Nonlinear Analysis: Real World Applications 61 (2021) 103323

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

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Homogenization of nonisothermal immiscible incompressible two-phase flow in double porosity media

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ARTICLE INFO

Article history:
Received 29 June 2020
Received in revised form 1 March 2021
Accepted 2 March 2021
Available online xxxx

Keywords:
Homogenization
Double porosity media
Nonisothermal two-phase flow
Nonlinear degenerate system
Two-scale convergence
Dilatation operator

ABSTRACT

In this paper, we establish a homogenization result for a nonlinear degenerate system arising from two-phase flow through fractured porous media with periodic microstructure taking into account the temperature effects. The mathematical model is given by a coupled system of two-phase flow equations, and an energy balance equation. The microscopic model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy–Muskat and the capillary pressure laws. The problem is written in terms of the phase formulation, i.e. the saturation of one phase, the pressure of the second phase and the temperature are primary unknowns. The fractured medium consists of periodically repeating homogeneous blocks and fractures, the permeability being rapidly oscillating discontinuous function. Over the matrix domain, the permeability is scaled by ε^2, where ε is the size of a typical porous block. Furthermore, we will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. The model involves highly oscillatory characteristics and internal nonlinear interface conditions accounting for discontinuous capillary pressures. We then show by a rigorous mathematical argument that the solution of this microscopic problem converges as ε tends to zero to the solution of a double-porosity model of the global macroscopic flow. Our techniques make use of the two-scale convergence method combined to extension and dilation operators in the homogenization context. The memory effects of usual double porosity media are reproduced by this model. We show how the effective coefficients of the porous medium are determined in a precise way by certain physical and geometric features of the microscopic fracture domain, the microscopic matrix blocks, and the interface between them.

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https://doi.org/10.1016/j.nonrwa.2021.103323
1468-1218/

1. Introduction

Modeling two-phase flow through fractured porous media is of interest for a wide range of science fields, including energy and environmental engineering. Examples include geothermal systems, oil reservoir engineering, ground-water hydrology, and thermal energy storage, see for instance [1,2]. More recently, modeling multiphase flow received an increasing attention in connection with gas migration in a nuclear waste repository and sequestration of CO_2. Furthermore, fractured rock domains corresponding to the so-called Excavation Damaged Zone (EDZ) receives increasing attention in connection with the behavior of geological isolation of radioactive waste after drilling the wells and shafts, see, e.g., [3]. Efficient heat exploitation strategies from geothermal systems demand for modeling of coupled flow-heat equations on large-scale heterogeneous fractured formation, see, e.g. [4,5] and the references therein.

Dual-porosity models are typically used to simulate multiphase flow in fractured formations. Naturally fractured reservoirs can be modeled by two superimposed continua, a connected fracture system and a system of topologically disconnected matrix blocks. The fracture system has low storage capacity but high conductivity, while the matrix block system has low conductivity and large storage capacity. The majority of fluid transport will occur along flow paths through the fissure system. When the system of fissures is so well developed the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. For more details on the physical formulation of such problems see, e.g., [6–8].

The study of two-phase fluid flow through fractured porous media is a challenging nonlinear multiscale problem with obvious multiphysics features. During the last decade, there appeared a significant body of literature devoted to the modeling of such problems. Many works have been devoted to perform upscaling of two-phase fluid flow in porous media by different approaches. Here we comment only on those publications which are related to the present work. Namely, we restrict ourselves to the mathematical homogenization methods of such models.

The mathematical analysis and the homogenization of the system describing the flow of isothermal single and incompressible two-phase flow in porous media are quite understood. A recent review of the mathematical homogenization methods developed for incompressible single phase flow, incompressible immiscible two-phase flow in porous media and compressible miscible flow in porous media can be viewed in [9–18] and the references therein. The situation is quite different for immiscible compressible two-phase flow in porous media, where, only recently, few results have been obtained, see for instance [19–22] and the references therein.

However, as reported in [23,24], all the aforementioned works do not include any temperature dependence and are restricted to the case where flows are under isothermal conditions, contrary to the present work. This assumption is too restrictive for some realistic problems, such as thermally enhanced oil recovery, geothermal energy production, high-level radioactive waste repositories. For such systems, the temperature dependence is essential. The present work was motivated by a need to incorporate the thermal behavior for such problems. The purpose of this paper is to carry out investigations of a generalized two-phase model for fractured porous media which accounts for varying reservoir temperature to capture flow physics accurately.

In a previous paper [23], we gave an existence result of weak solutions for such a model under some realistic assumptions on the data. A model fully coupling the two-phase flow and heat transfer was developed to investigate immiscible incompressible two-phase flow in porous media under nonisothermal conditions. The corresponding homogenization problem for a single rock type model was proposed and analyzed in [24]. We provided a rigorous derivation of an upscaled model by means of the two-scale convergence. To the best knowledge of the authors, the homogenization of such coupled models under nonisothermal conditions for fractured media is still missing. Closer to the present problem, recently homogenization for a Richard's model arising from the heat and moisture flow through a partially saturated porous medium was obtained in [25].

In [26], a model for nonisothermal single phase flow in double porosity media is constructed by the technique of homogenization. Concerning the numerical simulation of such upscaled models for nonisothermal flows in fractured media, we refer for instance to [5,27,28] and the references therein.

Here, we extend the model by developing a general approach that would allow us to incorporate the temperature effects into two-phase flow in double porosity media made of several types of rocks accounting for discontinuous capillary pressures. More precisely, the fluids are assumed immiscible and incompressible and the solid matrix is non-deformable. The fractured medium consists of periodically repeating homogeneous blocks and fractures, the permeability being highly discontinuous. Over the matrix domain, the permeability is scaled by ε^2, where ε is the size of a typical porous block. Furthermore, we will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. The mathematical model is given by a coupled system of two-phase flow equations, and an energy balance equation. The model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy–Muskat and the capillary pressure laws. The problem is written in terms of the phase formulation, i.e. the saturation of one phase, the pressure of the second phase and the temperature are primary unknowns. The model involves highly oscillatory characteristics and internal nonlinear interface conditions. This leads to a system of three coupled nonlinear partial differential equations, a degenerate parabolic two-phase flow system and a parabolic diffusion-convection one. As we include temperature effects in fluid flow, the resulting model is much more complex. Including temperature effects requires a new equation: energy conservation. The coupling between these equations raises several issues in the upscaling process that are to be elaborated on. Our aim is to study the macroscopic behavior of solutions of this system of equations as ε tends to zero and give a rigorous mathematical derivation of upscaled models by means of the two-scale convergence method combined with the dilation technique. The major difficulties related to this model are in the nonlinear degenerate structure of the two-phase flow equations, as well as in the coupling in the system and the transient fracture–matrix interactions. The equations in the matrix blocks are analyzed by the dilatation technique and the passage to the limit in these equations is achieved by an adaptation of the monotonicity argument developed in [29]. It should be emphasized that in the nonisothermal model both a priori estimates and passage to the limit in the two-phase flow equations is rather involved, especially in the part related to the fracture–matrix interactions. Thus, we extend the results of our previous paper [24] to the case of highly heterogeneous porous media with discontinuous capillary pressures for nonisothermal immiscible incompressible two-phase flow through fractured porous media.

The rest of the paper is organized as follows. Section 2 is devoted to the formulation of the homogenization problem considered in the paper. Then we recall the notion of the so-called nonisothermal global pressure. We also provide the assumptions on the data and we give the definition of a weak solution to our problem. In Section 3 we obtain the basic *a priori* estimates for a weak solution of the problem under consideration. Namely, for the phase pressures, the saturation, and the temperature. In Section 4 we establish the compactness and the two-scale convergence results which will be used in the proof of the main result of the paper. Namely, first, we extend the global pressure and the saturation functions defined in the fissure system. Then in Section 4.2 we obtain the compactness result for the family of the extended saturation functions and, using the ideas from [24], we also establish the compactness result for the family of temperature functions. In Section 4.3 we make use of the compactness results from the previous Subsections in order to prove rigorously the convergence of the homogenization process by means of two-scale convergence approach (see, e.g., [30]). Section 5 is devoted to the properties of the dilated functions defined in the matrix blocks. Namely, first, in Section 5.1 we introduce the definition of the dilation operator and describe its main properties. Then in Section 5.2 we obtain the equations for the dilated saturation and the global pressure functions, the corresponding uniform estimates and the convergence results. In Section 6 we formulate the main result of the paper and we complete its proof. The resulting homogenized problem is a dual-porosity type model that contains a term representing memory effects which could be seen as a source term or as a time delay. The proof is done in several steps. The main difficulty with the phase pressure functions is that they do not possess the uniform H^1-estimates. To overcome the difficulties, we pass to the equivalent formulation of the problem in terms of the global pressure, saturation, and the temperature functions. Then using the convergence and compactness results from Section 4 we pass to the limit in the corresponding equations. This is done in Section 6.1. The homogenized equations contain some additional nonlocal in time terms which depend on the saturation function in the matrix block. Section 6.2 is devoted to the derivation of the effective equations in terms of the global pressure, the saturation and the temperature. In order, to obtain the homogenized phase pressures we make use of the change of the unknown functions. Then we rewrite the limit system obtained in terms of the global pressure and the saturation in terms of the homogenized phase pressures (see Section 6.3). The passage to the limit in the matrix blocks makes use of the dilation operator. Then in Section 6.4 we pass to the equivalent problem for the imbibition equation and, finally, obtain the local problem in the matrix block. Lastly, some concluding remarks are forwarded.

2. Formulation of the problem

In this section we formulate the homogenization problem. First, in Section 2.1 we introduce the adimensionalized system of equations describing a nonisothermal immiscible incompressible two-phase flow in a reservoir with double porosity. Then in Section 2.2 we define the so-called *nonisothermal global pressure*. Section 2.3 provides the main assumptions on the data. Finally, in Section 2.4 we give the definition of a weak solution to our problem.

2.1. Governing equations

We consider a reservoir $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ which is assumed to be a bounded, connected Lipschitz domain with a periodic microstructure. More precisely, we scale a given periodic structure in $\mathbb{R}^d$ with a scaling parameter ε which represents the ratio of the cell size to the size of the whole region Ω. We assume that $0 < \varepsilon \ll 1$ that is ε is a small positive parameter tending to zero. Let $Y \stackrel{\text{def}}{=} (0,1)^d$ be a basic cell of a fractured porous medium. For the sake of simplicity and without loss of generality, we assume that Y is made up of two homogeneous porous media Y_{m} and Y_{f} corresponding to the parties of the mesoscopic domain occupied by the matrix block and the fracture, respectively. Thus $Y = Y_{\mathsf{m}} \cup Y_{\mathsf{f}} \cup \Gamma_{\mathsf{fm}}$, where Γ_{fm} denotes the interface between the two media. Let Ω_ℓ^ε with $\ell =$ "f" or "m" denotes the open set filled with the porous medium ℓ. Then $\Omega = \Omega_{\mathsf{m}}^\varepsilon \cup \Gamma_{\mathsf{fm}}^\varepsilon \cup \Omega_{\mathsf{f}}^\varepsilon$, where $\Gamma_{\mathsf{fm}}^\varepsilon \stackrel{\text{def}}{=} \partial\Omega_{\mathsf{f}}^\varepsilon \cap \partial\Omega_{\mathsf{m}}^\varepsilon \cap \Omega$ and the subscripts m and f refer to the matrix and fracture, respectively. For the sake of simplicity, we assume that $\Omega_{\mathsf{m}}^\varepsilon \cap \partial\Omega = \emptyset$. We also introduce the notation:

$$\Omega_{\mathcal{T}} \stackrel{\text{def}}{=} \Omega \times (0,\mathcal{T}), \quad \Omega_{\ell,\mathcal{T}}^\varepsilon \stackrel{\text{def}}{=} \Omega_\ell^\varepsilon \times (0,\mathcal{T}), \quad \Sigma_{\mathcal{T}}^\varepsilon \stackrel{\text{def}}{=} \Gamma_{\mathsf{fm}}^\varepsilon \times (0,\mathcal{T}) \quad (\mathcal{T} > 0 \text{ is fixed}). \tag{2.1}$$

We focus our attention on a model where both fluids are assumed incompressible, that is the densities of the wetting and non-wetting phases are strictly positive constants, and the skeleton density is also assumed to be a strictly positive constant. It is assumed that no exchange of mass between the two phases can take place and each phase remains homogeneous. Then the flow can be described in terms of the following adimensionalized characteristics: $\Phi^\varepsilon(x) = \Phi(x, \frac{x}{\varepsilon})$ is the porosity of the medium Ω; $K^\varepsilon(x) = K(x, \frac{x}{\varepsilon})$ is the absolute permeability tensor of Ω; ϱ_w, ϱ_n, and ϱ_s are the mass densities of the wetting and non-wetting phases, and the skeleton, respectively; $S_\ell^\varepsilon = S_\ell^\varepsilon(x,t)$ is the saturation of the wetting phase in $\Omega_{\ell,\mathcal{T}}^\varepsilon$; $k_{r,w}^{(\ell)}(S_\ell^\varepsilon)$ and $k_{r,n}^{(\ell)}(S_\ell^\varepsilon)$ are the relative permeabilities of the wetting and non-wetting phases in the medium $\Omega_{\ell,\mathcal{T}}^\varepsilon$ $(\ell = \mathsf{f}, \mathsf{m})$; $p_{\ell,w}^\varepsilon = p_{\ell,w}^\varepsilon(x,t)$, $p_{\ell,n}^\varepsilon = p_{\ell,n}^\varepsilon(x,t)$ are the pressures of wetting and non-wetting phases in $\Omega_{\ell,\mathcal{T}}^\varepsilon$;

$P_{\ell,c}(S^\varepsilon_\ell)$ is the capillary pressure in $\Omega^\varepsilon_{\ell,\mathfrak{T}}$; $T^\varepsilon = T^\varepsilon(x,t)$ is the temperature; $\mathbb{C}_w, \mathbb{C}_n$ are the constant heat capacities of the wetting and non-wetting phases, respectively; $\mathbb{C}^\varepsilon_s(x) = \mathbb{C}_s(\frac{x}{\varepsilon})$ is the heat capacity of the solid part; $\mu^\varepsilon_w = \mu_w(T^\varepsilon)$ and $\mu^\varepsilon_n = \mu_n(T^\varepsilon)$ are the viscosities of the wetting and non-wetting phases, respectively; $k^\varepsilon_T(x) = k_T(\frac{x}{\varepsilon})$ is the thermal conductivity of the combined three-phase system. For all $S, T \in \mathbb{R}$, the mobility functions $\lambda_{\ell,w}$, $\lambda_{\ell,n}$ are defined by:

$$\lambda_{\ell,w}(S,T) \stackrel{\text{def}}{=} \frac{k^{(\ell)}_{r,w}(S)}{\mu_w(T)}; \quad \lambda_{\ell,n}(S,T) \stackrel{\text{def}}{=} \frac{k^{(\ell)}_{r,n}(S)}{\mu_n(T)} \quad (\ell = \mathsf{f}, \mathsf{m}). \tag{2.2}$$

In what follows, each function $f^\varepsilon := S^\varepsilon, p^\varepsilon_w, p^\varepsilon_n, T^\varepsilon, \mathbb{C}^\varepsilon_s$ is defined as:

$$f^\varepsilon \stackrel{\text{def}}{=} f^\varepsilon_{\mathsf{f}}(x,t)\, \mathbf{1}^\varepsilon_{\mathsf{f}}(x) + f^\varepsilon_{\mathsf{m}}(x,t)\, \mathbf{1}^\varepsilon_{\mathsf{m}}(x),$$

where $\mathbf{1}^\varepsilon_\ell(x) = \mathbf{1}_\ell(\frac{x}{\varepsilon})$ is the characteristic function of the subdomain Ω^ε_ℓ for $\ell = \mathsf{f}, \mathsf{m}$. In a similar way, we define the functions $\lambda_w, \lambda_n, P_c$. Namely,

$$\lambda_\sigma\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right) \stackrel{\text{def}}{=} \lambda_{\mathsf{f},\sigma}\left(S^\varepsilon_{\mathsf{f}}, T^\varepsilon_{\mathsf{f}}\right)\, \mathbf{1}^\varepsilon_{\mathsf{f}}(x) + \lambda_{\mathsf{m},\sigma}\left(S^\varepsilon_{\mathsf{m}}, T^\varepsilon_{\mathsf{m}}\right)\, \mathbf{1}^\varepsilon_{\mathsf{m}}(x) \quad (\sigma = w, n);$$

$$P_c\left(\frac{x}{\varepsilon}, S^\varepsilon\right) \stackrel{\text{def}}{=} P_{\mathsf{f},c}\left(S^\varepsilon_{\mathsf{f}}\right)\, \mathbf{1}^\varepsilon_{\mathsf{f}}(x) + P_{\mathsf{m},c}\left(S^\varepsilon_{\mathsf{m}}\right)\, \mathbf{1}^\varepsilon_{\mathsf{m}}(x).$$

In what follows, for the sake of presentation simplicity, we neglect the source terms. Then the conservation of mass in each phase and conservation of energy relations read (see, e.g., [31–33]):

$$\begin{cases} 0 \leqslant S^\varepsilon \leqslant 1 \quad \text{in } \Omega_{\mathfrak{T}}; \\ \Phi^\varepsilon \frac{\partial S^\varepsilon}{\partial t} - \operatorname{div}\Big\{ K^\varepsilon \lambda_w\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)\left(\nabla p^\varepsilon_w - \vec{r}_w\right)\Big\} = 0 \quad \text{in } \Omega_{\mathfrak{T}}; \\ -\Phi^\varepsilon \frac{\partial S^\varepsilon}{\partial t} - \operatorname{div}\Big\{ K^\varepsilon \lambda_n\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)\left(\nabla p^\varepsilon_n - \vec{r}_n\right)\Big\} = 0 \quad \text{in } \Omega_{\mathfrak{T}}; \\ \frac{\partial \mathbf{\Psi}^\varepsilon}{\partial t} - \operatorname{div}\Big\{ K^\varepsilon T^\varepsilon \left[\mathbb{C}_w \lambda_w\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)\left(\nabla p^\varepsilon_w - \vec{r}_w\right) + \mathbb{C}_n \lambda_n\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)\left(\nabla p^\varepsilon_n - \vec{r}_n\right)\right]\Big\} \\ \qquad\qquad -\operatorname{div}\Big\{ k^\varepsilon_T \nabla T^\varepsilon \Big\} = 0 \quad \text{in } \Omega_{\mathfrak{T}}; \\ P_c\left(\frac{x}{\varepsilon}, S^\varepsilon\right) = p^\varepsilon_n - p^\varepsilon_w \quad \text{in } \Omega_{\mathfrak{T}}, \end{cases} \tag{2.3}$$

where $\vec{r}_w \stackrel{\text{def}}{=} \varrho_w\, \vec{g}$, $\vec{r}_n \stackrel{\text{def}}{=} \varrho_n\, \vec{g}$ with $\vec{g}$ being the gravity vector, and

$$\mathbf{\Psi}^\varepsilon\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right) \stackrel{\text{def}}{=} \Big\{\left(\mathbb{C}_w S^\varepsilon + \mathbb{C}_n[1 - S^\varepsilon]\right)\Phi^\varepsilon + \mathbb{C}^\varepsilon_s\left[1 - \Phi^\varepsilon\right]\Big\}\, T^\varepsilon. \tag{2.4}$$

The exact form of the porosity function, the function $\mathbb{C}^\varepsilon_s$, the thermal conductivity, and the absolute permeability tensor corresponding to the double porosity model studied in this paper will be specified in conditions (A.1), (A.2), (A.3), and (A.4) in Section 2.3.

The model (2.3) has to be completed with appropriate interface, boundary and initial conditions.

The interface conditions on $\Sigma^\varepsilon_{\mathfrak{T}}$ are the continuity of the phase fluxes, the phase pressures, and the temperature:

$$\begin{cases} \vec{q}^{\,\varepsilon}_{\mathsf{f},w} \cdot \vec{\nu} = \vec{q}^{\,\varepsilon}_{\mathsf{m},w} \cdot \vec{\nu} \quad \text{and} \quad \vec{q}^{\,\varepsilon}_{\mathsf{f},n} \cdot \vec{\nu} = \vec{q}^{\,\varepsilon}_{\mathsf{m},n} \cdot \vec{\nu} \quad \text{on } \Sigma^\varepsilon_{\mathfrak{T}}; \\ k^\varepsilon_T \nabla T^\varepsilon_{\mathsf{f}} \cdot \vec{\nu} = k^\varepsilon_T \nabla T^\varepsilon_{\mathsf{m}} \cdot \vec{\nu} \quad \text{on } \Sigma^\varepsilon_{\mathfrak{T}}; \\ p^\varepsilon_{\mathsf{f},w} = p^\varepsilon_{\mathsf{m},w} \quad \text{and} \quad p^\varepsilon_{\mathsf{f},n} = p^\varepsilon_{\mathsf{m},n} \quad \text{on } \Sigma^\varepsilon_{\mathfrak{T}}, \\ T^\varepsilon_{\mathsf{f}} = T^\varepsilon_{\mathsf{m}} \quad \text{on } \Sigma^\varepsilon_{\mathfrak{T}}, \end{cases} \tag{2.5}$$

where $\Sigma^\varepsilon_{\mathfrak{T}}$ is defined in (2.1), $\vec{\nu}$ is the unit outer normal on $\Gamma^\varepsilon_{\mathsf{fm}}$, and the fluxes $\vec{q}^{\,\varepsilon}_{\ell,w}, \vec{q}^{\,\varepsilon}_{\ell,n}$ are given by:

$$\vec{q}^{\,\varepsilon}_{\ell,\sigma} \stackrel{\text{def}}{=} -K^\varepsilon \lambda_{\ell,\sigma}(S^\varepsilon_{\ell,\sigma}, T^\varepsilon_\ell)\left[\nabla p^\varepsilon_{\ell,\sigma} - \vec{r}_\sigma\right] \quad (\ell = \mathsf{f}, \mathsf{m};\ \sigma = w, n).$$

The boundary $\partial\Omega$ consists of two parts Γ_D and Γ_N such that $\Gamma_D \cap \Gamma_N = \emptyset$, $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $|\Gamma_D| > 0$. Here Γ_D, Γ_N are Lipschitz subsets of $\partial\Omega$. On Γ_D the pressures and the temperature satisfy homogeneous

Dirichlet boundary condition while on Γ_N the corresponding fluxes through the boundary are equal to zero, that is:

$$\begin{cases} p_n^\varepsilon(x,t) = p_w^\varepsilon(x,t) = T^\varepsilon(x,t) = 0 \quad \text{on } \Gamma_D \times (0,\mathfrak{T}); \\ \vec{q}_w^{\,\varepsilon} \cdot \vec{\nu} = \vec{q}_n^{\,\varepsilon} \cdot \vec{\nu} = k_T^\varepsilon \nabla T^\varepsilon \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_N \times (0,\mathfrak{T}), \end{cases} \tag{2.6}$$

where the fluxes $\vec{q}_w^{\,\varepsilon}, \vec{q}_n^{\,\varepsilon}$ are defined as follows:

$$\vec{q}_\sigma^{\,\varepsilon} \stackrel{\text{def}}{=} -K^\varepsilon(x)\lambda_\sigma\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)\left(\nabla p_\sigma^\varepsilon - \vec{r}_\sigma\right) \quad (\sigma = w, n).$$

The initial conditions read:

$$p_w^\varepsilon(x,0) = p_w^0(x), \quad p_n^\varepsilon(x,0) = p_n^0(x), \quad T^\varepsilon(x,0) = T^0(x) \quad \text{in } \Omega. \tag{2.7}$$

2.2. *The concept of nonisothermal global pressure*

In the sequel, we deal with a formulation of problem (2.2)–(2.7) obtained after a proper change of unknown functions. This transformation uses the concept of the so-called *nonisothermal global pressure*. For the isothermal incompressible immiscible two-phase flow, this concept was introduced for the first time in [34,35]. Then it was generalized to the nonisothermal case in [36]. This concept plays a crucial mathematical role for *a priori* estimates and compactness results. Notice that in contrast to the gradients of phase pressures which do not have uniform estimates with respect to ε because of the degeneration of the mobilities (see Lemma 3.1 below), the gradients of the nonisothermal global pressure possess the corresponding uniform estimates (see Lemma 3.4 below). This fact is then used in the proof of the main result of the paper given in Section 6.1. Following [36], for any subdomain Ω_ℓ^ε ($\ell = \mathsf{f}, \mathsf{m}$), we define the nonisothermal global pressure $\mathsf{P}_\ell^\varepsilon$ as follows:

$$p_{\ell,n}^\varepsilon = \mathsf{P}_\ell^\varepsilon + \int_1^{S_\ell^\varepsilon} \frac{\lambda_{\ell,w}}{\lambda_\ell}(\xi, T_\ell^\varepsilon)\, P_{\ell,c}'(\xi)\, d\xi \stackrel{\text{def}}{=} \mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,n}(S_\ell^\varepsilon, T_\ell^\varepsilon), \tag{2.8}$$

where

$$\lambda_\ell(S_\ell^\varepsilon, T_\ell^\varepsilon) \stackrel{\text{def}}{=} \lambda_{\ell,w}(S_\ell^\varepsilon, T_\ell^\varepsilon) + \lambda_{\ell,n}(S_\ell^\varepsilon, T_\ell^\varepsilon).$$

Then using the capillary pressure relation $(2.3)_5$, one can easily calculate that

$$p_{\ell,w}^\varepsilon = \mathsf{P}_\ell^\varepsilon - \int_1^{S_\ell^\varepsilon} \frac{\lambda_{\ell,n}}{\lambda_\ell}(\xi, T_\ell^\varepsilon)\, P_{\ell,c}'(\xi)\, d\xi \stackrel{\text{def}}{=} \mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,w}(S_\ell^\varepsilon, T_\ell^\varepsilon). \tag{2.9}$$

It is easy to see that

$$\nabla p_{\ell,n}^\varepsilon = \nabla \mathsf{P}_\ell^\varepsilon + \frac{\lambda_{\ell,w}}{\lambda_\ell}(S_\ell^\varepsilon, T_\ell^\varepsilon)\, \nabla P_{\ell,c}(S_\ell^\varepsilon) + \mathsf{B}_\ell^\varepsilon\, \nabla T_\ell^\varepsilon$$

and

$$\nabla p_{\ell,w}^\varepsilon = \nabla \mathsf{P}_\ell^\varepsilon - \frac{\lambda_{\ell,n}}{\lambda_\ell}(S_\ell^\varepsilon, T_\ell^\varepsilon)\, \nabla P_{\ell,c}(S_\ell^\varepsilon) + \mathsf{B}_\ell^\varepsilon\, \nabla T_\ell^\varepsilon,$$

where

$$\mathsf{B}_\ell^\varepsilon = \mathsf{B}_\ell(S_\ell^\varepsilon, T_\ell^\varepsilon) \stackrel{\text{def}}{=} \int_1^{S_\ell^\varepsilon} \frac{\partial}{\partial T}\left[\frac{\lambda_{\ell,w}}{\lambda_\ell}(\xi, T_\ell^\varepsilon)\right] P_{\ell,c}'(\xi)\, d\xi. \tag{2.10}$$

As in [23], we introduce the following functions that depend on the saturation only:

$$\beta_\ell(S_\ell^\varepsilon) \stackrel{\text{def}}{=} \int_0^{S_\ell^\varepsilon} \alpha_\ell(\xi)\, d\xi \quad \text{with } \alpha_\ell(\xi) \stackrel{\text{def}}{=} \left(\frac{\frac{k_{r,w}^{(\ell)}(\xi)}{\mathsf{M}_w} \cdot \frac{k_{r,n}^{(\ell)}(\xi)}{\mathsf{M}_n}}{\frac{k_{r,w}^{(\ell)}(\xi)}{\mathsf{m}_w} + \frac{k_{r,n}^{(\ell)}(\xi)}{\mathsf{m}_n}}\right)^{1/2} |P_{\ell,c}'(\xi)|, \tag{2.11}$$

where the constants $\mathsf{M}_w, \mathsf{M}_n, \mathsf{m}_w, \mathsf{m}_n$ are defined in condition (A.7) below. Furthermore, we set

$$\Lambda_0^{(\ell)}(S_\ell^\varepsilon, T_\ell^\varepsilon) \stackrel{\text{def}}{=} \frac{\mathsf{M}_n \mathsf{M}_w}{\mathsf{m}_n \mathsf{m}_w} \frac{k_{r,n}^{(\ell)}(S_\ell^\varepsilon)\mathsf{m}_w + k_{r,w}^{(\ell)}(S_\ell^\varepsilon)\mathsf{m}_n}{k_{r,n}^{(\ell)}(S_\ell^\varepsilon)\mu_w(T_\ell^\varepsilon) + k_{r,w}^{(\ell)}(S_\ell^\varepsilon)\mu_n(T_\ell^\varepsilon)}; \tag{2.12}$$

$$\Lambda_1^{(\ell)}(S_\ell^\varepsilon, T_\ell^\varepsilon) \stackrel{\text{def}}{=} \sqrt{\Lambda_0^{(\ell)}(S_\ell^\varepsilon, T_\ell^\varepsilon)}\sqrt{\frac{\lambda_{\ell,w}(S_\ell^\varepsilon, T_\ell^\varepsilon)\lambda_{\ell,n}(S_\ell^\varepsilon, T_\ell^\varepsilon)}{\lambda_\ell(S_\ell^\varepsilon, T_\ell^\varepsilon)}}. \tag{2.13}$$

Due to (A.6) and (A.7), the function $\Lambda_0^{(\ell)}$ ($\ell = \mathsf{f}, \mathsf{m}$) satisfies the estimates

$$0 < \Lambda_{0,min} \leqslant \Lambda_0^{(\ell)}(S_\ell^\varepsilon, T_\ell^\varepsilon) \leqslant \Lambda_{0,max} < +\infty, \tag{2.14}$$

with some constants $\Lambda_{0,min}$ and $\Lambda_{0,max}$. The function $\Lambda_1^{(\ell)}$ keeps the degenerations in (2.3) as it is zero for $S_\ell^\varepsilon = 0$ and $S_\ell^\varepsilon = 1$. With these new functions we can write:

$$\lambda_{\ell,n}\nabla p_{\ell,n}^\varepsilon = \lambda_{\ell,n}\nabla \mathsf{P}_\ell^\varepsilon - \Lambda_1^{(\ell)}\,\nabla\beta_\ell(S_\ell^\varepsilon) + \lambda_{\ell,n}\mathsf{B}_\ell^\varepsilon\,\nabla T_\ell^\varepsilon; \tag{2.15}$$

$$\lambda_{\ell,w}\nabla p_{\ell,w}^\varepsilon = \lambda_{\ell,w}\nabla \mathsf{P}_\ell^\varepsilon + \Lambda_1^{(\ell)}\,\nabla\beta_\ell(S_\ell^\varepsilon) + \lambda_{\ell,w}\mathsf{B}_\ell^\varepsilon\,\nabla T_\ell^\varepsilon; \tag{2.16}$$

$$\lambda_{\ell,n}\,|\nabla p_{\ell,n}^\varepsilon|^2 + \lambda_{\ell,w}\,|\nabla p_{\ell,w}^\varepsilon|^2 = \lambda_\ell\,|\nabla \mathsf{P}_\ell^\varepsilon|^2 + \Lambda_0^{(\ell)}|\nabla\beta_\ell(S_\ell^\varepsilon)|^2 + \lambda_\ell\,[\mathsf{B}_\ell^\varepsilon]^2\,|\nabla T_\ell^\varepsilon|^2 + 2\,\lambda_\ell\,\mathsf{B}_\ell^\varepsilon\,\nabla \mathsf{P}_\ell^\varepsilon\cdot\nabla T_\ell^\varepsilon. \tag{2.17}$$

In what follows we also make use of the function $\widehat{\beta}$ defined by

$$\widehat{\beta}(s) \stackrel{\text{def}}{=} \int_0^s \widehat{\alpha}(\xi)\,d\xi \quad \text{with } \widehat{\alpha}(s) \stackrel{\text{def}}{=} \min\Big\{\alpha_{\mathsf{f}}(s), \alpha_{\mathsf{m}}\left(\mathcal{P}^{-1}(s)\right)\Big\}, \tag{2.18}$$

where

$$\mathcal{P}(s) \stackrel{\text{def}}{=} (P_{\mathsf{f},c}^{-1}\circ P_{\mathsf{m},c})(s), \quad s\in[0,1]. \tag{2.19}$$

Remark 1. Notice that due to the properties of the capillary pressure functions $P_{\mathsf{f},c}$ and $P_{\mathsf{m},c}$ (see condition (A.5) below), the function $\mathcal{P}$ is a smooth, increasing function with a bounded derivative. Moreover, $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = 1$.

2.3. Main assumptions

The main assumptions on the data (A.1)–(A.10) are listed below. In the rest of this paper we assume that these assumptions hold.

(A.1) The porosity function Φ^ε is given by:

$$\Phi^\varepsilon(x) \stackrel{\text{def}}{=} \Phi_{\mathsf{f}}^\varepsilon(x)\,\mathbf{1}_{\mathsf{f}}^\varepsilon(x) + \Phi_{\mathsf{m}}\left(\frac{x}{\varepsilon}\right)\,\mathbf{1}_{\mathsf{m}}^\varepsilon(x),$$

where $\Phi_{\mathsf{f}}^\varepsilon \in L^\infty(\Omega)$ and there are positive constants $\phi_-^{\mathsf{f}}, \phi_+^{\mathsf{f}}$ independent of ε such that $0 < \phi_-^{\mathsf{f}} \leqslant \Phi_{\mathsf{f}}^\varepsilon(x) \leqslant \phi_+^{\mathsf{f}} < 1$ a.e. in Ω. Moreover,

$$\Phi_{\mathsf{f}}^\varepsilon \longrightarrow \Phi_{\mathsf{f}}^{\mathrm{H}} \quad \text{strongly in } L^2(\Omega).$$

$\Phi_{\mathsf{m}} = \Phi_{\mathsf{m}}(y)$ is Y-periodic, $\Phi_{\mathsf{m}} \in L^\infty(Y)$ and there are positive constants $\phi_-^{\mathsf{m}}, \phi_+^{\mathsf{m}}$ independent of ε and such that $0 < \phi_-^{\mathsf{m}} \leqslant \Phi_{\mathsf{m}}(y) \leqslant \phi_+^{\mathsf{m}} < 1$ a.e. in Y.

(A.2) The absolute permeability tensor $K^\varepsilon(x) = K^\varepsilon(x, \frac{x}{\varepsilon})$ is defined as

$$K^\varepsilon(x,y) \stackrel{\text{def}}{=} K_{\mathsf{f}}(x,y)\,\mathbf{1}_{\mathsf{f}}^\varepsilon(x) + \varepsilon^2\,K_{\mathsf{m}}(x,y)\,\mathbf{1}_{\mathsf{m}}^\varepsilon(x),$$

where $K_\ell \in (L^\infty(\Omega; C_\#(Y)))^{d\times d}$ with $\ell = \mathsf{f}, \mathsf{m}$; the subindex $\#$ indicates that $K_\ell(x,y)$ is periodic in y. Moreover, there exist constants $k_{\min}, k^{\max}$ such that $0 < k_{\min} \le k^{\max}$ and

$$k_{\min}|\xi|^2 \leqslant (K_\ell(x,y)\,\xi,\xi) \leqslant k^{\max}|\xi|^2 \text{ for all } \xi\in\mathbb{R}^d, \text{ a.e. in } \Omega\times Y.$$

(A.3) The heat capacity of the solid part is given by $\mathbb{C}_s^\varepsilon(x) \stackrel{\text{def}}{=} \mathbb{C}_s(\frac{x}{\varepsilon})$ where $\mathbb{C}_s$ is a Y-periodic function,

$$\mathbb{C}_s(y) \stackrel{\text{def}}{=} \mathbb{C}_{\mathsf{f},s}\,\mathbf{1}_{\mathsf{f}}(y) + \mathbb{C}_{\mathsf{m},s}\,\mathbf{1}_{\mathsf{m}}(y) \quad \text{with } 0 < \mathbb{C}_{\mathsf{f},s}, \mathbb{C}_{\mathsf{m},s} < +\infty,$$

where the constants $\mathbb{C}_{\mathsf{f},s}, \mathbb{C}_{\mathsf{m},s}$ do not depend on ε. The fluid heat capacities $\mathbb{C}_w$ and $\mathbb{C}_n$ are strictly positive constants.

(A.4) The thermal conductivity tensor k_T^ε is given by:

$$k_T^\varepsilon(x) \stackrel{\text{def}}{=} k_{\mathsf{f},T}\,\mathbf{1}_{\mathsf{f}}^\varepsilon(x)\,\mathbb{I} + k_{\mathsf{m},T}\,\mathbf{1}_{\mathsf{m}}^\varepsilon(x)\,\mathbb{I},$$

where $\mathbb{I}$ is the unit tensor and $k_{\mathsf{f},T}$, $k_{\mathsf{m},T}$ are positive parameters that do not depend on ε.

(A.5) The capillary pressure function $P_{\ell,c}(s) \in C^1([0,1];\mathbb{R}^+)$ $(\ell = \mathsf{f}, \mathsf{m})$. Moreover, $P'_{\ell,c}(s) < 0$ in $[0,1]$, $P_{\ell,c}(1) = 0$ and $P_{\mathsf{f},c}(0) = P_{\mathsf{m},c}(0)$.

(A.6) The functions $k_{r,w}^{(\ell)}, k_{r,n}^{(\ell)}$, belong to the space $C^1(\mathbb{R})$ and satisfy the following properties: **(i)** $0 \leqslant k_{r,w}^{(\ell)}, k_{r,n}^{(\ell)} \leqslant 1$ on $\mathbb{R}$; **(ii)** $k_{r,w}^{(\ell)}(S) = 0$ for $S \leqslant 0$ and $k_{r,n}^{(\ell)}(S) = 0$ for $S \geqslant 1$; $k_{r,w}^{(\ell)}(S) = 1$ for $S \geqslant 1$ and $k_{r,n}^{(\ell)}(S) = 1$ for $S \leqslant 0$; **(iii)** there is a positive constant k_0 such that $k_{r,w}^{(\ell)}(S) + k_{r,n}^{(\ell)}(S) \geqslant k_0 > 0$ for all $S \in \mathbb{R}$.

(A.7) The viscosities $\mu_w, \mu_n \in C^1(\mathbb{R})$ are functions of the temperature T. Moreover, these functions, for any $T \in \mathbb{R}$, satisfy the following bounds:

$$0 < \mathsf{m}_\sigma \leqslant \mu_\sigma(T) \leqslant \mathsf{M}_\sigma, \quad |\mu'_\sigma(T)| \leqslant \mathsf{M}_\sigma < +\infty \quad (\sigma = w, n).$$

(A.8) The function α_ℓ defined in (2.11) is such that $\alpha_\ell \in C^1([0,1];\mathbb{R}^+)$. Moreover, $\alpha_\ell(0) = \alpha_\ell(1) = 0$ and $\alpha_\ell > 0$ in $(0,1)$. In addition, there exists a constant $C > 0$ such that

$$\alpha_{\mathsf{m}}(s) \leqslant C\alpha_{\mathsf{f}}(s) \quad \text{for all } s \in [0,1]. \tag{2.20}$$

(A.9) The function $\widehat{\beta}^{-1}$, inverse of $\widehat{\beta}$ defined in (2.18), is a Hölder function of order θ on the interval $[0, \widehat{\beta}(1)]$ with $\theta \in (0,1)$. That is there exists a positive constant C_β such that for all $u_1, u_2 \in [0, \widehat{\beta}(1)]$ the following inequality holds:

$$\left|\widehat{\beta}^{-1}(u_1) - \widehat{\beta}^{-1}(u_2)\right| \leqslant C_\beta\,|u_1 - u_2|^\theta.$$

(A.10) The initial data for the phase pressures are such that $p_n^0, p_w^0 \in L^2(\Omega)$ and $0 \leqslant p_n^0 - p_w^0 \leqslant P_c(0)$. The initial data for the saturation $0 \leqslant S^{0,\varepsilon} \leqslant 1$ is defined by the capillary pressure law: $p_n^0 - p_w^0 = P_c(x/\varepsilon, S^{0,\varepsilon})$. The initial temperature $T^0 \in L^\infty(\Omega)$ satisfies the bounds $T_m \leqslant T^0(x) \leqslant T_M$ a.e. in Ω for some constants T_m and T_M, $T_m \leqslant 0 \leqslant T_M$.

Remark 2. According to (A.6) and (A.7) the mobility functions $\lambda_{\ell,w}, \lambda_{\ell,n}$ defined in (2.2) belong to the space $C(\mathbb{R} \times \mathbb{R};\mathbb{R}^+)$ and satisfy the following properties:

(i) $\lambda_{\ell,w}(0,T) = 0$ and $\lambda_{\ell,n}(1,T) = 0$ for all $T \in \mathbb{R}$;

(ii) there is a positive constant L_0 such that

$$\lambda_\ell(S,T) \stackrel{\text{def}}{=} \lambda_{\ell,w}(S,T) + \lambda_{\ell,n}(S,T) \geqslant L_0 \stackrel{\text{def}}{=} \min\{\mathsf{m}_n, \mathsf{m}_w\}\,\frac{k_0}{\mathsf{M}_w\,\mathsf{M}_n} > 0 \quad \text{for all } S, T \in \mathbb{R}. \tag{2.21}$$

It also easily follows from conditions (A.6), (A.7) that

$$\lambda_\ell(S,T) = \frac{k_{r,w}^{(\ell)}(S)}{\mu_w(T)} + \frac{k_{r,n}^{(\ell)}(S)}{\mu_n(T)} \leqslant \frac{1}{\mathsf{m}_w} + \frac{1}{\mathsf{m}_n} \stackrel{\text{def}}{=} L_1. \tag{2.22}$$

Remark 3. Notice that the initial data for the nonisothermal global pressure function, $\mathsf{P}^{0,\varepsilon}$, can be calculated with the help of the definition (2.8) and the initial data for the phase pressures and saturation function determined in condition (A.10).

Remark 4. Since the derivative of the capillary pressure functions is bounded from below and from above (see condition (A.5)), the function $\mathcal{P}$ has a strictly positive derivative, i.e., $\min_{s\in[0,1]} \mathcal{P}'(s) > 0$. Then it holds,

$$C_\beta \left|\widehat{\beta}(\mathcal{P}(S_1)) - \widehat{\beta}(\mathcal{P}(S_2))\right|^\theta \geqslant |\mathcal{P}(S_1) - \mathcal{P}(S_2)| \geqslant \min_{s\in[0,1]} \mathcal{P}'(s)\,|S_1 - S_2|\,,$$

where the function $\widehat{\beta}$ is defined in (2.18). This inequality shows that the inverse of the function $\mathcal{C}(s) \stackrel{\text{def}}{=} (\widehat{\beta}\circ\mathcal{P})(s)$ is Hölder continuous with the same exponent θ as for the inverse function of $\widehat{\beta}$ (see condition (A.9)). The same fact is true for the unbounded capillary pressure functions, with, possibly, smaller exponent θ, if we assume that the inverse of the capillary function $P_{\mathsf{f},c}$ is Hölder continuous.

We also note that the function β_{f} (see definition (2.11)) is Hölder continuous with the exponent θ since

$$C_\beta \left|\beta_f(S_1) - \beta_f(S_2)\right|^\theta = C_\beta \left|\int_{S_2}^{S_1} \alpha_f(s)\,ds\right|^\theta \geqslant C_\beta \left|\int_{S_2}^{S_1} \widehat{\alpha}(s)\,ds\right|^\theta = C_\beta \left|\widehat{\beta}(S_1) - \widehat{\beta}(S_2)\right|^\theta \geqslant |S_1 - S_2|\,,$$

and therefore the same bound is valid for $\beta_{\mathsf{f}}\circ\mathcal{P}$.

Remark 5. If we define $S_\ell^0 \stackrel{\text{def}}{=} P_{\ell,c}^{-1}(p_{\ell,n}^0 - p_{\ell,w}^0)$, for $\ell \in \{\mathsf{f},\mathsf{m}\}$, then the initial saturation defined in condition (**A.10**) is given by

$$S^{0,\varepsilon}(x) = S_{\mathsf{f}}^0(x)\,\mathbf{1}_{\mathsf{f}}^\varepsilon(x) + S_{\mathsf{m}}^0(x)\,\mathbf{1}_{\mathsf{m}}^\varepsilon(x).$$

Remark 6. The assumptions (A.1)–(A.10) are classical and physically meaningful for two-phase flow in porous media. They are similar to the assumptions made in our previous work [23] that dealt with the existence of a weak solution for the studied problem.

2.4. Definition of a weak solution

In order to define a weak solution to problem (2.3)–(2.7) we introduce the following Sobolev space:

$$H^1_{\Gamma_D}(\Omega) \stackrel{\text{def}}{=} \left\{u \in H^1(\Omega)\,:\, u = 0 \text{ on } \Gamma_D\right\}.$$

The space $H^1_{\Gamma_D}(\Omega)$ is a Hilbert space when it is equipped with the norm $\|u\|_{H^1_{\Gamma_D}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

Definition 2.1. We say that a quadruple function $\langle p_w^\varepsilon, p_n^\varepsilon, S^\varepsilon, T^\varepsilon\rangle$ is a weak solution to problem (2.3)–(2.7) if, for any $\varepsilon > 0$,

(i) $0 \leqslant S^\varepsilon \leqslant 1$ a.e. in $\Omega_{\mathcal{T}}$.
(ii) $T_m \leqslant T^\varepsilon \leqslant T_M$ a.e. in $\Omega_{\mathcal{T}}$.
(iii) The functions $p_n^\varepsilon, p_w^\varepsilon, S^\varepsilon, T^\varepsilon$ have the following regularity properties:

$$p_w^\varepsilon, p_n^\varepsilon \in L^2(\Omega_{\mathcal{T}}) \quad \text{and} \quad \sqrt{\lambda_w\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)}\,\nabla p_w^\varepsilon,\ \sqrt{\lambda_n\left(\frac{x}{\varepsilon}, S^\varepsilon, T^\varepsilon\right)}\,\nabla p_n^\varepsilon \in L^2(\Omega_{\mathcal{T}}); \tag{2.23}$$

$$T^\varepsilon \in L^2(0,\mathcal{T}; H^1_{\Gamma_D}(\Omega)). \tag{2.24}$$

(iv) For any $\varphi_w, \varphi_n, \varphi_T \in C^1([0,\mathcal{T}]; H^1(\Omega))$ satisfying $\varphi_w = \varphi_n = \varphi_T = 0$ on $\Gamma_D \times (0,\mathcal{T})$ and $\varphi_w(x,\mathcal{T}) = \varphi_n(x,\mathcal{T}) = \varphi_T(x,\mathcal{T}) = 0$, we have:

Wetting phase pressure equation:

$$
\begin{aligned}
-\int_{\Omega_{\mathcal{T}}} \Phi^{\varepsilon}(x) S^{\varepsilon} \frac{\partial \varphi_w}{\partial t}\, dx\, dt - \int_{\Omega} \Phi^{\varepsilon}(x) S^{0,\varepsilon}(x) \varphi_w(x,0)\, dx \\
+ \int_{\Omega_{\mathcal{T}}} K^{\varepsilon}(x) \lambda_w \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) \left[\nabla p_w^{\varepsilon} - \vec{r}_w\right] \cdot \nabla \varphi_w\, dx\, dt = 0;
\end{aligned} \tag{2.25}
$$

Non-wetting phase pressure equation:

$$
\begin{aligned}
\int_{\Omega_{\mathcal{T}}} \Phi^{\varepsilon}(x) S^{\varepsilon} \frac{\partial \varphi_n}{\partial t}\, dx\, dt + \int_{\Omega} \Phi^{\varepsilon}(x) S^{0,\varepsilon}(x) \varphi_n(x,0)\, dx \\
+ \int_{\Omega_{\mathcal{T}}} K^{\varepsilon}(x) \lambda_n \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) \left[\nabla p_n^{\varepsilon} - \vec{r}_n\right] \cdot \nabla \varphi_n\, dx\, dt = 0;
\end{aligned} \tag{2.26}
$$

Temperature equation:

$$
-\int_{\Omega_{\mathcal{T}}} \mathbf{\Psi}^{\varepsilon} \frac{\partial \varphi_T}{\partial t}\, dx\, dt - \int_{\Omega} \mathbf{\Psi}^{0,\varepsilon} \varphi_T(x,0)\, dx + \int_{\Omega_{\mathcal{T}}} k_T^{\varepsilon}(x) \nabla T^{\varepsilon} \cdot \nabla \varphi_T\, dx\, dt \tag{2.27}
$$

$$
+ \int_{\Omega_{\mathcal{T}}} \left\{ T^{\varepsilon} K^{\varepsilon}(x) \Big[\mathbb{C}_w \lambda_w \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) (\nabla p_w^{\varepsilon} - \vec{r}_w) + \mathbb{C}_n \lambda_n \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) (\nabla p_n^{\varepsilon} - \vec{r}_n) \Big] \right\} \cdot \nabla \varphi_T\, dx\, dt = 0,
$$

where the function $\mathbf{\Psi}^{\varepsilon}$ is defined in (2.4) and

$$
\mathbf{\Psi}^{0,\varepsilon} \stackrel{\text{def}}{=} \left\{ \left(\mathbb{C}_w S^{0,\varepsilon} + \mathbb{C}_n [1 - S^{0,\varepsilon}]\right) \Phi^{\varepsilon} + \mathbb{C}_s^{\varepsilon} \left[1 - \Phi^{\varepsilon}\right] \right\} T^0. \tag{2.28}
$$

According to [23] and its direct generalization to the case of multiple rock types, under assumptions (A.1)–(A.10) problem (2.3)–(2.7) has at least one weak solution.

Notational convention. From now on $C, C_1, \mathsf{C}_0, \ldots$ stand for the generic constants that do not depend on ε.

3. Uniform estimates for a solution to problem (2.3)–(2.7)

In this section we obtain the *a priori* estimates for a solution to problem (2.3)–(2.7). Inspired by [23,24] we obtain the following results.

Lemma 3.1. *Let a quadruple function* $\langle p_w^{\varepsilon}, p_n^{\varepsilon}, S^{\varepsilon}, T^{\varepsilon} \rangle$ *be a weak solution to (2.3)–(2.7). Then*

$$
\int_{\Omega_{\mathcal{T}}} K^{\varepsilon}(x) \left\{ \lambda_w \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) |\nabla p_w^{\varepsilon}|^2 + \lambda_n \left(\frac{x}{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\right) |\nabla p_n^{\varepsilon}|^2 \right\} dx\, dt \leqslant \mathsf{C}_0. \tag{3.1}
$$

Corollary 3.1. *Taking into account condition (A.2) the bound (3.1) implies that*

$$
\begin{aligned}
\int_{\Omega_{\mathsf{f},\mathcal{T}}^{\varepsilon}} \left\{ \lambda_{\mathsf{f},w}(S_{\mathsf{f}}^{\varepsilon}, T^{\varepsilon}) |\nabla p_{\mathsf{f},w}^{\varepsilon}|^2 + \lambda_{\mathsf{f},n}(S_{\mathsf{f}}^{\varepsilon}, T^{\varepsilon}) |\nabla p_{\mathsf{f},n}^{\varepsilon}|^2 \right\} dx\, dt \\
+ \varepsilon^2 \int_{\Omega_{\mathsf{m},\mathcal{T}}^{\varepsilon}} \left\{ \lambda_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon}, T^{\varepsilon}) |\nabla p_{\mathsf{m},w}^{\varepsilon}|^2 + \lambda_{\mathsf{m},n}(S_{\mathsf{m}}^{\varepsilon}, T^{\varepsilon}) |\nabla p_{\mathsf{m},n}^{\varepsilon}|^2 \right\} dx\, dt \leqslant \mathsf{C}_0.
\end{aligned} \tag{3.2}
$$

Lemma 3.2. *Let a quadruple function* $\langle p_w^{\varepsilon}, p_n^{\varepsilon}, S^{\varepsilon}, T^{\varepsilon} \rangle$ *be a weak solution to (2.3)–(2.7). Then*

$$
\int_{\Omega_{\mathcal{T}}} |\nabla T^{\varepsilon}|^2\, dx\, dt \leqslant \mathsf{C}_1. \tag{3.3}
$$

Remark 7. In the derivation of (3.3) the crucial role plays condition (A.4) where the thermal symmetric conductivity tensor k_T^ε (in contrast to the global permeability tensor K^ε) is of order one with respect to ε both in the matrix part and the fissures system.

Now we turn to the estimates of the nonisothermal global pressure $\mathsf{P}_\ell^\varepsilon$ and the function $\beta_\ell(S_\ell^\varepsilon)$ ($\ell = \mathsf{f}, \mathsf{m}$) defined above in Section 2.2. To this end we make use of the relation (2.17) in which we first estimate the quantity $\mathsf{B}_\ell^\varepsilon$. The following result holds true.

Lemma 3.3. *Let a quadruple function $\langle p_n^\varepsilon, p_w^\varepsilon, S^\varepsilon, T^\varepsilon\rangle$ be a weak solution to (2.3)–(2.7). Then*

$$|\mathsf{B}_\ell^\varepsilon| \leqslant \mathsf{C}_B \quad \text{with } \mathsf{C}_B \stackrel{\text{def}}{=} P_{\ell,c}(0)\left[\frac{\mathsf{M}_n}{\mathsf{m}_n} + \frac{\mathsf{M}_w}{\mathsf{m}_w}\right], \tag{3.4}$$

where the constants $\mathsf{M}_n, \mathsf{m}_n, \mathsf{M}_w, \mathsf{m}_w$ are defined in condition (A.7).

The **Proof of** Lemma 3.3 is presented in Lemma 3.3 from [24]. □

The gradients of the nonisothermal global pressure $\mathsf{P}_\ell^\varepsilon$ and the function $\beta_\ell(S_\ell^\varepsilon)$ ($\ell = \mathsf{f}, \mathsf{m}$) admit the following estimates.

Lemma 3.4. *Let a quadruple function $\langle p_n^\varepsilon, p_w^\varepsilon, S^\varepsilon, T^\varepsilon\rangle$ be a weak solution to (2.3)–(2.7). Then*

$$\int_{\Omega_{\mathsf{f},\mathcal{T}}^\varepsilon} |\nabla \mathsf{P}_\mathsf{f}^\varepsilon|^2\, dx\, dt + \varepsilon^2 \int_{\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon} |\nabla \mathsf{P}_\mathsf{m}^\varepsilon|^2\, dx\, dt \leqslant C_1; \tag{3.5}$$

$$\int_{\Omega_{\mathsf{f},\mathcal{T}}^\varepsilon} |\nabla \beta_\mathsf{f}(S_\mathsf{f}^\varepsilon)|^2\, dx\, dt + \varepsilon^2 \int_{\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon} |\nabla \beta_\mathsf{m}(S_\mathsf{m}^\varepsilon)|^2\, dx\, dt \leqslant C_2. \tag{3.6}$$

where the function $\beta_\ell(S_\ell^\varepsilon)$ is defined in (2.11).

The **Proof of** Lemma 3.4 can be done by the arguments similar to those from Lemma 3.4 in [24]. □

Lemma 3.5. *Let a quadruple function $\langle p_n^\varepsilon, p_w^\varepsilon, S^\varepsilon, T^\varepsilon\rangle$ be a weak solution to (2.3)–(2.7). Then under assumptions (A.1)–(A.10) the following uniform in ε estimates hold true:*

$$\|\mathsf{P}_\mathsf{f}^\varepsilon\|_{L^2(\Omega_{\mathsf{f},\mathcal{T}}^\varepsilon)} \leqslant C \quad \text{and} \quad \|\mathsf{P}_\mathsf{m}^\varepsilon\|_{L^2(\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon)} \leqslant C. \tag{3.7}$$

Proof of Lemma 3.5. In the proof of the lemma we follow the lines of the proof of Lemma 3.2 from [9]. The first bound in (3.7) follows immediately from Friedrichs' inequality and the uniform estimate (3.5).

Now we turn to the derivation of the second bound in (3.7). Since the global pressure is a discontinuous function on the interface $\Gamma_{\mathsf{fm}}^\varepsilon$ (see (2.1) for the definition), then we make use of the ideas from [37,38]. For $\mathsf{P}_\mathsf{m}^\varepsilon \in L^2(0,\mathcal{T}; H^1(\Omega_\mathsf{m}^\varepsilon))$ and $\mathsf{P}_\mathsf{f}^\varepsilon - \mathsf{P}_{\Gamma_D} \in L^2(0,\mathcal{T}; H^1_{\Gamma_D}(\Omega_\mathsf{f}^\varepsilon))$ it is proved in [38] (see estimates (3.10), (3.11) in Theorem 3.1) that there exists a constant C independent of ε such that

$$\|\mathsf{P}_\mathsf{m}^\varepsilon\|_{L^2(\Omega_{m,\mathcal{T}}^\varepsilon)} \leqslant C\left[\varepsilon\, \|\nabla \mathsf{P}_\mathsf{m}^\varepsilon\|_{L^2(\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon)} + \sqrt{\varepsilon}\, \|\mathsf{P}_\mathsf{m}^\varepsilon\|_{L^2(\Sigma_\mathcal{T}^\varepsilon)}\right], \tag{3.8}$$

$$\sqrt{\varepsilon}\, \|\mathsf{P}_\mathsf{f}^\varepsilon\|_{L^2(\Sigma_\mathcal{T}^\varepsilon)} \leqslant C\left[\varepsilon\, \|\nabla \mathsf{P}_\mathsf{f}^\varepsilon\|_{L^2(\Omega_{\mathsf{f},\mathcal{T}}^\varepsilon)} + \|\mathsf{P}_\mathsf{f}^\varepsilon\|_{L^2(\Omega_{\mathsf{f},\mathcal{T}}^\varepsilon)}\right]. \tag{3.9}$$

Note that inequalities (3.8) and (3.9) follow from the Poincaré inequality, the trace inequality and the scaling argument.

Then due to the definition of the global pressure $\mathsf{P}_{\mathsf{m}}^{\varepsilon}$, (2.8), and the interface condition (2.5) written in terms of the global pressure, one obtains the following estimate:

$$\begin{aligned}\|\mathsf{P}_{\mathsf{m}}^{\varepsilon}\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} &\leqslant \|\mathsf{P}_{\mathsf{m}}^{\varepsilon} + \mathsf{G}_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon}, T_{\mathsf{m}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} + \|\mathsf{G}_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon}, T_{\mathsf{m}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} \\ &= \|\mathsf{P}_{\mathsf{f}}^{\varepsilon} + \mathsf{G}_{\mathsf{f},w}(S_{\mathsf{f}}^{\varepsilon}, T_{\mathsf{f}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} + \|\mathsf{G}_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon}, T_{\mathsf{m}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} \\ &\leqslant \|\mathsf{P}_{\mathsf{f}}^{\varepsilon}\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} + \|\mathsf{G}_{\mathsf{f},w}(S_{\mathsf{f}}^{\varepsilon}, T_{\mathsf{f}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})} + \|\mathsf{G}_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon}, T_{\mathsf{m}}^{\varepsilon})\|_{L^2(\Sigma_{\mathcal{T}}^{\varepsilon})}.\end{aligned} \tag{3.10}$$

Now, taking into account the boundedness of $\mathsf{G}_{\ell,w}(S_{\ell}^{\varepsilon}, T_{\ell}^{\varepsilon})$, the geometry of $\Omega_{\mathsf{m},\mathcal{T}}^{\varepsilon}$, (3.8)–(3.10), we obtain:

$$\|\mathsf{P}_{\mathsf{m}}^{\varepsilon}\|_{L^2(\Omega_{\mathsf{m},\mathcal{T}}^{\varepsilon})} \leqslant C\left(\varepsilon\|\nabla \mathsf{P}_{\mathsf{m}}^{\varepsilon}\|_{L^2(\Omega_{\mathsf{m},\mathcal{T}}^{\varepsilon})} + 1\right). \tag{3.11}$$

By using (3.5), from (3.11) we get the desired inequality (3.7). Lemma 3.5 is proved. □

Let us pass to the uniform bounds for the time derivatives of S^{ε}. In a standard way (see, e.g., [39]) we get:

Lemma 3.6. *Let a quadruple function $\langle p_n^{\varepsilon}, p_w^{\varepsilon}, S^{\varepsilon}, T^{\varepsilon}\rangle$ be a weak solution to (2.3)–(2.7). Then under assumptions (A.1)–(A.10) the following uniform in ε estimates hold true:*

$$\{\partial_t(\Phi_{\ell}^{\varepsilon} S_{\ell}^{\varepsilon})\}_{\varepsilon>0} \quad \text{is uniformly bounded in } L^2(0,\mathcal{T}; H^{-1}(\Omega_{\ell}^{\varepsilon})); \tag{3.12}$$

$$\{\partial_t \mathbf{\Psi}_{\ell}^{\varepsilon}\}_{\varepsilon>0} \quad \text{is uniformly bounded in } L^2(0,\mathcal{T}; H^{-1}(\Omega_{\ell}^{\varepsilon})); \tag{3.13}$$

where

$$\mathbf{\Psi}_{\ell}^{\varepsilon} = \mathbf{\Psi}_{\ell}^{\varepsilon}(S_{\ell}^{\varepsilon}, T_{\ell}^{\varepsilon}) \stackrel{\text{def}}{=} \left\{\left(\mathbb{C}_w S_{\ell}^{\varepsilon} + \mathbb{C}_n[1 - S_{\ell}^{\varepsilon}]\right)\Phi_{\ell}^{\varepsilon} + \mathbb{C}_s^{\varepsilon}\left[1 - \Phi_{\ell}^{\varepsilon}\right]\right\} T_{\ell}^{\varepsilon} \tag{3.14}$$

and where $\Phi_{\mathsf{f}}^{\varepsilon}$, $\Phi_{\mathsf{m}}^{\varepsilon}$ are defined in condition (A.1).

4. Compactness and convergence results

The outline of this section is as follows. First, in Section 4.1 we extend the functions $\mathsf{P}_{\mathsf{f}}^{\varepsilon}$ and $S_{\mathsf{f}}^{\varepsilon}$ from the subdomain $\Omega_{\mathsf{f}}^{\varepsilon}$ to the whole Ω and obtain uniform estimates for the extended functions. Then in Section 4.2, using these uniform estimates we prove the compactness result for the family $\{\widetilde{S}_{\mathsf{f}}^{\varepsilon}\}_{\varepsilon>0}$. In Section 4.2 we establish the compactness result for the sequence of functions $\{T^{\varepsilon}\}_{\varepsilon>0}$.

4.1. Extensions of the functions $\mathsf{P}_{\mathsf{f}}^{\varepsilon}$, $S_{\mathsf{f}}^{\varepsilon}$

The goal of this subsection is to extend the functions $\mathsf{P}_{\mathsf{f}}^{\varepsilon}$, $S_{\mathsf{f}}^{\varepsilon}$ defined in the subdomain $\Omega_{\mathsf{f}}^{\varepsilon}$ to the whole Ω and to derive the uniform in ε estimates for the extended functions.
Extension of the function $\mathsf{P}_{\mathsf{f}}^{\varepsilon}$. First, we introduce the extension by reflection operator from the subdomain $\Omega_{\mathsf{f}}^{\varepsilon}$ to the whole Ω. Taking into account the results of [40] we conclude that there exists a linear continuous extension operator $\Pi^{\varepsilon} : H^1(\Omega_{\mathsf{f}}^{\varepsilon}) \longrightarrow H^1(\Omega)$ such that: (i) $\Pi^{\varepsilon} u = u$ in $\Omega_{\mathsf{f}}^{\varepsilon}$ and (ii) for any $u \in H^1(\Omega_{\mathsf{f}}^{\varepsilon})$,

$$\|\Pi^{\varepsilon} u\|_{L^2(\Omega)} \leqslant C\,\|u\|_{L^2(\Omega_{\mathsf{f}}^{\varepsilon})} \quad \text{and} \quad \|\nabla(\Pi^{\varepsilon} u)\|_{L^2(\Omega)} \leqslant C\,\|\nabla u\|_{L^2(\Omega_{\mathsf{f}}^{\varepsilon})}, \tag{4.1}$$

where C is a constant that does not depend on u and ε. Now it follows from (3.5), (3.7) that

$$\|\nabla(\Pi^{\varepsilon}\mathsf{P}_{\mathsf{f}}^{\varepsilon})\|_{L^2(\Omega_{\mathcal{T}})} + \|\Pi^{\varepsilon}\mathsf{P}_{\mathsf{f}}^{\varepsilon}\|_{L^2(\Omega_{\mathcal{T}})} \leqslant C. \tag{4.2}$$

Notational convention. In what follows we will denote $\widetilde{\mathsf{P}}_{\mathsf{f}}^{\varepsilon} = \Pi^{\varepsilon}\mathsf{P}_{\mathsf{f}}^{\varepsilon}$.

Extension of the function $S^\varepsilon_{\mathsf{f}}$. In order to extend $S^\varepsilon_{\mathsf{f}}$, following the ideas of [29], we make use of the function $\widehat{\beta}$ defined in (2.18). It is evident that $\widehat{\beta}$ is a monotone function of s. Let $\overline{S}^{\,\varepsilon}$ be the function defined by:

$$\overline{S}^{\,\varepsilon} \stackrel{\text{def}}{=} \begin{cases} S^\varepsilon_{\mathsf{f}} & \text{in } \Omega^\varepsilon_{\mathsf{f}} \\ \mathcal{P}(S^\varepsilon_{\mathsf{m}}) & \text{in } \Omega^\varepsilon_{\mathsf{m}}, \end{cases} \tag{4.3}$$

where the function $\mathcal{P}$ is defined in (2.19). Then we introduce $L^2(0,\mathcal{T}; H^1_{\Gamma_D}(\Omega))$ function:

$$\widehat{\beta}^\varepsilon(x,t) \stackrel{\text{def}}{=} \widehat{\beta}(\overline{S}^{\,\varepsilon}) = \int_0^{\overline{S}^{\,\varepsilon}} \widehat{\alpha}(\varsigma)\, d\varsigma, \tag{4.4}$$

where the function $\widehat{\alpha}$ is defined in (2.18). As in [37] (see Lemma 4.1) using (3.6) and the definition (2.18) of the function $\widehat{\beta}$, one can show that there is a constant C which does not depend on ε such that

$$\|\nabla \widehat{\beta}(S^\varepsilon_{\mathsf{f}})\|_{L^2(\Omega^\varepsilon_{\mathsf{f},\mathcal{T}})} = \|\nabla \widehat{\beta}(\overline{S}^{\,\varepsilon})\|_{L^2(\Omega^\varepsilon_{\mathsf{f},\mathcal{T}})} \leqslant C \quad \text{and} \quad \varepsilon \|\nabla \widehat{\beta}(\overline{S}^{\,\varepsilon})\|_{L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})} \leqslant C. \tag{4.5}$$

Let now the function $\mathcal{B}^\varepsilon$ be the extension of the function $\widehat{\beta}(S^\varepsilon_{\mathsf{f}})$ to the whole domain Ω, i.e.,

$$\mathcal{B}^\varepsilon \stackrel{\text{def}}{=} \Pi^\varepsilon \widehat{\beta}(S^\varepsilon_{\mathsf{f}}) \text{ in } \Omega_{\mathcal{T}} \text{ with } \mathcal{B}^\varepsilon = \widehat{\beta}(S^\varepsilon_{\mathsf{f}}) \text{ in } \Omega^\varepsilon_{\mathsf{f},\mathcal{T}}.$$

Then it follows from the extension by reflection and (4.5) that

$$0 \leqslant \mathcal{B}^\varepsilon \leqslant \max_{s\in[0,1]} \widehat{\beta}(s) \text{ a.e. in } \Omega_{\mathcal{T}} \quad \text{and} \quad \|\nabla \mathcal{B}^\varepsilon\|_{L^2(\Omega_{\mathcal{T}})} \leqslant C, \tag{4.6}$$

Now we can extend $S^\varepsilon_{\mathsf{f}}$ from $\Omega^\varepsilon_{\mathsf{f}}$ to the whole Ω. We denote this extension by $\widetilde{S}^\varepsilon$ and define it as follows:

$$\widetilde{S}^{\,\varepsilon} \stackrel{\text{def}}{=} \widehat{\beta}^{-1}(\mathcal{B}^\varepsilon). \tag{4.7}$$

This implies that

$$\int_{\Omega_{\mathcal{T}}} \left|\nabla \widehat{\beta}\big(\widetilde{S}^{\,\varepsilon}\big)\right|^2 dx\, dt \leqslant C \quad \text{and} \quad 0 \leqslant \widetilde{S}^\varepsilon \leqslant 1 \text{ a.e. in } \Omega_{\mathcal{T}}. \tag{4.8}$$

4.2. Compactness results for the sequences $\{\widetilde{S}^\varepsilon\}_{\varepsilon>0}$ *and* $\{T^\varepsilon\}_{\varepsilon>0}$

The main goal of this subsection is to establish the compactness and corresponding convergence results for the sequence $\{\widetilde{S}^\varepsilon\}_{\varepsilon>0}$ constructed in the previous section (see the definition (4.7)). To this end we obtain an auxiliary estimate of the modulus of continuity with respect to time variable for the saturation function S^ε. This result is also used below in the proof of the compactness and convergence results for the temperature function T^ε. In this section we apply the ideas of the papers [41] and [23]. The main result of the section is then given in Proposition 4.2.

Lemma 4.1. *Under our standing assumptions, for h sufficiently small,*

$$\int_{\Omega^{\varepsilon,h}_{\mathsf{f},\mathcal{T}}} \left[S^\varepsilon_{\mathsf{f}}(t) - S^\varepsilon_{\mathsf{f}}(t-h)\right] \left[\widehat{\beta}(S^\varepsilon_{\mathsf{f}})(t) - \widehat{\beta}(S^\varepsilon_{\mathsf{f}})(t-h)\right] dx\, dt \leqslant C\, h; \tag{4.9}$$

$$\int_{\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}}} \left[S^\varepsilon_{\mathsf{m}}(t) - S^\varepsilon_{\mathsf{m}}(t-h)\right] \left[\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}))(t) - \widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}))(t-h)\right] dx\, dt \leqslant C\, h, \tag{4.10}$$

where $\Omega^{\varepsilon,h}_{\mathsf{f},\mathcal{T}} \stackrel{\text{def}}{=} \Omega^\varepsilon_{\mathsf{f}} \times (h,\mathcal{T})$, $\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}} \stackrel{\text{def}}{=} \Omega^\varepsilon_{\mathsf{m}} \times (h,\mathcal{T})$ *and C is a constant that does not depend on* ε *and h.*

Proof of Lemma 4.1. First, we rewrite Eq. (2.25) as follows. For any function $\varphi_w \in L^2(0,\mathcal{T};H^1(\Omega))$ such that $\varphi_w = 0$ on $\Gamma_D \times (0,\mathcal{T})$ it holds:

$$\int_{\Omega_{\mathcal{T}}} \Phi^\varepsilon(x)\frac{\partial S^\varepsilon}{\partial t}\,\varphi_w\,dx\,dt + \int_{\Omega^\varepsilon_{\mathsf{f},\mathcal{T}}} K_{\mathsf{f}}\left(x,\frac{x}{\varepsilon}\right)\lambda_{\mathsf{f},w}(S^\varepsilon_{\mathsf{f}},T^\varepsilon)\big(\nabla p^\varepsilon_{\mathsf{f},w} - \vec{r}_w\big)\cdot\nabla\varphi_w\,dx\,dt \\ + \varepsilon^2\int_{\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}} K_{\mathsf{m}}\left(x,\frac{x}{\varepsilon}\right)\lambda_{\mathsf{m},w}(S^\varepsilon_{\mathsf{m}},T^\varepsilon)\big(\nabla p^\varepsilon_{\mathsf{m},w} - \vec{r}_w\big)\cdot\nabla\varphi_w\,dx\,dt = 0. \tag{4.11}$$

Following the ideas of the proof of Lemma 6.3 from [41], we introduce the function χ^ε:

$$\chi^\varepsilon(x,t) \stackrel{\text{def}}{=} \int_{\max\{t,h\}}^{\min\{t+h,\mathcal{T}\}} h\left[\partial^h\widehat{\beta}(\overline{S}^{\,\varepsilon})\right](x,\tau)\,d\tau \quad \text{with } \partial^h u \stackrel{\text{def}}{=} \frac{u(t)-u(t-h)}{h}. \tag{4.12}$$

Then, due to Lemma 3.4 and the boundary conditions for the function $\widehat{\beta}(\overline{S}^{\,\varepsilon})$, we have $\chi^\varepsilon \in L^2(0,\mathcal{T};H^1_{\Gamma_D}(\Omega))$ for any $\varepsilon > 0$. Setting $\varphi_w = \chi^\varepsilon$ in (4.11), by the Fubini theorem we have:

$$\int_{\Omega_{\mathcal{T}}} \Phi^\varepsilon(x)\frac{\partial S^\varepsilon}{\partial t}\,\chi^\varepsilon\,dxdt = \int_h^{\mathcal{T}}\int_\Omega \Phi^\varepsilon(x)\,h^2\left[\partial^h S^\varepsilon\right]\left[\partial^h\widehat{\beta}(\overline{S}^{\,\varepsilon})\right]dx\,d\tau. \tag{4.13}$$

Then from (4.11) with $\varphi_w = \chi^\varepsilon$ and relation (4.13) we obtain the following relation:

$$\int_h^{\mathcal{T}}\int_\Omega \Phi^\varepsilon(x)\,h^2\left[\partial^h S^\varepsilon\right]\left[\partial^h\widehat{\beta}(\overline{S}^{\,\varepsilon})\right]dx\,d\tau = \mathcal{I}^\varepsilon[\chi^\varepsilon],$$

where

$$\mathcal{I}^\varepsilon[\chi^\varepsilon] \stackrel{\text{def}}{=} -\int_{\Omega^\varepsilon_{\mathsf{f},\mathcal{T}}} K_{\mathsf{f}}\left(x,\frac{x}{\varepsilon}\right)\lambda_{\mathsf{f},w}(S^\varepsilon_{\mathsf{f}},T^\varepsilon)\big(\nabla p^\varepsilon_{\mathsf{f},w} - \vec{r}_w\big)\cdot\nabla\chi^\varepsilon\,dx\,dt \\ - \varepsilon^2\int_{\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}} K_{\mathsf{m}}\left(x,\frac{x}{\varepsilon}\right)\lambda_{\mathsf{m},w}(S^\varepsilon_{\mathsf{m}},T^\varepsilon)\big(\nabla p^\varepsilon_{\mathsf{m},w} - \vec{r}_w\big)\cdot\nabla\chi^\varepsilon\,dx\,dt.$$

Now by Lemma (3.1), estimate (4.5) and Cauchy's inequality we obtain that

$$\left|\mathcal{I}^\varepsilon[\chi^\varepsilon]\right| \leqslant C\,h,$$

where C is a constant that does not depend on ε, h. It is clear that

$$\int_h^{\mathcal{T}}\int_\Omega \Phi^\varepsilon h^2\left[\partial^h S^\varepsilon\right]\left[\partial^h\widehat{\beta}(\overline{S}^{\,\varepsilon})\right]dx\,d\tau \\ = \int_{\Omega^\varepsilon_{\mathsf{f},\mathcal{T}}} \Phi^\varepsilon_{\mathsf{f}}(x)(S^\varepsilon_{\mathsf{f}}(x,t) - S^\varepsilon_{\mathsf{f}}(x,t-h))\left(\widehat{\beta}(S^\varepsilon_{\mathsf{f}}(x,t)) - \widehat{\beta}(S^\varepsilon_{\mathsf{f}}(x,t-h))\right)dxdt \\ + \int_{\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}} \Phi^\varepsilon_{\mathsf{m}}(x)(S^\varepsilon_{\mathsf{m}}(x,t) - S^\varepsilon_{\mathsf{m}}(x,t-h))\left(\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}(x,t))) - \widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}(x,t-h)))\right)dxdt.$$

Since $\widehat{\beta}$ and $\mathcal{P}$ are monotonically increasing functions, then both integrals on the right hand side are positive and we find the bounds (4.9) and (4.10). This completes the proof of Lemma 4.1. □

Corollary 4.1. *Under our standing assumptions, for h sufficiently small,*

$$\int_{\Omega^h_{\mathcal{T}}} \left|\widehat{\beta}(\widetilde{S}^\varepsilon)(t) - \widehat{\beta}(\widetilde{S}^\varepsilon)(t-h)\right|^2 dx\,dt \leqslant C\,h; \tag{4.14}$$

$$\int_{\Omega^h_{\mathcal{T}}} \left|\widetilde{S}^\varepsilon(t) - \widetilde{S}^\varepsilon(t-h)\right|^{2/\theta} dx\,dt \leqslant C\,h; \tag{4.15}$$

$$\int_{\Omega^h_{\mathcal{T}}} \left|S^\varepsilon(t) - S^\varepsilon(t-h)\right|^{2/\theta} dx\, dt \leqslant C\, h. \tag{4.16}$$

Here $\Omega^h_{\mathcal{T}} \stackrel{\text{def}}{=} \Omega \times (h, \mathcal{T})$; $\widetilde{S}^\varepsilon$ *is the extension of the function* $S^\varepsilon_{\mathsf{f}}$ *to the whole* Ω *defined by* (4.7); θ *is defined in condition (A.9);* C *is a constant that does not depend on* ε *and* h.

Proof of Corollary 4.1. It follows from the definition (2.18) of the function $\widehat{\beta}$ and condition (A.8) that

$$\left|\widehat{\beta}(S^\varepsilon(t)) - \widehat{\beta}(S^\varepsilon(t-h))\right| \leqslant \max_{s\in[0,1]} \widehat{\alpha}(s)\, |S^\varepsilon(t) - S^\varepsilon(t-h)|.$$

$$\left|\widehat{\beta}(\mathcal{P}(S^\varepsilon(t))) - \widehat{\beta}(\mathcal{P}(S^\varepsilon(t-h)))\right| \leqslant \max_{s\in[0,1]} \{\widehat{\alpha}(\mathcal{P}(s))\} \max_{s\in[0,1]} \{\mathcal{P}'(s)\}\, |S^\varepsilon(t) - S^\varepsilon(t-h)|.$$

Then from (4.9) we get:

$$\begin{aligned} &\int_{\Omega^{\varepsilon,h}_{\mathsf{f},\mathcal{T}}} \left|\widehat{\beta}(S^\varepsilon_{\mathsf{f}}(t)) - \widehat{\beta}(S^\varepsilon_{\mathsf{f}}(t-h))\right|^2 dx\, dt \\ &\qquad\leqslant C \int_{\Omega^{\varepsilon,h}_{\mathsf{f},\mathcal{T}}} \left[S^\varepsilon_{\mathsf{f}}(t) - S^\varepsilon_{\mathsf{f}}(t-h)\right] \left[\widehat{\beta}(S^\varepsilon_{\mathsf{f}}(t)) - \widehat{\beta}(S^\varepsilon_{\mathsf{f}}(t-h))\right] dx\, dt \leqslant C\, h, \end{aligned} \tag{4.17}$$

where C is a constant that does not depend on ε, h. In the same way we obtain that

$$\int_{\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}}} \left|\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}(t))) - \widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}}(t-h)))\right|^2 dx\, dt \leqslant C\, h. \tag{4.18}$$

The inequality (4.14) stated in the whole domain $\Omega^h_{\mathcal{T}}$ is a consequence of the bound (4.17) and the properties (4.1) of the extension operator Π^ε.

From condition (A.9) we have:

$$\begin{aligned} \int_{\Omega^h_{\mathcal{T}}} \left|\widetilde{S}^\varepsilon(t) - \widetilde{S}^\varepsilon(t-h)\right|^{2/\theta} dx\, dt &= \int_{\Omega^h_{\mathcal{T}}} \left|\widehat{\beta}^{-1}(\widehat{\beta}(\widetilde{S}^\varepsilon))(t) - \widehat{\beta}^{-1}(\widehat{\beta}(\widetilde{S}^\varepsilon))(t-h)\right|^{2/\theta} dx\, dt \\ &\leqslant C \int_{\Omega^h_{\mathcal{T}}} \left|\widehat{\beta}(\widetilde{S}^\varepsilon)(t) - \widehat{\beta}(\widetilde{S}^\varepsilon)(t-h)\right|^2 dx\, dt. \end{aligned}$$

Then from (4.14) we obtain the desired bound (4.15). Taking into account Remark 4 we know that the inverse of the function $\widehat{\beta} \circ \mathcal{P}$ is Hölder continuous with the same exponent θ and therefore we get:

$$\begin{aligned} \int_{\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}}} \left|S^\varepsilon_{\mathsf{m}}(t) - S^\varepsilon_{\mathsf{m}}(t-h)\right|^{2/\theta} dx\, dt &= \int_{\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}}} \left|(\widehat{\beta}\circ\mathcal{P})^{-1}(\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}})))(t) - (\widehat{\beta}\circ\mathcal{P})^{-1}(\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}})))(t-h)\right|^{2/\theta} dx\, dt \\ &\leqslant \widehat{C} \int_{\Omega^{\varepsilon,h}_{\mathsf{m},\mathcal{T}}} \left|\widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}})(t)) - \widehat{\beta}(\mathcal{P}(S^\varepsilon_{\mathsf{m}})(t-h))\right|^2 dx\, dt. \end{aligned}$$

Combining this inequality with (4.18) and the restriction of (4.15) to $\Omega^{\varepsilon,h}_{\mathsf{f},\mathcal{T}}$ we obtain (4.16). Corollary 4.1 is proved. □

The main result of the section reads.

Proposition 4.2. *Under our standing assumptions there is a function* S *such that* $0 \leqslant S \leqslant 1$ *in* $\Omega_{\mathcal{T}}$ *and, up to a subsequence,*

$$\widetilde{S}^\varepsilon \to S \;\text{ strongly in } L^q(\Omega_{\mathcal{T}}) \quad \text{for any } q \geqslant 1. \tag{4.19}$$

Proof of Proposition 4.2. By (4.8) the sequence $\{\widehat{\beta}(\widetilde{S}^\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,\mathcal{T};H^1(\Omega))$. Since this sequence also satisfies (4.14), it follows from [42] that $\{\widehat{\beta}(\widetilde{S}^\varepsilon)\}_{\varepsilon>0}$ is a relatively compact set in the space $L^2(\Omega_{\mathcal{T}})$. Therefore, for a subsequence, $\widehat{\beta}(\widetilde{S}^\varepsilon) \to \widehat{\beta}^\star$ strongly in the space $L^2(\Omega_{\mathcal{T}})$. Letting $S = \widehat{\beta}^{-1}(\widehat{\beta}^\star)$ we get $\widetilde{S}^\varepsilon \to S$ strongly in $L^{2/\theta}(\Omega_{\mathcal{T}})$. In view of the uniform boundedness of the functions $\widehat{\beta}(\widetilde{S}^\varepsilon)$ this implies the strong convergence in the space $L^q(\Omega_{\mathcal{T}})$ for any $1 \leqslant q < \infty$. This completes the proof of Proposition 4.2. □

Relying on (4.16) one can repeat the proof of Proposition 4.4 in [24] and prove the following compactness for the temperature:

Proposition 4.3. *Under our standing assumptions there is a function T such that $T_m \leqslant T \leqslant T_M$ in $\Omega_{\mathcal{T}}$ and, up to a subsequence,*

$$T^\varepsilon \to T \text{ strongly in } L^q(\Omega_{\mathcal{T}}) \quad \text{for any } q \geqslant 1. \tag{4.20}$$

4.3. Two-scale convergence results

In this Subsection, taking into account the compactness results from the previous section, we formulate the convergence results for the sequences $\{\widetilde{P}^\varepsilon_{\mathsf{f}}\}_{\varepsilon>0}$, $\{\widetilde{S}^\varepsilon\}_{\varepsilon>0}$, and $\{T^\varepsilon\}_{\varepsilon>0}$. In this paper the homogenization process is rigorously obtained by using the two-scale approach (see, e.g., [30]). For reader's convenience, we recall the definitions of the two-scale convergence.

Definition 4.4. A sequence of functions $\{v^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega_{\mathcal{T}})$ two-scale converges to $v \in L^2(\Omega_{\mathcal{T}} \times Y)$ if $\|v^\varepsilon\|_{L^2(\Omega_{\mathcal{T}})} \leqslant C$, and for any test function $\varphi \in C^\infty(\overline{\Omega_{\mathcal{T}}}; C_\#(Y))$ the following relation holds:

$$\lim_{\varepsilon\to 0} \int_{\Omega_{\mathcal{T}}} v^\varepsilon(x,t)\, \varphi\left(x,t,\frac{x}{\varepsilon}\right)\, dx\, dt = \int_{\Omega_{\mathcal{T}}\times Y} v(x,t,y)\, \varphi(x,t,y)\, dy\, dx\, dt.$$

This convergence is denoted by $v^\varepsilon(x,t) \xrightarrow{2s} v(x,t,y)$.

Now we summarize the convergence results for the sequences $\{\widetilde{P}^\varepsilon_{\mathsf{f}}\}_{\varepsilon>0}$, $\{\widetilde{S}^\varepsilon\}_{\varepsilon>0}$ and $\{T^\varepsilon\}_{\varepsilon>0}$. We have:

Lemma 4.2. *There exist a function S such that $0 \leqslant S \leqslant 1$ a.e. in $\Omega_{\mathcal{T}}$, $\beta_{\mathsf{f}}(S) \in L^2(0,\mathcal{T};H^1_{\Gamma_D}(\Omega))$, and functions $\mathsf{P}, T \in L^2(0,\mathcal{T};H^1_{\Gamma_D}(\Omega))$, $\mathsf{w}_p, \mathsf{w}_s, \mathsf{w}_T \in L^2(\Omega_{\mathcal{T}};H^1_{per}(Y))$ such that up to a subsequence:*

$$\widetilde{S}^\varepsilon(x,t) \longrightarrow S(x,t) \text{ strongly in } L^q(\Omega_{\mathcal{T}})\ \forall\ 1 \leqslant q < +\infty; \tag{4.21}$$

$$\widetilde{\mathsf{P}}^\varepsilon_{\mathsf{f}}(x,t) \rightharpoonup \mathsf{P}(x,t) \text{ weakly in } L^2(0,\mathcal{T};H^1(\Omega)); \tag{4.22}$$

$$\nabla\widetilde{\mathsf{P}}^\varepsilon_{\mathsf{f}}(x,t) \xrightarrow{2s} \nabla\mathsf{P}(x,t) + \nabla_y \mathsf{w}_p(x,t,y); \tag{4.23}$$

$$\beta_{\mathsf{f}}(\widetilde{S}^\varepsilon) \longrightarrow \beta_{\mathsf{f}}(S) \text{ strongly in } L^q(\Omega_{\mathcal{T}})\ \forall\ 1 \leqslant q < +\infty; \tag{4.24}$$

$$\nabla\beta_{\mathsf{f}}(\widetilde{S}^\varepsilon)(x,t) \xrightarrow{2s} \nabla\beta_{\mathsf{f}}(S)(x,t) + \nabla_y \mathsf{w}_s(x,t,y); \tag{4.25}$$

$$T^\varepsilon(x,t) \longrightarrow T(x,t) \text{ strongly in } L^q(\Omega_{\mathcal{T}})\ \forall\ 1 \leqslant q < +\infty; \tag{4.26}$$

$$T^\varepsilon(x,t) \rightharpoonup T(x,t) \text{ weakly in } L^2(0,\mathcal{T};H^1(\Omega)); \tag{4.27}$$

$$\nabla T^\varepsilon(x,t) \xrightarrow{2s} \nabla T(x,t) + \nabla_y \mathsf{w}_T(x,t,y); \tag{4.28}$$

Proof of Lemma 4.2. The proof of Lemma 4.2 is based on the *a priori estimates* for the functions $\beta_{\mathsf{f}}(S^\varepsilon_{\mathsf{f}})$, $\mathsf{P}^\varepsilon_{\mathsf{f}}$ and T^ε obtained in Section 3, the extension results from Section 4.1, Propositions 4.2 and 4.3. The two-scale convergence results are obtained by arguments similar to those in [30]. Lemma 4.2 is proved. □

5. Dilation operator and convergence results

It is known that due to the nonlinearities and the strong coupling of the problem, the two-scale convergence does not provide an explicit form for the source terms appearing in the homogenized model, see for instance [13,29,41]. To overcome this difficulty the authors make use of the dilation operator. Here we refer to [10,13,29,41] for the definition and main properties of the dilation operator. Let us also notice that the notion of the dilation operator is closely related to the notion of the unfolding operator. We refer here, e.g., to [43] for the definition and the properties of this operator.

The outline of this section is as follows. First, in Section 5.1 we introduce the definition of the dilation operator and describe its main properties. Then in Section 5.2 we obtain the equations for the dilated saturation and the global pressure functions, the corresponding uniform estimates and the convergence results.

5.1. Definition and preliminary results

Definition 5.1. For a given $\varepsilon > 0$, we define a dilation operator $\mathfrak{D}^\varepsilon$ mapping measurable functions defined in $\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}$ to measurable functions defined in $\Omega_\mathcal{T} \times Y_\mathsf{m}$ by

$$(\mathfrak{D}^\varepsilon \varphi)(x,t,y) \overset{\text{def}}{=} \begin{cases} \varphi\left(c^\varepsilon(x) + \varepsilon\, y, t\right), & \text{if } c^\varepsilon(x) + \varepsilon\, y \in \Omega^\varepsilon_\mathsf{m}; \\ 0, & \text{elsewhere,} \end{cases} \tag{5.1}$$

where $c^\varepsilon(x) \overset{\text{def}}{=} \varepsilon\, k$ if $x \in \varepsilon\,(Y + k)$ with $k \in \mathbb{Z}^d$ denotes the lattice translation point of the ε-cell domain containing x.

The basic properties of the dilation operator are given by the following lemma (see, e.g., [10,41]).

Lemma 5.1. *Let $\varphi, \psi \in L^2(0,\mathcal{T}; H^1(\Omega^\varepsilon_\mathsf{m}))$. Then we have:*

$$\nabla_y \mathfrak{D}^\varepsilon \varphi = \varepsilon\, \mathfrak{D}^\varepsilon(\nabla_x \varphi) \quad \text{a.e. in } \Omega_\mathcal{T} \times Y_\mathsf{m}; \tag{5.2}$$

$$\|\mathfrak{D}^\varepsilon \varphi\|_{L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})} = \|\varphi\|_{L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})};$$

$$\|\nabla_y \mathfrak{D}^\varepsilon \varphi\|_{L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})} = \varepsilon\,\|\mathfrak{D}^\varepsilon \nabla_x\, \varphi\|_{L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})} = \varepsilon\,\|\nabla_x\,\varphi\|_{L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})};$$

$$(\mathfrak{D}^\varepsilon \varphi, \mathfrak{D}^\varepsilon \psi)_{L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})} = (\varphi,\psi)_{L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})}.$$

The following lemma gives the link between the two-scale and the weak convergence (see, e.g., [29]).

Lemma 5.2. *Let $\{\varphi^\varepsilon\}_{\varepsilon>0}$ be a uniformly bounded sequence in $L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})$ satisfying: (i) $\mathfrak{D}^\varepsilon\varphi^\varepsilon \rightharpoonup \varphi^0$ weakly in $L^2(\Omega_\mathcal{T}; L^2_{per}(Y_\mathsf{m}))$; (ii) $\mathbf{1}^\varepsilon_\mathsf{m}(x)\varphi^\varepsilon \overset{2s}{\rightharpoonup} \varphi^\star \in L^2(\Omega_\mathcal{T}; L^2_{per}(Y_\mathsf{m}))$. Then $\varphi^0 = \varphi^\star$ a.e. in $\Omega_\mathcal{T} \times Y_\mathsf{m}$.*

Finally, we also have the following result (see, e.g., [13,41]).

Lemma 5.3. *If $\varphi^\varepsilon \in L^2(\Omega^\varepsilon_{\mathsf{m},\mathcal{T}})$ and $\mathbf{1}^\varepsilon_\mathsf{m}(x)\varphi^\varepsilon \overset{2s}{\to} \varphi \in L^2(\Omega_\mathcal{T}; L^2_{per}(Y_\mathsf{m}))$ then $\mathfrak{D}^\varepsilon\varphi^\varepsilon$ converges to φ strongly in $L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})$. Here $\overset{2s}{\to}$ denotes the strong two-scale convergence. If $\varphi \in L^2(\Omega_\mathcal{T})$ is considered as an element of $L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})$ constant in y, then $\mathfrak{D}^\varepsilon\varphi$ converges strongly to φ in $L^2(\Omega_\mathcal{T}\times Y_\mathsf{m})$.*

5.2. The dilated functions $\mathfrak{D}^\varepsilon S^\varepsilon_{\mathsf{m}}$, $\mathfrak{D}^\varepsilon P^\varepsilon_{\mathsf{m}}$, $\mathfrak{D}^\varepsilon T^\varepsilon_{\mathsf{m}}$ and their properties

In this section we derive the equations for the dilated functions $\mathfrak{D}^\varepsilon S^\varepsilon_{\mathsf{m}}, \mathfrak{D}^\varepsilon P^\varepsilon_{\mathsf{m}}, \mathfrak{D}^\varepsilon T^\varepsilon_{\mathsf{m}}$ and obtain the corresponding uniform estimates. In what follows we also make use of the notation:

$$\mathfrak{D}^\varepsilon S^\varepsilon_{\mathsf{m}} \stackrel{\text{def}}{=} s^\varepsilon_{\mathsf{m}}, \quad \mathfrak{D}^\varepsilon \mathsf{P}^\varepsilon_{\mathsf{m}} \stackrel{\text{def}}{=} p^\varepsilon_{\mathsf{m}} \quad \text{and} \quad \mathfrak{D}^\varepsilon \mathsf{T}^\varepsilon_{\mathsf{m}} \stackrel{\text{def}}{=} \theta^\varepsilon_{\mathsf{m}}. \tag{5.3}$$

The equations for the dilated functions $s^\varepsilon_{\mathsf{m}}, p^\varepsilon_{\mathsf{m}}$ are given by the following lemma.

Lemma 5.4. *For $x \in \Omega$, the functions $s^\varepsilon_{\mathsf{m}}, p^\varepsilon_{\mathsf{m}}$ satisfy in the space $L^2(0, \mathcal{T}; H^{-1}(Y_{\mathsf{m}}))$ the following system of equations:*

$$\begin{aligned}\Phi_{\mathsf{m}}(y)\frac{\partial s^\varepsilon_{\mathsf{m}}}{\partial t} - \operatorname{div}_y \Big\{ K_{\mathsf{m}}(x,y)\Big[\lambda_{\mathsf{m},w}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})[\nabla_y p^\varepsilon_{\mathsf{m}} - \varepsilon\vec{r}_w] + \Lambda^{(\mathsf{m})}_1(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}) \\ + \lambda_{\mathsf{m},w} B_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})\nabla_y \theta^\varepsilon\Big]\Big\} = 0;\end{aligned} \tag{5.4}$$

$$\begin{aligned}-\Phi_{\mathsf{m}}(y)\frac{\partial s^\varepsilon_{\mathsf{m}}}{\partial t} - \operatorname{div}_y \Big\{ K_{\mathsf{m}}(x,y)\Big[\lambda_{\mathsf{m},n}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})[\nabla_y p^\varepsilon_{\mathsf{m}} - \varepsilon\vec{r}_n] - \Lambda^{(\mathsf{m})}_1(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}) \\ + \lambda_{\mathsf{m},n} B_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})\nabla_y \theta^\varepsilon\Big]\Big\} = 0.\end{aligned} \tag{5.5}$$

The **Proof of** Lemma 5.4 is given in [29,41]. The system of equations (5.4)–(5.5) is provided with the following *boundary conditions*:

$$\beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}) = \mathcal{M}(\beta_{\mathsf{f}}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon)) \quad \text{on } \Gamma_{\mathsf{fm}} \text{ for } (x,t) \in \Omega^\varepsilon_{\mathsf{m},\mathcal{T}}, \tag{5.6}$$

where

$$\mathcal{M} \stackrel{\text{def}}{=} \beta_{\mathsf{m}} \circ (P_{\mathsf{m},c})^{-1} \circ P_{\mathsf{f},c} \circ (\beta_{\mathsf{f}})^{-1} = \beta_{\mathsf{m}} \circ (\beta_{\mathsf{f}} \circ \mathcal{P})^{-1}. \tag{5.7}$$

We also have

$$\begin{aligned}&\mathsf{p}^\varepsilon_{\mathsf{m}} + \mathsf{G}_{\mathsf{m},w}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}) = \mathfrak{D}^\varepsilon \widetilde{\mathsf{P}}^\varepsilon_{\mathsf{f}} + \mathsf{G}_{\mathsf{f},w}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon, \mathfrak{D}^\varepsilon T^\varepsilon_{\mathsf{f}}) \quad \text{on } \Gamma_{\mathsf{fm}} \text{ for } (x,t) \in \Omega^\varepsilon_{\mathsf{m},\mathcal{T}};\\ &\mathsf{p}^\varepsilon_{\mathsf{m}} + \mathsf{G}_{\mathsf{m},n}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}) = \mathfrak{D}^\varepsilon \widetilde{\mathsf{P}}^\varepsilon_{\mathsf{f}} + \mathsf{G}_{\mathsf{f},n}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon, \mathfrak{D}^\varepsilon T^\varepsilon_{\mathsf{f}}) \quad \text{on } \Gamma_{\mathsf{fm}} \text{ for } (x,t) \in \Omega^\varepsilon_{\mathsf{m},\mathcal{T}}.\end{aligned}$$

The *initial conditions* are

$$s^\varepsilon_{\mathsf{m}}(x,y,0) = (\mathfrak{D}^\varepsilon S^0_{\mathsf{m}})(x,y) \quad \text{and} \quad p^\varepsilon_{\mathsf{m}}(x,y,0) = (\mathfrak{D}^\varepsilon \mathsf{P}^0_{\mathsf{m}})(x,y) \quad \text{in } \Omega^\varepsilon_{\mathsf{m}} \times Y_{\mathsf{m}}, \tag{5.8}$$

where $S^0_{\mathsf{m}}, \mathsf{P}^0_{\mathsf{m}}$ are the functions defined in (A.10) and Remark 5.

The dilations of the functions defined on the fracture system can be defined in a way similar to that already used for the functions defined on the matrix part.

Now we establish *a priori* estimates for the functions $s^\varepsilon_{\mathsf{m}}, p^\varepsilon_{\mathsf{m}}$. They are given by the following lemma.

Lemma 5.5. *Let $\langle s^\varepsilon_{\mathsf{m}}, p^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}\rangle$ be dilated functions defined in (5.3). Then:*

$$0 \leqslant s^\varepsilon_{\mathsf{m}} \leqslant 1, \quad \text{a.e. in } \Omega_{\mathcal{T}} \times Y_{\mathsf{m}}, \tag{5.9}$$

$$\|\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}})\|_{L^2(\Omega_{\mathcal{T}}; L^2_{per}(Y_{\mathsf{m}}))} \leqslant C, \tag{5.10}$$

$$\|\theta^\varepsilon_{\mathsf{m}}\|_{L^2(\Omega_{\mathcal{T}}; L^2_{per}(Y_{\mathsf{m}}))} \leqslant C, \quad \|\nabla_y \theta^\varepsilon_{\mathsf{m}}\|_{L^2(\Omega_{\mathcal{T}}; L^2_{per}(Y_{\mathsf{m}}))} \leqslant C\varepsilon. \tag{5.11}$$

Proof of Lemma 5.5. Statement (5.9) is evident. The uniform estimate for the gradient of the function $\beta_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon})$ follows from the uniform bounds of Lemmas 3.4 and 5.1. Finally, the uniform estimates (5.11) follow from Lemmas 3.2 and 5.1. Lemma 5.5 is proved. □

Since our fluid system is incompressible we can show that the global pressure in the matrix blocks converges to a constant, independent of the fast variable y. This is the subject of the following lemma.

Lemma 5.6. *For all $\phi \in L^2(\Omega_{\mathcal{T}}; H_0^1(Y_{\mathsf{m}}))$ it holds*

$$\left| \int_{\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\, \lambda_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y p_{\mathsf{m}}^{\varepsilon} \cdot \nabla_y \phi \, dy\, dx\, dt \right| \leqslant C\,\varepsilon \int_{\Omega_{\mathcal{T}}} \|\nabla_y \phi\|_{L^2(Y_{\mathsf{m}})}\, dx\, dt. \tag{5.12}$$

Proof of Lemma 5.6. From (5.4) and (5.5), by summing the equations we eliminate the saturation and obtain:

$$\mathrm{div}_y \left\{ K_{\mathsf{m}}(x,y) \left[\lambda_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y p_{\mathsf{m}}^{\varepsilon} - \varepsilon \lambda_{\mathsf{m},w}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \vec{r}_w - \varepsilon \lambda_{\mathsf{m},n}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \vec{r}_n + \lambda_{\mathsf{m}} B_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y \theta_{\mathsf{m}}^{\varepsilon} \right] \right\} = 0,$$

where $\lambda_{\mathsf{m}} = \lambda_{\mathsf{m},w} + \lambda_{\mathsf{m},n}$. The weak formulation of this equation is as follows: for any $\phi \in L^2(0, \mathcal{T}; H_0^1(\Omega))$,

$$\int_{\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}} \left\{ K_{\mathsf{m}}(x,y) \lambda_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y p_{\mathsf{m}}^{\varepsilon} \cdot \nabla_y \phi \right\} dy\, dx\, dt$$

$$= \int_{\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}} \left\{ K_{\mathsf{m}}(x,y) \left[\varepsilon \lambda_{\mathsf{m},w}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \vec{r}_w + \varepsilon \lambda_{\mathsf{m},n}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \vec{r}_n - \lambda_{\mathsf{m}} B_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y \theta_{\mathsf{m}}^{\varepsilon} \right] \cdot \nabla_y \phi \right\} dy\, dx\, dt.$$

Using the boundedness of the functions $\lambda_{\mathsf{m},w}$, $\lambda_{\mathsf{m},n}$ and $\lambda_{\mathsf{m}} B_{\mathsf{m}}$, together with the estimate (5.11) we conclude that there is a constant C which does not depend on ε, such that (5.12) holds. Lemma 5.6 is proved. □

Lemma 5.2 imply the following convergence results.

Lemma 5.7. *Let $\langle s_{\mathsf{m}}^{\varepsilon}, p_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon} \rangle$ be the dilated functions defined in (5.3). Then, up to a subsequence, we have:*

$$\begin{aligned} \mathbf{1}_{\mathsf{m}}^{\varepsilon}(x) S_{\mathsf{m}}^{\varepsilon} &\xrightarrow{2s} s \in L^2(\Omega_{\mathcal{T}}; L_{per}^2(Y_{\mathsf{m}})); \\ s_{\mathsf{m}}^{\varepsilon} &\rightharpoonup s \text{ weakly in } L^2(\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}). \end{aligned} \tag{5.13}$$

Lemma 5.8. *The weak formulation of Eq. (5.4) for the dilated matrix saturation $s_{\mathsf{m}}^{\varepsilon}$ has the form:*

$$\int_{\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y) \frac{\partial s_{\mathsf{m}}^{\varepsilon}}{\partial t} \varphi \, dy\, dx\, dt + \int_{\Omega_{\mathcal{T}} \times Y_{\mathsf{m}}} \{ K_{\mathsf{m}}(x,y) \nabla_y \beta_{\mathsf{m}}^*(s_{\mathsf{m}}^{\varepsilon}, T) + F^{\varepsilon} \} \cdot \nabla_y \varphi \, dy\, dx\, dt = 0, \tag{5.14}$$

for all $\varphi \in L^2(\Omega_{\mathcal{T}}; H_0^1(Y_{\mathsf{m}}))$, where β_{m}^* is the function introduced in (6.6), the temperature $T = T(x,t)$ is given in Proposition 4.3, and

$$\begin{aligned} F^{\varepsilon} = K_{\mathsf{m}}(x,y) \Big[& \lambda_{\mathsf{m},w}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) [\nabla_y p_{\mathsf{m}}^{\varepsilon} - \varepsilon \vec{r}_w] \\ & + [\Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) - \Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^{\varepsilon}, T)] \nabla_y \beta_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}) + \lambda_{\mathsf{m},w} B_{\mathsf{m}}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) \nabla_y \theta_{\mathsf{m}}^{\varepsilon} \Big]. \end{aligned} \tag{5.15}$$

Furthermore, we have

$$|\Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^{\varepsilon}, \theta_{\mathsf{m}}^{\varepsilon}) - \Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^{\varepsilon}, T)| \to 0 \quad \text{a.e. in } \Omega_{\mathcal{T}} \times Y_m \text{ as } \varepsilon \to 0. \tag{5.16}$$

Proof of Lemma 5.8. From direct calculation we get

$$\Lambda_1^{(\ell)}(S,T) = \sqrt{\frac{\mathsf{M}_n\mathsf{M}_w}{\mathsf{m}_n\mathsf{m}_w}}\,\frac{\sqrt{[k_{r,n}^{(\ell)}(S)\mathsf{m}_w + k_{r,w}^{(\ell)}(S)\mathsf{m}_n]k_{r,w}^{(\ell)}(S)k_{r,n}^{(\ell)}(S)}}{k_{r,n}^{(\ell)}(S)\mu_w(T) + k_{r,w}^{(\ell)}(S)\mu_n(T)},$$

and consequently by (A.7) there is a constant $C > 0$ such that

$$|\Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^\varepsilon, \theta_{\mathsf{m}}^\varepsilon) - \Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^\varepsilon, T)| \leqslant C\left[|\mu_w(\theta_{\mathsf{m}}^\varepsilon) - \mu_w(T)| + |\mu_n(\theta_{\mathsf{m}}^\varepsilon) - \mu_n(T)|\right].$$

Due to the strong convergence given in Proposition 4.3 and Lemma 5.3, we get (5.16). Furthermore, using the fact that $T = T(x,t)$, we introduce the function $\beta_{\mathsf{m}}^* = \beta_{\mathsf{m}}^*(S,T)$ given by the relation

$$\frac{d}{dS}\beta_{\mathsf{m}}^*(S,T) = \Lambda_1^{(\mathsf{m})}(S,T)\beta_{\mathsf{m}}'(S)$$

Then we have

$$\Lambda_1^{(\mathsf{m})}(s_{\mathsf{m}}^\varepsilon, T)\nabla_y\beta_{\mathsf{m}}(s_{\mathsf{m}}^\varepsilon) = \nabla_y\beta_{\mathsf{m}}^*(s_{\mathsf{m}}^\varepsilon, T). \tag{5.17}$$

One can easily verify that

$$\Lambda_1^{(\mathsf{m})}(S,T)\beta_{\mathsf{m}}'(S) = \frac{\lambda_{\mathsf{m},w}(S,T)\lambda_{\mathsf{m},n}(S,T)}{\lambda_{\mathsf{m},w}(S,T) + \lambda_{\mathsf{m},n}(S,T)}.$$

Therefore, the function β_{m}^* introduced in (5.17) coincides with the function β_{m}^* defined by (6.6) below in Section 6. The weak formulationn Eq. (5.14)–(5.15) follow now directly from (5.4) and (5.17). Lemma 5.8 is proved. □

6. Statement of the homogenization result

In this section we formulate the main result of the paper and we complete its proof.

First, we introduce the notation. By S, P_w, P_n and T we denote the homogenized wetting phase saturation, wetting phase pressure, nonwetting phase pressure and temperature, respectively.

- $\Phi^\star = \Phi^\star(x)$ denotes the effective porosity and it is given by:

$$\Phi^\star(x) \stackrel{\text{def}}{=} \Phi_{\mathsf{f}}^{\mathrm{H}}(x)\,|Y_{\mathsf{f}}|/|Y_{\mathsf{m}}|, \tag{6.1}$$

  where the function $\Phi_{\mathsf{f}}^{\mathrm{H}}$ is defined in condition (A.1) and $|Y_\ell|$ is the measure of the set Y_ℓ ($\ell = \mathsf{f}, \mathsf{m}$).
- $\mathbb{K}^\star$ is the homogenized permeability tensor with the entries $(\mathbb{K}^\star)_{i,j}$ defined by:

$$(\mathbb{K}^\star)_{i,j}(x) \stackrel{\text{def}}{=} \frac{1}{|Y_{\mathsf{m}}|}\int_{Y_{\mathsf{f}}} K_{\mathsf{f}}(x,y)\,[\nabla_y\xi_i + \vec{e}_i]\cdot[\nabla_y\xi_j + \vec{e}_j]\,dy, \tag{6.2}$$

  where $\xi_j = \xi_j(x,y)$ $(j = 1,\dots,d)$ is a Y-periodic solution to the auxiliary cell problem:

$$\begin{cases} -\mathrm{div}_y\{K_{\mathsf{f}}(x,y)(\nabla_y\xi_j + \vec{e}_j)\} = 0 & \text{in } Y_{\mathsf{f}}; \\ \nabla_y\xi_j\cdot\vec{\nu}_y = -\vec{e}_j\cdot\vec{\nu}_y & \text{on } \Gamma_{\mathsf{fm}}; \\ y \mapsto \xi_j(y) & Y-\text{periodic}. \end{cases} \tag{6.3}$$

- $\mathcal{K}_T^\star$ is the homogenized thermal conductivity tensor with the entries $(\mathcal{K}_T^\star)_{ij}$ defined by:

$$(\mathcal{K}_T^\star)_{i,j} \stackrel{\text{def}}{=} \frac{1}{|Y_{\mathsf{m}}|}\int_Y k_T(y)[\nabla_y\eta_j + \vec{e}_j]\cdot[\nabla_y\eta_i + \vec{e}_i]dy. \tag{6.4}$$

  where $\eta_j = \eta_j(y)$ $(j = 1,\dots,d)$ is a Y-periodic solution to the auxiliary cell problem:

$$\begin{cases} -\mathrm{div}_y\left\{k_T(y)(\nabla_y\eta_j + \vec{e}_j)\right\} = 0 & \text{in } Y; \\ y \mapsto \eta_j(y) & Y-\text{periodic}. \end{cases} \tag{6.5}$$

- For a fixed value of the temperature function $T \in \mathbb{R}$ we define:

$$\beta^\star_m(s,T) = \int_0^s \frac{\lambda_{\mathsf{m},w}(\varsigma,T)\lambda_{\mathsf{m},n}(\varsigma,T)}{\lambda_{\mathsf{m}}(\varsigma,T)} |\, P'_{\mathsf{m},c}(\varsigma)|\, d\varsigma. \tag{6.6}$$

We study the asymptotic behavior of the solution to problem (2.3)–(2.7) as $\varepsilon \to 0$. In particular, we are going to show that the effective model, expressed in terms of the homogenized phase pressures and the temperature function reads:

$$\begin{cases} 0 \leqslant S \leqslant 1 \quad \text{in } \Omega_{\mathcal{T}}; \\ \Phi^\star \dfrac{\partial S}{\partial t} - \operatorname{div}\Big\{\mathbb{K}^\star \lambda_{\mathsf{f},w}(S,T)(\nabla P_w - \vec{r}_w)\Big\} = \dfrac{\partial \mathcal{Q}_w}{\partial t} \quad \text{in } \Omega_{\mathcal{T}}; \\ -\Phi^\star \dfrac{\partial S}{\partial t} - \operatorname{div}\Big\{\mathbb{K}^\star \lambda_{\mathsf{f},n}(S,T)(\nabla P_n - \vec{r}_n)\Big\} = \dfrac{\partial \mathcal{Q}_n}{\partial t} \quad \text{in } \Omega_{\mathcal{T}}; \\ \dfrac{\partial \mathbf{\Psi}^\star}{\partial t} - \operatorname{div}\Big\{\mathbb{C}_w T\, \mathbb{K}^\star \lambda_{\mathsf{f},w}(S,T)[\nabla P_w - \vec{r}_w] + \mathbb{C}_n T\, \mathbb{K}^\star \lambda_{\mathsf{f},n}(S,T)[\nabla P_n - \vec{r}_n]\Big\} \\ \qquad - \operatorname{div}\big\{\mathcal{K}^\star_T \nabla T\big\} = (\mathbb{C}_w - \mathbb{C}_n)\dfrac{\partial}{\partial t}(\mathcal{Q}_w T) \quad \text{in } \Omega_{\mathcal{T}}; \\ P_{\mathsf{f},c}(S) = P_n - P_w \quad \text{in } \Omega_{\mathcal{T}} \end{cases} \tag{6.7}$$

with

$$\mathbf{\Psi}^\star(S,T) \overset{\text{def}}{=} \big[\big(\mathbb{C}_w\, S + \mathbb{C}_n[1-S]\big)\, \Phi^\star + \mathbb{C}^\star_{\mathsf{m},s} + \mathbb{C}^\star_{\mathsf{f},s} + \mathbb{C}_n\, \overline{\Phi}_{\mathsf{m}}\big]\; T, \tag{6.8}$$

where $\overline{\Phi}_{\mathsf{m}}$ is the mean value of the function Φ_{m} over Y_{m} and the constants $\mathbb{C}^\star_{\mathsf{m},s}, \mathbb{C}^\star_{\mathsf{f},s}$ are given by:

$$\mathbb{C}^\star_{\mathsf{m},s} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{m},s}\big[1 - \overline{\Phi}_{\mathsf{m}}\big] \quad \text{and} \quad \mathbb{C}^\star_{\mathsf{f},s} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},s}\big[|Y_{\mathsf{f}}|/|Y_{\mathsf{m}}| - \Phi^\star(x)\big]. \tag{6.9}$$

For almost every point $x \in \Omega$ the system for flow in a matrix block $Y_{\mathsf{m}} \subset \mathbb{R}^d$ is given by the so-called *imbibition equation*:

$$\begin{cases} \Phi_{\mathsf{m}}(y)\dfrac{\partial s}{\partial t} - \operatorname{div}_y\big\{K_{\mathsf{m}}(x,y)\nabla_y \beta^\star_{\mathsf{m}}(s, T(x,t))\big\} = 0 \quad \text{in } Y_{\mathsf{m}} \times \Omega_{\mathcal{T}}; \\ \qquad s(x,y,t) = \mathcal{P}^{-1}(S(x,t)) \;\; \text{on } \Gamma_{\mathsf{fm}} \times \Omega_{\mathcal{T}}; \\ \qquad s(x,y,0) = S^0_{\mathsf{m}}(x) \quad \text{in } Y_{\mathsf{m}} \times \Omega, \end{cases} \tag{6.10}$$

where $\mathcal{P}(S)$ is defined in (2.19), s denotes the wetting liquid saturation in the matrix block Y_{m} and S^0_{m} is defined in Remark 5.

Remark 8. The inequalities from Remark 4 show that the parabolic operator in the imbibition equation (6.10) is degenerate and thus important qualitative properties (such as, for instance, the finite speed of propagation) remain true even in the nonisothermal case. This can be justified by means of the local in space energy methods such as given in [44].

For any $x \in \Omega$ and $t > 0$, the matrix–fracture sources have the form:

$$\mathcal{Q}_w \overset{\text{def}}{=} -\frac{1}{|Y_{\mathsf{m}}|} \int_{Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\, s(x,y,t)\, dy = -\mathcal{Q}_n. \tag{6.11}$$

The boundary conditions for the effective system (6.7) are given by:

$$\begin{cases} P_w = P_n = T = 0 \quad \text{on } \Gamma_D \times (0,\mathcal{T}); \\ \mathbb{K}^\star\, \lambda_n(S,T)\,(\nabla P_w - \vec{g}) \cdot \vec{\nu} = \mathbb{K}^\star\, \lambda_w(S,T)(\nabla P_n - \vec{g}) \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_N \times (0,\mathcal{T}); \\ \mathcal{K}^\star_T \nabla T \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_N \times (0,\mathcal{T}). \end{cases} \tag{6.12}$$

Finally, the initial conditions read:

$$S(x,0)=S_{\mathsf{f}}^{0}(x)\quad\text{and}\quad T(x,0)=T^{0}(x)\quad\text{in }\Omega. \tag{6.13}$$

The main result of the paper is given by the following theorem.

Theorem 6.1. *Let assumptions (A.1)–(A.10) be fulfilled. Then the solution of the* initial problem (2.3)–(2.7) *converges (up to a subsequence) in the two-scale sense to a weak solution of the* homogenized problem (6.7), (6.10), (6.11)–(6.13).

Proof of Theorem 6.1. The proof is done in several steps. We start our analysis by considering the system (2.3). The main difficulty with the initial unknown functions $p_w^\varepsilon, p_n^\varepsilon$ in this system is that they do not possess the uniform H^1-estimates (see Lemma 3.1). To overcome the difficulties appearing due to the absence of the uniform H^1-estimates, we pass to the equivalent formulation of the problem in terms of the global pressure, saturation, and the temperature function. Then using the convergence and compactness results from Section 4 we pass to the limit in Eqs. (2.25), (2.26), (2.27). This is done in Section 6.1. In order, to pass to the *homogenized phase pressures* we make use of the change of the unknown functions. Namely, we set, by the definition of the global pressure: $P_w \overset{\text{def}}{=} \mathsf{P}+\mathsf{G}_{\mathsf{f},w}(S,T)$ and $P_n \overset{\text{def}}{=} \mathsf{P}+\mathsf{G}_{\mathsf{f},n}(S,T)$. Then we rewrite the limit system obtained in terms of the global pressure and saturation in terms of the homogenized phase pressures (see Section 6.3). The passage to the limit in the matrix blocks makes use of the dilation operator (see Section 5). Then in Section 6.4 we pass to the equivalent problem for the imbibition equation and, finally, obtain the local problem (6.10).

6.1. Passage to the limit in Eqs. (2.25), (2.26), (2.27)

In this section we pass to the limit, as $\varepsilon\to 0$, in Eqs. (2.25), (2.26), (2.27). This will be done in the following way. We replace the gradients of the functions $p_w^\varepsilon, p_n^\varepsilon$ by their representations in terms of the global pressure and saturation (see (2.15), (2.16)) and then pass to the limit in this equivalent formulation. The homogenized equations are then obtained in terms of the homogenized global pressure, the homogenized saturation, and the homogenized temperature function.

Passage to the limit in Eq. (2.25). In order to pass to the limit in (2.25), we make use of the relation (2.16) for the gradient of the function p_w. Then we set:

$$\varphi_w\left(x,t,\frac{x}{\varepsilon}\right)\overset{\text{def}}{=}\varphi(x,t)+\varepsilon\,\zeta^\varepsilon\left(x,t,\frac{x}{\varepsilon}\right)=\varphi(x,t)+\varepsilon\,\zeta_1(x,t)\,\zeta_2\left(\frac{x}{\varepsilon}\right), \tag{6.14}$$

where $\varphi\in\mathcal{D}(\Omega\times[0,\mathcal{T})),\zeta_1\in\mathcal{D}(\Omega_{\mathcal{T}}),\zeta_2\in C_{per}^\infty(Y)$, and plug the function φ_w in the equivalent form of (2.25) in terms of the global pressure and saturation. This yields:

$$\begin{aligned}
&-\int_{\Omega_{\mathcal{T}}}\Phi_{\mathsf{f}}^\varepsilon(x)\,\widetilde{S}^\varepsilon\left[\frac{\partial\varphi}{\partial t}+\varepsilon\frac{\partial\zeta^\varepsilon}{\partial t}\right]\mathbf{1}_{\mathsf{f}}^\varepsilon(x)\,dx\,dt-\int_\Omega\Phi_{\mathsf{f}}^\varepsilon(x)\,S_{\mathsf{f}}^0\mathbf{1}_{\mathsf{f}}^\varepsilon(x)\,\varphi(x,0)\,dx\\
&+\int_{\Omega_{\mathcal{T}}}K_{\mathsf{f}}^\varepsilon(x)\Big\{\lambda_{\mathsf{f},w}(\widetilde{S}^\varepsilon,T^\varepsilon)\left(\nabla\widetilde{\mathsf{P}}_{\mathsf{f}}^\varepsilon-\vec{r}_w\right)+\Lambda_1^{(\mathsf{f})}(\widetilde{S}^\varepsilon,T^\varepsilon)\nabla\beta_{\mathsf{f}}(\widetilde{S}^\varepsilon)+\lambda_{\mathsf{f},w}(\widetilde{S}^\varepsilon,T^\varepsilon)\,\mathsf{B}_{\mathsf{f}}\,(\widetilde{S}^\varepsilon,T^\varepsilon)\nabla T^\varepsilon\Big\}\\
&\qquad\cdot\Big\{\nabla\varphi+\varepsilon\nabla_x\zeta^\varepsilon+\nabla_y\zeta^\varepsilon\Big\}\,\mathbf{1}_{\mathsf{f}}^\varepsilon(x)\,dx\,dt\\
&-\int_{\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon}\Phi_{\mathsf{m}}\left(\frac{x}{\varepsilon}\right)S_{\mathsf{m}}^\varepsilon\left[\frac{\partial\varphi}{\partial t}+\varepsilon\frac{\partial\zeta^\varepsilon}{\partial t}\right]dx\,dt-\int_{\Omega_{\mathsf{m}}^\varepsilon}\Phi_{\mathsf{m}}^\varepsilon(x)\,S_{\mathsf{m}}^0\varphi(x,0)\,dx\\
&+\varepsilon^2\int_{\Omega_{\mathsf{m},\mathcal{T}}^\varepsilon}K_{\mathsf{m}}^\varepsilon(x)\,\Big\{\,\lambda_{\mathsf{m},w}(S_{\mathsf{m}}^\varepsilon,T^\varepsilon)\,(\nabla\mathsf{P}_{\mathsf{m}}^\varepsilon-\vec{r}_w)+\Lambda_1^{(\mathsf{m})}(S_{\mathsf{m}}^\varepsilon,T^\varepsilon)\nabla\beta_{\mathsf{m}}(S_{\mathsf{m}}^\varepsilon)\\
&\qquad+\lambda_{\mathsf{m},w}(S_{\mathsf{m}}^\varepsilon,T^\varepsilon)\,\mathsf{B}_{\mathsf{m}}(S_{\mathsf{m}}^\varepsilon,T^\varepsilon)\nabla T^\varepsilon\,\Big\}\cdot\Big\{\nabla\varphi+\varepsilon\nabla_x\zeta^\varepsilon+\nabla_y\zeta^\varepsilon\Big\}\,dx\,dt=0,
\end{aligned} \tag{6.15}$$

where $K_{\mathsf{f}}^{\varepsilon}(x)=K(x,\frac{x}{\varepsilon})\mathbf{1}_{\mathsf{f}}^{\varepsilon}(x)$, $K_{\mathsf{m}}^{\varepsilon}(x)=K(x,\frac{x}{\varepsilon})\mathbf{1}_{\mathsf{m}}^{\varepsilon}(x)$, and $\widetilde{S}^{\varepsilon}$, $\widetilde{\mathsf{P}}_{\mathsf{f}}^{\varepsilon}$ are the extensions of the functions $S_{\mathsf{f}}^{\varepsilon}$, $\mathsf{P}_{\mathsf{f}}^{\varepsilon}$ from $\Omega_{\mathsf{f}}^{\varepsilon}$ to the whole Ω that were defined in Section 4.

Now taking into account the uniform bounds given in Lemmata 3.2, 3.4 and the convergence results of Lemmata 4.2, 5.7 we pass to the limit in (6.15) as $\varepsilon\to 0$ and obtain the following homogenized equation:

$$\begin{aligned}
&-|Y_{\mathsf{f}}|\int_{\Omega_{\mathfrak{T}}}\Phi_{\mathsf{f}}^{\mathrm{H}}S\frac{\partial\varphi}{\partial t}dxdt-|Y_{\mathsf{f}}|\int_{\Omega}\Phi_{\mathsf{f}}^{\mathrm{H}}S_{\mathsf{f}}^{0}\varphi(x,0)dx+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{f}}}K_{\mathsf{f}}(x,y)\Big\{\lambda_{\mathsf{f},w}(S,T)\big[\nabla\mathsf{P}+\nabla_y\mathsf{w}_p-\vec{r}_w\big]\\
&+\Lambda_1^{(\mathsf{f})}(S,T)[\nabla\beta_{\mathsf{f}}(S)+\nabla_y\mathsf{w}_s]+\lambda_{\mathsf{f},w}(S,T)\mathsf{B}_{\mathsf{f}}(S,T)[\nabla T+\nabla_y\mathsf{w}_T]\Big\}\cdot\Big\{\nabla\varphi+\zeta_1\nabla_y\zeta_2\Big\}\,dy\,dx\,dt\\
&=\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\,s(x,y,t)\frac{\partial\varphi}{\partial t}\,dy\,dx\,dt+\int_{\Omega\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\,S_{\mathsf{m}}^{0}(x)\varphi(x,0)\,dy\,dx.
\end{aligned}\tag{6.16}$$

Passage to the limit in Eq. (2.26). In a similar way, using relation (2.15) for the gradient of the function p_n, we obtain the second homogenized equation. It reads:

$$\begin{aligned}
&|Y_{\mathsf{f}}|\int_{\Omega_{\mathfrak{T}}}\Phi_{\mathsf{f}}^{\mathrm{H}}S\,\frac{\partial\varphi}{\partial t}\,dx\,dt+|Y_{\mathsf{f}}|\int_{\Omega}\Phi_{\mathsf{f}}^{\mathrm{H}}S_{\mathsf{f}}^{0}\varphi(x,0)dx+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{f}}}K_{\mathsf{f}}(x,y)\Big\{\lambda_{\mathsf{f},n}(S,T)\big[\nabla\mathsf{P}+\nabla_y\mathsf{w}_p-\vec{r}_n\big]\\
&-\Lambda_1^{(\mathsf{f})}(S,T)[\nabla\beta_{\mathsf{f}}(S)+\nabla_y\mathsf{w}_s]+\lambda_{\mathsf{f},n}(S,T)\mathsf{B}_{\mathsf{f}}(S,T)[\nabla T+\nabla_y\mathsf{w}_T]\Big\}\cdot\Big\{\nabla\varphi+\zeta_1\nabla_y\zeta_2\Big\}\,dy\,dx\,dt\\
&=-\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)s(x,y,t)\frac{\partial\varphi}{\partial t}dy\,dx\,dt-\int_{\Omega\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\,S_{\mathsf{m}}^{0}(x)\varphi(x,0)\,dy\,dx.
\end{aligned}\tag{6.17}$$

Here in (6.16), (6.17) the function s is defined in (5.13).

Passage to the limit in Eq. (2.27). Taking into account the relations (2.15), (2.16) and then using the test function (6.14), after passing to the limit as $\varepsilon\to 0$, we get:

$$\begin{aligned}
&-\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{f}}}\Big\{\Phi_{\mathsf{f}}^{H}(x)\left(\mathbb{C}_wS+\mathbb{C}_n[1-S]\right)+\left(1-\Phi_{\mathsf{f}}^{H}(x)\right)\mathbb{C}_s(y)\Big\}\,T\,\frac{\partial\varphi}{\partial t}\,dx\,dy\,dt\\
&-\int_{\Omega\times Y_{\mathsf{f}}}\Big\{\Phi_{\mathsf{f}}^{H}(x)(\mathbb{C}_wS_{\mathsf{f}}^{0}+\mathbb{C}_n[1-S_{\mathsf{f}}^{0}])+(1-\Phi_{\mathsf{f}}^{H}(x))\,\mathbb{C}_s(y)\Big\}\,T^{0}\varphi(x,0)\,dx\,dy\\
&+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{f}}}\mathbb{C}_wTK_{\mathsf{f}}(x,y)\Big\{\lambda_{\mathsf{f},w}(S,T)[\nabla\mathsf{P}+\nabla_y\mathsf{w}_p-\vec{r}_w]+\Lambda_1^{(\mathsf{f})}(S,T)[\nabla\beta_{\mathsf{f}}(S)+\nabla_y\mathsf{w}_s]\\
&\qquad+\lambda_{\mathsf{f},w}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,[\nabla T+\nabla_y\mathsf{w}_T]\Big\}\cdot\Big\{\nabla\varphi+\zeta_1\nabla_y\zeta_2\Big\}\,dx\,dy\,dt\\
&+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{f}}}\mathbb{C}_nTK_{\mathsf{f}}(x,y)\Big\{\lambda_{\mathsf{f},n}(S,T)[\nabla\mathsf{P}+\nabla_y\mathsf{w}_p-\vec{r}_n]-\Lambda_1^{(\mathsf{f})}(S,T)[\nabla\beta_{\mathsf{f}}(S)+\nabla_y\mathsf{w}_s]\\
&\qquad+\lambda_{\mathsf{f},n}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)[\nabla T+\nabla_y\mathsf{w}_T]\Big\}\cdot\Big\{\nabla\varphi+\zeta_1\nabla_y\zeta_2\Big\}\,dx\,dy\,dt\\
&-\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Big\{\Phi_{\mathsf{m}}(y)(\mathbb{C}_ws+\mathbb{C}_n[1-s])+(1-\Phi_{\mathsf{m}}(y))\mathbb{C}_s(y)\Big\}\,T\,\frac{\partial\varphi}{\partial t}\,dx\,dy\,dt+\\
&-\int_{\Omega\times Y_{\mathsf{m}}}\Big\{\Phi_{\mathsf{m}}(y)(\mathbb{C}_wS_{\mathsf{m}}^{0}+\mathbb{C}_n[1-S_{\mathsf{m}}^{0}])+(1-\Phi_{\mathsf{m}}(y))\,\mathbb{C}_s(y)\Big\}\,T^{0}\varphi(x,0)\,dx\,dy\\
&\qquad+\int_{\Omega_{\mathfrak{T}}\times Y}k_T(y)\,[\nabla T+\nabla_y\mathsf{w}_T]\cdot\Big\{\nabla\varphi+\zeta_1\nabla_y\zeta_2\Big\}\,dx\,dy\,dt=0.
\end{aligned}\tag{6.18}$$

It remains to identify the corrector functions w_p, w_s, w_T appearing in Eqs. (6.16), (6.17), (6.18) in the standard way (see, e.g., [16]). By setting $\varphi\equiv 0$ in these equations and by multiplying (6.16) and (6.17) by $\mathbb{C}_w T$ and $\mathbb{C}_n T$, respectively, and subtracting from (6.18), and taking into account the fact that the temperature T does not depend on the fast variable y, we get

$$\mathsf{w}_T(x,y,t) = \sum_{j=1}^{d} \eta_j(y) \frac{\partial T}{\partial x_j}(x,t), \tag{6.19}$$

where $\eta_j = \eta_j(y)$ $(j = 1,\dots,d)$ are the Y-periodic solution of the auxiliary cell problem (6.5).

Adding Eqs. (6.16), (6.17) and dividing by $\lambda_{\mathsf{f}}(S,T)$, which does not depend on y and is strictly positive due to (2.21), we obtain:

$$\mathsf{w}_p(x,y,t) = \sum_{j=1}^{d} \left(\frac{\partial \mathsf{P}}{\partial x_j}(x,t) - r_{w,j} \right) \xi_j(y) + \mathsf{B}_{\mathsf{f}}(S,T) \sum_{j=1}^{d} \frac{\partial T}{\partial x_j}(x,t) \chi_j(y), \tag{6.20}$$

where the functions $\xi_j(x,y)$ satisfy the local problems (6.3) and $\chi_j(x,y)$ satisfy the following local problems:

$$\begin{cases} -\mathrm{div}_y \left\{ K_{\mathsf{f}}(x,y) \nabla_y \chi_j \right\} = \mathrm{div}_y \left\{ K_{\mathsf{f}}(x,y)(\nabla_y \eta_j + \vec{e}_j) \right\} \quad \text{in } Y_{\mathsf{f}}; \\ \nabla_y \chi_j \cdot \vec{\nu}_y = -(\nabla_y \eta_j + \vec{e}_j) \cdot \vec{\nu}_y \quad \text{on } \Gamma_{\mathsf{fm}}; \\ y \mapsto \chi_j(y) \quad Y - \text{periodic}. \end{cases} \tag{6.21}$$

Note that from the uniqueness of the solution to problem (6.21), we have

$$\chi_j + \eta_j = \xi_j \tag{6.22}$$

up to an additive constant.

Finally, we can identify

$$\mathsf{w}_s(x,y,t) = \sum_{j=1}^{d} \xi_j(y) \frac{\partial \beta_{\mathsf{f}}(S)}{\partial x_j}(x,t). \tag{6.23}$$

6.2. Effective equations in terms of the global pressure and saturation

In this section we derive the homogenized equations for the wetting, nonwetting phases, and the temperature. By setting $\zeta_2 = 0$ in (6.16)–(6.18) and using the representations of the corrector functions obtained in the previous section, we derive the following homogenized equations:

$$\begin{aligned} &- \int_{\Omega_{\mathfrak{T}}} \Phi^\star\, S \frac{\partial \varphi}{\partial t}\, dx\, dt - \int_{\Omega} \Phi^\star S_{\mathsf{f}}^0 \varphi(x,0) dx \\ &+ \int_{\Omega_{\mathfrak{T}}} \mathbb{K}^\star(x) \left\{ \lambda_{\mathsf{f},w}(S,T)\, (\nabla \mathsf{P} - \vec{r}_w) + \Lambda_1^{(\mathsf{f})}(S,T) \nabla \beta_{\mathsf{f}}(S) + \lambda_{\mathsf{f},w}(S,T)\, \mathsf{B}_{\mathsf{f}}(S,T)\, \nabla T \right\} \cdot \nabla \varphi\, dx\, dt \\ &= - \int_{\Omega_{\mathfrak{T}}} \mathcal{Q}_w \frac{\partial \varphi}{\partial t}\, dx\, dt + \int_{\Omega} \overline{\Phi}_{\mathsf{m}} S_{\mathsf{m}}^0(x) \varphi(x,0)\, dx. \end{aligned} \tag{6.24}$$

where the homogenized porosity $\Phi^\star$ is given by (6.1), $\overline{\Phi}_{\mathsf{m}}$ is the mean value of the function Φ_{m} over Y_{m}, the homogenized permeability tensor $\mathbb{K}^\star$ is defined in (6.2), $\mathcal{Q}_w$ stands for the source term given by (6.11) and the function $s = s(x,y,t)$ is defined in (5.13).

Homogenized nonwetting phase equation is given by:

$$\begin{aligned} &\int_{\Omega_{\mathfrak{T}}} \Phi^\star\, S \frac{\partial \varphi}{\partial t}\, dx\, dt + \int_{\Omega} \Phi^\star S_{\mathsf{f}}^0 \varphi(x,0) dx \\ &+ \int_{\Omega_{\mathfrak{T}}} \mathbb{K}^\star(x) \left\{ \lambda_{\mathsf{f},n}(S,T)\, (\nabla \mathsf{P} - \vec{r}_n) - \Lambda_1^{(\mathsf{f})}(S,T)\, \nabla \beta_{\mathsf{f}}(S) + \lambda_{\mathsf{f},n}(S,T)\, \mathsf{B}_{\mathsf{f}}(S,T)\, \nabla T \right\} \cdot \nabla \varphi\, dx\, dt \\ &= - \int_{\Omega_{\mathfrak{T}}} \mathcal{Q}_n \frac{\partial \varphi}{\partial t}\, dx\, dt - \int_{\Omega} \overline{\Phi}_{\mathsf{m}} S_{\mathsf{m}}^0(x) \varphi(x,0)\, dx, \end{aligned} \tag{6.25}$$

where $\mathcal{Q}_w = -\mathcal{Q}_n$ (see (6.11)).

Homogenized equation for the temperature is given by:

$$\begin{aligned}
&-\int_{\Omega_{\mathfrak{T}}}\Big\{\Phi^\star(\mathbb{C}_w\,S+\mathbb{C}_n[1-S])+\mathbb{C}^\star_{\mathsf{m},s}+\mathbb{C}^\star_{\mathsf{f},s}+\mathbb{C}_n\,\overline{\Phi}_{\mathsf{m}}\Big\}\,T\,\frac{\partial\varphi}{\partial t}dxdt\\
&-\int_{\Omega}\Big\{\Phi^\star(\mathbb{C}_w S^0_{\mathsf{f}}+\mathbb{C}_n[1-S^0_{\mathsf{f}}])+\mathbb{C}^\star_{\mathsf{m},s}+\mathbb{C}^\star_{\mathsf{f},s}+\mathbb{C}_n\,\overline{\Phi}_{\mathsf{m}}\Big\}\,T^0\varphi(x,0)\,dx\\
&+\int_{\Omega_{\mathfrak{T}}}\mathbb{C}_w\,T\,\mathbb{K}^\star\Big\{\lambda_{\mathsf{f},w}(S,T)[\nabla\mathsf{P}-\vec{r}_w]+\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},w}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\Big\}\cdot\nabla\varphi\,dx\,dt\\
&+\int_{\Omega_{\mathfrak{T}}}\mathbb{C}_n\,T\,\mathbb{K}^\star\Big\{\lambda_{\mathsf{f},n}(S,T)[\nabla\mathsf{P}-\vec{r}_n]-\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},n}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\Big\}\cdot\nabla\varphi\,dx\,dt\\
&+\int_{\Omega_{\mathfrak{T}}}\mathcal{K}^\star_T\,\nabla T\cdot\nabla\varphi dx\,dt=-\int_{\Omega_{\mathfrak{T}}}(\mathbb{C}_w-\mathbb{C}_n)\mathcal{Q}_w\,T\,\frac{\partial\varphi}{\partial t}dxdt+\overline{\Phi}_{\mathsf{m}}(\mathbb{C}_w-\mathbb{C}_n)\int_{\Omega}S^0_{\mathsf{m}}T^0\varphi(x,0)\,dx,
\end{aligned}\tag{6.26}$$

where $\mathbb{C}^\star_{\mathsf{m},s}$ and $\mathbb{C}^\star_{\mathsf{f},s}$ are given by (6.9) and $\mathcal{K}^\star_T$ is the homogenized thermal conductivity tensor defined in (6.4)–(6.5).

6.3. Effective equations in terms of the phase pressures

The variational equations (6.24)–(6.26) can be written in the differential form as follows:

$$\begin{aligned}
\Phi^\star\,\frac{\partial S}{\partial t}-\operatorname{div}\Big\{\,\mathbb{K}^\star\,\Big[\,\lambda_{\mathsf{f},w}(S,T)(\nabla\mathsf{P}-\vec{r}_w)+\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},w}(S,T)\mathsf{B}_{\mathsf{f}}(S,T)\nabla T\,\Big]\Big\}\\
=\frac{\partial\mathcal{Q}_w}{\partial t}.
\end{aligned}\tag{6.27}$$

$$\begin{aligned}
-\Phi^\star\,\frac{\partial S}{\partial t}-\operatorname{div}\Big\{\,\mathbb{K}^\star\,\Big[\,\lambda_{\mathsf{f},n}(S,T)(\nabla\mathsf{P}-\vec{r}_n)-\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},n}(S,T)\mathsf{B}_{\mathsf{f}}(S,T)\nabla T\,\Big]\Big\}\\
=\frac{\partial\mathcal{Q}_n}{\partial t}.
\end{aligned}\tag{6.28}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{\Big[\,\Phi^\star(\mathbb{C}_w\,S+\mathbb{C}_n[1-S])+\mathbb{C}^\star_{\mathsf{m},s}+\mathbb{C}^\star_{\mathsf{f},s}+\mathbb{C}_n\,\overline{\Phi}_{\mathsf{m}}\,\Big]\,T\,\Big\}\\
&-\operatorname{div}\Big\{\,\mathbb{C}_w\,T\,\mathbb{K}^\star\,\Big\{\,\lambda_{\mathsf{f},w}(S,T)\,[\nabla\mathsf{P}-\vec{r}_w]+\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},w}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\,\Big\}\Big\}\\
&-\operatorname{div}\Big\{\,\mathbb{C}_n\,T\,\mathbb{K}^\star\,\Big\{\,\lambda_{\mathsf{f},n}(S,T)\,[\nabla\mathsf{P}-\vec{r}_n]-\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},n}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\,\Big\}\Big\}\\
&-\operatorname{div}\big\{\mathcal{K}^\star_T\,\nabla T\big\}=\big(\mathbb{C}_w-\mathbb{C}_n\big)\,\frac{\partial}{\partial t}\big(\mathcal{Q}_w\,T\big),
\end{aligned}\tag{6.29}$$

with the following boundary conditions:

$$\mathsf{P}=T=0,\quad S=1\quad\text{on }\Gamma_D\times(0,\mathfrak{T});\tag{6.30}$$

and the Neumann boundary conditions on $\Gamma_N\times(0,\mathfrak{T})$:

$$\mathbb{K}^\star\Big\{\lambda_{\mathsf{f},w}(S,T)\,[\nabla\mathsf{P}-\vec{r}_w]+\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},w}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\Big\}\cdot\vec{\nu}=0,\tag{6.31}$$

$$\mathbb{K}^\star\Big\{\lambda_{\mathsf{f},n}(S,T)\,[\nabla\mathsf{P}-\vec{r}_n]-\Lambda^{(\mathsf{f})}_1(S,T)\nabla\beta_{\mathsf{f}}(S)+\lambda_{\mathsf{f},n}(S,T)\,\mathsf{B}_{\mathsf{f}}(S,T)\,\nabla T\Big\}\cdot\vec{\nu}=0,\tag{6.32}$$

$$\mathcal{K}^\star_T\,\nabla T\cdot\vec{\nu}=0.\tag{6.33}$$

It is straightforward to show that the initial conditions read,

$$\Phi^\star S(x,0)+\frac{1}{|Y_{\mathsf{f}}|}\int_{Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\,s(x,y,0)\,dy=\Phi^\star S^0_{\mathsf{f}}(x)+\overline{\Phi}_{\mathsf{m}}S^0_{\mathsf{m}}(x),\quad T(x,0)=T^0(x)\quad\text{for }x\in\Omega.\tag{6.34}$$

Let us introduce now the functions

$$p_w \overset{\text{def}}{=} \mathsf{P} + \mathsf{G}_{\mathsf{f},w}(S,T) \quad \text{and} \quad p_n \overset{\text{def}}{=} \mathsf{P} + \mathsf{G}_{\mathsf{f},n}(S,T), \tag{6.35}$$

where the functions $\mathsf{G}_{\mathsf{f},w}, \mathsf{G}_{\mathsf{f},n}$ are defined in Section 2.2. We call these functions *homogenized phase pressures*. Then using relations (2.15) and (2.16) it is easy to see that the homogenized equations (6.27), (6.28), and (6.29) become the desired equations $(6.7)_2$–$(6.7)_4$. The boundary conditions (6.12) follow from (6.30)–(6.33).

6.4. Flow equations in the matrix block

In preceding section we have shown that the homogenized phase pressures p_w and p_n, homogenized wetting phase saturation S and homogenized temperature T satisfy the homogenized equations $(6.7)_2$–$(6.7)_4$ and the boundary conditions (6.12). The source terms $\mathcal{Q}_w$ and $\mathcal{Q}_n$ are given by (6.11) with the function $s(x,y,t)$ given as a weak limit of the dilated functions $s^\varepsilon_{\mathsf{m}}$ (see Lemma 5.7). Furthermore, only a combination of the functions appearing on the left hand side in (6.34) has enough regularity to satisfy corresponding initial conditions. In this section it is shown by the asymptotic analysis of Eqs. (5.4) and (5.5) satisfied by dilated functions $s^\varepsilon_{\mathsf{m}}$, $p^\varepsilon_{\mathsf{m}}$ and $\theta^\varepsilon_{\mathsf{m}}$, that homogenized matrix block saturation s, given as a weak limit of $s^\varepsilon_{\mathsf{m}}$, satisfies the imbibition equation (6.10), which completes the homogenized model. In that way it is also shown that the function s has sufficient regularity in temporal variable and satisfies the initial condition $(6.10)_3$, which together with (6.34) gives the initial condition

$$S(x,0) = S^0_{\mathsf{f}}(x) \quad \text{for } x \in \Omega. \tag{6.36}$$

In this section we proceed with the monotonicity arguments similar to ones from [29]. Let us temporarily denote by $s^* = s^*(x,y,t)$ a solution to problem (6.10), Then, for any $\varphi \in L^2(\Omega_{\mathcal{T}}; H^1_0(Y_{\mathsf{m}}))$,

$$\int_{\Omega_{\mathcal{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\,\frac{\partial s^*}{\partial t}\,\varphi\,dy\,dx\,dt + \int_{\Omega_{\mathcal{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\nabla_y\beta^*_{\mathsf{m}}(s^*,T)\cdot\nabla_y\varphi\,dy\,dx\,dt = 0.$$

Subtracting this equation from (5.14) we get:

$$\begin{aligned}&\int_{\Omega_{\mathcal{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\frac{\partial}{\partial t}(s^\varepsilon_{\mathsf{m}} - s^*)\,\varphi\,dy\,dx\,dt \\ &\quad + \int_{\Omega_{\mathcal{T}}\times Y_{\mathsf{m}}} \Big\{ K_{\mathsf{m}}(x,y)\,[\nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T) - \nabla_y\beta^*_{\mathsf{m}}(s^*,T)]\cdot\nabla_y\varphi + F^\varepsilon\cdot\nabla_y\varphi \Big\}\,dy\,dx\,dt = 0,\end{aligned} \tag{6.37}$$

where F^ε is given in (5.15).

In order to pass to the limit as $\varepsilon \to 0$ in (6.37) we consider the following auxiliary problem:

$$\begin{cases} -\mathrm{div}_y \Big\{ K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon \Big\} = \Phi_{\mathsf{m}}(y)\,\big[s^\varepsilon_{\mathsf{m}}(x,y,t) - s^*(x,y,t)\big] & \text{in } Y_{\mathsf{m}}\,; \\ w^\varepsilon = 0 \quad \text{on } \Gamma_{\mathsf{m}}, \end{cases} \tag{6.38}$$

for all $t \in (0,\mathcal{T})$. From (6.38) we obtain that for a.e. $(x,t) \in \Omega_{\mathcal{T}}$,

$$\|\nabla_y w^\varepsilon(x,\cdot,t)\|_{L^2(Y_{\mathsf{m}})} \leqslant C\|s^\varepsilon_{\mathsf{m}}(x,\cdot,t) - s^*(x,\cdot,t)\|_{L^2(Y_{\mathsf{m}})}, \tag{6.39}$$

where C is a constant that does not depend on (x,t).

Since $s^\varepsilon_{\mathsf{m}}(x,y,0) = (\mathfrak{D}^\varepsilon S^0_{\mathsf{m}})(x,y)$ and $s^*(x,y,0) = S^0_{\mathsf{m}}(x)$, for $t = 0$,

$$\begin{cases} -\mathrm{div}_y \Big\{ K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon \Big\} = \Phi_{\mathsf{m}}(y)\,\big[\mathfrak{D}^\varepsilon S^0_{\mathsf{m}}(x,y) - S^0_{\mathsf{m}}(x)\big] & \text{in } Y_{\mathsf{m}}\,; \\ w^\varepsilon = 0 \quad \text{on } \ \Gamma_{\mathsf{m}}. \end{cases} \tag{6.40}$$

Lemma 6.1. *Let $s^\varepsilon_{\mathsf{m}}$ be dilated matrix saturation defined in (5.3) and let s^* be the solution to (6.10). Then,*

$$\begin{aligned}0 \leqslant &\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\,\left[\,\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T) - \beta^*_{\mathsf{m}}(s^*,T)\,\right]\,(s^\varepsilon_{\mathsf{m}} - s^*)\,dy\,dx\,dt \\ &\leqslant -\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,\nabla_y\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt \\ &\quad+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T) - \beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S),T)\right](s^\varepsilon_{\mathsf{m}} - s^*)\,dy\,dx\,dt \\ &\quad+\frac{1}{2}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)|\nabla_y w^\varepsilon(x,y,0)|^2\,dx\,dy + C\,\varepsilon\int_{\Omega_{\mathfrak{T}}}\|\nabla_y w^\varepsilon\|_{L^2(Y_{\mathsf{m}})}\,dx\,dt \\ &\quad+C\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\left|\Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}},\theta^\varepsilon_{\mathsf{m}}) - \Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}},T)\right||\nabla_y\beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}})|\,|\nabla_y w^\varepsilon|\,dy\,dx\,dt.\end{aligned} \tag{6.41}$$

Proof of Lemma 6.1. We plug w^ε in (6.37) as the test function. We get:

$$\begin{aligned}&\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\frac{\partial}{\partial t}(s^\varepsilon_{\mathsf{m}} - s^*)\,w^\varepsilon\,dy\,dx\,dt \\ &\quad+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\left\{K_{\mathsf{m}}(x,y)\left[\nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T) - \nabla_y\beta^*_{\mathsf{m}}(s^*,T)\right]\cdot\nabla_y w^\varepsilon + F^\varepsilon\cdot\nabla_y w^\varepsilon\right\}dy\,dx\,dt = 0.\end{aligned} \tag{6.42}$$

Now we rearrange the terms in this equation. This is done in several steps.

Step 1. Integration by parts with respect to time in the first term of (6.42). Using (6.38) we get:

$$\begin{aligned}&\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\frac{\partial}{\partial t}(s^\varepsilon_{\mathsf{m}} - s^*)\,w^\varepsilon\,dx\,dt\,dy = -\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[s^\varepsilon_{\mathsf{m}} - s^*\right]\frac{\partial w^\varepsilon}{\partial t}\,dy\,dx\,dt \\ &\quad+\int_{\Omega\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[s^\varepsilon_{\mathsf{m}}(x,y,\mathfrak{T}) - s^*(x,y,\mathfrak{T})\right]w^\varepsilon(x,y,\mathfrak{T})\,dy\,dx \\ &-\int_{\Omega\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[s^\varepsilon_{\mathsf{m}}(x,y,0) - s^*(x,y,0)\right]w^\varepsilon(x,y,0)\,dy\,dx = -\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[s^\varepsilon_{\mathsf{m}} - s^*\right]\frac{\partial w^\varepsilon}{\partial t}\,dy\,dx\,dt \\ &\quad+\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,\mathfrak{T})|^2\,dy\,dx - \int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,0)|^2\,dy\,dx.\end{aligned}$$

Using integration by parts with respect to time variable in (6.38) we get:

$$\begin{aligned}&\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y)\left[s^\varepsilon_{\mathsf{m}} - s^*\right]\frac{\partial w^\varepsilon}{\partial t}\,dx\,dt\,dy = \int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,\nabla_y w^\varepsilon(x,y,t)\cdot\nabla_y\frac{\partial w^\varepsilon}{\partial t}(x,y,t)\,dy\,dx\,dt \\ &= \int_0^{\mathfrak{T}}\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,t)|^2\,dy\,dx\,dt \\ &= -\frac{1}{2}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,0)|^2\,dy\,dx + \frac{1}{2}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,\mathfrak{T})|^2\,dy\,dx.\end{aligned}$$

Taking into account this two transformations in (6.42) we get:

$$\begin{aligned}&\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\left[\nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T) - \nabla_y\beta^*_{\mathsf{m}}(s^*,T)\right]\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt \\ &\quad= -\frac{1}{2}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,\mathfrak{T})|^2\,dx\,dy \\ &\quad+\frac{1}{2}\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\,|\nabla_y w^\varepsilon(x,y,0)|^2\,dy\,dx - \int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} F^\varepsilon\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt.\end{aligned} \tag{6.43}$$

Step 2. Integration by parts with respect to space variable in (6.38). Taking into account the auxiliary problem (6.38) we get:

$$\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\ \big[\ \nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\nabla_y\beta^*_{\mathsf{m}}(s^*,T)\ \big]\ \cdot\nabla_y w^\varepsilon\,dy\,dx\,dt$$
$$=\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} \mathrm{div}_y\Big\{\big[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\big]\,K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon\Big\}\,dy\,dx\,dt$$
$$-\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\big[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\big]\,\mathrm{div}_y\Big\{K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon\Big\}\,dy\,dx\,dt$$
$$=\int_{\Omega_{\mathfrak{T}}\times \partial Y_{\mathsf{m}}}\big[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\big]\,K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon\cdot\mathbf{n}\,dS_y\,dx\,dt$$
$$+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\big[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\big](s^\varepsilon_{\mathsf{m}}-s^*)\,dy\,dx\,dt.$$

Note that the boundary condition (5.6) implies that $s^\varepsilon_{\mathsf{m}}=\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon)$ a.e. on $\partial Y_{\mathsf{m}}\times\Omega_{\mathfrak{T}}$ and, therefore,

$$\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)=\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\quad\text{on }\ \Gamma_{\mathsf{fm}}\ \text{ for }\ (x,t)\in\Omega^\varepsilon_{\mathsf{m},\mathfrak{T}}.$$

In the same way, since $s^*=\mathcal{P}^{-1}(S)$ a.e. on $\partial Y_{\mathsf{m}}\times\Omega_{\mathfrak{T}}$, we have that

$$\beta^*_{\mathsf{m}}(s^*,T)=\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S),T)\quad\text{on }\ \Gamma_{\mathsf{fm}}\ \text{ for }\ (x,t)\in\Omega^\varepsilon_{\mathsf{m},\mathfrak{T}}.$$

Since the boundary values are well defined in the whole $\Omega_{\mathfrak{T}}\times Y$ we can substitute them in the integral over ∂Y_{m} and perform again the integration by parts. We get:

$$\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\big[\nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\nabla_y\beta^*_{\mathsf{m}}(s^*,T)\big]\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt$$
$$=\int_{\Omega_{\mathfrak{T}}\times \partial Y_{\mathsf{m}}}\big[\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)-\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S),T)\big]\,K_{\mathsf{m}}(x,y)\nabla_y w^\varepsilon\cdot\mathbf{n}\,dS_y\,dx\,dt$$
$$+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\big[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\big](s^\varepsilon_{\mathsf{m}}-s^*)\,dy\,dx\,dt.$$

The integration by parts now gives:

$$\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)[\nabla_y\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\nabla_y\beta^*_{\mathsf{m}}(s^*,T)]\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt$$
$$=\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\nabla_y\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\cdot\nabla_y w^\varepsilon\,dy\,dx\,dt$$
$$-\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\ \big[\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)-\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S),T)\big]\ (s^\varepsilon_{\mathsf{m}}-s^*)\,dy\,dx\,dt$$
$$+\int_{\Omega_{\mathfrak{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)](s^\varepsilon_{\mathsf{m}}-s^*)\,dy\,dx\,dt,$$

where we have taken into account the auxiliary problem (6.38) and also the fact that $\nabla_y \beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S), T) = 0$. Plugging now this equality into (6.43) we get:

$$
\begin{aligned}
0 \leqslant \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} & \Phi_{\mathsf{m}}(y) \left[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}}, T) - \beta^*_{\mathsf{m}}(s^*, T)\right] (s^\varepsilon_{\mathsf{m}} - s^*)\, dy\, dt\, dt \\
& \leqslant - \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\, \nabla_y \beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon), Tx) \cdot \nabla_y w^\varepsilon\, dy\, dx\, dt \\
& \quad + \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y) \left[\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon), T) - \beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S), T)\right] (s^\varepsilon_{\mathsf{m}} - s^*)\, dy\, dx\, dt \\
& \quad - \frac{1}{2} \int_{\Omega \times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y) \left|\nabla_y w^\varepsilon(x,y,\mathfrak{T})\right|^2 dy\, dx + \frac{1}{2} \int_{\Omega \times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y) \left|\nabla_y w^\varepsilon(x,y,0)\right|^2 dy\, dx \\
& \quad - \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} F^\varepsilon \cdot \nabla_y w^\varepsilon\, dy\, dx\, dt,
\end{aligned}
\tag{6.44}
$$

where we used the monotonicity of the function $s \mapsto \beta^*_{\mathsf{m}}(s, T)$.

The boundedness of the functions $\lambda_{\mathsf{m},w}$, $\lambda_{\mathsf{m},n}$, $\lambda_{\mathsf{m}} B_{\mathsf{m}}$ and estimate (5.11) imply that

$$
\left| \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} F^\varepsilon \cdot \nabla_y w^\varepsilon\, dy\, dx\, dt \right| \leqslant \left| \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\, \lambda_{\mathsf{m},w}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}})\, \nabla_y p^\varepsilon_{\mathsf{m}} \cdot \nabla_y w^\varepsilon\, dy\, dx\, dt \right|
$$

$$
+ C \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} \left|\Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}) - \Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, T)\right| \left|\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}})\right| \left|\nabla_y w^\varepsilon\right| dy\, dx\, dt + C\,\varepsilon \int_{\Omega_{\mathfrak{T}}} \|\nabla_y w^\varepsilon\|_{L^2(Y_{\mathsf{m}})}\, dx\, dt.
$$

Then, from (5.12) it follows the estimate:

$$
\begin{aligned}
\left| \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} F^\varepsilon \cdot \nabla_y w^\varepsilon\, dy\, dx\, dt \right| & \leqslant C\,\varepsilon \int_{\Omega_{\mathfrak{T}}} \|\nabla_y w^\varepsilon\|_{L^2(Y_{\mathsf{m}})}\, dx\, dt \\
+ C \int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} & \left|\Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}) - \Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, T)\right| \left|\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}})\right| \left|\nabla_y w^\varepsilon\right| dy\, dx\, dt.
\end{aligned}
\tag{6.45}
$$

Finally, the bound (6.41) immediately follows from (6.44) and (6.45). Lemma 6.1 is proved. □

Now we turn to the main result of the section. This is done by studying the asymptotic behavior of the right-hand side of (6.41) as $\varepsilon \to 0$. We have.

Lemma 6.2. *Let $s \in L^2(\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}})$ be the limit function from (5.13) and let s^* be the weak solution of the imbibition equation (6.10). Then $s = s^*$ a.e. in $\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}$.*

Proof of Lemma 6.2. We will show that all terms on the right hand side in (6.41) converge to zero as $\varepsilon \to 0$. First note that by (6.39) the functions w^ε are uniformly bounded in $L^2(\Omega_{\mathfrak{T}}; H^1(Y_{\mathsf{m}}))$ and, therefore,

$$
\varepsilon \int_{\Omega_{\mathfrak{T}}} \|\nabla_y w^\varepsilon\|_{L^2(Y_{\mathsf{m}})}\, dx\, dt \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.46}
$$

By Lemma 5.8 and the bound (5.10) we have:

$$
\int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} \left|\Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, \theta^\varepsilon_{\mathsf{m}}) - \Lambda_1^{(\mathsf{m})}(s^\varepsilon_{\mathsf{m}}, T)\right| \left|\nabla_y \beta_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}})\right| \left|\nabla_y w^\varepsilon\right| dy\, dx\, dt \to 0 \quad \text{as } \varepsilon \to 0.
$$

Proposition 4.2 implies that $\widetilde{S}^\varepsilon \to S$ strongly in $L^2(\Omega_{\mathfrak{T}})$ and from Lemma 5.3 we have that $\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon \to S$ strongly in $L^2(\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}})$. This leads to

$$
\int_{\Omega_{\mathfrak{T}} \times Y_{\mathsf{m}}} \Phi_{\mathsf{m}}(y) \left[\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon \widetilde{S}^\varepsilon), T) - \beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(S), T)\right] (s^\varepsilon_{\mathsf{m}} - s^*)\, dy\, dx\, dt \to 0 \quad \text{as } \varepsilon \to 0.
$$

From (6.39), (6.40), and Lemma 5.3 we get:

$$\int_{\Omega\times Y_{\mathsf{m}}} K_{\mathsf{m}}(x,y)\left|\nabla_y w^\varepsilon(x,y,0)\right|^2 dy\,dx \leqslant C\int_\Omega \|\mathfrak{D}^\varepsilon S^0_{\mathsf{m}}(x,\cdot)-S^0_{\mathsf{m}}(x)\|^2_{L^2(Y_{\mathsf{m}})}\,dx \to 0 \quad \text{as } \varepsilon\to 0.$$

In order to estimate the first integral on the right hand side in (6.41) we note that

$$\left|\nabla_y\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\right| = \left|\Lambda_1^{(\mathsf{m})}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\nabla_y\beta_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon))\right| \leqslant C\left|\nabla_y\mathfrak{D}^\varepsilon\beta_{\mathsf{m}}(\mathcal{P}^{-1}(\widetilde{S}^\varepsilon))\right|.$$

With the help of the inequality (2.20) in condition (A.8) and (4.8), from the last inequality we obtain that

$$\int_{\Omega_{\mathcal{T}}}\|\nabla_y\beta^*_{\mathsf{m}}(\mathcal{P}^{-1}(\mathfrak{D}^\varepsilon\widetilde{S}^\varepsilon),T)\|^2_{L^2(Y_m)}\,dx\,dt \leqslant C\int_{\Omega_{\mathcal{T}}}\|\nabla_y\mathfrak{D}^\varepsilon\beta_{\mathsf{m}}(\mathcal{P}^{-1}(\widetilde{S}^\varepsilon))\|^2_{L^2(Y_m)}\,dx\,dt$$

$$= C\,\varepsilon^2\int_{\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}}\left|\nabla_x\beta_{\mathsf{m}}(\mathcal{P}^{-1}(\widetilde{S}^\varepsilon))\right|^2 dx\,dt \leqslant C\,\varepsilon^2\int_{\Omega^\varepsilon_{\mathsf{m},\mathcal{T}}}\left|\nabla_x\widehat{\beta}(\widetilde{S}^\varepsilon)\right|^2 dx\,dt \to 0 \quad \text{as } \varepsilon\to 0.$$

The above convergence result and (6.41) imply that

$$0\leqslant\int_{\Omega_{\mathcal{T}}\times Y_{\mathsf{m}}}\Phi_{\mathsf{m}}(y)\left[\beta^*_{\mathsf{m}}(s^\varepsilon_{\mathsf{m}},T)-\beta^*_{\mathsf{m}}(s^*,T)\right](s^\varepsilon_{\mathsf{m}}-s^*)\,dx\,dt\,dy\to 0 \quad \text{as } \varepsilon\to 0.$$

This enables us to conclude that the limit of $s^\varepsilon_{\mathsf{m}}$ is equal to s^*, which is the solution of problem (6.10). Lemma 6.2 is proved. □

Let us also note that by standard arguments (see, e.g., [45]) we have the uniqueness of a solution to the imbibition equation (6.10) for a fixed value of the temperature T.

This completes the proof of Theorem 6.1. □

7. Concluding remarks

We have presented a homogenization result for a degenerate system modeling nonisothermal immiscible incompressible two-phase flow through a double porosity medium made of several types of rocks. We have assumed that the porosity, the absolute permeability, the capillary and relative permeabilities curves are different in each type of porous medium. This leads to nonlinear transmission conditions representing the continuity of some physical characteristics such as wetting and nonwetting pressures, at the interfaces that separate different media. Then the saturation and some other characteristics become discontinuous at the interface. We proved the homogenized results by using the two-scale convergence method combined with the dilation technique. This homogenization result improves previous results that were obtained for isothermal model in highly heterogeneous porous media with discontinuous capillary pressures. The study still needs to be improved by developing a general approach that would allow us to incorporate the cases of compressible phases and double porosity media. These more complicated cases appear in various applications. Further work on these important issues is in progress, in particular the homogenization of nonisothermal immiscible compressible two-phase flow through a double porosity medium as well as the corresponding existence results for such kind of flow.

Acknowledgments

The work of M. Jurak was funded by the Croatian Science Foundation project no 3955. The work of L. Pankratov was supported by the Russian Foundation for Basic Research, Grant N 19-01-00592. Most of the work on this paper was done when M. Jurak, L. Pankratov and A. Piatnitski were visiting the Applied Mathematics Laboratory of the University of Pau & CNRS. They are grateful for the invitations and the hospitality. This work was partially supported by the Carnot Institute, ISIFoR project (Institute for the sustainable engineering of fossil resources). This support is gratefully acknowledged. The authors gratefully thank the anonymous referees for their insightful comments and suggestions.

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