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HOMOGENIZATION OF IMMISCIBLE COMPRESSIBLE TWO-PHASE FLOW IN POROUS MEDIA: APPLICATION TO GAS MIGRATION IN A NUCLEAR WASTE REPOSITORY*

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Abstract. This paper is devoted to the homogenization of a coupled system of diffusion-convection equations in a domain with periodic microstructure, modeling the flow and transport of immiscible compressible, such as water-gas, fluids through porous media. The problem is formulated in terms of a nonlinear parabolic equation for the nonwetting phase pressure and a nonlinear degenerate parabolic diffusion-convection equation for the wetting saturation phase with rapidly oscillating porosity function and absolute permeability tensor. We obtain a nonlinear homogenized problem with effective coefficients which are computed via a cell problem. We rigorously justify this homogenization process for the problem by using two-scale convergence. In order to pass to the limit in nonlinear terms, we also obtain compactness results which are nontrivial due to the degeneracy of the system.

Key words. compressible immiscible, heterogeneous porous media, homogenization, nuclear waste, two-phase flow, water-hydrogen

AMS subject classifications. 35B27, 35K65, 76S05, 76T05, 76T10

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1. Introduction. The modeling of multiphase flow in porous formations is important for both the management of petroleum reservoirs and environmental remediation. Petroleum engineers need to model multiphase flow for production of hydrocarbons from petroleum reservoirs. Hydrologists and soil scientists are concerned with underground water flow in connection with applications to civil and agricultural engineering, and, of course, the design and evaluation of remediation technologies in water quality control rely on the properties of underground fluid flow. More recently, modeling multiphase flow received increasing attention in connection with the disposal of radioactive waste and sequestration of CO_2.

The paper focuses on the modeling of immiscible compressible two-phase flow in heterogeneous porous media in the framework of the geological disposal of radioactive waste. As a matter of fact, one of the solutions envisaged for managing waste produced by the nuclear industry is to dispose of the waste in deep geological formations chosen for their ability to prevent and attenuate possible releases of radionuclides in the geosphere. In the frame of designing nuclear waste geological repositories appears

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a problem of possible two-phase flow of water and gas. Multiple recent studies have established that, in such installations, important amounts of gases are expected to be produced, in particular, due to the corrosion of metallic components used in the repository design. The creation and transport of a gas phase is an issue of concern with regard to the capability of the engineered and natural barriers to evacuate the gas phase and avoid overpressure, thus preventing mechanical damage. It has become necessary to carefully evaluate those issues while assessing the performance of a geological repository; see, for instance, [36]. As mentioned above, the most important source of gas is the corrosion phenomena of metallic components (e.g., steel lines, waste containers). The second source, generally less important depending on the type of waste, is the water radiolysis by radiation issued from nuclear waste. Both processes would produce mainly hydrogen. Furthermore, the microbial activity will generate some methane and carbon dioxide and also would transform some hydrogen into methane. Hydrogen is expected to represent more than 90% of the total mass of produced gases.

In the subsurface, these processes are complicated by the effects of permeability heterogeneity on the flow and transport. Simulation models, if they are to provide realistic predictions, must accurately account for these effects. However, because permeability heterogeneity occurs at many different length scales, numerical flow models cannot, in general, resolve all of the variation of scales. Therefore, approaches are needed for representing the effects of subgrid scale variations on larger scale flow results which are more appropriate for reservoir simulations.

The upscaling or homogenization of multiphase flow through heterogeneous porous media has been a problem of interest for many years, and many methods have been developed. There is an extensive literature on this subject. We will not attempt a literature review here but will merely mention a few references. Here we restrict ourselves to the mathematical homogenization method as described in [29] for flow and transport in porous media. We refer, for instance, to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 24, 29, 31, 32, 34, 35, 37, 40, 41] for more information on the homogenization of incompressible immiscible two-phase flow in porous media and to [4, 23] on the homogenization of compressible miscible flow in porous media and the references therein. To our knowledge, convergence results on the homogenization of immiscible compressible two-phase flow in porous media are still missing.

In this paper, we will be concerned with a nonlinear system of diffusion-convection equations in a domain modeling the flow and transport of immiscible compressible fluids through heterogeneous porous media, taking into account capillary and gravity effects. We restrict our attention to water (incompressible) and gas such as hydrogen (compressible) in the context of gas migration through engineered and geological barriers for a deep repository for radioactive waste. For more details on the formulation of such problems see, e.g., the Couplex–Gas benchmark (http://www.gdrmomas.org/ex_qualifications.html) which was proposed by ANDRA, the French National Agency for Radioactive Waste Management, and the French research group MoMaS [5] to improve the simulation of the migration of hydrogen produced by the corrosion of nuclear waste packages in a deep repository for radioactive waste. This is a system of two-phase (water-hydrogen) flow in a porous medium.

We start with a microscopic model defined on a domain with periodic microstructure. Here we consider a single rock-type model. Namely, we consider that the porosity and the absolute permeability are rapidly oscillating functions of the microscopic scale $y = x/\varepsilon$, where x is the macroscopic scale and ε is a small parameter which characterizes the periodicity of the blocks. The model to be presented herein is formulated in terms of the wetting (water) saturation phase and the nonwetting (gas) pressure phase. The governing equations are derived from the mass conservation laws of both fluids, along with constitutive relations relating the velocities to the pressure gradients and gravitational effects. Traditionally, the standard Darcy–Muskat law provides this relationship. This formulation leads to a coupled system consisting of a nonlinear parabolic equation for the gas pressure and a nonlinear degenerate parabolic diffusion-convection equation for the water saturation, subject to appropriate boundary and initial conditions. There are two kinds of degeneracy in the studied system. The first one is the classical degeneracy of the diffusion operator. This degeneracy is due to the capillary effects; it can be observed even in the case of incompressible two-phase flow. The second one represents the evolution term degeneracy. It occurs in the region where the gas saturation vanishes: the gas density cannot be determined by its evolution and has no physical meaning since the gas phase is missing. In both cases the presence of degeneracy weakens the energy estimates and makes a proof of compactness results more involved. Our aim is to study the macroscopic behavior of solutions of this system of equations as ε tends to zero.

The degeneracy and strong coupling of these differential equations makes it very hard to study them. In particular, the degeneracy of the relative permeability implies that we have no uniform estimates for the gradients of the phase pressures. This is the reason why we have to pass to the formulation of our problem in terms of the global pressure and saturation. But even in this formulation, we have no uniform estimates for the gradient of the saturation. This creates the difficulties in the proof of the compactness results (see Proposition 4.5 below). Also, due to the degeneracy and strong coupling, the solutions do not have much regularity. Only recently some mathematical properties, in particular the existence of weak solutions to these equations, for immiscible compressible fluids have been obtained; see [18, 25, 26, 27, 28, 33, 38, 39].

The outline of the paper is as follows. In section 2 we give a short description of the mathematical and physical models used in this study for immiscible compressible two-phase flow in porous media. Then the resulting equations are written in a fractional flow formulation (see [19, 22]) which employs the saturation of one of the phases and a global pressure as independent variables. The fractional flow approach treats the two-phase flow problem as a total fluid flow of a single mixed fluid and then describes the individual phases as fractions of the total flow. This approach leads to a less strong coupling between the two coupled equations: the global pressure equation and the saturation equation. Then we formulate the assumptions on data and give a weak formulation of the problem. Section 3 is devoted to the presentation of the main result of the paper. We prove this result in section 4 using the approach based on the two-scale convergence technique. Our analysis relies essentially on the compactness result which is rather involved due to the degeneracy and the nonlinearity of the system. The macroscopic behavior of the flow is then described in terms of a global or effective model of an equivalent homogeneous medium. It approximates well the global behavior of the flow and is appropriate for reservoir simulation. However, the price to pay for this simplification is the necessity to describe the local structure of the medium and to solve additional problems formulated with respect to the microscopic variable in a basic cell which could be done as in [2, 3]. Lastly, some concluding remarks are forwarded.

2. Formulation of the problem.

2.1. Physical-mathematical model. In this subsection, we derive the mathematical model describing two-phase flow in a periodically heterogeneous porous medium. For a more detailed and more general derivation, we refer to [6, 19, 22, 30].

We consider an immiscible compressible two-phase flow process in porous media. We focus on the phases water and gas, but the consideration below is also valid for a general wetting phase and a nonwetting phase, each consisting of a component. For simplicity, we assume no source/sink term.

Let us define the microscopic model. We consider a reservoir $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ to be a bounded, connected Lipschitz domain with a periodic structure. More precisely, we scale this periodic structure by a parameter ε which represents the ratio of the cell size to the whole region Ω, and we assume that $0 < \varepsilon \ll 1$ is a small parameter tending to zero. Let $Y =]0, 1[^d$ represent the microscopic domain of the basic cell. Before describing the equations of the model, we give some notation. Let $\Phi^\varepsilon(x) = \Phi(x/\varepsilon)$ be the porosity of Ω; $K^\varepsilon(x) = K(x/\varepsilon)$ be the absolute permeability tensor of Ω; $S^\varepsilon_w = S^\varepsilon_w(x, t)$, $S^\varepsilon_g = S^\varepsilon_g(x, t)$ be the saturations of water and gas, respectively; $k_{r,w} = k_{r,w}(S^\varepsilon_w)$, $k_{r,g} = k_{r,g}(S^\varepsilon_g)$ be the relative permeabilities of water and gas, respectively; $p^\varepsilon_w = p^\varepsilon_w(x, t)$, $p^\varepsilon_g = p^\varepsilon_g(x, t)$ be the pressures of water and gas, respectively; and ϱ_w, ϱ_g be the densities of water and gas, respectively. The mass balance of each phase is described by

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S^\varepsilon_w\, \varrho_w(p^\varepsilon_w)) + \operatorname{div}\left(\varrho_w(p^\varepsilon_w)\, \vec{q}_w^{\,\varepsilon}\right) = 0 \quad \text{in } \Omega_T, \\ \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S^\varepsilon_g\, \varrho_g(p^\varepsilon_g)) + \operatorname{div}\left(\varrho_g(p^\varepsilon_g)\, \vec{q}_g^{\,\varepsilon}\right) = 0 \quad \text{in } \Omega_T, \end{cases} \tag{2.1}$$

where $T > 0$ is fixed, $\Omega_T \overset{\text{def}}{=} \Omega\times]0, T[$, and $\vec{q}_w^{\,\varepsilon}$ and $\vec{q}_g^{\,\varepsilon}$ are defined by Darcy–Muskat's law

$$\vec{q}_w^{\,\varepsilon} = -K^\varepsilon(x)\lambda_w(S^\varepsilon_w)\left(\nabla p^\varepsilon_w - \varrho_w(p^\varepsilon_w)\vec{g}\right), \quad \lambda_w(S^\varepsilon_w) = \frac{k_{r,w}}{\mu_w}(S^\varepsilon_w), \tag{2.2}$$

$$\vec{q}_g^{\,\varepsilon} = -K^\varepsilon(x)\widetilde{\lambda}_g(S^\varepsilon_g)\left(\nabla p^\varepsilon_g - \varrho_g(p^\varepsilon_g)\vec{g}\right), \quad \widetilde{\lambda}_g(S^\varepsilon_g) = \frac{k_{r,g}}{\mu_g}(S^\varepsilon_g) \tag{2.3}$$

with $\vec{g}$, μ_w, μ_g being the gravity vector and the viscosities of the water and gas, respectively. From now on, we assume that the density of the water is constant, which, for the sake of simplicity, will be taken equal to one, i.e., $\varrho_w(p^\varepsilon_w) = \text{Const} = 1$, and the gas density ϱ_g is a monotone smooth function such that

$$\begin{gathered} \varrho_g(p) = \varrho_{\min} \quad \text{for } p \leqslant p_{\min}, \qquad \varrho_g(p) = \varrho_{\max} \quad \text{for } p \geqslant p_{\max}, \\ \varrho_{\min} < \varrho_g(p) < \varrho_{\max} \quad \text{for } p_{\min} < p < p_{\max}; \end{gathered} \tag{2.4}$$

here the pairs of constants $\varrho_{\min}, \varrho_{\max}$ and $p_{\min}, p_{\max}$ satisfy the bounds

$$0 < \varrho_{\min} < \varrho_{\max} < +\infty \quad \text{and} \quad 0 < p_{\min} < p_{\max} < +\infty. \tag{2.5}$$

We also suppose that ϱ_g is a monotone function.

To close the system, we need two additional supplementary equations. The first is the saturation balance

$$S^\varepsilon_w + S^\varepsilon_g = 1 \quad \text{with } S^\varepsilon_w, S^\varepsilon_g \geqslant 0, \tag{2.6}$$

and the second describes the relation between the pressures

$$P_c\left(S^\varepsilon\right) = p^\varepsilon_g - p^\varepsilon_w \quad \text{with } P'_c(S^\varepsilon) < 0, \tag{2.7}$$

where P_c is a given capillary pressure-saturation function and $P'_c(s)$ denotes the derivative of the function $P_c(s)$ with respect to the variable s.

To simplify the notation, we denote

$$S^\varepsilon \stackrel{\text{def}}{=} S_w^\varepsilon. \tag{2.8}$$

Due to (2.2), (2.3), (2.8), and the assumption on the density of the water phase, we rewrite system (2.1) as follows:

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial S^\varepsilon}{\partial t} - \text{div}\,\left(K^\varepsilon(x)\lambda_w(S^\varepsilon)\left(\nabla p_w^\varepsilon - \vec{g}\right)\right) = 0 \quad \text{in } \Omega_T, \\ \Phi^\varepsilon(x)\dfrac{\partial(\varrho_g(p_g^\varepsilon)\left(1-S^\varepsilon\right))}{\partial t} - \text{div}\,\left(K^\varepsilon(x)\lambda_g(S^\varepsilon)\varrho_g(p_g^\varepsilon)\left(\nabla p_g^\varepsilon - \varrho_g^\varepsilon\vec{g}\right)\right) = 0 \quad \text{in } \Omega_T, \\ P_c\left(S^\varepsilon\right) = p_g^\varepsilon - p_w^\varepsilon \quad \text{in } \Omega_T, \end{cases} \tag{2.9}$$

where

$$\lambda_g(S^\varepsilon) \stackrel{\text{def}}{=} \widetilde{\lambda}_g(1-S^\varepsilon).$$

Now we specify the boundary and initial conditions. We suppose that the boundary $\partial\Omega$ consists of two parts Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The boundary conditions are given by

$$S^\varepsilon(x,t) = S^{\mathbf{1}}(x,t) \text{ and } p_g^\varepsilon(x,t) = p_g^{\mathbf{1}}(x,t) \quad \text{on } \Gamma_1\times]0,T[, \tag{2.10}$$

$$\vec{q}_w^{\,\varepsilon}\cdot\vec{\nu} = \vec{q}_g^{\,\varepsilon}\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_2\times]0,T[. \tag{2.11}$$

Finally, the initial conditions read

$$S^\varepsilon(x,0) = S^{\mathbf{0}}(x) \text{ and } p_g^\varepsilon(x,0) = p_g^{\mathbf{0}}(x) \quad \text{in } \Omega. \tag{2.12}$$

The missing data, i.e., S_g^ε, p_w^ε on Γ_1 and $S_g^\varepsilon(x,0)$, $p_w^\varepsilon(x,0)$ are deduced from (2.10) and (2.12) by virtue of (2.6) and (2.7).

2.2. A fractional flow formulation. In the sequel, we use a formulation obtained after transformation using the concept of global pressure (see [6, 19, 22]). This form is more suitable for mathematical purposes and will allow us to get a priori estimates for the solution of the problem. Now following the ideas from [6, 19, 22, 27], we transform system (2.9) to a problem formulated as a parabolic equation for the global pressure P^ε and a degenerate diffusion-convection equation for the saturation S^ε. The idea of introducing the global pressure can be interpreted as follows. We want to replace the water-gas flow by a flow of a fictive fluid obeying the Darcy law with a nondegenerating coefficient. Namely, we are looking for a pressure P^ε and the coefficient $\gamma(S^\varepsilon)$ such that $\gamma(S^\varepsilon) > 0$ in $[0,1]$ and

$$\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon + \lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \gamma(S^\varepsilon)\nabla\mathsf{P}^\varepsilon. \tag{2.13}$$

Now we define the global pressure as follows:

$$p_w^\varepsilon \stackrel{\text{def}}{=} \mathsf{P}^\varepsilon + \mathsf{G}_w(S^\varepsilon) \quad \text{and} \quad p_g^\varepsilon \stackrel{\text{def}}{=} \mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon), \tag{2.14}$$

where the functions $\mathsf{G}_w(S^\varepsilon)$, $\mathsf{G}_g(S^\varepsilon)$ will be specified later. Then it is easy to see that

$$\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon + \lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \lambda(S^\varepsilon)\nabla\mathsf{P}^\varepsilon + \left\{\lambda_g(S^\varepsilon)\nabla\mathsf{G}_g(S^\varepsilon) + \lambda_w(S^\varepsilon)\nabla\mathsf{G}_w(S^\varepsilon)\right\},$$

where

$$\lambda(s) \stackrel{\text{def}}{=} \lambda_w(s) + \lambda_g(s). \tag{2.15}$$

We choose the functions $\mathsf{G}_w(S^\varepsilon)$, $\mathsf{G}_g(S^\varepsilon)$ in order to realize the identity

$$\lambda_g(S^\varepsilon)\nabla\mathsf{G}_g(S^\varepsilon) + \lambda_w(S^\varepsilon)\nabla\mathsf{G}_w(S^\varepsilon) = 0, \tag{2.16}$$

where G_g is given by

$$\mathsf{G}_g(S^\varepsilon) \stackrel{\text{def}}{=} \mathsf{G}_g(0) + \int_0^{S^\varepsilon} \frac{\lambda_w(s)}{\lambda(s)}\, P_c'(s)\, ds. \tag{2.17}$$

The function G_w is then defined by

$$\mathsf{G}_w(S^\varepsilon) \stackrel{\text{def}}{=} \mathsf{G}_g(S^\varepsilon) - P_c\left(S^\varepsilon\right). \tag{2.18}$$

Moreover, it is easy to see that

$$\nabla\mathsf{G}_w(S^\varepsilon) = -\frac{\lambda_g(S^\varepsilon)}{\lambda(S^\varepsilon)} P_c'(S^\varepsilon)\,\nabla S^\varepsilon. \tag{2.19}$$

It is clear that $\gamma(S^\varepsilon) \equiv \lambda(S^\varepsilon)$. The standard assumption on the function $\lambda(s)$ is that $\lambda(s) > 0$ for $s \in [0,1]$ (see condition (A.5) below). Thus relation (2.13) is established.

Now we link the capillary pressure and the mobilities in the following way. We define two scalar functions $\mathsf{A}_g, \mathsf{A}_w$ as follows:

$$\sqrt{\lambda_g(S^\varepsilon)}\ \mathsf{G}_g'(S^\varepsilon) = \mathsf{A}_g'(S^\varepsilon) \quad \text{and} \quad \sqrt{\lambda_w(S^\varepsilon)}\ \mathsf{G}_w'(S^\varepsilon) = \mathsf{A}_w'(S^\varepsilon). \tag{2.20}$$

Notice that due to (2.14), (2.16), (2.15), and (2.20), we have the following identity:

$$\lambda_g(S^\varepsilon)|\nabla p_g^\varepsilon|^2 + \lambda_w(S^\varepsilon)|\nabla p_w^\varepsilon|^2 = \lambda(S^\varepsilon)|\nabla\mathsf{P}^\varepsilon|^2 + |\nabla\mathsf{A}_g(S^\varepsilon)|^2 + |\nabla\mathsf{A}_w(S^\varepsilon)|^2. \tag{2.21}$$

If we use the global pressure and the saturation as new unknown functions, then the first and second equations in (2.9) read

$$\Phi^\varepsilon\frac{\partial S^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\left(\lambda_w(S^\varepsilon)\nabla\mathsf{P}^\varepsilon + \alpha(S^\varepsilon)\nabla S^\varepsilon - \lambda_w(S^\varepsilon)\vec{g}\right)\right) = 0 \text{ in } \Omega_T \tag{2.22}$$

and

$$\Phi^\varepsilon\frac{\partial \Theta^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\tilde{\varrho}_g^\varepsilon\left(\lambda_g(S^\varepsilon)\nabla\mathsf{P}^\varepsilon - \alpha(S^\varepsilon)\nabla S^\varepsilon - \lambda_g(S^\varepsilon)\tilde{\varrho}_g^\varepsilon\,\vec{g}\right)\right) = 0 \text{ in } \Omega_T, \tag{2.23}$$

where, for brevity, we introduced the notation

$$\tilde{\varrho}_g^\varepsilon = \varrho_g(\mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon)), \tag{2.24}$$

$$\alpha(s) \stackrel{\text{def}}{=} \frac{\lambda_g(s)\,\lambda_w(s)}{\lambda(s)}\left|P_c'(s)\right|, \tag{2.25}$$

and

$$\Theta^\varepsilon = \Theta^\varepsilon(S^\varepsilon, \mathsf{P}^\varepsilon) \stackrel{\text{def}}{=} \varrho_g(\mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon))(1 - S^\varepsilon). \tag{2.26}$$

Notice that the relations (2.19) and (2.25) imply

$$\lambda_w(s)\nabla \mathsf{G}_w(s) = \alpha(s)\nabla s \quad \text{and} \quad \lambda_g(s)\nabla \mathsf{G}_g(s) = -\alpha(s)\nabla s. \tag{2.27}$$

The system (2.22)–(2.23) is completed by the following boundary and initial conditions:

$$S^\varepsilon(x,t) = S^{\mathbf{1}}(x,t) \text{ and } \mathsf{P}^\varepsilon(x,t) = \mathsf{P}^{\mathbf{1}}(x,t) \quad \text{on } \Gamma_1\times]0,T[, \tag{2.28}$$

$$\vec{q}_w^{\,\varepsilon}\cdot\vec{\nu} = \vec{q}_g^{\,\varepsilon}\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_2\times]0,T[, \tag{2.29}$$

$$S^\varepsilon(x,0) = S^{\mathbf{0}}(x) \text{ and } \mathsf{P}^\varepsilon(x,0) = \mathsf{P}^{\mathbf{0}}(x) \quad \text{in } \Omega. \tag{2.30}$$

Here the boundary and initial data for the global pressure, i.e., $\mathsf{P}^{\mathbf{1}}, \mathsf{P}^{\mathbf{0}}$, are expressed in terms of the corresponding data for p_g^ε using relations (2.14) and (2.17).

Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling, the degeneracy of the diffusion term in the saturation equation, and the degeneracy of the temporal term in the global pressure equation.

2.3. A weak formulation of the problem. Let us begin this section by stating the following assumptions.

- (A.1) The function $\Phi = \Phi(y)$ is a Y-periodic function, $\Phi \in L^\infty(Y)$, and there are positive constants ϕ_1, ϕ_2 such that $0 < \phi_1 \leqslant \Phi(y) \leqslant \phi_2 < 1$ a.e. in Y.
- (A.2) The tensor $K = K(y)$ is a Y-periodic function, and it belongs to $(L^\infty(Y))^{d\times d}$; moreover, there exist positive constants κ_0, κ_∞ such that

  $$\kappa_0|\xi|^2 \leqslant (K(y)\xi,\xi) \leqslant \kappa_\infty|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{\mathrm{d}} \text{ a.e. in } Y. \tag{2.31}$$
- (A.3) The function $\varrho_g = \varrho_g(p)$ given by (2.4) is a monotone C^1–function in $\mathbb{R}$.
- (A.4) The capillary pressure function $s \mapsto P_c(s)$ is positive and locally Lipschitz continuous in $[0,1]$. We also suppose that $P_c'(s) < 0$ in $]0,1[$.
- (A.5) The function $\lambda_w \in C([0,1])$ and satisfies the following properties: $0 \leqslant \lambda_w(s) \leqslant 1$ for all $s \in [0,1]$ and $\lambda_w(0) = 0$. The function $\lambda_g \in C([0,1])$ and satisfies the following properties: $0 \leqslant \lambda_g(s) \leqslant 1$ for all $s \in [0,1]$ and $\lambda_g(1) = 0$. In addition, there is a positive constant L_0 such that $\lambda(s) = \lambda_w(s) + \lambda_g(s) \geqslant L_0 > 0$ for all $s \in [0,1]$.
- (A.6) The function α given by (2.25) is a continuous function in $[0,1]$. Moreover, $\alpha(0) = \alpha(1) = 0$, and $\alpha > 0$ in $]0,1[$.
- (A.7) The functions $\mathsf{A}_g, \mathsf{A}_w$ are invertible, and $\mathsf{A}_g^{-1}, \mathsf{A}_w^{-1}$ are assumed to be θ–Hölder functions with $0 < \theta \leqslant 1$.
- (A.8) The functions $\mathsf{G}_g, \mathsf{G}_w$ defined in (2.16) and (2.18) belong to the space $C^1([0,1])$.
- (A.9) The initial data for the saturation, i.e., $S^{\mathbf{0}}$, $S^{\mathbf{1}}$, satisfy the bound $0 \leqslant S^{\mathbf{0}}, S^{\mathbf{1}} \leqslant 1$. The initial data for the pressure is such that $p_g^{\mathbf{0}} \in L^2(\Omega)$.

Definition of a weak solution for the problem. In order to define a weak solution to problem (2.9)–(2.12), we introduce the following Sobolev space:

$$H^1_{\Gamma_1}(\Omega) \stackrel{\text{def}}{=} \left\{u \in H^1(\Omega) \,:\, u = 0 \text{ on } \Gamma_1\right\}.$$

The space $H^1_{\Gamma_1}(\Omega)$ is a Hilbert space when it is equipped with the norm $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

Definition 2.1. *We say that the pair of functions $\langle \mathsf{P}^\varepsilon, S^\varepsilon\rangle$ is a weak solution of* problem (2.22)–(2.30) *if the following are true:*

(i) $0 \leqslant S^\varepsilon \leqslant 1$ *a.e. in* Ω_T, $S^\varepsilon \in L^{2/\theta}(0,T; W^{\tau,2/\theta}(\Omega))$, *where* θ *is defined in condition* (A.7) *and* $0 < \tau < 1$.
(ii) $\mathsf{P}^\varepsilon - \mathsf{P}^{\mathbf{1}} \in L^2(0,T; H^1_{\Gamma_1}(\Omega))$.
(iii) *The boundary conditions* (2.28)–(2.29) *are satisfied.*
(iv) *For any* $\varphi_w, \varphi_g \in C^1([0,T]; H^1(\Omega))$ *satisfying* $\varphi_w = \varphi_g = 0$ *on* $\Gamma_1 \times]0,T[$ *and* $\varphi_w(T) = \varphi_g(T) = 0$, *we have*

$$
\begin{aligned}
(2.32)\qquad & -\int_{\Omega_T} \Phi^\varepsilon(x) S^\varepsilon \frac{\partial \varphi_w}{\partial t}\, dxdt + \int_\Omega \Phi^\varepsilon(x) S^{\mathbf{0}}(x)\varphi_w(0,x)\, dx \\
& + \int_{\Omega_T} K^\varepsilon(x)\lambda_w(S^\varepsilon)\nabla \mathsf{P}^\varepsilon \cdot \nabla\varphi_w\, dxdt \\
& + \int_{\Omega_T} K^\varepsilon(x)\alpha(S^\varepsilon)\nabla S^\varepsilon \cdot \nabla\varphi_w\ dx\, dt - \int_{\Omega_T} K^\varepsilon(x)\lambda_w(S^\varepsilon)\vec{g}\cdot\nabla\varphi_w\ dx\, dt = 0,
\end{aligned}
$$

$$
\begin{aligned}
(2.33)\qquad & \\
& -\int_{\Omega_T} \Phi^\varepsilon(x)\, \Theta^\varepsilon \frac{\partial \varphi_g}{\partial t}\ dx\, dt + \int_\Omega \Phi^\varepsilon(x)(1 - S^{\mathbf{0}})\varrho_g(\mathsf{P}^{\mathbf{0}} + \mathsf{G}_g(S^{\mathbf{0}}))\varphi_g(0,x)\ dx \\
& + \int_{\Omega_T} K^\varepsilon(x)\lambda_g(S^\varepsilon)\tilde{\varrho}_g^\varepsilon \nabla\mathsf{P}^\varepsilon \cdot \nabla\varphi_g\ dx\, dt - \int_{\Omega_T} K^\varepsilon(x)\, \tilde{\varrho}_g^\varepsilon\, \alpha(S^\varepsilon)\nabla S^\varepsilon \cdot \nabla\varphi_g\ dx\, dt \\
& - \int_{\Omega_T} K^\varepsilon(x)\lambda_g(S^\varepsilon)\left[\tilde{\varrho}_g^\varepsilon\right]^2 \vec{g}\cdot\nabla\varphi_g\ dx\, dt = 0,
\end{aligned}
$$

where $\tilde{\varrho}_g^\varepsilon$ *has been defined in* (2.24).

If $\Phi \in W^{1,\infty}(Y)$ and $K \in (W^{1,\infty}(Y))^{d\times d}$, then, according to [27], under conditions (A.1)–(A.9), for each $\varepsilon > 0$, problem (2.22)–(2.30) has at least one weak solution.

The existence result remains valid for $\Phi \in L^\infty(Y)$ and $K \in (L^\infty(Y))^{d\times d}$. To show this, one can combine the proof developed in [27] with the compactness arguments used in the proof of Proposition 4.5 below.

Notational convention. In what follows $C, C_1, \ldots$ denote generic constants that do not depend on ε.

3. Statement of the main result. We study the asymptotic behavior of the solution to problem (2.22)–(2.30) as $\varepsilon \to 0$. In particular, we are going to show that the effective model reads

$$
(3.1)\quad
\begin{cases}
\langle\Phi\rangle \dfrac{\partial S}{\partial t} - \operatorname{div}_x\left(\mathbb{K}^\star\left(\lambda_w(S)\,\nabla\mathsf{P} + \alpha(S)\nabla S\right) + \mathbf{F}_w\right) = 0 \quad \text{in } \Omega_T, \\[2ex]
\langle\Phi\rangle \dfrac{\partial}{\partial t}\big((1-S)\varrho_g(\mathsf{P} + G_g(S))\big) \\[1ex]
\qquad - \operatorname{div}_x\left(\varrho_g(\mathsf{P} + G_g(S))\mathbb{K}^\star\left(\lambda_g(S)\,\nabla\mathsf{P} - \alpha(S)\nabla S\right) + \mathbf{F}_g\right) = 0 \text{ in } \Omega_T.
\end{cases}
$$

Here $\langle\cdot\rangle$ stands for the mean value of the corresponding function over the cell Y; $\mathbb{K}^\star$ is the homogenized tensor, with the entries $\mathbb{K}^\star_{ij}$, defined by

$$
(3.2)\qquad \mathbb{K}^\star_{ij} \stackrel{\text{def}}{=} \int_Y K(y)\,[\nabla_y\xi_i + \vec{e}_i]\ [\nabla_y\xi_j + \vec{e}_j]\ dy,
$$

where the function ξ_j is a Y-periodic solution of

$$\begin{cases} -\mathrm{div}_y \Big(K(y)\,[\nabla_y \xi_j + \vec{e}_j] \Big) = 0 \quad \text{in } Y, \\ y \longmapsto \xi_j(y) \quad Y\text{-periodic} \end{cases} \tag{3.3}$$

with $\vec{e}_j$ being the j-th coordinate vector; the functions $\mathbf{F}_w = \mathbf{F}_w(S, \mathsf{P}), \mathbf{F}_g = \mathbf{F}_g(S, \mathsf{P})$ denote the lower order term in (3.1) and are given by

$$\mathbf{F}_w(S, \mathsf{P}) \stackrel{\text{def}}{=} \lambda_w(S)\langle K\,\nabla_y \mathsf{f}_p\rangle + \langle K\,\nabla_y \mathsf{f}_s\rangle - \lambda_w(S)\langle K\,\vec{g}\rangle; \tag{3.4}$$

$$\begin{aligned} \mathbf{F}_g(S, \mathsf{P}) \stackrel{\text{def}}{=} {} & \lambda_g(S)\,\varrho_g(\mathsf{P} + G_g(S))\langle K\,\nabla_y \mathsf{f}_p\rangle \\ & - \varrho_g(\mathsf{P} + G_g(S))\langle K\,\nabla_y \mathsf{f}_s\rangle - \lambda_g(S)\,[\varrho_g(\mathsf{P} + G_g(S))]^2\,\langle K\,\vec{g}\rangle, \end{aligned} \tag{3.5}$$

where the functions $\mathsf{f}_p = \mathsf{f}_p(y, S, \mathsf{P}), \mathsf{f}_s = \mathsf{f}_s(y, S, \mathsf{P})$ are solutions of the following:

$$\begin{cases} \mathrm{div}_y\,(K(y)\,\nabla_y \mathsf{f}_p) = \dfrac{\lambda_w(S) + \varrho_g(\mathsf{P} + G_g(S))\lambda_g(S)}{\lambda(S)}\,\mathrm{div}_y(K(y)\vec{g}) \;\text{ in } \Omega_T \times Y, \\ y \longmapsto \mathsf{f}_p(y, S, \mathsf{P}) \quad Y\text{-periodic} \end{cases} \tag{3.6}$$

and

$$\begin{cases} \mathrm{div}_y\,(K(y)\,\nabla_y \mathsf{f}_s) = \dfrac{\lambda_w(S)\lambda_g(S)}{\lambda(S)}\,[1 - \varrho_g(\mathsf{P} + G_g(S))]\,\mathrm{div}_y(K(y)\vec{g}) \text{ in } \Omega_T \times Y, \\ y \longmapsto \mathsf{f}_s(y, S, \mathsf{P}) \quad Y\text{-periodic.} \end{cases} \tag{3.7}$$

The system (3.1)–(3.7) has to be completed by the following boundary and initial conditions:

$$S(x,t) = S^{\mathbf{1}}(x,t) \text{ and } \mathsf{P}(x,t) = \mathsf{P}^{\mathbf{1}}(x,t) \quad \text{on } \Gamma_1 \times]0,T[, \tag{3.8}$$

$$\vec{q}_w \cdot \vec{\nu} = \vec{q}_g \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_2 \times]0,T[\tag{3.9}$$

with

$$\vec{q}_w = -\mathbb{K}^\star\,(\lambda_w(S)\,\nabla \mathsf{P} + \alpha(S)\nabla S) + \mathbf{F}_w,$$

$$\vec{q}_g = -\mathbb{K}^\star\,(\lambda_g(S)\,\nabla \mathsf{P} - \alpha(S)\nabla S) + \frac{\mathbf{F}_g}{\varrho_g^H},$$

$$S^\varepsilon(x,0) = S^{\mathbf{0}}(x) \quad \text{and} \quad \mathsf{P}^\varepsilon(x,0) = \mathsf{P}^{\mathbf{0}}(x) \quad \text{in } \Omega. \tag{3.10}$$

In what follows, for the sake of brevity we use the notation

$$\tilde{\varrho}_g^H \stackrel{\text{def}}{=} \varrho_g(\mathsf{P} + \mathsf{G}_g(S)). \tag{3.11}$$

The effective model described above could be obtained formally by the technique of two-scale asymptotic expansions. Here the homogenization process for the problem is rigorously obtained by using the two-scale approach, see, e.g., [1]. For the reader's convenience, let us recall the definition of the two-scale convergence.

DEFINITION 3.1. *A function,* $\varphi \in L^2(\Omega_T; C^2_\#(Y))$*, which is* Y*-periodic in* y *and which satisfies*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\left|\varphi\left(x,\frac{x}{\varepsilon},t\right)\right|^2 dx\,dt = \int_{\Omega_T\times Y}|\varphi(x,y,t)|^2\,dy\,dx\,dt,$$

is called an admissible test function.

Here $L^2(\Omega_T; C^2_\#(Y))$ is the space of functions $\phi = \phi(x,y,t)$ periodic and two times continuously differentiable in y for a.e. $(x,t)\in\Omega_T$ with the norm

$$\|\phi\|^2_{L^2(\Omega_T;C^2_\#(Y))} = \int_{\Omega_T}\|\phi(x,\cdot,t)\|^2_{C^2(Y)}dxdt.$$

DEFINITION 3.2. *A sequence of functions* $v^\varepsilon \in L^2(\Omega_T)$ *two-scale converges to* $v\in L^2(\Omega_T\times Y)$ *if, for any admissible test function* $\varphi(x,y,t)$*,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} v^\varepsilon(x,t)\varphi\left(x,\frac{x}{\varepsilon},t\right)dx\,dt = \int_{\Omega_T\times Y} v(x,y,t)\varphi(x,y,t)\,dy\,dx\,dt.$$

This convergence is denoted by $v^\varepsilon(x,t) \xrightarrow{2s} v(x,y,t)$.

Finally, we introduce the notation

$$\Upsilon^\varepsilon \stackrel{\text{def}}{=} \Upsilon(S^\varepsilon) \quad \text{with} \quad \Upsilon(s) \stackrel{\text{def}}{=} \int_0^s \alpha(\xi)\,d\xi, \tag{3.12}$$

where the function α is defined in (2.25).

The main result of the paper is the following theorem.

THEOREM 3.3. *Let assumptions* (A.1)–(A.9) *be fulfilled, and let the pair of functions* $\langle \mathsf{P}^\varepsilon, S^\varepsilon\rangle$ *be a weak solution of* (2.22)–(2.30)*. Then there exists a subsequence (still denoted by* ε*) such that*

$$S^\varepsilon(x,t)\to S(x,t) \text{ strongly in } L^q(\Omega_T), \quad 1\leqslant q<+\infty, \tag{3.13}$$

$$\mathsf{P}^\varepsilon(x,t)\to \mathsf{P}(x,t) \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{3.14}$$

$$\nabla\mathsf{P}^\varepsilon(x,t) \xrightarrow{2s} \nabla\mathsf{P}(x,t)+\nabla_y \mathsf{w}_p(x,t,y), \tag{3.15}$$

$$\Upsilon^\varepsilon\to\Upsilon \text{ strongly in } L^q(\Omega_T), \qquad 1\leqslant q<+\infty, \tag{3.16}$$

$$\nabla\Upsilon^\varepsilon(x,t) \xrightarrow{2s} \nabla\Upsilon(x,t)+\nabla_y \mathsf{w}_s(x,t,y), \tag{3.17}$$

$$\Theta^\varepsilon \to (1-S)\tilde{\varrho}^H_g \text{ strongly in } L^2(\Omega_T). \tag{3.18}$$

Here

$$\mathsf{w}_p = \sum_{j=1}^d \xi_j(y)\frac{\partial\,\mathsf{P}}{\partial x_j}(x,t)+\mathsf{f}_p \quad\text{and}\quad \mathsf{w}_s = \sum_{j=1}^d \xi_j(y)\frac{\partial\,\Upsilon}{\partial x_j}(x,t)+\mathsf{f}_s,$$

where ξ_j *is a* Y*-periodic solution of* (3.3) *and where the functions* f_p *and* f_s *are defined in* (3.6) *and* (3.7)*, respectively. The function* $\tilde{\varrho}^H_g$ *is defined in* (3.11)*, and* $\langle\mathsf{P},S\rangle$ *is a weak solution to* (3.1)–(3.10).

The proof of Theorem 3.3 is given below in section 4.

4. Proof of Theorem 3.3. The outline of the proof is as follows. First in section 4.1 we obtain appropriate uniform estimates, and then in section 4.2 we pass to the limit, as $\varepsilon \to 0$, in (2.1)–(2.33). The important part of the proof is the compactness result given by Propositions 4.5 and 4.7. The proof of these propositions allows us to weaken the conditions imposed in [25, 26, 27, 28] on the porosity function and the absolute permeability tensor.

4.1. A priori estimates. In this section we obtain the a priori estimates for the solution of problem (2.9)–(2.12) (or the equivalent problem (2.22)–(2.30)). It will be done in two main steps. At the first step, following the ideas of [27], we establish the *energy equality* and get the first group of the a priori estimates with respect to the space variable. Then in the second step, we establish the compactness result that will be used below in section 4.2. In this section, for the sake of simplicity, we suppose that $p_g^\varepsilon, p_w^\varepsilon = 0$ on $\Gamma_1\times]0,T[$.

Step 1. Energy equality. To obtain the *energy equality* for the weak solution of problem (2.9)–(2.12), we introduce the functions

$$\mathsf{R}_w(p_w^\varepsilon) \stackrel{\text{def}}{=} \int_0^{p_w^\varepsilon} d\xi = p_w^\varepsilon, \text{ and } \mathsf{R}_g(p_g^\varepsilon) \stackrel{\text{def}}{=} \int_0^{p_g^\varepsilon} \frac{d\xi}{\varrho_g(\xi)}. \tag{4.1}$$

Moreover,

$$\nabla \mathsf{R}_w(p_w^\varepsilon) = \nabla p_w^\varepsilon \quad \text{and} \quad \nabla \mathsf{R}_g(p_g^\varepsilon) = \frac{1}{\tilde{\varrho}_g^\varepsilon} \nabla p_g^\varepsilon \text{ with } \tilde{\varrho}_g^\varepsilon = \varrho_g(p_g^\varepsilon). \tag{4.2}$$

Multiplying now the first equation in (2.9) by $\mathsf{R}_w(p_w^\varepsilon)$ and the second by $\mathsf{R}_g(p_g^\varepsilon)$, integrating by parts, we get

$$\begin{aligned}
&\int_\Omega \Phi^\varepsilon(x) \frac{\partial S^\varepsilon}{\partial t} p_w^\varepsilon \, dx\, dt + \int_\Omega \Phi^\varepsilon(x) \frac{\partial}{\partial t}\left[\tilde{\varrho}_g^\varepsilon (1-S^\varepsilon)\right] \mathsf{R}_g(p_g^\varepsilon)\, dx\, dt \\
&+ \int_\Omega K^\varepsilon(x)\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon \cdot \nabla p_w^\varepsilon \, dx\, dt + \int_\Omega K^\varepsilon(x)\lambda_g(S^\varepsilon)\tilde{\varrho}_g^\varepsilon \nabla p_g^\varepsilon \cdot \frac{1}{\tilde{\varrho}_g^\varepsilon}\nabla p_g^\varepsilon \, dx\, dt \\
&- \int_\Omega K^\varepsilon(x)\lambda_w(S^\varepsilon)\vec{g}\cdot\nabla p_w^\varepsilon\, dx\, dt - \int_\Omega K^\varepsilon(x)\lambda_g(S^\varepsilon)\left[\tilde{\varrho}_g^\varepsilon\right]^2 \vec{g}\cdot\frac{1}{\tilde{\varrho}_g^\varepsilon}\nabla p_g^\varepsilon\, dx\, dt = 0.
\end{aligned} \tag{4.3}$$

Let us rearrange the first two terms on the left-hand side of (4.3). First we have that

$$\frac{\partial S^\varepsilon}{\partial t} p_w^\varepsilon = \frac{\partial}{\partial t}\left\{S^\varepsilon p_w^\varepsilon\right\} - S^\varepsilon \frac{\partial p_w^\varepsilon}{\partial t}, \tag{4.4}$$

$$\frac{\partial}{\partial t}\left[\tilde{\varrho}_g^\varepsilon(1-S^\varepsilon)\right]\mathsf{R}_g(p_g^\varepsilon) = \frac{\partial}{\partial t}\left[\tilde{\varrho}_g^\varepsilon(1-S^\varepsilon)\mathsf{R}_g(p_g^\varepsilon)\right] - (1-S^\varepsilon)\frac{\partial p_g^\varepsilon}{\partial t}. \tag{4.5}$$

Then from the definition of the capillary pressure (2.7), we get

$$S^\varepsilon \frac{\partial p_w^\varepsilon}{\partial t} + (1-S^\varepsilon)\frac{\partial p_g^\varepsilon}{\partial t} = \frac{\partial}{\partial t}\left\{S^\varepsilon p_w^\varepsilon + (1-S^\varepsilon)p_g^\varepsilon + \mathcal{F}(S^\varepsilon)\right\}, \tag{4.6}$$

where

$$\mathcal{F}(s) \stackrel{\text{def}}{=} \int_1^s P_c(\xi)\, d\xi.$$

Now it follows from (4.4)–(4.6) that

$$\frac{\partial S^\varepsilon}{\partial t} p_w^\varepsilon + \frac{\partial}{\partial t}\left[p_g^\varepsilon(1-S^\varepsilon)\right]\mathsf{R}_g(p_g^\varepsilon) = \frac{\partial \mathcal{E}^\varepsilon}{\partial t}, \tag{4.7}$$

where

$$\mathcal{E}^\varepsilon \stackrel{\text{def}}{=} \varrho_g(p_g^\varepsilon)(1-S^\varepsilon)\mathsf{R}_g(p_g^\varepsilon) - (1-S^\varepsilon)p_g^\varepsilon - \mathcal{F}(S^\varepsilon) \quad \text{with } \mathcal{F}(s) \stackrel{\text{def}}{=} \int_1^s P_c(\xi)\,d\xi. \tag{4.8}$$

In order to verify (4.7), it suffices to differentiate the right-hand side of (4.8) and to exploit (2.7) and (4.1). It is important to notice that $\mathcal{F}(s) \leqslant 0$ in $]0,1]$.

In what follows, we make use of the uniform boundedness of the function $\mathcal{E}^\varepsilon$ from below. Namely, the following estimate holds.

LEMMA 4.1. *The function $\mathcal{E}^\varepsilon$ satisfies the bound*

$$\mathcal{E}^\varepsilon \geqslant -p_{\max}\left(1-\frac{\varrho_{\min}}{\varrho_{\max}}\right). \tag{4.9}$$

Proof. Since the function $\mathcal{F}(S^\varepsilon) \leqslant 0$, then we immediately get

$$\mathcal{E}^\varepsilon \geqslant \varrho_g(p_g^\varepsilon)(1-S^\varepsilon)\mathsf{R}_g(p_g^\varepsilon) - (1-S^\varepsilon)p_g^\varepsilon. \tag{4.10}$$

Notice that the results of [27] do not allow us to conclude that $p_g^\varepsilon \geqslant 0$. Therefore, we have to define the function ϱ_g for all $-\infty < p_g^\varepsilon < +\infty$; see (2.4).

Suppose first that $p_g^\varepsilon \geqslant 0$, and estimate the right-hand side of (4.10) from below. By the definition of function $\mathsf{R}_g(p_g^\varepsilon)$, we have

$$\begin{aligned}\varrho_g(p_g^\varepsilon)(1-S^\varepsilon)\mathsf{R}_g(p_g^\varepsilon) - (1-S^\varepsilon)p_g^\varepsilon &\geqslant (1-S^\varepsilon)\left[\varrho_g(p_g^\varepsilon)\int_0^{p_g^\varepsilon}\frac{d\xi}{\varrho_{\max}} - p_g^\varepsilon\right] \\ &= (1-S^\varepsilon)\,p_g^\varepsilon\left(\frac{\varrho_g(p_g^\varepsilon)}{\varrho_{\max}}-1\right).\end{aligned} \tag{4.11}$$

If $p_g^\varepsilon \geqslant p_{\max}$, then by the definition of function ϱ_g, we have that $\varrho_g(p_g^\varepsilon) = \varrho_{\max}$, and thus

$$(1-S^\varepsilon)\,p_g^\varepsilon\left(\frac{\varrho_g(p_g^\varepsilon)}{\varrho_{\max}}-1\right) = 0. \tag{4.12}$$

If $0 < p_g^\varepsilon \leqslant p_{\max}$, then

$$(1-S^\varepsilon)\,p_g^\varepsilon\left(\frac{\varrho_g(p_g^\varepsilon)}{\varrho_{\max}}-1\right) \geqslant (1-S^\varepsilon)\,p_g^\varepsilon\left(\frac{\varrho_{\min}}{\varrho_{\max}}-1\right) \geqslant -p_{\max}\left(1-\frac{\varrho_{\min}}{\varrho_{\max}}\right). \tag{4.13}$$

Let now $p_g^\varepsilon < 0$. Then by the definitions of functions $\mathsf{R}_g(p_g^\varepsilon)$ and ϱ_g, we have

$$\begin{aligned}&\varrho_g(p_g^\varepsilon)(1-S^\varepsilon)\mathsf{R}_g(p_g^\varepsilon) - (1-S^\varepsilon)p_g^\varepsilon \\ &\geqslant (1-S^\varepsilon)\left[\left|p_g^\varepsilon\right| - \varrho_{\min}\int_0^{|p_g^\varepsilon|}\frac{d\xi}{\varrho_g(-\xi)}\right] = (1-S^\varepsilon)\left[\left|p_g^\varepsilon\right| - \varrho_{\min}\int_0^{|p_g^\varepsilon|}\frac{d\xi}{\varrho_{\min}}\right] = 0.\end{aligned} \tag{4.14}$$

Now the statement of the lemma immediately follows from (4.10)–(4.14). ∎

We turn to (4.3). It can be rewritten as follows:

$$\int_\Omega \Phi^\varepsilon(x)\frac{\partial \mathcal{E}^\varepsilon}{\partial t}\,dx + \int_\Omega \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon\cdot\nabla p_w^\varepsilon + K^\varepsilon(x)\lambda_g(S^\varepsilon)\nabla p_g^\varepsilon\cdot\nabla p_g^\varepsilon\right\}\,dx$$
$$-\int_\Omega \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\vec{g}\cdot\nabla p_w^\varepsilon + K^\varepsilon(x)\lambda_g(S^\varepsilon)\tilde{\varrho}_g^\varepsilon\,\vec{g}\cdot\nabla p_g^\varepsilon\right\}\,dx = 0. \tag{4.15}$$

Integrating relation (4.15) over the interval $]0,t[$, with $t\in]0,T]$, we get

$$\int_\Omega \Phi^\varepsilon(x)\mathcal{E}^\varepsilon\,dx + \int_{\Omega_t} \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon\cdot\nabla p_w^\varepsilon + K^\varepsilon(x)\lambda_g(S^\varepsilon)\nabla p_g^\varepsilon\cdot\nabla p_g^\varepsilon\right\}\,dx\,d\tau$$
$$-\int_{\Omega_t} \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\vec{g}\cdot\nabla p_w^\varepsilon + K^\varepsilon(x)\lambda_g(S^\varepsilon)\tilde{\varrho}_g^\varepsilon\vec{g}\cdot\nabla p_g^\varepsilon\right\}\,dx\,d\tau = \int_\Omega \Phi^\varepsilon(x)\mathcal{E}^{\mathbf{0}}\,dx, \tag{4.16}$$

where $\Omega_t \stackrel{\text{def}}{=} \Omega\times(0,t)$ and

$$\mathcal{E}^{\mathbf{0}} \stackrel{\text{def}}{=} \varrho_g(p_g^{\mathbf{0}})(1-S^{\mathbf{0}})\mathsf{R}_g(p_g^{\mathbf{0}}) - (1-S^{\mathbf{0}})p_g^{\mathbf{0}} - \digamma(S^{\mathbf{0}}). \tag{4.17}$$

It follows from condition (A.1) and (4.9) that the first term on the left-hand side of (4.16) satisfies the bound

$$\int_\Omega \Phi^\varepsilon(x)\mathcal{E}^\varepsilon\,dx \geqslant -\text{meas}\,\Omega\,\phi_2\,p_{\max}\left(1-\frac{\varrho_{\min}}{\varrho_{\max}}\right). \tag{4.18}$$

For the second and third terms on the left-hand side of (4.16), by Cauchy's inequality, condition (A.2), and (2.4), we obtain the following bound:

$$\int_{\Omega_t} \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\nabla p_w^\varepsilon\cdot\nabla p_w^\varepsilon + K^\varepsilon(x)\lambda_g(S^\varepsilon)\nabla p_g^\varepsilon\cdot\nabla p_g^\varepsilon\right\}\,dx\,d\tau$$
$$-\int_{\Omega_t} \left\{K^\varepsilon(x)\lambda_w(S^\varepsilon)\vec{g}\cdot\nabla p_w^\varepsilon + K(x)\lambda_g(S^\varepsilon)\vec{g}\cdot\varrho_g(p_g^\varepsilon)\nabla p_g^\varepsilon\right\}\,dx\,d\tau$$
$$\geqslant -C + \frac{\kappa_0}{2}\int_{\Omega_t}\lambda_w(S^\varepsilon)\left|\nabla p_w^\varepsilon\right|^2\,dx\,d\tau + \frac{\kappa_0}{2}\int_{\Omega_t}\lambda_g(S^\varepsilon)\left|\nabla p_g^\varepsilon\right|^2\,dx\,d\tau. \tag{4.19}$$

Consider the right-hand side of (4.16). It follows from the definition of $\mathcal{E}^{\mathbf{0}}$, (4.17), and (A.1) and (A.9) that

$$\left|\int_\Omega \Phi^\varepsilon(x)\mathcal{E}^{\mathbf{0}}\,dx\right| \leqslant C. \tag{4.20}$$

Finally, (4.16), (4.18), (4.19), and (4.20) imply that

$$\int_{\Omega_T}\lambda_w(S^\varepsilon)\left|\nabla p_w^\varepsilon\right|^2\,dx\,d\tau + \int_{\Omega_T}\lambda_g(S^\varepsilon)\left|\nabla p_g^\varepsilon\right|^2\,dx\,d\tau \leqslant C. \tag{4.21}$$

This inequality along with (2.21) yields

$$\int_{\Omega_T} \lambda(S^\varepsilon)|\nabla \mathsf{P}^\varepsilon|^2\, dx\, dt + \int_{\Omega_T} |\nabla \mathsf{A}_g(S^\varepsilon)|^2\, dx\, dt + \int_{\Omega_T} |\nabla \mathsf{A}_w(S^\varepsilon)|^2\, dx\, dt \leqslant C. \tag{4.22}$$

Due to the definition of $\mathsf{A}_g, \mathsf{A}_w$ (see 2.20), we have

$$|\nabla \mathsf{A}_w(S^\varepsilon)|^2 = \lambda_w(S^\varepsilon)|\nabla \mathsf{G}_w(S^\varepsilon)|^2 \quad \text{and} \quad |\nabla \mathsf{A}_g(S^\varepsilon)|^2 = \lambda_g(S^\varepsilon)|\nabla \mathsf{G}_g(S^\varepsilon)|^2.$$

Therefore, condition (A.5) and (4.22) imply that

$$\int_{\Omega_T} |\nabla \mathsf{P}^\varepsilon|^2\, dxdt + \int_{\Omega_T} \lambda_w(S^\varepsilon)|\nabla \mathsf{G}_w(S^\varepsilon)|^2\, dxdt + \int_{\Omega_T} \lambda_g(S^\varepsilon)|\nabla \mathsf{G}_g(S^\varepsilon)|^2\, dxdt \leqslant C. \tag{4.23}$$

Our next goal is to obtain an additional uniform estimate for $\alpha(S^\varepsilon)|\nabla S^\varepsilon|$. From condition (A.5), (2.27), and (4.23), we have

$$\int_{\Omega_T} \alpha^2(S^\varepsilon)|\nabla S^\varepsilon|^2 dx\, dt = \int_{\Omega_T} \lambda_w^2(S^\varepsilon)|\nabla \mathsf{G}_w(S^\varepsilon)|^2 dxdt \leqslant \int_{\Omega_T} \lambda_w(S^\varepsilon)|\nabla \mathsf{G}_w(S^\varepsilon)|^2 dxdt \leqslant C.$$

Thus, we get

$$\int_{\Omega_T} \alpha^2(S^\varepsilon)|\nabla S^\varepsilon|^2\, dx\, dt \leqslant C. \tag{4.24}$$

In what follows, we will also make use of the a priori information for the function Υ^ε defined in (3.12). For the reader's convenience, we recall that

$$\Upsilon^\varepsilon \stackrel{\text{def}}{=} \Upsilon(S^\varepsilon) \quad \text{with} \quad \Upsilon(s) \stackrel{\text{def}}{=} \int_0^s \alpha(\xi)\, d\xi.$$

Notice that Υ is a monotone function of s. For the function Υ^ε from condition (A.6) we have,

$$0 \leqslant \Upsilon^\varepsilon \leqslant \max_{s\in[0,1]} \alpha(s) \quad \text{a.e. in } \Omega_T. \tag{4.25}$$

It is also clear that since $\nabla\Upsilon^\varepsilon = \alpha(S^\varepsilon)\, \nabla S^\varepsilon$, then it follows from (4.24) that

$$\int_{\Omega_T} |\nabla \Upsilon^\varepsilon|^2\, dx\, dt \leqslant C. \tag{4.26}$$

In addition, there exists a function $\omega(\xi) \geqslant 0$ such that

$$\left|\Upsilon^{-1}(\Upsilon_1) - \Upsilon^{-1}(\Upsilon_2)\right| \leqslant \omega(|\Upsilon_1 - \Upsilon_2|) \text{ with } \omega(\xi) \to 0 \text{ as } \xi \to 0. \tag{4.27}$$

Step 2. Compactness results for the sequences $\{S^\varepsilon\}_{\varepsilon>0}$, $\{\Theta^\varepsilon\}_{\varepsilon>0}$. We start this section by obtaining the following compactness lemma.

LEMMA 4.2 (compactness lemma). *Let the function $\Phi = \Phi(y)$ be a Y-periodic function, $\Phi \in L^\infty(Y)$, and there are positive constants ϕ_1, ϕ_2 such that $0 < \phi_1 \leqslant \Phi(y) \leqslant \phi_2 < 1$ a.e. in Y, and let $\{v^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega_T)$ be a family of functions satisfying the following properties:*

1. *The function v^ε is uniformly bounded in the space $L^\infty(\Omega_T)$, i.e.,*

$$0 \leqslant v^\varepsilon \leqslant C. \tag{4.28}$$

2. *There exists a function ϖ such that $\varpi(\xi) \to 0$ as $\xi \to 0$, and the following inequality holds true:*

$$\int_{\Omega_T} |v^\varepsilon(x+\Delta x,\tau) - v^\varepsilon(x,\tau)|^2 \, dx\, d\tau \leqslant C\,\varpi(|\Delta x|). \tag{4.29}$$

3. *The function v^ε is such that*

$$\left\| \frac{\partial}{\partial t}(\Phi^\varepsilon v^\varepsilon) \right\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C. \tag{4.30}$$

Then the family $\{v^\varepsilon\}_{\varepsilon>0}$ is a compact set in $L^2(\Omega_T)$.

Remark 1. In the formulation of the compactness lemma, the periodicity of Φ can be replaced with the assumption that $\Phi^\varepsilon \rightharpoonup 1$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0$.

Proof. From now on, without loss of generality, we assume that $\langle \Phi \rangle = 1$. Then

$$(\Phi^\varepsilon - 1) \rightharpoonup 0 \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0. \tag{4.31}$$

Denote $Q \stackrel{\text{def}}{=} (0,2\pi)^d$. Without loss of generality, we assume that $\overline{\Omega} \subset Q$. Then we extend the function v^ε in $(Q \setminus \Omega) \times (0,T)$ by setting $v^\varepsilon(x,t) = 0$ for $x \in Q \setminus \Omega$. Then, as an easy consequence of (4.29), we have

$$\int_{Q_T} |v^\varepsilon(x+y,t) - v^\varepsilon(x,t)|^2 \, dx\, dt \leqslant \omega_1(y) \tag{4.32}$$

perhaps with a new function ω_1 which still satisfies the limit relation $\lim_{|y|\to 0} \omega_1(y) = 0$. Here $Q_T \stackrel{\text{def}}{=} Q \times (0,T)$.

Letting $\omega_2(s) \stackrel{\text{def}}{=} \sup_{|y|\leqslant s} \omega_1(y)$, one can easily check that $\omega_2(s)$ is monotone, $\omega_1(y) \leqslant \omega_2(|y|)$, and $\lim_{s\to 0} \omega_2(s) = 0$.

In the space $L^2(Q)$, we introduce the standard orthonormal basis $\{\psi_j\}$, where

$$\psi_j(x) = (2\pi)^{-d} \exp(\imath j x) \quad \text{with } j \in \mathbb{Z}^d \text{ and } \imath = \sqrt{-1}.$$

Then $\{|1+|j|^2|^{-1/2}\,\psi_j\}$ is an orthonormal basis in $H^1_{\text{per}}(Q)$, and $\{|1+|j|^2|^{1/2}\,\psi_j\}$ is an orthonormal basis in $H^{-1}_{\text{per}}(Q)$ if these spaces are equipped with the following norms:

$$\|w\|^2_{H^1_{\text{per}}(Q)} = \sum_j (w,\psi_j)^2 (1+|j|^2), \qquad \|w\|^2_{H^{-1}_{\text{per}}(Q)} = \sum_j < w, \psi_j >^2 \, (1+|j|^2)^{-1}.$$

LEMMA 4.3. *For any $\delta > 0$, there exists $N(\delta) \in \mathbb{Z}^+$ such that*

$$\frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} dy \int_Q \left|\psi_j(x+y) - \psi_j(x)\right|^2 dx \geqslant \frac{1}{2(2\pi)^d} \quad \text{for all } j \text{ with } |j| \geqslant N(\delta).$$

Proof. We have

$$\frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} dy \int_Q \left|\exp\left(\imath j(x+y)\right) - \exp(\imath j x)\right|^2 dx = \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} dy \int_Q \left|\exp(\imath j y) - 1\right|^2 dx$$

$$= \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} \left|\exp(\imath j y) - 1\right|^2 dy = \frac{1}{2^d} \int_{[-1,1]^d} \left|\exp(\imath j \delta y) - 1\right|^2 dy.$$

The integral in the right-hand side is greater than $1/2$ if $|j| > 2\,d\,\delta^{-1}$, and Lemma 4.3 is proved. ▯

Now we represent the function v^ε by its Fourier series in Q with coefficients depending on time t:

$$v^\varepsilon(x,t) = \sum_{j\in\mathbb{Z}^d} \beta_j^\varepsilon(t)\psi_j(x) \quad \text{with} \quad \beta_j^\varepsilon(t) = \int_Q v^\varepsilon(x,t)\psi_j(x)\,dx.$$

Then

$$\|v^\varepsilon\|^2_{L^2(Q_T)} = \sum_{j\in\mathbb{Z}^d} \int_0^T \left|\beta_j^\varepsilon(t)\right|^2 dt.$$

This allows us to obtain the following bound for the function ω_2:

$$\omega_2(\delta) \geqslant \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} dy \int_{Q_T} \left|v^\varepsilon(x+y) - v^\varepsilon(x)\right|^2 \,dx\,dt$$

$$= \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} dy \int_{Q_T} \left|\beta_j^\varepsilon(t)\right|^2 \left|\psi_j(x+y) - \psi_j(x)\right|^2 \,dx\,dt \geqslant \frac{1}{2(2\pi)^d} \sum_{|j|\geqslant N(\delta)} \int_0^T \left|\beta_j^\varepsilon(t)\right|^2 dt.$$

Since $\omega_2(\delta) \to 0$ as $\delta \to 0$, we conclude that, for any $\gamma > 0$, there exists, $N(\gamma) > 0$ such that

$$\left\|\sum_{|j|\geqslant N(\gamma)} \beta_j^\varepsilon(t)\psi_j(x)\right\|^2_{L^2(Q_T)} = \sum_{|j|\geqslant N(\gamma)} \int_0^T |\beta_j^\varepsilon(t)|^2 dt < \gamma.$$

This implies that

$$v^\varepsilon = \sum_{|j|\leqslant N(\gamma)} \beta_j^\varepsilon(t)\psi_j(x) \,+\, \Theta^\varepsilon_{\mathrm{res},N} = \Theta^\varepsilon_N + \Theta^\varepsilon_{\mathrm{res},N} \tag{4.33}$$

with

$$\|\Theta^\varepsilon_{\mathrm{res},N}\|^2_{L^2(Q_T)} < \gamma. \tag{4.34}$$

Next we exploit the upper bound (4.30). Let us write the Fourier series of $(\Phi^\varepsilon v^\varepsilon)$ in the variable x. We have

$$\Phi^\varepsilon(x)v^\varepsilon(x,t) = \sum_{j\in\mathbb{Z}^d} \widetilde{\beta}_j^\varepsilon(t)\psi_j(x) \quad \text{and} \quad \partial_t\big(\Phi^\varepsilon(x)v^\varepsilon(x,t)\big) = \sum_{j\in\mathbb{Z}^d} \frac{d}{dt}\widetilde{\beta}_j^\varepsilon(t)\psi_j(x).$$

Since $\psi_j(\cdot)$ are orthogonal in $H^{-1}_{\mathrm{per}}(Q)$, then we obtain

$$\sum_{j\in\mathbb{Z}^d} \int_0^T \left|\frac{d}{dt}\widetilde{\beta}_j^\varepsilon(t)\right|^2 (1+|j|^2)^{-1} \leqslant C. \tag{4.35}$$

Let us estimate the difference

$$\int_0^T \left|\beta_j^\varepsilon(t) - \widetilde{\beta}_j^\varepsilon(t)\right|^2 dt = \int_0^T \left|\int_Q (\Phi^\varepsilon(x) - 1)v^\varepsilon(x)\psi_j(x)\,dx\right|^2 dt.$$

LEMMA 4.4. *For any $j \in \mathbb{Z}^d$,*

$$\lim_{\varepsilon\to 0}\int_0^T \left|\int_Q (\Phi^\varepsilon(x)-1)v^\varepsilon(x)\psi_j(x)\,dx\right|^2 dt \,=0. \tag{4.36}$$

Proof. For any $\gamma>0$, we choose $N(\gamma)$ so that (4.34) holds true. Then

$$\int_0^T \left|\int_Q (\Phi^\varepsilon(x)-1)\Theta^\varepsilon_{\mathrm{res},N}(x)\psi_j(x)\,dx\right|^2 dt \,\leqslant C\,\gamma\,(1+\|\Phi\|^2_{L^\infty(Y)}). \tag{4.37}$$

Next we want to estimate the contribution of Θ^ε_N. We have

$$\begin{aligned}&\int_0^T \left|\int_Q (\Phi^\varepsilon(x)-1)\left(\sum_{|k|\leqslant N(\gamma)}\beta^\varepsilon_k(t)\psi_k(x)\right)\psi_j(x)\,dx\right|^2 dt\\ &\leqslant N(\gamma)\sum_{|k|\leqslant N(\gamma)}\int_0^T |\beta^\varepsilon_k(t)|^2\left|\int_Q (\Phi^\varepsilon(x)-1)\psi_k(x)\psi_j(x)\,dx\right|^2 dt.\end{aligned}$$

Since $(\Phi^\varepsilon(x)-1)$ tends to zero weakly in $L^2(Q)$ as $\varepsilon\to 0$ and $\psi_k(x)\psi_j(x)$ does not depend on ε, then

$$\lim_{\varepsilon\to 0}\int_Q \psi_k(x)\psi_j(x)(\Phi^\varepsilon(x)-1)\,dx \,=0.$$

Combining this limit relation with the bound

$$\sum_k \int_0^T |\beta^\varepsilon_k(t)|^2\,dt \leqslant C$$

yields

$$\lim_{\varepsilon\to 0}\int_0^T \left|\int_Q (\Phi^\varepsilon(x)-1)\left(\sum_{|k|\leqslant N(\gamma)}\beta^\varepsilon_k(t)\psi_k(x)\right)\psi_j(x)\,dx\right|^2 dt = 0.$$

Considering (4.37), we deduce that

$$\lim_{\varepsilon\to 0}\int_0^T \left|\int_Q (\Phi^\varepsilon(x)-1)v^\varepsilon(x)\psi_j(x)\,dx\right|^2 dt \,\leqslant C\,\gamma$$

for any $\gamma>0$. This implies (4.36), and Lemma 4.4 is proved. ∎

Denote

$$\widetilde{\Theta}^\varepsilon_N = \sum_{|j|\leqslant N(\gamma)}\widetilde{\beta}^\varepsilon_j(t)\psi_j(x).$$

By Lemma 4.4, for any $\gamma>0$, there is $\varepsilon_0=\varepsilon_0(\gamma)$ such that, for all $\varepsilon<\varepsilon_0$, it holds that

$$\|\Theta^\varepsilon_N - \widetilde{\Theta}^\varepsilon_N\|_{L^2(\Omega_T)} \leqslant 3\gamma. \tag{4.38}$$

Due to (4.35) and the evident bound

$$\sum_j \int_0^T |\widetilde{\beta}_j^\varepsilon(t)|^2 \, dt \leqslant C,$$

the family $\{\widetilde{\beta}_j^\varepsilon(\cdot)\,,\, |j| \leqslant N\}$ is a compact set in $(L^2(0,T))^M$, where M is the number of $j \in \mathbb{Z}^d$ such that $|j| \leqslant N$. Therefore, the family

$$\left\{ \widetilde{\Theta}_N^\varepsilon = \sum_{|j| \leqslant N(\gamma)} \widetilde{\beta}_j^\varepsilon(t) \psi_j(x) \right\}_{\varepsilon>0}$$

is a compact set in the space $L^2(\Omega_T)$. Thus there exist a finite γ-net for this family, denoting its elements by $\zeta_1, \zeta_2, \ldots, \zeta_K$. Taking into account (4.34) and (4.38), we conclude that, for sufficiently small $\varepsilon > 0$, the functions $\zeta_1, \zeta_2, \ldots, \zeta_K$ form a 5γ-net for $\{v^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega_T)$. This implies the desired compactness, and Lemma 4.2 is proved. ☐

Now we turn to the compactness result for the family $\{\Theta^\varepsilon\}_{\varepsilon>0}$. It is assured by the following statement.

PROPOSITION 4.5. *Under our standing assumptions, the family* $\{\Theta^\varepsilon\}_{\varepsilon>0}$ *is a compact set in the space* $L^2(\Omega_T)$.

Proof. The idea of the proof of Proposition 4.5 is to apply the compactness lemma. To this end, we check the conditions of the lemma. First it follows from (2.4) and (2.26) that

$$0 \leqslant \Theta^\varepsilon = \varrho_g(\mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon))(1 - S^\varepsilon) \leqslant \varrho_{\max} < +\infty. \tag{4.39}$$

It also follows from (4.23) and (4.27) that

$$\int_{\Omega_T} \left|\Theta^\varepsilon(x + \Delta x, \tau) - \Theta^\varepsilon(x, \tau)\right|^2 \, dx\, d\tau \leqslant C\, \omega_1(|\Delta x|) \text{ with } \omega_1(\xi) \to 0 \text{ as } \xi \to 0, \tag{4.40}$$

where we suppose that $\Theta^\varepsilon = 0$ for $x + \Delta x \notin \Omega$.

Finally we obtain the uniform estimates for the time derivative of the function Θ^ε. To this end, we consider (2.23). It reads

$$\Phi^\varepsilon(x) \frac{\partial \Theta^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x) \left(\lambda_g(S^\varepsilon)\, \tilde{\varrho}_g^\varepsilon \, \nabla \mathsf{P}^\varepsilon - \, \tilde{\varrho}_g^\varepsilon \, \nabla \Upsilon^\varepsilon - \lambda_g(S^\varepsilon) \left[\tilde{\varrho}_g^\varepsilon\right]^2 \vec{g} \right) \right) = 0. \tag{4.41}$$

Multiplying (4.41) by $\varphi_g \in \mathcal{D}(\Omega_T)$ and integrating by parts, we get

$$\begin{aligned} - \int_{\Omega_T} \Phi^\varepsilon(x) \Theta^\varepsilon \frac{\partial \varphi_g}{\partial t} \, dx\, dt &= \int_{\Omega_T} K^\varepsilon(x) \lambda_g(S^\varepsilon) \tilde{\varrho}_g^\varepsilon \, \nabla \mathsf{P}^\varepsilon \cdot \nabla \varphi_g \, dx\, dt \\ &- \int_{\Omega_T} K^\varepsilon(x)\, \tilde{\varrho}_g^\varepsilon \, \nabla \Upsilon^\varepsilon \cdot \nabla \varphi_g \, dx\, dt - \int_{\Omega_T} K^\varepsilon(x) \lambda_g(S^\varepsilon) \left(\tilde{\varrho}_g^\varepsilon\right)^2 \vec{g} \cdot \nabla \varphi_g \, dx\, dt. \end{aligned} \tag{4.42}$$

Then it follows from Cauchy's inequality, the definition of the function ϱ_g, and condition (A.6) that

$$\left| \int_{\Omega_T} \Phi^\varepsilon(x)\, \Theta^\varepsilon \frac{\partial \varphi_g}{\partial t} \, dx\, dt \right| \leqslant C \, \left(1 + \|\nabla \mathsf{P}^\varepsilon\|_{L^2(\Omega_T)} + \|\nabla \Upsilon^\varepsilon\|_{L^2(\Omega_T)}\right) \, \|\nabla \varphi_g\|_{L^2(\Omega_T)}. \tag{4.43}$$

Inequality (4.43) along with the a priori estimates (4.24) and (4.23) implies that

$$\left\| \frac{\partial}{\partial t}(\Phi^\varepsilon \Theta^\varepsilon) \right\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C. \tag{4.44}$$

Now it is clear that the family $\{\Theta^\varepsilon\}_{\varepsilon>0}$ satisfies all the conditions of the compactness lemma. Thus the family $\{\Theta^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^2(\Omega_T)$, and Proposition 4.5 is proved. ◻

As a consequence of the uniform L^∞ bound for Θ^ε, we have the following result.

COROLLARY 4.6. *The family $\{\Theta^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_T)$ for all $q \in [1, +\infty[$.*

By similar arguments, we prove the compactness result for the family $\{S^\varepsilon\}_{\varepsilon>0}$. Namely, the following result holds.

PROPOSITION 4.7. *Under our standing assumptions, the family $\{S^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_T)$ for all $q \in [1, +\infty[$.*

4.2. Passage to the limit in (2.1)–(2.33). In this section, using the a priori estimates of the previous section, we obtain the compactness results (see Lemma 4.8 below) and pass to the limit in (2.1)–(2.33).

LEMMA 4.8. *There exists a function $S \in L^{2/\theta}(0,T; W^{\tau,2/\theta}(\Omega))$ with $0 \leqslant S \leqslant 1$ a.e. in Ω_T, θ defined in condition* (A.7) *and $0 < \tau < 1$, and functions, $\mathsf{P} \in L^2(0,T;H^1(\Omega))$, $\mathsf{w}_p, \mathsf{w}_s \in L^2(\Omega_T; H^1_\#(Y))$ such that, up to a subsequence,*

$$S^\varepsilon(x,t) \to S(x,t) \text{ strongly in } L^q(\Omega_T) \text{ for all } 1 \leqslant q < +\infty, \tag{4.45}$$

$$\mathsf{P}^\varepsilon(x,t) \to \mathsf{P}(x,t) \text{ weakly in } L^2(\Omega_T), \tag{4.46}$$

$$\nabla \mathsf{P}^\varepsilon(x,t) \xrightarrow{2s} \nabla \mathsf{P}(x,t) + \nabla_y \mathsf{w}_p(x,t,y), \tag{4.47}$$

$$\Upsilon^\varepsilon \to \Upsilon \text{ strongly in } L^q(\Omega_T) \text{ for all } 1 \leqslant q < +\infty, \tag{4.48}$$

$$\nabla \Upsilon^\varepsilon(x,t) \xrightarrow{2s} \nabla \Upsilon(x,t) + \nabla_y \mathsf{w}_s(x,t,y), \tag{4.49}$$

$$\Theta^\varepsilon \to (1-S)\tilde{\varrho}_g^H \text{ strongly in } L^2(\Omega_T), \tag{4.50}$$

where the function $\tilde{\varrho}_g^H$ is defined in (3.11).

Proof. The proof of Lemma 4.8 is based on the a priori estimates obtained in the previous section and two-scale convergence arguments similar to those in [1]. The only nontrivial convergence is that in (4.50). It follows from (4.45)–(4.46), the inequality $0 \leqslant S^\varepsilon \leqslant 1$, and the fact that ϱ_g is monotone. Indeed, for any $L^\infty(\Omega_T)$ function v, we have

$$\big(((\varrho_g(\mathsf{P}^\varepsilon + G_g(S^\varepsilon))(1-S^\varepsilon) - \varrho_g(v + G_g(S^\varepsilon))(1-S^\varepsilon)), (\mathsf{P}^\varepsilon - v)\big)_{L^2(\Omega_T)} \geqslant 0.$$

Denoting $\bar{\Theta}$ as the limit of Θ^ε and passing to the limit, as $\varepsilon \to 0$, in the last inequality, we obtain

$$\big((\bar{\Theta} - \varrho_g(v + G_g(S))(1-S)), (\mathsf{P} - v)\big)_{L^2(\Omega_T)} \geqslant 0.$$

Choosing $v = \mathsf{P} + \delta v_1$ and sending δ to zero yields

$$\left((\bar{\Theta} - \varrho_g(\mathsf{P} + G_g(S))(1-S)), v_1\right)_{L^2(\Omega_T)} \geqslant 0$$

for any $v_1 \in L^2(\Omega_T)$. This implies (4.50). ◻

Passage to the limit in (2.1). We set

$$\varphi_w\left(x, \frac{x}{\varepsilon}, t\right) \overset{\text{def}}{=} \varphi(x,t) + \varepsilon\, \zeta\left(x, \frac{x}{\varepsilon}, t\right) = \varphi(x,t) + \varepsilon\, \zeta_1(x,t)\, \zeta_2\left(\frac{x}{\varepsilon}\right), \tag{4.51}$$

where $\varphi \in \mathcal{D}(\Omega_T), \zeta_1 \in \mathcal{D}(\Omega_T)$, and $\zeta_2 \in C^\infty_\#(Y)$, and we plug the function φ_w in (2.1). This yields

$$-\int_{\Omega_T} \Phi^\varepsilon(x) S^\varepsilon \left[\frac{\partial \varphi}{\partial t} + \varepsilon \frac{\partial \zeta^\varepsilon}{\partial t}\right] \, dx\, dt \tag{4.52}$$

$$+ \int_{\Omega_T} K^\varepsilon(x) \Big\{ \lambda_w(S^\varepsilon) \nabla \mathsf{P}^\varepsilon + \nabla \Upsilon^\varepsilon - \lambda_w(S^\varepsilon)\vec{g} \Big\} \cdot \left[\nabla \varphi + \varepsilon \nabla_x \zeta^\varepsilon + \nabla_y \zeta^\varepsilon\right] \, dx\, dt = 0.$$

Passing now to the two-scale limit in (4.52), we get

$$-\int_{\Omega_T\times Y} \Phi(y)\, S(x,t)\, \frac{\partial \varphi}{\partial t} \, dy\, dx\, dt + \int_{\Omega_T\times Y} K(y) \Big\{ \lambda_w(S) \left[\nabla \mathsf{P} + \nabla_y \mathsf{w}_p\right] \tag{4.53}$$

$$+ \left[\nabla \Upsilon + \nabla_y \mathsf{w}_s\right] - \lambda_w(S)\vec{g} \Big\} \cdot \left[\nabla \varphi + \zeta_1 \nabla_y \zeta_2\right] \, dy\, dx\, dt = 0.$$

Passage to the limit in (2.33). We set

$$\varphi_g\left(x, \frac{x}{\varepsilon}, t\right) \overset{\text{def}}{=} \varphi(x,t) + \varepsilon\, \zeta\left(x, \frac{x}{\varepsilon}, t\right) = \varphi(x,t) + \varepsilon\, \zeta_1(x,t)\, \zeta_2\left(\frac{x}{\varepsilon}\right), \tag{4.54}$$

where $\varphi \in \mathcal{D}(\Omega_T), \zeta_1 \in \mathcal{D}(\Omega_T)$, and $\zeta_2 \in C^\infty_\#(Y)$, and we plug the function φ_g in (2.33). We obtain

$$-\int_{\Omega_T} \Phi^\varepsilon(x) \Theta^\varepsilon \left[\frac{\partial \varphi}{\partial t} + \varepsilon \frac{\partial \zeta^\varepsilon}{\partial t}\right] \, dx\, dt \tag{4.55}$$

$$+ \int_{\Omega_T} K^\varepsilon(x) \tilde{\varrho}_g^\varepsilon \Big\{ \lambda_g(S^\varepsilon) \nabla \mathsf{P}^\varepsilon - \nabla \Upsilon^\varepsilon - \tilde{\varrho}_g^\varepsilon \lambda_g(S^\varepsilon)\vec{g} \Big\} \cdot \left[\nabla \varphi + \varepsilon \nabla_x \zeta^\varepsilon + \nabla_y \zeta^\varepsilon\right] \, dx\, dt = 0.$$

Now passing to the two-scale limit in (4.55), we get

$$-\int_{\Omega_T\times Y} \Phi(y)\,(1-S)\, \tilde{\varrho}_g^H \frac{\partial \varphi}{\partial t} \, dy\, dx\, dt + \int_{\Omega_T\times Y} K(y) \tilde{\varrho}_g^H \Big\{ \lambda_g(S) \left[\nabla \mathsf{P} + \nabla_y \mathsf{w}_p\right] \tag{4.56}$$

$$- \left[\nabla \Upsilon + \nabla_y \mathsf{w}_s\right] - \lambda_g(S)\tilde{\varrho}_g^H \vec{g} \Big\} \cdot \left[\nabla \varphi + \zeta_1 \nabla_y \zeta_2\right] \, dy\, dx\, dt = 0,$$

where the function $\tilde{\varrho}_g^H$ is defined in (3.11); here we have also used the fact that, after taking a subsequence, the convergence in (4.50) implies the a.e. convergence of $\tilde{\varrho}_g^\varepsilon \lambda_g(S^\varepsilon)$ to $\tilde{\varrho}_g^H \lambda_g(S)$.

Equations for the functions $\mathsf{w}_p, \mathsf{w}_s$. Consider first (4.53). Choosing $\varphi = 0$ gives

$$-\int_Y \mathrm{div}_y\, (K(y)\lambda_w(S)\nabla_y \mathsf{w}_p)\,\zeta_2\, dy - \int_Y \mathrm{div}_y\, (K(y)\nabla_y \mathsf{w}_s)\,\zeta_2\, dy$$

$$= \int_Y \mathrm{div}_y\, (K(y)\lambda_w(S)\nabla \mathsf{P} + K(y)\nabla\Upsilon - K(y)\lambda_w(S)\vec{g})\,\zeta_2\, dy.$$

This equation leads to the first equation for the functions $\mathsf{w}_p, \mathsf{w}_s$. It reads

$$\mathrm{div}_y\, (K(y)\lambda_w(S)\nabla_y \mathsf{w}_p) + \mathrm{div}_y\, (K(y)\nabla_y \mathsf{w}_s) = \mathbf{F}_w^C, \tag{4.57}$$

where $\mathbf{F}_w^C$ does not depend on $\mathsf{w}_p, \mathsf{w}_s$ and is given by

$$\mathbf{F}_w^C \stackrel{\text{def}}{=} -\mathrm{div}_y\, (K(y)\lambda_w(S)\nabla \mathsf{P} + K(y)\nabla\Upsilon - K(y)\lambda_w(S)\vec{g})\,. \tag{4.58}$$

Consider now (4.56). Choosing $\varphi = 0$ gives

$$-\int_Y \mathrm{div}_y\, \left(K(y)\,\lambda_g(S)\,\tilde{\varrho}_g^H \nabla_y \mathsf{w}_p\right)\zeta_2\, dy + \int_Y \mathrm{div}_y\, \left(K(y)\,\tilde{\varrho}_g^H\,\nabla_y \mathsf{w}_s\right)\zeta_2\, dy =$$

$$= \int_Y \mathrm{div}_y\, \Big(K(y)\,\lambda_g(S)\,\tilde{\varrho}_g^H \nabla \mathsf{P} - K(y)\,\tilde{\varrho}_g^H\,\nabla\Upsilon - K(y)\,\lambda_g(S)\,\left[\tilde{\varrho}_g^H\right]^2 \vec{g}\Big)\,\zeta_2\, dy.$$

This equation leads to the second equation for the functions $\mathsf{w}_p, \mathsf{w}_s$,

$$\mathrm{div}_y\, \left(K(y)\lambda_g(S)\,\tilde{\varrho}_g^H \nabla_y \mathsf{w}_p\right) - \mathrm{div}_y\, \left(K(y)\,\tilde{\varrho}_g^H \nabla_y \mathsf{w}_s\right) = \mathbf{F}_g^C, \tag{4.59}$$

where $\mathbf{F}_g^C$ does not depend on $\mathsf{w}_p, \mathsf{w}_s$ and is given by

$$\mathbf{F}_g^C \stackrel{\text{def}}{=} -\mathrm{div}_y\, \Big(K(y)\lambda_g(S)\,\tilde{\varrho}_g^H \nabla \mathsf{P} - K(y)\,\tilde{\varrho}_g^H \nabla\Upsilon - K(y)\lambda_g(S)\,\left[\tilde{\varrho}_g^H\right]^2 \vec{g}\Big)\,. \tag{4.60}$$

(4.57) and (4.59) lead to the following coupled linear system of equations with respect to the variables $\mathbf{X} = \mathrm{div}_y\, (K(y)\nabla_y \mathsf{w}_p)$ and $\mathbf{Y} = \mathrm{div}_y\, (K(y)\nabla_y \mathsf{w}_s)$:

$$\left\{\begin{array}{l} \lambda_w(S)\,\mathbf{X} + \mathbf{Y} = \mathbf{F}_w^C, \\ \lambda_g(S)\,\tilde{\varrho}_g^H\,\mathbf{X} - \tilde{\varrho}_g^H\,\mathbf{Y} = \mathbf{F}_g^C. \end{array}\right. \tag{4.61}$$

Resolving system (4.61), we obtain two decoupled equations for the functions $\mathsf{w}_p, \mathsf{w}_s$:

$$\mathrm{div}_y\, (K(y)\,(\nabla \mathsf{P} + \nabla_y \mathsf{w}_p)) = \frac{\lambda_w(S) + \tilde{\varrho}_g^H\,\lambda_g(S)}{\lambda(S)}\,\mathrm{div}_y\,(K(y)\vec{g}) \quad \text{in } \Omega_T \times Y, \tag{4.62}$$

$$\mathrm{div}_y\, (K(y)\,(\nabla\Upsilon + \nabla_y \mathsf{w}_s)) = -\frac{\lambda_w(S)\lambda_g(S)}{\lambda(S)}\left(\tilde{\varrho}_g^H - 1\right)\mathrm{div}_y\,(K(y)\vec{g}) \text{ in } \Omega_T \times Y. \tag{4.63}$$

(4.62) and (4.63) allow us to represent the functions $\mathsf{w}_p, \mathsf{w}_s$ in the following way:

$$\mathsf{w}_p = \sum_{j=1}^d \xi_j(y)\frac{\partial\,\mathsf{P}}{\partial x_j}(x,t) + \mathsf{f}_p \quad \text{and} \quad \mathsf{w}_s = \sum_{j=1}^d \xi_j(y)\frac{\partial\,\Upsilon}{\partial x_j}(x,t) + \mathsf{f}_s, \tag{4.64}$$

where the function ξ_j is a Y-periodic solution of

$$\begin{cases} -\mathrm{div}_y \, (K(y) \, [\nabla_y \xi_j + \vec{e}_j]) = 0 \quad \text{in } Y, \\ \\ y \longmapsto \xi_j(y) \quad Y\text{-periodic} \end{cases}$$

with $\vec{e}_j$ being the j-th coordinate vector. The functions $\mathsf{f}_p = \mathsf{f}_p(y, S, \mathsf{P}), \mathsf{f}_s = \mathsf{f}_s(y, S, \mathsf{P})$ are then the solutions of the following equations:

$$\begin{cases} \mathrm{div}_y \, (K(y) \, \nabla_y \mathsf{f}_p) = \dfrac{\lambda_w(S) + \tilde{\varrho}_g^H \lambda_g(S)}{\lambda(S)} \, \mathrm{div}_y \, (K(y)\vec{g}) \text{ in } \Omega_T \times Y, \\ y \longmapsto \mathsf{f}_p(y, S, \mathsf{P}) \quad Y\text{-periodic}, \end{cases}$$

and

$$\begin{cases} \mathrm{div}_y \, (K(y) \, \nabla_y \mathsf{f}_s) = -\dfrac{\lambda_w(S)\lambda_g(S)}{\lambda(S)} \left(\tilde{\varrho}_g^H - 1\right) \, \mathrm{div}_y \, (K(y)\vec{g}) \text{ in } \Omega_T \times Y, \\ y \longmapsto \mathsf{f}_s(y, S, \mathsf{P}) \quad Y\text{-periodic}. \end{cases}$$

Homogenized equations (equations for the functions S, P). Consider first (4.53). Choosing $\zeta_2 = 0$ gives

$$- \int_{\Omega_T \times Y} \Phi(y) \, S \, \frac{\partial \varphi}{\partial t} \, dy \, dx \, dt \tag{4.65}$$

$$+ \int_{\Omega_T \times Y} K(y) \Big\{ \lambda_w(S) \, [\nabla \mathsf{P} + \nabla_y \mathsf{w}_p] + [\nabla \Upsilon + \nabla_y \mathsf{w}_s] - \lambda_w(S) \, \vec{g} \Big\} \cdot \nabla \varphi \, dy \, dx \, dt = 0.$$

Relation (4.65) leads to the following homogenized equation:

$$\langle \Phi \rangle \, \frac{\partial S}{\partial t} - \mathrm{div}_x \, \left\{ \lambda_w(S) \int_Y K(y) \, [\nabla \mathsf{P} + \nabla_y \mathsf{w}_p] \, dy \right\} \tag{4.66}$$

$$- \mathrm{div}_x \, \left\{ \int_Y K(y) \, [\nabla \Upsilon + \nabla_y \mathsf{w}_s] \, dy \right\} + \mathrm{div}_x \, \left\{ \lambda_w(S) \int_Y K(y) \, \vec{g} \, dy \right\} = 0,$$

where $\langle \cdot \rangle$ stands for the mean value of the corresponding function over the cell Y.

Using now the definition of the homogenized tensor $\mathbb{K}^\star$ given in (3.2), (3.3), and (4.64), we obtain the homogenized saturation equation

$$\langle \Phi \rangle \, \frac{\partial S}{\partial t} - \mathrm{div}_x \left(\mathbb{K}^\star \, (\lambda_w(S) \nabla \mathsf{P} + \nabla \Upsilon) + \mathbf{F}_w \right) = 0, \tag{4.67}$$

where the function $\mathbf{F}_w = \mathbf{F}_w(S, \mathsf{P})$ denotes the lower order term in (4.67) and is given by

$$\mathbf{F}_w(S, \mathsf{P}) \stackrel{\text{def}}{=} \lambda_w(S) \langle K \, \nabla_y \mathsf{f}_p \rangle + \langle K \, \nabla_y \mathsf{f}_s \rangle - \lambda_w(S) \, \langle K \, \vec{g} \, \rangle \, . \tag{4.68}$$

Consider now (4.56). Choosing $\zeta_2 = 0$ gives

$$- \int_{\Omega_T \times Y} \Phi(y) \, (1 - S) \, \tilde{\varrho}_g^H \, \frac{\partial \varphi}{\partial t} \, dy \, dx \, dt \tag{4.69}$$

$$+ \int_{\Omega_T \times Y} K(y)\tilde{\varrho}_g^H \Big\{ \lambda_g(S)\, [\nabla \mathsf{P} + \nabla_y \mathsf{w}_p] - [\nabla \Upsilon + \nabla_y \mathsf{w}_s] - \lambda_g(S) \tilde{\varrho}_g^H \vec{g} \Big\} \cdot \nabla \varphi \; dy\, dx\, dt = 0.$$

Relation (4.69) leads to the following homogenized equation:

$$\langle \Phi \rangle \frac{\partial}{\partial t} \left(\tilde{\varrho}_g^H \, (1 - S) \right) - \operatorname{div}_x \left\{ \lambda_g(S)\, \tilde{\varrho}_g^H \int_Y K(y)\, [\nabla \mathsf{P} + \nabla_y \mathsf{w}_p] \; dy \right\}$$
$$+ \operatorname{div}_x \left\{ \tilde{\varrho}_g^H \int_Y K(y)\, [\nabla \Upsilon + \nabla_y \mathsf{w}_s] \; dy \right\} + \operatorname{div}_x \left\{ \lambda_g(S) \left[\tilde{\varrho}_g^H \right]^2 \int_Y K(y)\, \vec{g}\, dy \right\} = 0. \tag{4.70}$$

As in the previous case, (4.70) leads to the homogenized global pressure equation

$$\langle \Phi \rangle \frac{\partial}{\partial t} \left(\tilde{\varrho}_g^H \, (1 - S) \right) - \operatorname{div}_x \left(\mathbb{K}^\star \, \left(\lambda_g(S)\, \tilde{\varrho}_g^H \nabla \mathsf{P} - \tilde{\varrho}_g^H \, \nabla \Upsilon \right) + \mathbf{F}_g \right) = 0, \tag{4.71}$$

where the function $\mathbf{F}_g = \mathbf{F}_g(S, \mathsf{P})$ denotes the lower order term in (4.71) and is given by

$$\mathbf{F}_g(S, \mathsf{P}) \stackrel{\text{def}}{=} \lambda_g(S)\, \tilde{\varrho}_g^H \langle K \, \nabla_y \mathsf{f}_p \rangle - \tilde{\varrho}_g^H \langle K \, \nabla_y \mathsf{f}_s \rangle - \lambda_g(S) \left[\tilde{\varrho}_g^H \right]^2 \langle K \, \vec{g} \rangle. \tag{4.72}$$

Theorem 3.3 is proved. □

5. Concluding remarks. This paper presents a homogenization result for an immiscible compressible two-phase flow in porous media in the case of a single rock-type model, i.e., we assume here that the capillary pressure and relative permeabilities depend solely on the saturation. Our future study will focus on extension of these homogenization results to the case of porous media with several rock types: capillary pressure and relative permeability curves being different in each type of porous media. This leads to nonlinear transmission conditions representing the continuity of some physical characteristics such as water and gas pressures at the interfaces that separate different media. Then the saturation and some other characteristics are getting discontinuous at the interfaces. It makes the upscaling procedure more complicated. In particular, we cannot expect, in this case, the compactness of a saturation function. Instead we will have to prove a two-scale compactness result which will ensure the strong two-scale convergence.

Another possible generalization of our result concerns the cases of unbounded capillary pressure and vanishing $\varrho_{\min}$. These more complicated cases appear in the applications. The first step in this direction is to prove the corresponding existence results.

It is also planned to test the upscaling method developed here by means of numerical simulations of water-gas flow through a heterogeneous reservoir.

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