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary and let $\{x^{\mathbf{k},\varepsilon}\}$, $x^{\mathbf{k},\varepsilon} = \varepsilon\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^n$, be an ε–periodic grid in the space $\mathbb{R}^n$. We cover $\mathbb{R}^n$ with cubes $K_\varepsilon^{\mathbf{k}} = \varepsilon\mathbf{k} + \varepsilon Y$, $Y = [-1/2, 1/2]^n$

Let $\Theta(x,y) : \overline{\Omega} \times Y \mapsto Y$ be a smooth function such that for each $x \in \overline{\Omega}$ the mapping $\Theta(x,\cdot) : Y \mapsto Y$ is a diffeomorphism which keeps all the points of ∂Y fixed. Given a smooth domain F, $\bar{F} \subset Y$, we introduce

$$\mathcal{F}_{\mathbf{k}}^\varepsilon = \left\{ x \in K_\varepsilon^{\mathbf{k}} \,:\, \Theta\Big(\varepsilon\mathbf{k}, \frac{x - \varepsilon\mathbf{k}}{\varepsilon}\Big) \in F \right\}, \qquad \mathbf{k} \in \mathbb{Z}^n.$$

Then, as in previous section, the symbol $\mathcal{K}_\varepsilon$ stands for the union of $\mathbf{k} \in \mathbb{Z}^n$ such that $K_\varepsilon^{\mathbf{k}} \subset \Omega$. We set

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{\mathbf{k}\in\mathcal{K}_\varepsilon} \mathcal{F}_{\mathbf{k}}^\varepsilon \tag{113}$$

and study the asymptotic behaviour of functionals J^ε defined in (95) and their minimizers, as $\varepsilon \to 0$.

Before we formulate the main result of this Section, we introduce some notation. Let $U^{\mathbf{b}} = U^{\mathbf{b}}(p,x,y)$ be a minimizer in $W^{1,p_0(\cdot)}_{\rm per}(Y^\star_x)$ of the following variational problem:

$$\int\limits_{Y^\star_x} \frac{1}{p}\left|\nabla_y U^{\mathbf{b}} - \mathbf{b}\right|^p\,dy \longrightarrow \min, \quad u \in W^{1,p}_{\rm per}(Y^\star_x). \tag{114}$$

where $Y^\star_x = \{y \in Y \,:\, \Theta(x,y) \in Y \setminus \overline{F}\}$, and $\mathbf{b}$ is a vector in $\mathbb{R}^n$. If $p \geqslant 2$, then $U^{\mathbf{b}}(p,x,y)$ solves the problem

$$\begin{cases} \operatorname{div}_y\left(\left|\nabla_y U^{\mathbf{b}}\right|^{p-2}\nabla_y U^{\mathbf{b}}\right) = 0 & \text{in } Y^\star_x;\\ \left(\left|\nabla_y U^{\mathbf{b}}\right|^{p-2}\nabla_y U^{\mathbf{b}} - \mathbf{b}, \vec{\nu}\right) = 0 & \text{on } \partial F_x;\\ y \to U^{\mathbf{b}}(y) \quad Y-\text{periodic}, \end{cases} \tag{115}$$

here $x \in \Omega$ is a parameter, $F_x = \{y \in Y \,:\, \Theta(x,y) \in F\}$, and $\vec{\nu}$ is the outward normal vector to ∂F_x.

The main result of this section is given by the following theorem.

Theorem 6.3. *The functional J^ε $\Gamma_{p_0(\cdot)}$–converges to the functional $J_{\rm hom}$: $W^{1,p_0(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ given by:*

$$J_{\rm hom}[u] \equiv \int\limits_\Omega \left\{T(x,\nabla u) + \frac{\rho(x)}{p_0(x)}|u|^{p_0(x)} - \rho(x)\,f(x)\,u\right\}dx, \tag{116}$$

where

$$\rho = \operatorname{meas} Y^\star_x \quad \text{and} \quad T(x,\mathbf{b}) = \int\limits_{Y^\star_x} \frac{1}{p_0(x)}\left|\nabla U^{\mathbf{b}}(p_0(x),x,y) - \mathbf{b}\right|^{p_0(x)}dy. \tag{117}$$

Moreover, the minimizer u^ε of the functional J^ε converges strongly in the space $L^{p_0(\cdot)}(\Omega^\varepsilon)$ to the minimizer of the homogenized functional $J_{\rm hom}$.

Proof of Theorem 6.3. The proof of Theorem 6.3 is similar to the proof of Theorem 6.1.

Appendix. Proof of Proposition 1. Let K^z_h be an open cube $z + (-h/2, h/2)^n$ centered at $z \in \Omega$ with $0 < \varepsilon \ll h < 1$. As for the condition **(C1)**, it is easy to show that

$$\operatorname{meas}(K^x_h \cap \Omega^\varepsilon) = \frac{h^n}{\varepsilon^n}\operatorname{meas}(\varepsilon\,Y^\star) + o(h^n). \tag{118}$$

Then the condition **(C1)** is satisfied and the function $\rho(x)$ is given by (101).

The fact that the extension condition **(C2)** holds in a disperse periodic medium is well known, see for instance [1].

It remains to prove that the condition **(C3)** is fulfilled with $T(x,\mathbf{b})$ given by formula (101). Let $\{p_\varepsilon(\cdot)\}$ be a family of continuous functions in $\bar\Omega$ which converges to $p_0(\cdot)$ uniformly in $\bar\Omega$. We recall the definition of $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})$. It reads:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) = \inf_{v^\varepsilon} \int\limits_{K^z_h\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon(x)}|\nabla v^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|v^\varepsilon - (x-z,\mathbf{b})|^{p_\varepsilon(x)}\right\}dx, \tag{119}$$

where $0 < \gamma < \mathsf{p}^-$, $\mathbf{b} \in \mathbb{R}^n$, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K^z_h \cap \Omega^\varepsilon)$.

Let $\mathsf{U}^{\mathbf{b}} = \mathsf{U}^{\mathbf{b}}(p,y)$ be a Y–periodic extension of the function $U^{\mathbf{b}}(p,y)$ (the solution of the cell problem (98)) on $\mathbb{R}^n \setminus \overline{\mathfrak{F}}$, where $\mathfrak{F} = \bigcup_{\mathbf{k}\in\mathbb{Z}^n} (k+F)$. The regularity properties of the function $\mathsf{U}^{\mathbf{b}}$ are given by the following lemma.

Lemma 6.4. *The function $\mathsf{U}^{\mathbf{b}}$ possesses the following properties:*

$$\mathsf{U}^{\mathbf{b}} \in L^q(Y^\star) \quad \text{and} \quad \nabla \mathsf{U}^{\mathbf{b}} \in L^{p+\delta}(Y^\star), \tag{120}$$

where $\delta > 0$ and

$$q = \begin{cases} \dfrac{pn}{n-p} & \text{if } p < n, \\ \text{any number} & \text{if } p \geqslant n. \end{cases}$$

In the cube K_h^z we introduce the function

$$W^\varepsilon(x) = (x-z,\mathbf{b}) - \varepsilon\, \mathsf{U}^{\mathbf{b}}\left(p_0(z), \frac{x}{\varepsilon}\right) \quad \text{in } K_h^z. \tag{121}$$

The property (120) of the function $\mathsf{U}^{\mathbf{b}}$ implies that $W^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega^\varepsilon)$ for small enough h. Then

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) \leqslant \mathsf{W}^{\varepsilon,h}(z,\mathbf{b}), \tag{122}$$

where

$$\mathsf{W}^{\varepsilon,h}(z,\mathbf{b}) = \int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} |\nabla W^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)} |W^\varepsilon - (x-z,\mathbf{b})|^{p_\varepsilon(x)} \right\} dx. \tag{123}$$

Using the definition of the function W^ε it is easy to obtain that, for ε sufficiently small,

$$\mathsf{W}^{\varepsilon,h}(z,\mathbf{b}) = h^n \int\limits_{Y^\star} \frac{1}{p_0(z)} \left|\nabla U^{\mathbf{b}}(p_0(z),y) - \mathbf{b}\right|^{p_0(z)} dy + o(h^n) \quad \text{as } h \to 0. \tag{124}$$

This yields the upper bound

$$T(z,\mathbf{b}) \leqslant \int\limits_{Y^\star} \frac{1}{p_0(z)} \left|\nabla U^{\mathbf{b}}(p_0(z),y) - \mathbf{b}\right|^{p_0(z)} dy.$$

In order to estimate the functional $c^{\varepsilon,h}(z,\mathbf{b})$ from below we denote $v^{\varepsilon,h}_{\min} = v^{\varepsilon,h}_{\min}(x)$ the minimizer of (119) and represent this function as follows:

$$v^{\varepsilon,h}_{\min}(x) = (x-z,\mathbf{b}) + \zeta_h^\varepsilon(x). \tag{125}$$

Clearly, the function $\zeta_h^\varepsilon(x)$ provides the minimum in the following problem

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) = \inf_{\zeta(x)} \int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} |\nabla\zeta - \mathbf{b}|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)} |\zeta|^{p_\varepsilon(x)} \right\} dx, \tag{126}$$

with $\zeta \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega^\varepsilon)$.

We want to show that in the definition of the local characteristics $T(z,\mathbf{b})$ one can assume that the test functions are equal to zero at the boundary of K_h^z. Denote

$$\hat{c}^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) = \inf_{\zeta(x)} \int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)} |\nabla\zeta - \mathbf{b}|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)} |\zeta|^{p_\varepsilon(x)} \right\} dx, \quad \zeta\big|_{K_h^z} = 0.$$

Lemma 6.5. *The following inequality holds true*

$$\varlimsup_{h\to 0}\varlimsup_{\varepsilon\to 0} h^{-n}\hat{c}^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) \leqslant \varlimsup_{h\to 0}\varlimsup_{\varepsilon\to 0} h^{-n}c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b})$$

Proof. We use the cut-off function $\phi_h^z(x)$ having the following properties: $\phi_h^z \in C_0^\infty(K_h^z)$; $\phi_h^z = 1$ for $x \in z + [-h/2 + h^{1+\gamma_0}, h/2 - h^{1+\gamma_0}]$, $0 \leqslant \phi_h^z \leqslant 1$, $|\nabla\phi_h^z| \leqslant 2h^{-1-\gamma_0}$. Substituting the function $\zeta(x) = \zeta_h^\varepsilon\phi_h^z(x)$ as a test function in (126) and choosing γ_0 sufficiently small one can easily obtain the estimate

$$\int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon(x)}|\nabla(\zeta_h^\varepsilon\phi_h^z) - \mathbf{b}|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|\zeta_h^\varepsilon\phi_h^z|^{p_\varepsilon(x)}\right\} dx \leqslant c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) + o(h^n)$$

This implies the desired statement. □

Denote $p_\varepsilon^h = \min\limits_{x\in K_h^z} p_\varepsilon(x)$. Notice that

$$\lim_{h\to 0}\varlimsup_{\varepsilon\to 0}\left(\max_{x\in K_h^z} p_\varepsilon(x) - \min_{x\in K_h^z} p_\varepsilon(x)\right) = 0.$$

Using the same arguments as in the proof of (64), we can show that

$$\varlimsup_{h\to 0}\varlimsup_{\varepsilon\to 0} h^{-n}\hat{c}^{\varepsilon,h}_{p_\varepsilon^h}(z,\mathbf{b}) \leqslant \varlimsup_{h\to 0}\varlimsup_{\varepsilon\to 0} h^{-n}\hat{c}^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\mathbf{b}) \tag{127}$$

with

$$\hat{c}^{\varepsilon,h}_{p_\varepsilon^h}(z,\mathbf{b}) = \inf_{\zeta(x)} \int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon^h}|\nabla\zeta - \mathbf{b}|^{p_\varepsilon^h} + h^{-\gamma-p_\varepsilon^h}|\zeta|^{p_\varepsilon^h}\right\} dx, \quad \zeta\big|_{K_h^z} = 0. \tag{128}$$

Without loss of generality we assume that $(h/2)$ is an integer multiplier of ε, so that K_h^z consists of integer number of solid periods. Denote $\hat{\zeta}_\varepsilon^h$ the function which provides the minimum in (128). We extend $\hat{\zeta}_\varepsilon^h$ periodically from K_h^z to the whole $\mathbb{R}^n$, the extended function has period $[-h/2, h/2]^n$. Moreover, since $\hat{\zeta}_\varepsilon^h\big|_{K_h^z} = 0$, the extended function belongs to $W^{1,p_\varepsilon^h}_{\mathrm{loc}}(\mathbb{R}^n)$. We keep for this function the same notation $\hat{\zeta}_\varepsilon^h$.

Now, letting

$$\mathcal{X}_\varepsilon^h(x) = \frac{\varepsilon^n}{h^n}\sum_{\mathbf{k}\in\mathbb{Z}^n\cap[0,h/\varepsilon]^n} \hat{\zeta}_\varepsilon^h(x + \varepsilon\mathbf{k})$$

and considering the convexity of the integrand in (128), we conclude that

$$\int\limits_{K_h^z\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon^h}|\nabla\mathcal{X}_\varepsilon^h - \mathbf{b}|^{p_\varepsilon^h} + h^{-\gamma-p_\varepsilon^h}|\mathcal{X}_\varepsilon^h|^{p_\varepsilon^h}\right\} dx \leqslant \hat{c}^{\varepsilon,h}_{p_\varepsilon^h}(z,\mathbf{b}).$$

It remains to notice that, by construction, the function $\mathcal{X}_\varepsilon^h$ is (εY)-periodic. Therefore,

$$\int\limits_{Y^\star} \frac{1}{p_\varepsilon^h}\left|\nabla U^{\mathbf{b}}(p_\varepsilon^h, y) - \mathbf{b}\right|^{p_\varepsilon^h} dy \leqslant h^{-n}\hat{c}^{\varepsilon,h}_{p_\varepsilon^h}(z,\mathbf{b}). \tag{129}$$

Since $\lim\limits_{h\to 0}\lim\limits_{\varepsilon\to 0} p_\varepsilon^h = p_0(z)$, and by Lemma 6.2 the function $\mathcal{T}(p,\mathbf{b})$ is continuous in p, then

$$\int\limits_{Y^\star} \frac{1}{p_0(z)}\left|\nabla U^{\mathbf{b}}(p_0(z), y) - \mathbf{b}\right|^{p_0(z)} dy = \lim_{h\to 0}\lim_{\varepsilon\to 0}\int\limits_{Y^\star} \frac{1}{p_\varepsilon^h}\left|\nabla U^{\mathbf{b}}(p_\varepsilon^h, y) - \mathbf{b}\right|^{p_\varepsilon^h} dy$$

Combining this relation with (129), (127) and Lemma 6.5, we obtain the desired inequality

$$\int_{Y^\star} \frac{1}{p_0(z)} \left|\nabla U^{\mathbf{b}}(p_0(z), y) - \mathbf{b}\right|^{p_0(z)} \, dy \leqslant T(z, \mathbf{b}).$$

This completes the proof of Proposition 1.

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