# HOMOGENIZATION OF A POROUS MEDIUM WITH RANDOMLY PULSATING MICROSTRUCTURE[*]

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**Abstract.** We study a parabolic operator in a perforated medium with random rapidly pulsating perforation. Assuming that the geometry of the perforations is spatially periodic and stationary random in time with good mixing properties, we show that this problem admits homogenization in moving coordinates, and derive the homogenized problem.

**1. Introduction.** This note deals with the homogenization problem for the heat equation stated in a perforated medium with periodic microstructure rapidly pulsating in time. It is assumed that the geometrical characteristics of the microstructure are random stationary ergodic rapidly oscillating functions of time.

These equations model the long-term behavior of artificial materials with a periodic microstructure whose characteristics depend on atmosphere temperature, pressure, etc.

Throughout this paper we denote by $\varepsilon$ the microscopic length scale of the medium. Our goal is to show that under proper mixing and regularity assumptions the studied problem admits "homogenization in law" in moving coordinates $(x', t) = (x + \frac{\bar{b}}{\varepsilon}t, t)$ with a constant deterministic vector $\bar{b}$, and that the homogenized equation is a stochastic partial differential equation (SPDE). Namely, we will prove that in the said moving coordinates a solution of the original problem converges in law, as $\varepsilon \to 0$, in the energy functional space to a solution of the homogenized SPDE. The homogenized problem is well posed and determines the limit measure uniquely.

It can be shown that the homogenized SPDE has in general a nontrivial covariance operator so that we cannot expect a.s. homogenization.

A similar problem in the case of periodically pulsating holes has been studied in our earlier work [2], where it was shown that the homogenization takes place on the background of a large convection.

In the existing literature there are examples of homogenization problems for random parabolic operators such that the corresponding homogenized models involve SPDEs. This phenomenon was observed in the works [3], [8] and [9] devoted to homogenization of nonstationary parabolic equations with large lower order terms. However, in all these examples the limit behavior is diffusive due to the presence of large lower order terms, while for the divergence form random parabolic equations stated either in a solid medium or in a perforated domain with time independent

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perforation, the a.s. homogenization result holds, and the limit equation is a standard parabolic PDE; see [6], [5].

Basic results on homogenization in perforated domains and random homogenization can be found for instance in [4] and [7], respectively.

In the case of an initial boundary problem posed in a bounded "perforated" cylinder, the asymptotic behavior of solutions depends crucially on whether $\bar{b} = 0$ or not. If $\bar{b} = 0$, then the result similar to that of Theorem 3.4 holds. However, if $\bar{b} \neq 0$, then the above-mentioned moving coordinates do not make sense in a bounded domain. In this case, if at the exterior boundary of the cylinder the homogeneous Dirichlet or Fourier boundary condition is posed, then for any initial function $u_0 \in L^2$ a solution of the studied initial boundary problem converges to zero as $\varepsilon \to 0$ for any positive time.

We now outline the techniques used in this work. To obtain the limit (homogenized) problem, in particular the value of the (large) effective convection coefficient, we apply the multiscale asymptotic expansion technique with the diffusive scaling of the "fast" spatial and temporal variables.

The structure of the perforation suggests that the terms of the expansion are to be periodic in the fast spatial variables and stationary in the fast temporal variable. By substituting the expansion in the original problem and equating like powers of $\varepsilon$, we obtain in a standard way a sequence of auxiliary parabolic problems (see (3.1) and (3.2) below). We then derive necessary and sufficient condition of the existence of a stationary solution to these problems; this is the subject of Lemmas 3.1 and 3.3 below.

In order to make the first nontrivial auxiliary problem solvable, we introduce moving coordinates of the form $(x', t) = \left( x + \frac{b}{\varepsilon} t + \frac{1}{\varepsilon} \beta \left( \frac{t}{\varepsilon^2} \right), t \right)$ with a constant vector $b$ and stationary zero mean value random process $\beta(s)$, the drift $\frac{1}{\varepsilon} \beta \left( \frac{t}{\varepsilon^2} \right)$ being responsible for the presence of a stochastic term in the limit equation.

This allows us to find formally two leading terms of the expansion. To justify the convergence we need one more term. At this point we face a technical difficulty, namely, the data of the corresponding auxiliary equation for the third term do not satisfy the compatibility conditions. In order to make this equation solvable we modify its right-hand side by adding an extra term. Then we have to show, and this is an essential part of the work, that the contribution of this "compensator" vanishes as $\varepsilon \to 0$.

Let us also note that a priori estimates for solutions of the original problem are not straightforward. To obtain them we use a solution of the adjoint auxiliary problem as a weight function in the energy estimates. Although this weight function is random and rapidly oscillating, it admits uniform positive lower and upper bounds so that we get uniform estimates for the $H^1$ norm of the solution.

**2. The setup.** We begin by describing the geometry. Given a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $F_t = F_{t,\omega}$, $t \in (-\infty, +\infty)$ be a random stationary field of diffeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ that have the following properties:

   1. Periodicity. For each $t \in \mathbb{R}$ and $\omega \in \Omega$ the mapping $F_t$ is compatible with $[0,1]^n$ periodic structures in $\mathbb{R}^n$, that is,

$$F_t(x + z) = F_t(x) + z \qquad \text{for all} \ \ x \in \mathbb{R}^n \text{ and} \ \ z \in \mathbb{Z}^n.$$

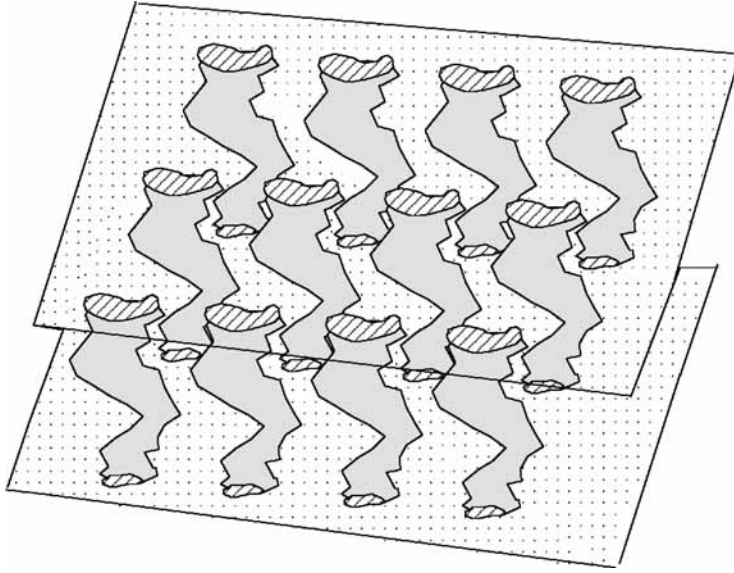   2. Stationarity. The random field $F_t$ is stationary in $t$.

FIG. 2.1. *Randomly pulsating perforation.*

3. Regularity.  The functions $F_t(x)$ and $F_t^{-1}(x)$ are a.s. continuously differentiable in $x$ and $t$, moreover,

$$\left|\frac{\partial F_t(x)}{\partial x}\right| \le C, \quad \left|\frac{\partial F_t^{-1}(x)}{\partial x}\right| \le C,$$

$$\left|\frac{\partial F_t(x)}{\partial t}\right| \le C, \quad \left|\frac{\partial F_t^{-1}(x)}{\partial t}\right| \le C$$

with a nonrandom constant $C$.

4. Mixing condition.  Denote by $\mathcal{F}_{\le 0}$ and $\mathcal{F}_{\ge r}$ the $\sigma$-algebras

$$\sigma\{F_s \, , \, s \le 0\}, \qquad \sigma\{F_s \, , \, s \ge r\},$$

respectively.  We suppose that the function

$$\alpha(r) = \sup_{\substack{A_1 \in \mathcal{F}_{\le 0} \\ A_2 \in \mathcal{F}_{\ge r}}} |\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)|$$

called *strong mixing coefficient*, satisfies the condition

(2.1)
$$\int_0^\infty \sqrt{\alpha(r)} dr \; < \; \infty.$$

Denote

$$B_0 = \{y \in \mathbb{R}^n \, : \, |y| \le \frac{1}{4}\}, \qquad B = \bigcup_{z \in \mathbb{Z}^n} (B_0 + z),$$

and let $G(s) = F_s(\mathbb{R}^n \setminus B)$.  By construction, $G(s)$ is a periodic connected set in $\mathbb{R}^n$; its geometric characteristics are random stationary in $s$.  We now introduce a randomly pulsating perforated medium (see Figure 2.1) as follows:

$$Q_T^\varepsilon = \left\{(x,t) \in \mathbb{R}^n \times [0,T] \, : \, x \in \varepsilon G\left(\frac{t}{\varepsilon^2}\right)\right\}.$$

In the domain $Q_T^\varepsilon$ we study a problem

$$\frac{\partial}{\partial t}u^\varepsilon = \Delta u^\varepsilon, \qquad (x,t) \in Q_T^\varepsilon,$$

(2.2)
$$\frac{\partial}{\partial n_x^\varepsilon}u^\varepsilon = 0 \qquad \text{on} \quad \left\{x \in \varepsilon\partial G\left(\frac{t}{\varepsilon^2}\right), 0 < t < T\right\},$$

$$u^\varepsilon(x,0) = u_0(x), \quad u_0 \in L^2(\mathbb{R}^n),$$

where $n_x^\varepsilon = n_x^\varepsilon(x,t)$ is an exterior unit normal to $\varepsilon\partial G\left(\frac{t}{\varepsilon^2}\right)$.

**3. Main results.** In order to formulate the main result of the note, we consider two auxiliary equations

$$\frac{\partial}{\partial s}\psi = \Delta\psi + f(y,s), \qquad y \in G(s), \ s \in \mathbb{R}$$

(3.1)
$$\frac{\partial}{\partial n}\psi = g(y,s) \qquad \text{on} \quad \partial G(s)$$

and

$$\frac{\partial}{\partial s}p + \Delta p = 0, \qquad y \in G(s), \ s \in \mathbb{R}$$

(3.2)
$$\frac{\partial}{\partial n}p + n_s p = 0 \qquad \text{on} \quad \partial G(s),$$

where $n_s$ is the $(n+1)$-th component of the unit normal vector on $\partial Q$ with $Q = \{(y,s) \in T^n \times (-\infty, +\infty) : y \in G(s)\}$. We are going to consider solutions of (3.1) and (3.2) (as well as of other auxiliary problems appearing in the paper) which are periodic in $y$.

Throughout this paper we will identify periodic functions and sets in $\mathbb{R}^n$ with the corresponding functions and sets on the standard torus $\mathbb{T}^n$.

LEMMA 3.1. *Equation* (3.2) *has a stationary, periodic in $y$ solution. Under the normalization condition*

(3.3)
$$\int_{T^n \cap G(0)} p(y,0)dy = 1$$

*this solution is unique and the estimate holds*

(3.4)
$$C_1 \leq p(y,s) \leq C_2, \qquad 0 < C_1 < C_2.$$

*Proof.* Consider two auxiliary Cauchy problems

$$\frac{\partial}{\partial s}\zeta - \Delta\zeta = 0, \qquad y \in (T^n \cap G(s)), \ s > s_0$$

(3.5)
$$\frac{\partial}{\partial n}\zeta = 0 \quad \text{on} \quad \partial G(s), \qquad \zeta(y,s_0) = \phi(y).$$

and

$$\frac{\partial}{\partial s}\eta + \Delta\eta = 0, \qquad y \in (T^n \cap G(s)), \ s < N$$

(3.6)
$$\frac{\partial}{\partial n}\eta + n_s\eta = 0 \quad \text{on} \quad \partial G(s), \qquad \eta(y,N) = \varphi(y).$$

Exploiting the strong maximum principle and compactness arguments one can show that for any continuous $\phi(y)$ the solution $\zeta$ of problem (3.5) satisfies the inequality

$$\underset{T^n \cap G(s_0+1)}{\operatorname{osc}} \zeta(s_0+1, \cdot) \leq (1-\gamma) \underset{T^n \cap G(s_0)}{\operatorname{osc}} \phi(\cdot),$$

where $\operatorname{osc}\varphi(\cdot) = \max\varphi(\cdot) - \min\varphi(\cdot)$, and $\gamma > 0$ is a deterministic constant that does not depend on $\phi(y)$, nor on $s_0$. Combining this with the standard parabolic estimates we conclude that

$$(3.7) \qquad |\zeta(y, s) - C_\phi| \leq C \exp(-\kappa(s-s_0))\|\phi\|_{L^2(T^n)}, \qquad s > s_0 + 1,$$

for some (random) constant $C_\phi$. Integrating the equation (3.6) by parts over the set $\{(y,s) \in T^n \times (s_1, s_2) : y \in G(s)\}$ one can easily show that

$$\int_{G(s_2)} \eta(y, s_2)dy = \int_{G(s_1)} \eta(y, s_1)dy,$$

for any $s_1 < s_2 \leq N$, and

$$\int_{G(s_2)} \zeta(y, s_2)\eta(y, s_2)dy = \int_{G(s_1)} \zeta(y, s_1)\eta(y, s_1)dy$$

for any $s_0 \leq s_1 < s_2 \leq N$ (see [2] for detailed computations). Together with (3.7) this yields

$$\left| \int_{G(s_0)} \phi(y)\eta(s_0, y)dy \right| = \left| \int_{G(s_0)} (\phi(y) - C_\phi)\eta(s_0, y)dy \right|$$

$$= \left| \int_{G(N)} (\zeta(y, N) - C_\phi)\varphi(y)dy \right| \leq C \exp(-\kappa(N-s_0))\|\phi\|_{L^2(G(s_0))}\|\varphi\|_{L^2(G(N))}$$

for any $\varphi \in L^2(G(N))$ such that $\int_{G(N)} \varphi(y)dy = 0$, and any $\phi \in L^2(G(s_0))$. Therefore,

$$(3.8) \qquad \|\eta(s_0, \cdot)\|_{L^2(G(s_0))} \leq C \exp(-\kappa(N-s_0))\|\varphi\|_{L^2(G(N))}.$$

Let $p_N$ be a solution of the Cauchy problem

$$(3.9) \qquad \begin{aligned} \frac{\partial}{\partial s}p_N + \Delta p_N &= 0, \qquad y \in (T^n \cap G(s)), \quad s < N, \\ \frac{\partial}{\partial n}p_N + n_s p_N &= 0 \quad \text{on } \partial G(s), \qquad p_N(y, N) = 1. \end{aligned}$$

From (3.8) it follows that $p_N$ converges, as $N \to \infty$, to a solution of (3.2) uniformly on compact sets. Clearly the function $p_{N+s}(y, s)$ converges to the same limit function denoted by $p(y, s)$. By construction, the function $p_{N+s}(y, s)$ is stationary; so is $p(y, s)$. The uniqueness of a stationary solution that satisfies (3.3) easily follows from the estimate (3.7), and the bounds (3.4) from the maximum principle.     □

    Denote

$$\bar{b} = \mathbf{E} \int_{\partial G(s)} p(y, s)n_s(y, s)\mathcal{H}^{n-1}(dy)$$

and

$$\beta(s) = \int_{\partial G(s)} p(y, s)n_s(y, s)\mathcal{H}^{n-1}(dy) - \bar{b},$$

where $\mathcal{H}^{n-1}(dy)$ is an element of surface volume on $\partial G(s)$. Notice that $\bar{b}$ is well defined due to the stationarity of $p$ and $G(s)$. We also introduce a matrix $\Lambda = \Lambda^{ij}$, such that

$$\{\Lambda\Lambda^*\}^{ij} = \frac{1}{2}\int_0^\infty \mathbf{E}(\beta^i(s)\beta^j(0) + \beta^i(0)\beta^j(s))ds.$$

The two statements below can be proved in the same way as Lemma 3 and Lemma 4 in [8].

LEMMA 3.2. *Under our standing assumptions the process $\beta(\cdot)$ satisfies the functional Central Limit Theorem (CLT) with correlation matrix $\Lambda\Lambda^*$, that is, the process*

$$\varepsilon \int_0^t \beta(\frac{s}{\varepsilon^2})ds$$

*converges in law, as $\varepsilon \to 0$, in the space $(C[0,T])^n$ to the process $\Lambda W_t$, where $W_t$ is a standard $n$-dimensional Wiener process.*

LEMMA 3.3. *Let $f(y,s)$ and $h(y,s)$, $(y,s) \in T^n \times (-\infty, \infty)$, be stationary, ergodic random functions such that*

$$\mathbf{E}(\|f(\cdot,s)\|_{L^2(T^n)} + \|h(\cdot,s)\|_{H^1(T^n)}) < \infty,$$

(3.10) $$\int_{G(s)} f(y,s)p(y,s)dy + \int_{\partial G(s)} h(y,s)p(y,s)\mathcal{H}^1(dx) = 0.$$

*Then the equation*

(3.11) 
$$\frac{\partial}{\partial s}\theta - \Delta\theta = f(y,s), \qquad y \in (T^n \cap G(s)),$$

$$\frac{\partial}{\partial n}\theta = h(y,s) \quad \text{on} \ \partial G(s)$$

*has a stationary ergodic solution. Under the normalization*

(3.12) $$\int_{G(s)} p(y,s)\theta(y,s)dy = 0,$$

*this solution is unique.*

We now proceed with the convergence result. It is convenient to extend a solution $u^\varepsilon$ of problem (2.2) inside the "holes" $(\mathbb{R}^n \times (0,T)) \setminus Q_T^\varepsilon$, the notation $u^\varepsilon$ being kept for the extended function. According to [1] there is an extension that satisfies the inequality

$$\|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|u^\varepsilon\|_{C(0,T;L^2(\mathbb{R}^n))}$$
$$\leq C\big(\|u^\varepsilon\|_{L^2(0,T;H^1(\varepsilon G(t/\varepsilon^2)))} + \|u^\varepsilon\|_{C(0,T;L^2(G(t/\varepsilon^2)))}\big)$$

with a constant $C$ which does not depend on $\varepsilon$. The notation $v^\varepsilon$ is used for $u^\varepsilon$ written in moving coordinates:

$$v^\varepsilon(x,t) = u^\varepsilon\left(x - \frac{\bar{b}}{\varepsilon}t, t\right).$$

Denote

$$V = L_w^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L_w^2(\mathbb{R}^n)),$$

where symbol $w$ indicates that the corresponding functional space is equipped with its weak topology.

The main result of this note is summarized in the following theorem.

THEOREM 3.4. *Under assumptions 1–4, a solution $v^\varepsilon$ of problem (2.2) converges in law, as $\varepsilon \to 0$, in the spaces $V$ and $L^2(\mathbb{R}^n \times (0,T))$ to a solution of the following SPDE*

$$(3.13) \qquad du = \hat{A}u \, dt + \Lambda \nabla u \, dW_t, \qquad u(x,0) = u_0(x)$$

*with*

$$\hat{A} = \hat{a}^{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

*and*

$$\hat{a} = \mathbf{E}\bigg( \int_{G(0)} (\mathbf{I} + \nabla\theta(y,0))(\mathbf{I} + \nabla\theta(y,0))^* p(y,0) dy \bigg) + \frac{1}{2}\Lambda\Lambda^*.$$

*Remark* 3.5. If the Neumann boundary condition at the border of perforation in problem (2.2) is replaced by a Dirichlet or Robin condition, then for any $u_0 \in L^2(\mathbb{R}^n)$ the solutions $u^\varepsilon$ would tend to zero, as $\varepsilon \to 0$, for all $t > 0$. In this case, since the $(n-1)$-dimensional volume of the perforation boundary tends to infinity, the boundary condition is getting increasingly dissipative as $\varepsilon \to 0$. One can try to divide $u^\varepsilon$ by a proper small parameter so that the ratio has a nontrivial finite limit, but this kind of analysis is not in the scope of the present work.

*Remark* 3.6. In our previous work [2] dealing with periodically pulsating perforation, we provided an example of a perforated structure in $\mathbb{R}^2$ for which $\bar{b} \neq 0$. In this example the shape of inclusions does not depend on time and is given by

$$S_0 = \left[-\frac{1}{6}, L\right] \times \left[-\frac{1}{3}, \frac{1}{3}\right] \setminus [0, L] \times \left[-\frac{1}{6}, \frac{1}{6}\right]$$

with big enough $L$, the cell of periodicity being $(0, L+1) \times (0, 1)$. This perforation just moves periodically forward and backward along the first coordinate axis.

Letting now

$$G(0) = \mathbb{R}^2 \setminus \bigcup_{i,j\in\mathbb{Z}} (S_0 + ie_1 + (L+1)je_2),$$

where $e_1$ and $e_2$ are the coordinate unit vectors, we introduce a randomly pulsating periodic perforation as follows:

$$G(t) = \begin{cases} G(0) + (-1)^{\xi_j}(t-j), & j \le t \le j + T, \\ G(0) + (-1)^{\xi_j}(2T + j - t), & j + T \le t \le j + 2T; \end{cases}$$

here $j = 0, \pm1, \pm2, \ldots$, and $\{\xi_j\}$ is a collection of independently and identically distributed (i.i.d.) random variables taking on the values 1 and 2 with probability 1/2.

Exactly in the same way as in [2], one can show that in this example for large enough $L$ and $T$, the first component of the vector $\bar{b}$ is not equal to zero.

*Proof of Theorem* 3.4. Let us first introduce a function

$$(3.14) \qquad z^\varepsilon(x,t) = u^\varepsilon\left(x + \frac{\bar{b}}{\varepsilon}t + \frac{1}{\varepsilon}\int_0^t \beta\left(\frac{s}{\varepsilon^2}\right), t\right).$$

We are going to show that this function converges a.s., as $\varepsilon \to 0$, to a solution of a deterministic parabolic equation with constant coefficients, i.e., that problem (2.2) admits a.s. homogenization in the randomly moving coordinates

$$(X_+^\varepsilon, t) = \left( x + \frac{\bar{b}}{\varepsilon} t + \frac{1}{\varepsilon} \int_0^t \beta \left( \frac{s}{\varepsilon^2} \right) ds, \, t \right).$$

To this end we substitute into (2.2) an ansatz of the form

$$(3.15) \quad \tilde{u}^\varepsilon = z^0(X_-^\varepsilon, t) + \varepsilon \chi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla z_0(X_-^\varepsilon, t) + \varepsilon^2 \psi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla \nabla z_0(X_-^\varepsilon, t)$$

with

$$X_-^\varepsilon = x - \frac{\bar{b}}{\varepsilon} t - \frac{1}{\varepsilon} \int_0^t \beta \left( \frac{s}{\varepsilon} \right) ds,$$

and collect like powers of $\varepsilon$ in the obtained equation. This yields

$$(3.16) \quad \begin{aligned} &\left( \frac{\partial}{\partial s} \chi^j \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) - \Delta_y \chi^j \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \frac{\partial}{\partial x_j} z^0(X_-^\varepsilon, t) \\ &+ \left( -\bar{b}^j - \beta^j \left( \frac{t}{\varepsilon^2} \right) \right) \frac{\partial}{\partial x_j} z^0(X_-^\varepsilon, t) = 0 \quad \text{in } Q_\varepsilon^T \end{aligned}$$

$$(3.17) \quad \left( n_j^\varepsilon + \frac{\partial}{\partial n^\varepsilon} \chi^j \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \frac{\partial}{\partial x_j} z^0(X_-^\varepsilon, t) = 0$$

$$\text{on } \{(x,t) \in \partial Q_\varepsilon^T \, : \, 0 < t < T\};$$

and

$$(3.18) \quad \begin{aligned} &\left( \frac{\partial}{\partial s} \psi^{ij} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) - \Delta_y \psi^{ij} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \frac{\partial^2}{\partial x_i \partial x_j} z^0(X_-^\varepsilon, t) \\ &+ \frac{\partial}{\partial t} z^0(X_-^\varepsilon, t) - \Delta z^0(X_-^\varepsilon, t) + \left( \bar{b}^j + \beta^j \left( \frac{t}{\varepsilon^2} \right) \right) \chi^i \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \frac{\partial^2}{\partial x_i \partial x_j} z^0(X_-^\varepsilon, t) \\ &+ 2 \frac{\partial}{\partial y_j} \chi^i \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \frac{\partial^2}{\partial x_i \partial x_j} z^0(X_-^\varepsilon, t) = 0, \end{aligned}$$

$$(3.19) \quad \left( n_i^\varepsilon \chi^j \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) + \frac{\partial}{\partial n^\varepsilon} \psi^{ij} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \frac{\partial^2}{\partial x_i \partial x_j} z^0(X_-^\varepsilon, t) = 0$$

$$\text{on } \{(x,t) \in \partial Q_\varepsilon^T \, : \, 0 < t < T\}.$$

By Lemma 3.3 and the definition of $\bar{b}$ and $\beta(s)$, the equation

$$(3.20) \quad \begin{aligned} \frac{\partial}{\partial s} \chi - \Delta \chi &= \bar{b} - \beta(s), \qquad y \in (T^n \setminus G(s)), \\ \frac{\partial}{\partial n} \chi &= -n(y, s) \quad \text{on } \partial G(s) \end{aligned}$$

has a stationary ergodic solution, which is uniquely defined by the normalization (3.12). Under this choice of $\chi$ the equation (3.16)–(3.17) is satisfied for any function $z^0$.

We now turn to problem (3.18)–(3.19). Considering fast and slow arguments as independent, and writing down an evident necessary condition of the existence of a stationary solution in (3.18)–(3.19), one has

$$
\frac{\partial}{\partial t} z^0(X_-^\varepsilon, t) - \Delta z^0(X_-^\varepsilon, t)
$$

$$
(3.21) \quad + \left\{ \mathbf{E} \int_{G(s)} \left( \bar{b}^j + \beta^j(s) \right) \chi^i(y,s) p(y,s) dy + 2\mathbf{E} \int_{G(s)} p(y,s) \frac{\partial}{\partial y_j} \chi^i(y,s) dy \right.
$$

$$
\left. + \mathbf{E} \int_{\partial G(s)} n_i(y,s) \chi^i(y,s) p(y,s) \mathcal{H}^{n-1}(dy) \right\} \frac{\partial^2}{\partial x_i \partial x_j} z^0(X_-^\varepsilon, t) = 0.
$$

The first integral in the figure brackets is equal to zero due to the normalization condition on $\chi$. Taking into account the definition of $p(y,s)$ and $\chi(y,s)$, one can show that

$$
\mathbf{E} \int_{G(s)} p(y,s) \left( \delta^{ij} + 2\frac{\partial}{\partial y_j} \chi^i(y,s) \right) dy + \mathbf{E} \int_{\partial G(s)} n_i(y,s) \chi^i(y,s) p(y,s) \mathcal{H}^{n-1}(dy)
$$

$$
= \mathbf{E} \int_{G(s)} (\mathbf{Id} + \nabla\chi) p(y,s) (\mathbf{Id} + \nabla\chi)^*.
$$

Therefore, the matrix of coefficients of (3.21) is positive definite and coincides with $\hat{a} - \frac{1}{2}\Lambda\Lambda^*$. Denote this matrix by $\bar{a}$.

We choose the function $z^0(x,t)$ to be a solution of the problem

$$
\frac{\partial}{\partial t} z^0 = \bar{a}^{ij} \frac{\partial^2}{\partial x_i \partial x_j} z^0, \qquad z^0(x,0) = u_0.
$$

Then the equation (3.18)–(3.19) takes the form

$$
(3.22) \quad \frac{\partial}{\partial s} \psi^{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - \Delta_y \psi^{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \left(\bar{b}^j + \beta^j\left(\frac{t}{\varepsilon^2}\right)\right) \chi^i\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)
$$

$$
+ 2\frac{\partial}{\partial y_j} \chi^i\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) - \lambda^{ij} = 0 \qquad \text{in } Q_\varepsilon^T,
$$

$$
n_i^\varepsilon \chi^j\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \frac{\partial}{\partial n^\varepsilon} \psi^{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) = 0 \qquad \text{on } \{(x,t) \in \partial Q_\varepsilon^T\},
$$

where

$$
\lambda^{ij} = \mathbf{E}\left\{ 2\int_{G(s)} p(y,s) \frac{\partial}{\partial y_j} \chi^i(y,s) dy + \int_{\partial G(s)} p(y,s) n(y,s) \mathcal{H}^{n-1}(dy) \right\}.
$$

The latter problem does not satisfy the conditions of Lemma 3.3, thus we cannot claim the existence of its stationary periodic in $y = \frac{x}{\varepsilon}$ solution. If we let now

$$
\mu^{ij}(s) = 2\int_{G(s)} p(y,s) \frac{\partial}{\partial y_j} \chi^i(y,s) dy + \int_{\partial G(s)} p(y,s) n(y,s) \mathcal{H}^{n-1}(dy) - \lambda^{ij},
$$

then $\mu(s)$ is a stationary ergodic bounded zero average process. The functions $\psi^{ij}$ are introduced as solutions of the following modified problem

$$\frac{\partial}{\partial s}\psi^{ij}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) - \Delta_y\psi^{ij}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \left(\bar{b}^j + \beta^j\left(\frac{t}{\varepsilon^2}\right)\right)\chi^i\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \mu^{ij}\left(\frac{t}{\varepsilon^2}\right)$$

$$+ 2\frac{\partial}{\partial y_j}\chi^i\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) - \lambda^{ij} = 0, \quad \left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \in \left\{\mathbb{T}^n \times \mathbb{R}^1 : \frac{x}{\varepsilon} \in G\left(\frac{t}{\varepsilon^2}\right)\right\},$$

$$n_i^\varepsilon\chi^j\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \frac{\partial}{\partial n^\varepsilon}\psi^{ij}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) = 0, \quad \frac{x}{\varepsilon} \in \partial G\left(\frac{t}{\varepsilon^2}\right).$$

(3.23)

By the definitions of $\mu^{ij}(s)$, $\lambda^{ij}$ and $\beta^j(s)$, we have

$$\int_{G(s)} p(y,s)\left(2\frac{\partial}{\partial y_j}\chi^i(y,s) + (\bar{b}^j + \beta^j(s))\chi^i(y,s) + \mu^{ij}(s) - \lambda^{ij}\right)dy$$

$$+ \int_{\partial G(s)} p(y,s)n(y,s)\mathcal{H}^{n-1}(dy) = 0,$$

Therefore, Lemma 3.3 applies and the last problem has a stationary ergodic periodic in $y$ matrix valued solution $\psi(y,s)$.

All the terms in the expression (3.15) are now defined. The estimate of the discrepancy $(u^\varepsilon - \tilde{u}^\varepsilon)$ is based on the following statements whose proof is similar to that of Proposition 3.1 in [2].

LEMMA 3.7. *A solution $v^\varepsilon$ of a Cauchy problem*

$$\frac{\partial}{\partial t}v^\varepsilon = \Delta v^\varepsilon + f(x,t), \qquad (x,t) \in Q_T^\varepsilon,$$

(3.24)

$$\frac{\partial}{\partial n_x^\varepsilon}v^\varepsilon = g(x,t) \qquad \text{on } \left\{x \in \varepsilon\partial G\left(\frac{t}{\varepsilon^2}\right), 0 < t < T\right\},$$

$$v^\varepsilon(x,0) = v_0(x)$$

*obeys the estimate*

$$\int_{Q_\varepsilon^T} |\nabla v^\varepsilon(x,t)|^2 dxdt + \sup_{0 \le t \le T}\int_{\mathbb{R}^n \setminus G(\frac{t}{\varepsilon^2})} |v^\varepsilon(x,t)|^2 dx$$

$$\le C(\|f\|_{L^2(Q_\varepsilon^T)}^2 + \|v_0\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon^{-1}\|g\|_{L^2(\partial Q_\varepsilon^T)}^2)$$

We should also estimate the contribution of the additional term $\mu^{ij}$ on the right-hand side of (3.23).

LEMMA 3.8. *Let $u_0$ be a $C_0^\infty$ function, and let $V^\varepsilon$ be a solution of the following Cauchy problem*

$$\frac{\partial}{\partial t}V^\varepsilon = \Delta V^\varepsilon + \mu^{ij}\left(\frac{t}{\varepsilon^2}\right)\frac{\partial}{\partial x_i\partial x_j}z^0(X_-^\varepsilon,t), \qquad (x,t) \in Q_T^\varepsilon,$$

(3.25)

$$\frac{\partial}{\partial n_x^\varepsilon}V^\varepsilon = 0 \quad \text{on } \left\{x \in \varepsilon\partial G\left(\frac{t}{\varepsilon^2}\right), 0 < t < T\right\},$$

$$V^\varepsilon(x,0) = 0.$$

*Then the expression*

$$\mathbf{E}\left( \int_{Q_\varepsilon^T} |\nabla V^\varepsilon(x,t)|^2 dx\, dt + \sup_{0 \le t \le T} \int_{\mathbb{R}^n \setminus G(\frac{t}{\varepsilon^2})} |V^\varepsilon(x,t)|^2\, dx \right)$$

*tends to zero, as $\varepsilon \to 0$.*

*Proof.* The function $\varepsilon F_{t/\varepsilon^2}^{-1}(\frac{x}{\varepsilon})$ with $F_t$ introduced in the beginning of section 2, maps $\{x \in \varepsilon \partial G\left(\frac{t}{\varepsilon^2}\right)\}$ onto $\mathbb{R}^n \setminus \varepsilon B$ for all $t$. Denote $\tilde{Q}_\varepsilon^T = (\mathbb{R}^n \setminus \varepsilon B) \times (0,T)$. In the coordinates $(\zeta, t) = (\varepsilon F_{t/\varepsilon^2}^{-1}(\frac{x}{\varepsilon}), t)$, problem (3.25) reads

$$\frac{\partial}{\partial t} V^\varepsilon = \mathcal{A}^\varepsilon V^\varepsilon + \mu\left(\frac{t}{\varepsilon^2}\right) \tilde{Z}_\varepsilon^0(x(\zeta,t),t) \quad \text{in } \tilde{Q}_\varepsilon^T,$$

(3.26)
$$\frac{\partial}{\partial n_\zeta^{a,\varepsilon}} V^\varepsilon = 0 \quad \text{on } \varepsilon \partial B \times (0,T),$$

$$V^\varepsilon(\zeta,0) = 0,$$

where $\tilde{Z}_\varepsilon^0(x(\zeta,t),t) = \dfrac{\partial}{\partial x_i \partial x_j} z^0(X_-^\varepsilon(x(\zeta,t),t))$,

$$\mathcal{A}^\varepsilon = a^{ij}\left(\frac{\zeta}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial^2}{\partial \zeta_i} \partial \zeta_j V^\varepsilon + \frac{1}{\varepsilon} b^i\left(\frac{\zeta}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial}{\partial \zeta_i} V^\varepsilon,$$

and $a(y,s)$ and $b(y,s)$ are defined by

$$a^{ij}(\zeta,t) = \sum_{k=1}^n \frac{\partial (F_t^{-1})^i}{\partial x_k} \frac{\partial (F_t^{-1})^j}{\partial x_k}, \qquad b^i(\zeta,t) = \sum_{k=1}^n \frac{\partial^2 (F_t^{-1})^i}{\partial x_k^2}.$$

Under our assumptions $z^0(x,t)$ is a $C^\infty$ Schwartz class function. Therefore,

$$|\tilde{Z}^0(x(\zeta,t),t) - \tilde{Z}^0(\zeta,t)| \le C\varepsilon,$$

and due to Lemma 3.7 one can replace the function $\tilde{Z}^0(X_-^\varepsilon(x(\zeta,t),t))$ on the right-hand side of (3.26) by $\tilde{Z}^0(X_-^\varepsilon(\zeta,t))$.

Denote $t_m^\varepsilon = m\varepsilon^{3/2}$. We represent $V^\varepsilon(x,t)$ as the sum

(3.27)
$$V^\varepsilon(x,t) = \sum_{m=0}^{[T/\varepsilon^{3/2}]} v_m^\varepsilon(x,t),$$

where $v_m^\varepsilon$ solves the problem

$$\frac{\partial}{\partial t} v_m^\varepsilon = \mathcal{A}^\varepsilon v_m^\varepsilon + \mu\left(\frac{t}{\varepsilon^2}\right) \tilde{Z}_\varepsilon^0(\zeta,t) \mathbf{1}_{t_m^\varepsilon \le t \le t_{m+1}^\varepsilon} \quad \text{in } \tilde{Q}_\varepsilon^T,$$

(3.28)
$$\frac{\partial}{\partial n_\zeta^{a,\varepsilon}} v_m^\varepsilon = 0 \quad \text{on } \varepsilon \partial B \times (0,T),$$

$$v_m^\varepsilon(\zeta,0) = 0.$$

Further analysis is based on the following statement.

PROPOSITION 3.9. *The relation holds*

(3.29)   $$v_m^\varepsilon(\zeta, t_{m+1}^\varepsilon) = \varepsilon^{3/2} \int_m^{m+1} \mu\left(\frac{s}{\varepsilon^{1/2}}\right) ds\, \tilde{Z}^0(X_-^\varepsilon(\zeta, t_m^\varepsilon)) + r^\varepsilon(\zeta, t_{m+1}^\varepsilon),$$

*where*

$$\|r^\varepsilon(\cdot, t^\varepsilon_{m+1})\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^2,$$

*and $C$ is a nonrandom constant.*

We will proof this proposition later on.

Now to complete the proof of Lemma 3.8 we notice that by the maximum principle

$$|v^\varepsilon_m(\zeta,t)| < \left|\varepsilon^{3/2}\int_m^{m+1}\mu\left(\frac{s}{\varepsilon^{1/2}}\right)ds\right||\tilde{Z}^0(X^\varepsilon_-(\zeta,t^\varepsilon_m))| + C\varepsilon^2$$

for all $t \ge t^\varepsilon_{m+1}$. This implies

$$\sup_{t\ge t^\varepsilon_{m+1}}\|v^\varepsilon_m(\cdot,t)\|_{L^\infty} < \left|\varepsilon^{3/2}\int_m^{m+1}\mu\left(\frac{s}{\varepsilon^{1/2}}\right)ds\right|\|\tilde{Z}^0\|_{L^\infty} + C\varepsilon^2.$$

Combining this with evident relations

$$v_m(\zeta,t) = 0 \quad \text{for} \quad t \le t^\varepsilon_m, \qquad |v_m(\zeta,t)| \le C\varepsilon^{3/2},$$

and taking into account (3.27) and the stationarity of $\mu(s)$, we get

$$\mathbf{E}\|V^\varepsilon\|_{L^\infty(Q^\varepsilon_T)} \le \|\tilde{Z}^0\|_{L^\infty}\mathbf{E}\left|\int_0^1\mu\left(\frac{s}{\varepsilon^{1/2}}\right)ds\right| + C\varepsilon^{1/2}.$$

The first term on the right-hand side tends to zero, as $\varepsilon \to 0$ by the Birkhoff ergodic theorem.

*Remark* 3.10. More careful analysis shows that $L^\infty$ norm of $V^\varepsilon$ vanishes a.s. as $\varepsilon \to 0$. We will not use this convergence.

Multiplying now (3.25) by $p\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)V^\varepsilon(x,t)$ and integrating the result over the set $Q^T_\varepsilon$, after multiple integration by parts we obtain

$$\int_{Q^T_\varepsilon}p\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)|\nabla V^\varepsilon(x,t)|^2dxdt + \int_{\mathbb{R}^n\setminus G(\frac{T}{\varepsilon^2})}p\left(\frac{x}{\varepsilon},\frac{T}{\varepsilon^2}\right)|V^\varepsilon(x,T)|^2dx$$

$$= \int_{Q^T_\varepsilon}p\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\mu\left(\frac{t}{\varepsilon^2}\right)\frac{\partial^2}{\partial x_i\partial x_j}z^0(X^\varepsilon_-,t)V^\varepsilon(x,t)dxdt.$$

The integral on the right-hand side here admits the bound

$$\left|\int_{Q^T_\varepsilon}p\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\mu\left(\frac{t}{\varepsilon^2}\right)\frac{\partial^2}{\partial x_i\partial x_j}z^0(X^\varepsilon_-,t)V^\varepsilon(x,t)dxdt\right| \le C\|V^\varepsilon\|_{L^\infty(Q^T_\varepsilon)},$$

and the desired estimate follows. $\square$

*Proof of Proposition* 3.9. Fix an arbitrary point $\zeta_0$ and represent the right-hand side in (3.28) as follows:

$$\mu\left(\frac{t}{\varepsilon^2}\right)\tilde{Z}^0(X^\varepsilon_-(\zeta,t)) = \mu\left(\frac{t}{\varepsilon^2}\right)\tilde{Z}^0(X^\varepsilon_-(\zeta_0,t^\varepsilon_m))$$

$$+ \mu\left(\frac{t}{\varepsilon^2}\right)\left(\tilde{Z}^0(X^\varepsilon_-(\zeta,t)) - \tilde{Z}^0(X^\varepsilon_-(\zeta_0,t^\varepsilon_m))\right).$$

Clearly, the first term on the right-hand side here gives us a solution which coincides with the first term on the right-hand side in (3.29). The remainder $r^\varepsilon(\zeta, t)$ satisfies the problem

(3.30)
$$\frac{\partial}{\partial t} r^\varepsilon = \mathcal{A}^\varepsilon r^\varepsilon + \mu\left(\frac{t}{\varepsilon^2}\right)(\tilde{Z}^0(\zeta, t) - \tilde{Z}^0(\zeta_0, t_m^\varepsilon))\mathbf{1}_{t_m^\varepsilon \leq t \leq t_{m+1}^\varepsilon} \quad \text{in } \tilde{Q}_\varepsilon^T,$$
$$\frac{\partial}{\partial n_\zeta^{a,\varepsilon}} r^\varepsilon = 0 \quad \text{on } \varepsilon \partial B \times (0, T),$$
$$r^\varepsilon(\zeta, 0) = 0.$$

We make use of the probabilistic representation of $r^\varepsilon(\zeta, t)$. Denote by $\xi_s^{\varepsilon, \zeta_0}$ the diffusion process in $\mathbb{R}^n \setminus \varepsilon B$ with reflection at $\varepsilon \partial B$, whose Kolmogorov equation is

$$\frac{\partial}{\partial s}\Psi = a^{ij}\left(\frac{\zeta}{\varepsilon}, \frac{t_{m+1}^\varepsilon - s}{\varepsilon^2}\right)\frac{\partial^2}{\partial \zeta_i \partial \zeta_j}\Psi + \frac{1}{\varepsilon}b^i\left(\frac{\zeta}{\varepsilon}, \frac{t_{m+1}^\varepsilon - s}{\varepsilon^2}\right)\frac{\partial}{\partial \zeta_i}\Psi;$$

the index $\zeta_0$ indicates the initial condition $\xi_0^{\varepsilon, \zeta_0} = \zeta_0$. Denote by $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$, respectively, the probability and the expectation related to $\xi$. Then

$$r^\varepsilon(\zeta_0, t_{m+1}^\varepsilon) = \tilde{\mathbf{E}}\int_0^{\varepsilon^{3/2}} \mu\left(\frac{t_{m+1}^\varepsilon - s}{\varepsilon^2}\right)\left(\tilde{Z}^0(\xi_s^\varepsilon, t_{m+1}^\varepsilon - s) - \tilde{Z}^0(\zeta_0, t_m^\varepsilon)\right)ds.$$

Since $a(\zeta, s)$ and $b(\zeta, s)$ are uniformly bounded, for sufficiently small $\varepsilon$ the inequality holds

$$\tilde{\mathbf{P}}\{\sup_{t_m^\varepsilon \leq s \leq t_{m+1}^\varepsilon} |\xi_s^{\varepsilon, \zeta_0} - \zeta_0| \geq \varepsilon^{1/2}\} < \varepsilon.$$

Therefore,

$$|r^\varepsilon(\zeta_0, t_{m+1}^\varepsilon)| \leq (C\varepsilon^{1/2} + C\varepsilon)\varepsilon^{3/2}$$

with a nonrandom constant $C$.  □

In order to justify the expansion (3.15), assume for a while that $u_0$ is $C_0^\infty$ function. Then, taking into account the definition of $\tilde{u}^\varepsilon$, one has by Lemma 3.7

$$\|u^\varepsilon - \tilde{u}^\varepsilon - V^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|u^\varepsilon - \tilde{u}^\varepsilon - V^\varepsilon\|_{C(0,T;L^2(\mathbb{R}^n))} \leq C\varepsilon$$

with a nonrandom constant $C$. Combining this bound with the statement of the last lemma yields

$$\lim_{\varepsilon \to 0} \mathbf{E}\left(\|u^\varepsilon - \tilde{u}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|u^\varepsilon - \tilde{u}^\varepsilon\|_{C(0,T;L^2(\mathbb{R}^n))}\right) = 0.$$

This implies the estimate

(3.31)
$$\lim_{\varepsilon \to 0} \mathbf{E}\|u^\varepsilon - z^0(X_-^\varepsilon, t)\|_{L^2((0,T)\times\mathbb{R}^n)} = 0.$$

In order to obtain this estimate for a general $u_0 \in L^2(\mathbb{R}^n)$ one can approximate $u_0$ by a sequence of $C_0^\infty$ functions and apply Lemma 3.7.

Notice that (3.31) is equivalent to the bound

(3.32) $\lim_{\varepsilon \to 0} \mathbf{E}\|u^\varepsilon\left(x + \frac{\bar{b}}{\varepsilon}t, t\right) - z^0\left(x - \frac{1}{\varepsilon}\int_0^t \beta\left(\frac{s}{\varepsilon^2}\right)ds, t\right)\|_{L^2((0,T)\times\mathbb{R}^n)} = 0.$

From Lemma 3.2 it follows that $z^0(x - \frac{1}{\varepsilon}\int_0^t \beta(\frac{s}{\varepsilon^2})ds, t)$ converges in law, as $\varepsilon \to 0$, in the space $L^2((0, T) \times \mathbb{R}^n)$ to the function $z^0(x - \Lambda W_t, t)$, where $W_t$ is a standard $n$-dimensional Wiener process. Together with (3.32) this implies that $u^\varepsilon(x + \frac{\bar{b}}{\varepsilon}t, t)$ converges in law in $L^2((0, T) \times \mathbb{R}^n)$ to $z^0(x - \Lambda W_t, t)$.

It remains to apply Ito's formula to the function $z^0(x - \Lambda W_t, t)$ in order to obtain the limit SPDE (3.13). Indeed,

$$dz_0(x - \Lambda W_t, t) = \frac{\partial}{\partial t}z_0(X, t)|_{X=x-\Lambda W_y} + \frac{1}{2}\Lambda\Lambda^*\frac{\partial^2}{\partial x_i \partial x_j}z(x - \Lambda W_t, t)$$

$$-\Lambda\frac{\partial}{\partial x}z_0(x - \Lambda W_t, t)dW_t = (\hat{A}z_0)(x - \Lambda W_t, t) + \Lambda\nabla z_0(x - \Lambda W_t, t)dW_t. \qquad \Box$$

## REFERENCES

[1] E. Acerbi, V. Chiado Piat, G. Dal Maso, and D. Percivale, *An extension theorem from connected sets, and homogenization in general periodic domains*, Nonlinear Anal., 18 (1992), pp. 481–496.

[2] D. Cioranescu and A. Piatnitski, *Homogenization in perforated domain with rapidly pulsing perforation*, ESAIM Control, Optim. Calc. Var., 9 (2003), pp. 461–483.

[3] F. Campillo, M. Kleptsyna, and A. Piatnitski, *Homogenization of random parabolic operator with large potential*, Stochastic Process. Appl., 93 (2001), pp. 57–85.

[4] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Appl. Math. Sci. 136, Springer-Verlag, New York, 1999.

[5] Y. Efendiev and A. Pankov, *Homogenization of nonlinear random parabolic operators*, submitted.

[6] V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik, *Averaging of parabolic operators*, Trudy Moskov. Mat. Obshch., 45 (1982), pp. 182–236.

[7] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.

[8] M. L. Kleptsyna and A. L. Piatnitski, *Homogenization of a random non-stationary convection-diffusion problem*, Russian Math. Surveys, 57 (2003), pp. 95–118.

[9] E. Pardoux and A. Piatnitski, *Homogenization of a nonlinear random parabolic partial differential equation*, Stochastic Process. Appl., 104 (2003), pp. 1–27.