Nonlinear flow through double porosity media in variable exponent Sobolev spaces

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A R T I C L E   I N F O

A B S T R A C T

We studied the asymptotic behavior of the solution of a nonlinear parabolic equation with nonstandard growth in a ε-periodic fractured medium, where ε is the parameter that characterizes the scale of the microstructure tending to zero. We consider a double porosity type model describing the flow of a compressible fluid in a heterogeneous anisotropic porous medium obeying the nonlinear Darcy law. We assume that the permeability ratio of matrix blocks to fractures is of order $\varepsilon^{p_\varepsilon(x)}$, where p_ε is a continuous positive function. We obtained the convergence of the solution and a macroscopic model of the problem was constructed using the notion of two-scale convergence combined with the variational homogenization method in the framework of Sobolev spaces with variable exponents.

1. Introduction

Motivation for the following mathematical problem arises in the area of modeling flow and transport in fractured porous media for problems related to the environment and energy. Many difficult problems arise in numerical simulation in fractured porous media of fluid processes in reservoir simulation, subsurface contaminant transport and remediation, sequestration of CO_2 and other applications. Exact mesoscopic models of flow in a fractured medium customarily treat the fissures and the matrix systems as two porous media with different physical parameters. Discontinuities in the parameter values across the matrix-fissure interfaces are severe, with the ratios of their values in the fissures and blocks usually being of some orders of magnitude; moreover, the characteristic width of the fissures will be very small in comparison with the size of the blocks. Consequently, the exact mesoscopic model, written as a classical interface problem, is numerically and analytically intractable. A common technique used to overcome this difficulty is to construct models which describe the flow on two scales, macroscopic and mesoscopic (see, e.g., [27–29]). In this paper, we focus our attention on the homogenization of a double porosity type model describing the flow of a compressible fluid in a heterogeneous anisotropic porous medium obeying the nonlinear Darcy law (see, e.g., [8]).

Modeling of flow in fractured media is a subject of intensive research in many engineering disciplines, such as petroleum engineering, water resources management, civil engineering (see for instance [14,29,33]). More recently, fractured rock domains corresponding to the so-called Excavation Damaged Zone (EDZ) receives increasing attention in connection with the behavior of geological isolation of radioactive waste after the drilling of wells or shafts [19]. A fractured medium is a

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structure consisting of a porous and permeable matrix which is interlaced on a fine scale by a system of highly permeable fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume and storage capacity of the porous matrix is much larger than that of the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. For more details on the physical formulation of such problems see, e.g., [14,29,33]. Therefore the large-scale description will have to incorporate the two different flow mechanisms. For some permeability ratios and fissure widths, the large-scale description is achieved by introducing the so-called double porosity model. It was first introduced for describing the global behavior of fractured porous media by Barenblatt et al. [13]. It has been since used in a wide range of engineering specialties related to geohydrology, petroleum reservoir engineering, civil engineering or soil science. The usual linear double porosity model assumes that the width of the fracture containing highly permeable porous media is of the same order as the block size. The related homogenization problem was first studied in [11], and was then revisited in the mathematical literature by many other authors (see, e.g., [2,15,25–29,32,37] and references therein). Let us mention that results on the rate of convergence for the linear double porosity model, for a large range of contrast, were obtained in [32]. Linear double porosity models with thin fissures were studied in [4,5,7,16,31]. Also some nonlinear models were treated, see for instance [6,18, 21,23,24,30,34] and references therein. A general non-periodic model and a random model were considered in [17,20], respectively.

The goal of the present paper is to investigate, by mean of mathematical homogenization, the global behavior for the flow of a single phase, compressible fluid, in a fractured medium obeying the nonlinear Darcy law. We shall apply general ideas of homogenization (see [2,28,38]) and the specific framework introduced in [12]) for modeling of flows in fractured media. More precisely, let ε be the size ratio of the matrix blocks to the whole medium and let the width of the fracture planes and the porous block diameter be in the same order. We assume that the permeability ratio of matrix blocks to fracture planes is of order $\varepsilon^{p_\varepsilon(x)}$, where p_ε is a smooth positive oscillating function satisfying some conditions which will be specified later. The nonlinear Darcy law combined with the continuity equation lead to the following equation [8]:

$$\omega^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial t} - \text{div}\left(k^\varepsilon(x)\nabla u^\varepsilon|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\right) = g(t, x) \quad \text{in }]0, T[\times\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$), $T > 0$ is given, u^ε is the density of the fluid, ω^ε, k^ε are the porosity and the permeability of the medium and g is a source term. We consider the fissured part to be a porous medium with permeability of order 1, and the porous blocks (or matrix) made of porous material with a small permeability of order $\varepsilon^{p_\varepsilon(x)}$. This ratio is exactly the one leading to the dual-porosity model. An exterior boundary condition and an initial condition must also be specified, but they do not enter into the derivation of the limit model.

Such types of equations are called $p_\varepsilon(x)$-Laplacian equations with nonstandard growth conditions. Here, problem (1.1) is stated in the framework of Sobolev spaces with variable exponents which will be briefly described in Section 3. In recent years, increasing attention has been paid to the study of elliptic and parabolic equations with nonstandard growth condition when there is no dependence on the small parameter motivated by their applications to mathematical modeling in continuum mechanics. Such equations arise, for example, from modeling non-Newtonian fluids with thermo-convective effects, modeling electro-rheological fluids, the thermistor problem, the problem of image recovery, and the motion of a compressible fluid in a heterogeneous anisotropic porous medium obeying the nonlinear Darcy law. There is an extensive literature on this subject. We will not attempt a review of the literature here, but merely refer to [9,10] and references therein. Recently, there also appeared a research group on variable exponent Lebesgue and Sobolev spaces; we refer to their web page http://www.math.helsinki.fi/analysis/varsobgroup/.

The homogenization and Γ-convergence problems for Lagrangians with variable exponents $p(x)$ in the space $W^{1,r}$ with constant r were considered in [35,36,38]. It was shown that the homogenized Lagrangians might be distinct for different values of r (the so called Lavrentiev phenomenon). Let us also mention that homogenization of the Dirichlet elliptic problem for Lagrangians of nonstandard growth in Sobolev spaces with variable exponent has been studied in [3]. To our knowledge, the homogenization problems for parabolic equations with nonstandard growth have not been studied before. In this paper, we study the asymptotic behavior of the solution of problem (1.1) as ε tends to zero. We derive the homogenized model by combining the technique of two-scale convergence (see, e.g., [2,37]) and the variational homogenization method (see, e.g., [22,28,38] and references therein) in the framework of Sobolev spaces with variable exponents.

The outline of the rest of the paper is as follows. In Section 2 all necessary mathematical notation is defined, the mesoscopic problem is formulated, and the general assumptions are stated. In Section 3, for the sake of completeness, we recall the definition and main results on the Lebesgue and Sobolev spaces with variable exponents and the two-scale convergence which will be used in the sequel. A priori estimates for the solutions of the mesoscopic problem and some preliminary results are proven. The proof of our main results on convergence of the homogenization process is carried out is Section 4. The function p_ε is assumed to be a continuous positive function in $\overline{\Omega}$ satisfying some standard conditions and such that $\gamma_\varepsilon(x) := p_\varepsilon(x) - 2$ is a positive function which converges uniformly to zero in $\overline{\Omega}$. The macroscopic models depend on the behavior of the function $\alpha(x) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma_\varepsilon(x)}$, $x \in \overline{\Omega}$. It is shown that if $\alpha \not\equiv 0$ the resulting homogenized model is of dual-porosity type with effective coefficients depending on the function α, but if $\alpha \equiv 0$ it is a single porosity model.

2. Statement of the problem and assumptions

In this section, we describe a mesoscopic double porosity model in a periodic fractured medium. We consider a reservoir $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) to be a bounded connected domain with a periodic structure. More precisely, we will scale this periodic structure by a parameter ε which represents the ratio of the cell size to the size of the whole region Ω and we will assume that ε is a parameter tending to zero. Let $Y = \;]0, 1[^n$ represent the mesoscopic domain of the basic cell of a fractured porous medium. For the sake of simplicity and without loss of generality, we assume that Y is made up of two homogeneous porous media $M \Subset Y$ and F corresponding to parties of the mesoscopic domain occupied by the matrix block and the fracture, respectively. Thus $Y = M \cup \Gamma_{m,f} \cup F$, where $\Gamma_{m,f}$ denotes the interface between the two media and the subscripts m and f refer to the matrix and fracture, respectively. Let Ω_i^ε with $i = m$ or f denote the open set filled with the porous medium i. Then $\Omega = \Omega_m^\varepsilon \cup \Gamma_{m,f}^\varepsilon \cup \Omega_f^\varepsilon$, where $\Gamma_{m,f}^\varepsilon = \partial\Omega_m^\varepsilon \cap \partial\Omega_f^\varepsilon$. For the sake of simplicity, we will assume that $\partial\Omega \cap \Omega_m^\varepsilon = \emptyset$.

Let us introduce the nonstandard growth function used in this paper. We assume that a family of continuous functions $p_\varepsilon = p_\varepsilon(x)$, $\varepsilon > 0$, is defined in $\overline{\Omega}$ and satisfies the following conditions:

(i) functions p_ε are bounded from below such that:

$$p_\varepsilon(x) \geq 2 \quad \text{in } \overline{\Omega};$$

(ii) for any $x, y \in \Omega$ and any $\varepsilon > 0$, we have

$$|p_\varepsilon(x) - p_\varepsilon(y)| \leq \varpi_\varepsilon(|x - y|) \quad \text{with } \overline{\lim_{\tau \to 0}} \varpi_\varepsilon(\tau) \ln\left(\frac{1}{\tau}\right) \leq C;$$

(iii) the function $\gamma_\varepsilon(x) = p_\varepsilon(x) - 2$ converges uniformly to zero in $\overline{\Omega}$.

Now let us introduce the permeability coefficient and the porosity of the porous medium Ω. We set

$$K^\varepsilon(x) = k_m \varepsilon^{p_\varepsilon(x)} \mathbf{1}_m^\varepsilon(x) + k_f \mathbf{1}_f^\varepsilon(x) \quad \text{and} \quad \omega^\varepsilon(x) = \omega_m \mathbf{1}_m^\varepsilon(x) + \omega_f \mathbf{1}_f^\varepsilon(x),$$

where k_f is the permeability or the hydraulic conductivity of fissures, k_m is the permeability or the hydraulic conductivity of blocks, ω_f is the porosity of fissures, ω_m is the porosity of blocks; $\mathbf{1}_f^\varepsilon = \mathbf{1}_f^\varepsilon(x)$ and $\mathbf{1}_m^\varepsilon = \mathbf{1}_m^\varepsilon(x)$ denote the (periodic) characteristic functions of the sets Ω_f^ε and Ω_m^ε, respectively. Here $0 < k_f, k_m, \omega_f, \omega_m < +\infty$.

We consider the following initial boundary value problem for the function $u^\varepsilon : Q \mapsto \mathbb{R}$:

$$\begin{cases} \omega^\varepsilon(x) \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}\left(k^\varepsilon(x) \nabla u^\varepsilon |\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\right) = g(t, x) & \text{in } Q; \\ u^\varepsilon = 0 \quad \text{on }]0, T[\times \partial\Omega; \\ u^\varepsilon(0, x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where Q denotes the cylinder $]0, T[\times \Omega$, $T > 0$ is given and g, u_0 are given functions.

For simplicity and without loss of generality, we restrict the presentation to a homogeneous Dirichlet boundary condition on $\partial\Omega$, but it is easy to see that all results also hold for other boundary conditions.

Throughout the paper, C will denote a generic positive constant, independent of ε and may take different values for different occurrences.

3. Preliminary results

The goal of this section is to obtain some a priori estimates on the solution u^ε of problem (2.1). For this let us define certain function spaces and notation. In what follows we use standard notation for Sobolev spaces. We refer to [9] and the bibliography therein for properties of Sobolev spaces with variable exponents. Following [9], for any $\varepsilon > 0$, we introduce the Sobolev space $W^{1,p_\varepsilon(\cdot)}(\Omega)$ with a variable exponent p_ε defined by

$$W^{1,p_\varepsilon(\cdot)}(\Omega) = \left\{ \phi \in L^{p_\varepsilon(\cdot)}(\Omega) : |\nabla\phi| \in L^{p_\varepsilon(\cdot)}(\Omega) \right\}.$$

Here by $L^{p_\varepsilon(\cdot)}(\Omega)$ we denote the space of measurable functions ϕ in Ω such that

$$A_{p_\varepsilon(\cdot)}(\phi) = \int_\Omega |\phi(x)|^{p_\varepsilon(x)} dx < +\infty. \tag{3.1}$$

This space equipped with the norm

$$\|\phi\|_{L^{p_\varepsilon(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 : A_{p_\varepsilon(\cdot)}\left(\frac{\phi}{\lambda}\right) \leq 1 \right\}$$

becomes a Banach space.

The existence and uniqueness result for problem (2.1) is given by the following result (see for instance [10]).

Theorem 3.1. *Let $g \in C(0, T; L^2(\Omega))$ and $u_0 \in H^2(\Omega)$. Then, for any $\varepsilon > 0$, there exists a unique solution $u^\varepsilon = u^\varepsilon(t, x)$ of the boundary value problem (2.1) in the space $L^\infty(0, T; W^{1,p_\varepsilon(\cdot)}(\Omega))$. Furthermore, this solution satisfies the following a priori estimates, for a.e. $t \in]0, T[$:*

$$\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^t dt \int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t} \right|^2 dx + \int_\Omega k^\varepsilon(x) |\nabla u^\varepsilon|^{p_\varepsilon(x)} dx \leq C; \tag{3.2}$$

and

$$\|u^\varepsilon(t + \delta t) - u^\varepsilon(t)\|_{L^2(\Omega)} \leq C (\delta t)^\kappa \quad \text{with } 0 < \kappa < 1, \tag{3.3}$$

where $\delta t > 0$ is a time step which tends to zero.

In what follows we make use of Hölder's inequality for Sobolev spaces with variable exponents. Let $\phi \in L^{p(\cdot)}(\Omega)$, $\psi \in L^{q(\cdot)}(\Omega)$ with

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad 1 < p^- \leq p(x) \leq p^+ < +\infty, 1 < q^- \leq q(x) \leq q^+ < +\infty,$$

then

$$\int_\Omega |\phi \psi| dx \leq 2 \|\phi\|_{L^{p(\cdot)}(\Omega)} \|\psi\|_{L^{q(\cdot)}(\Omega)}. \tag{3.4}$$

We also make use of the following results from the theory of Sobolev spaces with variable exponents $p = p(x)$. Let the function $p(x)$ satisfy the conditions $1 < p^- = \inf_\Omega p(x) \leq p(x) \leq \sup_\Omega p(x) = p^+ < +\infty$, and, for all $x, y \in \Omega$, $|p(x) - p(y)| \leq \varpi(|x - y|)$ with $\overline{\lim}_{\tau \to 0} \varpi(\tau) \ln\left(\frac{1}{\tau}\right) \leq C$. Then

$$\begin{cases} \min\left(\|\phi\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|\phi\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right) \leq A_{p(\cdot)}(\phi) \leq \max\left(\|\phi\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|\phi\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right); \\ \min\left(A_{p(\cdot)}^{\frac{1}{p^-}}(\phi), A_{p(\cdot)}^{\frac{1}{p^+}}(\phi) \right) \leq \|\phi\|_{L^{p(\cdot)}(\Omega)} \leq \max\left(A_{p(\cdot)}^{\frac{1}{p^-}}(\phi), A_{p(\cdot)}^{\frac{1}{p^+}}(\phi) \right), \end{cases} \tag{3.5}$$

where $A_{p(\cdot)}(\phi)$ is defined in (3.1).

We study the asymptotic behavior of the solution u^ε of problem (2.1) as $\varepsilon \to 0$. For this, it is convenient to introduce the following notation:

$$u^\varepsilon = \begin{cases} \rho^\varepsilon & \text{in } \Omega_f^\varepsilon; \\ \sigma^\varepsilon & \text{in } \Omega_m^\varepsilon; \end{cases}$$

and to rewrite the Eq. (2.1) separately in the domains Ω_f^ε, Ω_m^ε with appropriate interface conditions. Namely, in the domain Ω_f^ε the Eq. (2.1) reads:

$$\begin{cases} \omega_f \dfrac{\partial \rho^\varepsilon}{\partial t} - \text{div}\,(k_f \nabla \rho^\varepsilon |\nabla \rho^\varepsilon|^{p_\varepsilon(x)-2}) = g(t, x) & \text{in }]0, T[\times\Omega_f^\varepsilon; \\ k_f \nabla \rho^\varepsilon |\nabla \rho^\varepsilon|^{p_\varepsilon(x)-2} \cdot \vec{v} = k_m \varepsilon^{p_\varepsilon(x)} \nabla \sigma^\varepsilon |\nabla \sigma^\varepsilon|^{p_\varepsilon(x)-2} \cdot \vec{v} & \text{on }]0, T[\times\Gamma_{m,f}^\varepsilon; \\ \rho^\varepsilon = 0 & \text{on }]0, T[\times\partial\Omega; \\ \rho^\varepsilon(0, x) = u_0(x) & \text{in } \Omega_f^\varepsilon, \end{cases} \tag{3.6}$$

where $\vec{v}$ is the outward normal vector to Γ_{mf}^ε. In the domain Ω_m^ε the Eq. (2.1) reads:

$$\begin{cases} \omega_m \dfrac{\partial \sigma^\varepsilon}{\partial t} - \text{div}\,\left(k_m \varepsilon^{p_\varepsilon(x)} \nabla \sigma^\varepsilon |\nabla \sigma^\varepsilon|^{p_\varepsilon(x)-2} \right) = g(t, x) & \text{in }]0, T[\times\Omega_m^\varepsilon; \\ \sigma^\varepsilon = \rho^\varepsilon & \text{on }]0, T[\times\Gamma_{mf}^\varepsilon; \\ \sigma^\varepsilon(0, x) = u_0(x) & \text{in } \Omega_m^\varepsilon. \end{cases} \tag{3.7}$$

To establish a preliminary compactness result, first we notice that the *a priori* estimate (3.2), conditions (i), (iii) along with (4.1), and inequalities (3.4)–(3.5) imply the bound for a.e. $t \in]0, T[$:

$$\|u^\varepsilon(t)\|_{L^2(\Omega)} + \int_0^t dt \int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t} \right|^2 dx + \int_{\Omega_f^\varepsilon} |\nabla u^\varepsilon|^2 dx + \varepsilon^2 \int_{\Omega_m^\varepsilon} |\nabla u^\varepsilon|^2 dx \leq C. \tag{3.8}$$

Therefore, from (3.6)–(3.8), we have for a.e. $t \in]0, T[$:

$$\|\rho^\varepsilon(t)\|_{L^2(\Omega_f^\varepsilon)} + \int_0^t \mathrm{d}t \int_{\Omega_f^\varepsilon} \left|\frac{\partial \rho^\varepsilon}{\partial t}\right|^2 \mathrm{d}x + \int_{\Omega_f^\varepsilon} |\nabla \rho^\varepsilon|^2 \mathrm{d}x \le C; \tag{3.9}$$

$$\|\sigma^\varepsilon(t)\|_{L^2(\Omega_m^\varepsilon)} + \int_0^t \mathrm{d}t \int_{\Omega_m^\varepsilon} \left|\frac{\partial \sigma^\varepsilon}{\partial t}\right|^2 \mathrm{d}x + \varepsilon^2 \int_{\Omega_m^\varepsilon} |\nabla \sigma^\varepsilon|^2 \mathrm{d}x \le C. \tag{3.10}$$

The next result relies on the two-scale approach (see, e.g., [2]). For the reader's convenience, let us recall the definition of the two-scale convergence.

Definition 3.2. A sequence of functions $v^\varepsilon \in L^2(\Omega)$ two-scale converges to $v(x, y) \in L^2(\Omega \times Y)$ if, $\|v^\varepsilon\|_{L^2(\Omega)} \le C$ and for any function $\varphi(x, y) \in \mathcal{D}(\Omega; C_\#^\infty(Y))$, it holds

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} v(x, y)\varphi(x, y)\mathrm{d}x\mathrm{d}y.$$

This convergence is denoted by $v^\varepsilon(x) \overset{2s}{\rightharpoonup} v(x, y)$.

Now using the bounds (3.9)–(3.10) along with the extension result [1], it is easy to prove the following compactness result.

Lemma 3.1. Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of problem (2.1). Then there exists a subsequence, still denoted by $\{u^\varepsilon\}$, and functions $u_f = u_f(t, x)$, $v_f = v_f(x, y)$, $u_m = u_m(t, x, y)$ such that

(a.1) $u_f \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$, $v_f \in L^2(\Omega; H_\#^1(F) \setminus \mathbb{R})$;

(a.2) $u_m \in H^1(0, T; L^2(\Omega \times Y)) \cap L^\infty(0, T; L^2(\Omega; H^1(M)))$, with $u_m(t, x, y) = u_f(t, x)$ for $y \in \partial M$;

(b) for any $t \in]0, T[$, $\rho^\varepsilon \overset{2s}{\rightharpoonup} u_f$ and $\sigma^\varepsilon \overset{2s}{\rightharpoonup} u_m$;

(c) for any $\varphi \in L^\infty(0, T; L^2(\Omega; C_\#(Y)))$,

$$\int_0^T \mathrm{d}t \int_{\Omega_f^\varepsilon} \frac{\partial u^\varepsilon}{\partial t} \varphi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}x \longrightarrow \int_0^T \mathrm{d}t \int_{\Omega \times F} \frac{\partial u_f}{\partial t} \varphi(t, x, y)\mathrm{d}x\mathrm{d}y;$$

$$\int_0^T \mathrm{d}t \int_{\Omega_m^\varepsilon} \frac{\partial u^\varepsilon}{\partial t} \varphi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}x \longrightarrow \int_0^T \mathrm{d}t \int_{\Omega \times M} \frac{\partial u_m}{\partial t} \varphi(t, x, y)\mathrm{d}x\mathrm{d}y;$$

(d) for a.e. $t \in]0, T[$, $\nabla \rho^\varepsilon \overset{2s}{\rightharpoonup} (\nabla_x u_f + \nabla_y v_f)\mathbf{1}_f(y)$ and $\nabla \sigma^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y u_m \mathbf{1}_m(y)$.

4. Homogenization results

In this section, we formulate the main results of the paper. We present homogenization results for the problem (2.1). Convergence of the homogenization process is obtained by combining the technique of two-scale convergence (see, e.g., [2, 37]) and the variational homogenization method (see, e.g., [22,28,38] and references therein).

The idea of the proof is the following. First we will reduce our parabolic problem to an elliptic one depending on the time variable as a parameter. Then we introduce a functional corresponding to this elliptic problem and study the minimization problem for it in the limit of small ε. Then we obtain the limit functional corresponding to the homogenized problem. Regarding the variational technique, it is worth mentioning one trick used in the paper. In order to obtain the lower bound for the original functional, we first replace the original exponent $p_\varepsilon(x)$ by a new one $p_0 = 2$, and consider the corresponding family of auxiliary functionals. Then the lower semicontinuity property of convex functionals with respect to the two-scale convergence implies the desired inequality. Finally, it is not difficult to show that the limit functional for the auxiliary family does not exceed the limiting functional for the original one.

Now we are in position to formulate the first homogenization result of the paper.

Theorem 4.1. Let $u^\varepsilon = u^\varepsilon(t, x)$ be the solution of the boundary value problem (2.1) and let conditions (i)–(iii) be satisfied. Moreover, we assume that there exists a positive function $\alpha \in C(\overline{\Omega})$ such that, for any $x \in \overline{\Omega}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma_\varepsilon(x)} = \alpha(x). \tag{4.1}$$

Then, for a.e. $t \in]0, T[$, u^ε two-scale converges to $u^* \in L^2(]0, T[\times \Omega \times Y)$ such that

$$u^*(t, x, y) = \begin{cases} u_f(t, x) & \text{in } Q \times F; \\ u_m(t, x, y) & \text{in } Q \times M, \end{cases} \tag{4.2}$$

where the couple $(u_f, u_m) \in L^2(0, T; H^1(\Omega)) \times L^\infty(0, T; L^2(\Omega; H^1(M)))$ is the unique solution of the homogenized problem:

$$\begin{cases} \omega_f |F| \dfrac{\partial u_f}{\partial t} - \text{div}_x(K^* \nabla_x u_f) = \mathsf{S}(x, u_m) & \text{in } Q; \\ u_f = 0 \text{ on } (0, T) \times \partial \Omega \quad \text{and} \quad u_f(0, x) = u_0(x) \quad \text{in } \Omega; \\ \omega_m \dfrac{\partial u_m}{\partial t} - \widetilde{K}^*(x) \Delta_y u_m = g(t, x) \quad \text{in } Q \times M; \\ u_m(t, x, y) = u_f(t, x) \quad \text{on } Q \times \partial M \quad \text{and} \quad u_m(0, x, y) = u_0(x) \quad \text{in } \Omega \times M, \end{cases} \quad (4.3)$$

where $|F|$ is the measure of the set F and $K^* = \{k_{ij}^*\}$ is the homogenized permeability tensor defined by:

$$k_{ij}^* = k_f \int_F (\vec{e}_i + \nabla_y w_i) \cdot (\vec{e}_j + \nabla_y w_j) \mathrm{d}y \quad (4.4)$$

with $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ the canonical basis of $\mathbb{R}^n$ and w_i being the unique solution in $H_\#^1(F) \setminus \mathbb{R}$ of

$$\begin{cases} -k_f \Delta_y w_i = 0 & \text{in } F; \\ (\vec{e}_i + \nabla_y w_i) \cdot \vec{v} = 0 & \text{on } \partial M; \\ y \rightarrow w_i(x, y) & Y\text{-periodic}; \end{cases} \quad (4.5)$$

where $\vec{v} = \vec{v}(y)$ is the outer normal vector at ∂M and the effective coefficient $\widetilde{K}^*$ in the local problem is given by

$$\widetilde{K}^*(x) = \alpha(x) k_m; \quad (4.6)$$

and the effective source term $\mathsf{S}(x, u_m)$ is given by

$$\mathsf{S}(x, u_m) = |F| \, g(t, x) - \widetilde{K}^*(x) \int_{\partial M} (\nabla_y u_m \cdot \vec{v}) \mathrm{d}s_y. \quad (4.7)$$

Remark 1. The source term which appears in the right-hand side of the first equation in (4.3) is well defined, since $u_m \in L^\infty(0, T; L^2(\Omega; H^1(M)))$, and it follows from the third equation of (4.3) that

$$-\Delta_y u_m \in L^2(0, T; H^{-1/2}(\partial M))$$

which allows one to define $(\nabla_y u_m \cdot \vec{v})$ as an element of $L^2(0, T; H^{-1/2}(\partial M))$.

Proof. We consider our parabolic boundary value problem (2.1) as an elliptic one depending on the time variable t as a parameter. Namely, we consider the following boundary value problem, for a.e. $t \in]0, T[$,

$$\begin{cases} -\text{div}\left(k^\varepsilon(x) \nabla u^\varepsilon |\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\right) = G^\varepsilon & \text{in } Q; \\ u^\varepsilon = 0 & \text{on }]0, T[\times \partial \Omega, \end{cases} \quad (4.8)$$

where the function G^ε,

$$G^\varepsilon = G^\varepsilon(t, x) = g(t, x) - \omega^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial t}(t, x), \quad (4.9)$$

is considered as a given function. Then, for any $\Delta_t \subset [0, T]$, u^ε minimizes the functional:

$$J^\varepsilon[u] = \int_{\Delta_t} \mathrm{d}t \int_\Omega \left\{ \frac{k^\varepsilon(x)}{p_\varepsilon(x)} |\nabla u|^{p_\varepsilon(x)} - G^\varepsilon u \right\} \mathrm{d}x \quad (4.10)$$

over $u \in L^\infty(0, T; W^{1, p_\varepsilon(\cdot)}(\Omega))$.

In the following sections we study the minimization problem for the functional J^ε in the limit of small ε and obtain the homogenized functional.

4.1. Upper bound for the functional J^ε

We want to show that for any admissible

$$\phi_f \in L^\infty(0, T; C^1(\overline{\Omega})), \qquad \phi_m \in L^\infty(0, T; C^1(\overline{\Omega}; H^1(M) \cap C_\#^1(Y))), \quad \zeta \in L^\infty(0, T; C^1(\Omega; C_\#^1(Y)))$$

such that $\phi_m(t, x, y)|_{y \in \partial M} = \phi_f(t, x)$, the inequality holds:

$$\varlimsup_{\varepsilon \to 0} J^\varepsilon[u^\varepsilon] \leq J_{hom}[\phi_f, \phi_m], \quad (4.11)$$

where J_{hom} is the functional corresponding to the homogenized problem. It reads

$$J_{hom}[\phi_f, \phi_m] = \int_{\Delta_t} dt \int_{\Omega} \left\{ \frac{1}{2} \left(K^* \nabla_x \phi_f \cdot \nabla_x \phi_f \right) - |F| \left(g(t, x) - \omega_f \frac{\partial u_f}{\partial t} \right) \phi_f \right\} dx$$

$$+ \int_{\Delta_t} dt \int_{\Omega \times M} \left\{ \alpha(x) \frac{k_m}{2} \left| \nabla_y \phi_m \right|^2 - \left(g(t, x) - \omega_m \frac{\partial u_m}{\partial t} \right) \phi_m \right\} dx dy. \tag{4.12}$$

In order to prove (4.11), we introduce the test function

$$w^\varepsilon = \begin{cases} \phi_f(t, x) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) & \text{in } \Omega_f^\varepsilon; \\ \phi_m \left(t, x, \frac{x}{\varepsilon} \right) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) & \text{in } \Omega_m^\varepsilon. \end{cases} \tag{4.13}$$

It is clear that

$$\overline{\lim_{\varepsilon \to 0}} J^\varepsilon[u^\varepsilon] \leq \lim_{\varepsilon \to 0} J^\varepsilon[w^\varepsilon]. \tag{4.14}$$

Due to the regularity of the functions ϕ_f, ϕ_m, ζ and by condition (iii), for a.e. $t \in]0, T[$, we have

$$\int_\Omega \frac{k^\varepsilon(x)}{p_\varepsilon(x)} |\nabla w^\varepsilon|^{p_\varepsilon(x)} dx = k_f \int_{\Omega_f^\varepsilon} \frac{1}{p_\varepsilon(x)} \left| \nabla \left(\phi_f(t, x) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) \right|^{p_\varepsilon(x)} dx$$

$$+ k_m \int_{\Omega_m^\varepsilon} \frac{\varepsilon^{p_\varepsilon(x)}}{p_\varepsilon(x)} \left| \nabla \left(\phi_m \left(t, x, \frac{x}{\varepsilon} \right) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) \right|^{p_\varepsilon(x)} dx$$

$$\xrightarrow[\varepsilon \to 0]{} \frac{k_f}{2} \int_{\Omega \times F} \left| \nabla_x \phi_f + \nabla_y \zeta \right|^2 dx dy + \frac{k_m}{2} \int_{\Omega \times M} \alpha(x) \left| \nabla_y \phi_m \right|^2 dx dy. \tag{4.15}$$

In addition, assertion (c) of Lemma 3.1 implies that, as $\varepsilon \to 0$,

$$\int_{\Delta_t} dt \int_\Omega G^\varepsilon w^\varepsilon dx = \int_{\Delta_t} dt \int_{\Omega_f^\varepsilon} \left\{ g(t, x) \left(\phi_f(t, x) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) - \omega_f \frac{\partial \rho^\varepsilon}{\partial t} \left(\phi_f(t, x) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) \right\} dx$$

$$+ \int_{\Delta_t} dt \int_{\Omega_m^\varepsilon} \left\{ g(t, x) \left(\phi_m \left(t, x, \frac{x}{\varepsilon} \right) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) - \omega_m \frac{\partial \sigma^\varepsilon}{\partial t} \left(\phi_m \left(t, x, \frac{x}{\varepsilon} \right) + \varepsilon \zeta \left(t, x, \frac{x}{\varepsilon} \right) \right) \right\} dx$$

$$\longrightarrow \int_{\Delta_t} dt \left\{ \int_{\Omega \times F} \left(g(t, x) - \omega_f \frac{\partial u_f}{\partial t} \right) \phi_f dx dy + \int_{\Omega \times M} \left(g(t, x) - \omega_m \frac{\partial u_m}{\partial t} \right) \phi_m dx dy \right\}. \tag{4.16}$$

Then, it follows from (4.15), (4.16) and (4.1) that

$$\lim_{\varepsilon \to 0} J^\varepsilon[w^\varepsilon] = J[\phi_f, \phi_m, \zeta], \tag{4.17}$$

where

$$J[\phi_f, \phi_m, \zeta] = \int_{\Delta_t} dt \int_{\Omega \times F} \left\{ \frac{k_f}{2} \left| \nabla_x \phi_f + \nabla_y \zeta \right|^2 - \left(g(t, x) - \omega_f \frac{\partial u_f}{\partial t} \right) \phi_f \right\} dx dy$$

$$+ \int_{\Delta_t} dt \int_{\Omega \times M} \left\{ \alpha(x) \frac{k_m}{2} \left| \nabla_y \phi_m \right|^2 - \left(g(t, x) - \omega_m \frac{\partial u_m}{\partial t} \right) \phi_m \right\} dx dy. \tag{4.18}$$

Letting

$$\zeta(t, x, y) = \sum_{i=1}^{n} \frac{\partial \phi_f}{\partial x_i}(t, x) \, w_i(y), \tag{4.19}$$

where $w_i \, (i = 1, 2, \ldots, n)$ is the solution of the cell problem (4.5), and taking into account the regularity of w_i, we obtain the desired estimate (4.11).

It is clear that (4.11) holds for any $(\phi_f, \phi_m) \in L^\infty(0, T; H^1(\Omega)) \times L^\infty(0, T; L^2(\Omega; H^1(M)))$ such that $\phi_m(t, x, y) = \phi_f(t, x)$ for $y \in \partial M$.

4.2. Lower bound for the functional J^ε

The proof of the lower bound is done in two steps. At the first step we introduce an auxiliary functional $\widetilde{J}^\varepsilon$ and obtain a lower bound for this functional. This bound for the auxiliary functional implies the desired lower bound for the original functional J^ε. This will be justified at the second step.

Step 1. Define the functional:

$$\widetilde{J}^\varepsilon[u] = \int_{\Delta_t} dt \int_\Omega \left\{ \frac{k^\varepsilon(x)}{p_\varepsilon(x)} |\nabla u|^2 - G^\varepsilon u \right\} dx, \tag{4.20}$$

where G^ε is specified in (4.9). In the same way as in the proof of the upper bound for the functional J^ε, one can show that

$$\overline{\lim_{\varepsilon \to 0}} \widetilde{J}^\varepsilon[u^\varepsilon] \le J_{hom}[\phi_f, \phi_m] \tag{4.21}$$

for any pair of functions $(\phi_f, \phi_m) \in L^\infty(0, T; H^1(\Omega)) \times L^\infty(0, T; L^2(\Omega; H^1(M)))$ such that $\phi_m(t, x, y) = \phi_f(t, x)$ for $y \in \partial M$.

By Lemma 3.1, condition (4.1), and the lower semicontinuity property of convex functionals with respect to the two-scale convergence (see, e.g., [2]) we have:

$$\lim_{\varepsilon \to 0} \int_{\Delta_t} dt \int_\Omega \left\{ \frac{k^\varepsilon(x)}{p_\varepsilon(x)} |\nabla u^\varepsilon|^2 - g(t, x)u^\varepsilon \right\} dx \ge \int_{\Delta_t} dt \int_\Omega \left\{ \frac{1}{2} \left(K^* \nabla_x u_f \cdot \nabla_x u_f \right) - |F| g(t, x)u_f \right\} dx$$

$$+ \int_{\Delta_t} dt \int_{\Omega \times M} \left\{ \alpha(x) \frac{k_m}{2} |\nabla_y u_m|^2 - g(t, x)u_m \right\} dxdy. \tag{4.22}$$

Combining (4.21) and (4.22), we obtain:

$$\overline{\lim_{\varepsilon \to 0}} \int_{\Delta_t} dt \int_\Omega \frac{\partial u^\varepsilon}{\partial t} u^\varepsilon dx \le \omega_f |F| \int_{\Delta_t} dt \int_\Omega \frac{\partial u_f}{\partial t} u_f dx + \omega_m \int_{\Delta_t} dt \int_{\Omega \times M} \frac{\partial u_m}{\partial t} u_m dxdy. \tag{4.23}$$

On the other hand

$$\lim_{\varepsilon \to 0} \int_0^T dt \int_\Omega \frac{\partial u^\varepsilon}{\partial t} u^\varepsilon dx = \frac{1}{2} \lim_{\varepsilon \to 0} \left(\|u^\varepsilon(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)} \right)$$

$$= \frac{1}{2} \lim_{\varepsilon \to 0} \left(\|\rho^\varepsilon(T)\|^2_{L^2(\Omega_f^\varepsilon)} - \|u_0\|^2_{L^2(\Omega_f^\varepsilon)} \right) + \frac{1}{2} \lim_{\varepsilon \to 0} \left(\|\sigma^\varepsilon(T)\|^2_{L^2(\Omega_m^\varepsilon)} - \|u_0\|^2_{L^2(\Omega_m^\varepsilon)} \right)$$

$$\ge \frac{1}{2} \left(\|u_f(T)\|^2_{L^2(\Omega \times F)} - \|u_0\|^2_{L^2(\Omega \times F)} \right) + \frac{1}{2} \left(\|u_m(T)\|^2_{L^2(\Omega \times M)} - \|u_0\|^2_{L^2(\Omega \times M)} \right)$$

$$= \omega_f |F| \int_0^T dt \int_\Omega \frac{\partial u_f}{\partial t} u_f dx + \omega_m \int_0^T dt \int_{\Omega \times M} \frac{\partial u_m}{\partial t} u_m dxdy. \tag{4.24}$$

Comparing (4.23) and (4.24), we conclude that

$$\lim_{\varepsilon \to 0} \int_{\Delta_t} dt \int_\Omega \frac{\partial u^\varepsilon}{\partial t} u^\varepsilon dx = \omega_f |F| \int_{\Delta_t} dt \int_\Omega \frac{\partial u_f}{\partial t} u_f dx + \omega_m \int_{\Delta_t} dt \int_{\Omega \times M} \frac{\partial u_m}{\partial t} u_m dxdy.$$

This yields

$$\lim_{\varepsilon \to 0} \widetilde{J}^\varepsilon[u^\varepsilon] \ge J_{hom}[u_f, u_m]. \tag{4.25}$$

Step 2. Denote

$$I^\varepsilon[u^\varepsilon] = J^\varepsilon[u^\varepsilon] - \widetilde{J}^\varepsilon[u^\varepsilon] = \int_{\Delta_t} dt \int_\Omega \frac{k^\varepsilon(x)}{p_\varepsilon(x)} \left\{ |\nabla u^\varepsilon|^{p_\varepsilon(x)} - |\nabla u^\varepsilon|^2 \right\} dx. \tag{4.26}$$

It is clear that

$$I^\varepsilon[u^\varepsilon] \ge - \int_{\Delta_t} dt \int_{\{|\nabla u^\varepsilon| < 1\} \cap \Omega} \frac{k^\varepsilon(x)}{p_\varepsilon(x)} \left\{ |\nabla u^\varepsilon|^2 - |\nabla u^\varepsilon|^{p_\varepsilon(x)} \right\} dx. \tag{4.27}$$

Let us show that the function $\varphi^\varepsilon(\eta) = \eta^2 - \eta^{p_\varepsilon(\cdot)}$ converges uniformly to zero on the interval $[0, 1]$. To this end notice that the maximum of $\varphi^\varepsilon(\eta)$ is attained at

$$\eta^\varepsilon_{max} = \left(\frac{2}{p_\varepsilon(\cdot)} \right)^{\frac{1}{p_\varepsilon(\cdot) - 2}}.$$

Then we have:

$$\max_{0 \le \eta \le 1} \varphi^\varepsilon(\eta) = \varphi^\varepsilon(\eta^\varepsilon_{\max}) = \left(\frac{2}{p_\varepsilon(\cdot)}\right)^{\frac{2}{p_\varepsilon(\cdot)-2}} - \left(\frac{2}{p_\varepsilon(\cdot)}\right)^{\frac{p_\varepsilon(\cdot)}{p_\varepsilon(\cdot)-2}} \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \tag{4.28}$$

Now the inequalities (4.25) and (4.27), and the relations (4.26) and (4.28) immediately imply that

$$\varliminf_{\varepsilon \to 0} J^\varepsilon[u^\varepsilon] \ge J_{hom}[u_f, u_m]. \tag{4.29}$$

From the inequalities (4.11) and (4.29) it is easy to derive that

$$J_{hom}[u_f, u_m] \le J_{hom}[\phi_f, \phi_m]$$

for any pair of admissible functions (ϕ_f, ϕ_m). Therefore, (u_f, u_m) is the minimizer of the homogenized functional J_{hom}. The statement of Theorem 4.1 follows from the uniqueness of a solution to the corresponding Euler equation.

Theorem 4.1 is proved. $\quad\square$

The macroscopic model corresponding to the second situation is given by the following convergence result.

Theorem 4.2. *Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of the boundary value problem* (2.1) *and let conditions* (i)–(iii) *be satisfied. Moreover, we assume that for any $x \in \overline{\Omega}$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma_\varepsilon(x)} = 0. \tag{4.30}$$

Then, for a.e. $t \in]0, T[$,

1. *the function ρ^ε two-scale converges to $u_f \in L^2(0, T; H^1(\Omega))$, the solution of*

$$\begin{cases} \omega_f |F| \dfrac{\partial u_f}{\partial t} - \operatorname{div}_x(K^* \nabla_x u_f) = |F| g(t, x) & \text{in } Q; \\ u_f = 0 \ \text{ on }]0, T[\times \partial\Omega \quad \text{and} \quad u_f(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.31}$$

   where $K^* = \{k^*_{ij}\}$ is the homogenized permeability tensor defined in (4.4)–(4.5);
2. *the function σ^ε two-scale converges to $u_m \in L^2(Q)$ defined by*

$$u_m(t, x) = u_0(x) + \frac{1}{\omega_m} \int_0^t g(\tau, x) \mathrm{d}\tau \quad \text{in } Q. \tag{4.32}$$

Proof. The proof is exactly the same to that of Theorem 4.1 except that the third term in (4.12) is zero when $\alpha(x) = 0$. $\quad\square$

Remark 2. Notice that the structure of the limit problem depends crucially on the rate of convergence of $(p_\varepsilon(\cdot) - 2)$ to zero. The critical rate of convergence is

$$(p_\varepsilon(\cdot) - 2) \sim \frac{1}{|\ln \varepsilon|}.$$

More precisely, if

$$\varlimsup_{\varepsilon \to 0} |\ln \varepsilon|(p_\varepsilon(x) - 2) < +\infty,$$

then the limit model is of a double porosity type. If

$$\lim_{\varepsilon \to 0} |\ln \varepsilon|(p_\varepsilon(x) - 2) = +\infty,$$

then in the limit we obtain a single porosity model.

Remark 3. Having the statements of Theorems 4.1 and 4.2 proved, it is natural to raise a question on the rate of convergence for the solution u^ε. Clearly, without additional assumptions on the behavior of $|p_\varepsilon - p_0|$, as $\varepsilon \to 0$, we cannot expect any estimates for the rate of convergence of $|u^\varepsilon(t, x) - u^*(t, x, \frac{x}{\varepsilon})|$ to zero. The corresponding example can be easily constructed if we consider problem (2.1) with exponents p_ε independent of x. In this case the convergence rate for u^ε will be governed by that for p_ε.

The authors are not aware of any qualified estimates for the rate of convergence in the case of homogenization problems for equations with nonlinearity of non standard growth conditions.

We believe that fast enough convergence of p_ε to p_0 will imply good estimates for the discrepancy $|u^\varepsilon(t, x) - u^*(t, x, \frac{x}{\varepsilon})|$. This interesting problem is out of the scope of this work. It will be studied somewhere else.

Acknowledgements

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