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Averaging a transport equation with small diffusion and oscillating velocity

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SUMMARY

A complete asymptotic expansion is constructed for the transport equation with diffusion term small with respect to the convection. Error estimates are obtained by using matched asymptotic expansion technique and building all the boundary layer terms in time and in space, necessary for obtaining an accurate error estimate. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: asymptotic expansion; boundary layers; effective diffusion; dispersion

1. INTRODUCTION

Starting with the pioneering work of Aris [1] and Taylor [2] on flow through a tube, during the last decades there had been considerable discussion, in the engineering literature, for a better understanding of the effect of heterogeneities on dispersion in saturated porous media. This problem is relevant to a broad range of applications in chemical, civil, geological, mechanical, hydrological and petroleum engineering and was investigated by means of particles methods or volume averaging (see, for instance, References [3–5]).

One of the first rigorous mathematical investigation in this area [6] was made under the assumption that the convection field has zero mean value. Unfortunately, that assumption on the convection is not consistent with the model utilized for describing solute transport in porous media where the convection is coming from the Darcy law.

Our present approach is based on assuming two different scales, the first one L associated to the field scale and the second one l, associated to the local heterogeneity size, and on renormalizing the model by considering the ratio of these two scales $l/L=\varepsilon$. The local heterogeneity size (microscopic length scale) is much smaller than the characteristic field size (macroscopic scale) so that in the sequel ε is a small positive parameter, $\varepsilon \ll/1$.

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At the local level, we consider a linear convection–diffusion equation (see Equation (1)) describing, for instance, the transport of non-reacting non-adsorbing solutes in rigid saturated heterogeneous porous media. The linearity of the equation follows from the assumption that the mass density and the viscosity of the fluid are constant. The velocity field is given by Darcy's law (see Equation (2)) with the gravity term neglected. Then the equation for the pressure (see Equation (3)) is linear elliptic equation, completely decoupled from the convection–diffusion equation for the concentration of the solute (concentration equation). To simplify further, the turbulent diffusion is neglected and only the molecular diffusion is considered. In order to see the influence of local heterogeneities on the dispersion in the field scale transport equation, we assume that convection and diffusion have the same order at the local scale. Therefore, at the field scale, the diffusion is dominated by the convection introducing at the macroscopic scale a singular perturbation of order ε in the equation.

Our approach is based on the mathematical homogenization theory, as in [7–9], including the effect of boundary layers.

Since the main mechanism producing dispersion is known to come from the presence of a high number of local heterogeneities, for the sake of simplicity we are assuming spatial periodicity of the medium characteristics (molecular diffusion and rock permeability).

In Section 2, we introduce the equations describing the local level and provide the mathematical setting of the model.

In Section 3 we construct the asymptotic expansion for Darcy's velocity and give the error estimates in Theorem 3.3. This part being standard in the framework of homogenization theory, see References [7–11], we only describe briefly the procedure of constructing the asymptotic expansion including the boundary layer correctors. However, since we work in an unbounded domain we present details of the convergence proof.

In Section 4 the asymptotic expansion for the solution of the transport equation is developed. The expansion consists of the interior periodically oscillating terms plus time and spatial boundary layer correctors. It is shown that the leading term of the expansion satisfies the first-order equation which makes the choice of boundary conditions and construction of the boundary layers a rather non-standard problem.

In Section 5 the main results of this work are presented. We give an estimate of the error obtained by approximating the solution of the original problem by the sum of the first two leading terms in the asymptotic expansion.

Finally, in Section 6 we show that the homogenized solution satisfies an equation that involves a dispersion operating like a regularization of order ε for the limit first-order equation.

2. SETTING OF THE PROBLEM

For simplicity in the treatment of boundary effects we assume the domain Ω to be an $\mathbb{R}^d$ layer, i.e. $\Omega = \mathbb{R}^{d-1} \times (a,b)$ and for any $x \in \Omega$ we will use the notation $x = (x', x_d)$. We will also refer to $\mathbb{R}^{d-1} \times \{a\}$ and $\mathbb{R}^{d-1} \times \{b\}$ as the lower and the upper boundary of the layer Ω.

We are taking into account the presence of local heterogeneities by assuming the molecular diffusion tensor $\mathbf{D}$, the rock permeability tensor $\mathbf{K}$ and the rock porosity ϕ to be periodically oscillating, with period ε, $\varepsilon \ll 1$: $\mathbf{D} = \mathbf{D}(x/\varepsilon)$, $\mathbf{K} = \mathbf{K}(x/\varepsilon)$ and $\phi = \phi(x/\varepsilon)$. Hence the convection is driven by Darcy's velocity $\vec{q}^{\varepsilon}$ which then is fluctuating according to the permeability fluctuations.

In the transport equation, written at the field (or macroscopic) scale, the diffusion is assumed to be small compared to the convection. By choosing the characteristic convection time as the macroscopic characteristic time, we have then the diffusion of order ε in the field scale transport equation:

$$\phi\left(\frac{x}{\varepsilon}\right)\frac{\partial c^{\varepsilon}}{\partial t}+\vec{q}^{\varepsilon}\cdot\nabla c^{\varepsilon}=\varepsilon\,\mathrm{div}\left(\mathbf{D}\left(\frac{x}{\varepsilon}\right)\nabla c^{\varepsilon}\right)\quad \text{in } \Omega\times(0,T) \tag{1}$$

$$\vec{q}^{\varepsilon}=-\mathbf{K}\left(\frac{x}{\varepsilon}\right)\nabla P^{\varepsilon} \tag{2}$$

$$\mathrm{div}\,\vec{q}^{\varepsilon}=0\quad \text{in } \Omega \tag{3}$$

$$P^{\varepsilon}=P^{+},\quad c^{\varepsilon}=c_{+}\quad \text{for } x_d=a,\ t\in(0,T) \tag{4}$$

$$P^{\varepsilon}=P^{-},\ c^{\varepsilon}=c_{-}\quad \text{for } x_d=b,\ t\in(0,T) \tag{5}$$

$$c^{\varepsilon}=c_{\mathrm{init}}\quad \text{for } x\in\Omega,\ t=0 \tag{6}$$

Note that the pressure equation (3) is decoupled from the concentration equation (1) and that the diffusion tensor $\mathbf{D}$ is independent of the Darcy velocity. Then, in the sequel, we will consider first the asymptotic for (2)–(3) and later use it for the asymptotic related to the concentration.

Denoting the rescaled unit cell $Y=\{y\in(0,1)^d\subset\mathbb{R}^d\}$, where $y=x/\varepsilon$, we assume in (1)–(6) the following ellipticity and regularity conditions to hold: the porosity $\phi:\mathbb{R}^d\to\mathbb{R}^+$ is a strictly positive Y-periodic function and the diffusion $\mathbf{D}$ and permeability $\mathbf{K}$ are symmetric, Y-periodic matrix functions satisfying

$$\lambda|\xi|^2\leqslant\mathbf{D}(y)\xi\cdot\xi,\quad \mathbf{K}(y)\xi\cdot\xi\leqslant\Lambda|\xi|^2 \tag{7}$$

for all $\xi\in\mathbb{R}^d$, with $0<\lambda<\Lambda$;
there exists $\gamma>0$ such that

$$\mathbf{D},\mathbf{K}\in(C_{\mathrm{per}}^{1+\gamma}(\bar{Y}))^{d^2},\quad \phi\in C_{\mathrm{per}}^{\gamma}(\bar{Y}) \tag{8}$$

boundary and initial data (4)–(6) have at least the following regularity:

$$P^{+},P^{-}\in C^{2+\gamma}(\mathbb{R}^{d-1}) \tag{9}$$

$$c_{\mathrm{init}}\in C^{2+\gamma}(\bar{\Omega}),\quad c_{+},c_{-}\in C^{2+\gamma,1+\gamma/2}(\mathbb{R}^{d-1}\times[0,T]) \tag{10}$$

and verify the compatibility conditions of order one (see Reference [12, *Chapter*4]).

Under the above assumptions, problem (1)–(6) has a unique bounded solution $c^{\varepsilon}\in C^{2+\gamma,1+\gamma/2}(\bar{\Omega}\times[0,T])$, $P^{\varepsilon}\in C^{2+\gamma}(\bar{\Omega})$. In fact, for simplicity we will assume stronger compatibility conditions (see (54)–(56)) to hold. Further on, for simplicity we also assume that $(b-a)/\varepsilon$ is an integer.

3. ASYMPTOTIC EXPANSION OF DARCY'S VELOCITY

We denote $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)=(\alpha',\alpha_k)=(\alpha'',\alpha_{k-1},\alpha_k)\in\mathscr{I}^k$ with $\mathscr{I}=\{1,2,\ldots,d\}$. Then we denote by ∂_{α}^k or $\partial^k/\partial x^{\alpha}$ the kth order derivative $\partial/\partial x_{\alpha_1}\cdots\partial/\partial x_{\alpha_k}$ and for consistency we set

$\mathscr{I}^0 = \emptyset$. By $|\alpha|$ we denote the number of components of α. The mean value of a function f over the unit cell Y is denoted as usually by $\langle f \rangle$.

3.1. Structure of the expansion

The structure of Equations (1)–(6) and of the domain Ω suggests a microscopic periodic behaviour of the solution in the interior of the domain but perturbed in the vicinity of the Ω boundaries. This is why we seek an asymptotic expansion for the pressure P^ε as

$$P^\varepsilon(x) \approx P^0(x) + \varepsilon P^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 P^2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 P^3\left(x, \frac{x}{\varepsilon}\right) + \cdots \tag{11}$$

where the terms P^k, for $k \geqslant 1$, are (see for instance Reference [7]):

$$P^k\left(x, \frac{x}{\varepsilon}\right) = \sum_{i=0}^{k} \sum_{|\alpha|=i} \varphi_\alpha^i\left(\frac{x}{\varepsilon}\right) \partial_\alpha^i \bar{P}^{k-i}(x) \tag{12}$$

with φ_α^k being split in a Y-periodic part $\chi_\alpha^{k,\#}$ and two boundary layer parts $\chi_\alpha^{k,\pm}$:

$$\varphi_\alpha^k\left(\frac{x}{\varepsilon}\right) = \chi_\alpha^{k,\#}\left(\frac{x}{\varepsilon}\right) + \chi_\alpha^{k,+}\left(\frac{x'}{\varepsilon}, \frac{x_d - a}{\varepsilon}\right) + \chi_\alpha^{k,-}\left(\frac{x'}{\varepsilon}, \frac{x_d - b}{\varepsilon}\right), \quad \alpha \in \mathscr{I}^k, \ k \geqslant 1$$

For consistency in (12) for $k=0$ we have set $\chi^{0,\#}(y) = \chi_\emptyset^{0,\#}(y) \equiv 1$, $\chi^{0,\pm}(y) = \chi_\emptyset^{0,\pm}(y) \equiv 0$ and $P^0(x) = \bar{P}^0(x)$. If we introduce the new variables $\bar{y} = (y', \bar{\theta})$ and $\underline{y} = (y', \underline{\theta})$, $\bar{\theta} = (x_d - a)/\varepsilon$ and $\underline{\theta} = (x_d - b)/\varepsilon$, then the boundary layer functions $\chi_\alpha^{k,+}(y', \bar{\theta})$ and $\chi_\alpha^{k,-}(y', \underline{\theta})$ are Y'-periodic in the variable $y' = (y_1, \ldots, y_{d-1})$ and decay exponentially in the last variable (Figure 1). That is, if we define the cylindrical sets $\mathscr{Y}^+, \mathscr{Y}^- \subset \mathbb{R}^d$

$$\mathscr{Y}^+ = Y' \times (0, +\infty), \quad \mathscr{Y}^- = Y' \times (-\infty, 0) \quad \text{with } Y' = (0,1)^{d-1}$$

then

$$\chi_\alpha^{k,\pm} \colon \mathscr{Y}^\pm \to \mathbb{R}, \quad |\chi_\alpha^{k,+}(y', \bar{\theta})| \leqslant c_1 e^{-c_2 \bar{\theta}}, \quad |\chi_\alpha^{k,-}(y', \underline{\theta})| \leqslant c_1 e^{c_2 \underline{\theta}}, \quad c_1, c_2 > 0 \tag{13}$$

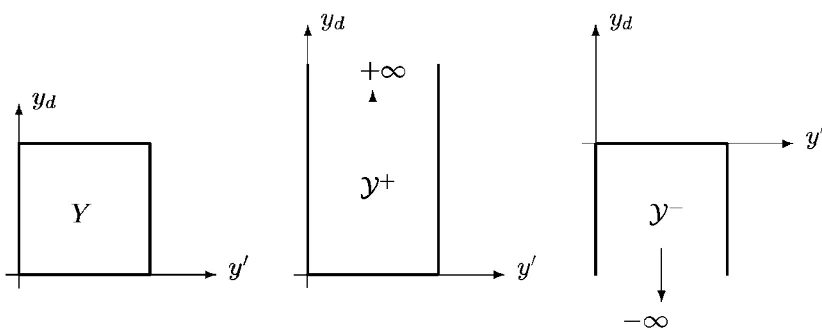

Figure 1.

The boundary layer functions $\chi_\alpha^{k,\pm}$ are introduced to cancel periodic oscillations in the term $P^k(x,x/\varepsilon)$ on the boundaries of the layer Ω since the boundary conditions in problem (1)–(6) do not oscillate. Therefore, we set

$$\chi_\alpha^{k,+}\left(\frac{x'}{\varepsilon},\frac{x_d-a}{\varepsilon}\right)\Big|_{x_d=a}=-\,\chi_\alpha^{k,\#}\left(\frac{x}{\varepsilon}\right)\Big|_{x_d=a}-v_\alpha^{k,+}\tag{14}$$

$$\chi_\alpha^{k,-}\left(\frac{x'}{\varepsilon},\frac{x_d-b}{\varepsilon}\right)\Big|_{x_d=b}=-\,\chi_\alpha^{k,\#}\left(\frac{x}{\varepsilon}\right)\Big|_{x_d=b}-v_\alpha^{k,-}\tag{15}$$

where the constants $v_\alpha^{k,\pm}$ will be determined so that $\chi_\alpha^{k,\pm}$ vanish at infinity. For the successive terms of expansion (11) the boundary conditions (4), (5) take the form $P^0=P^\pm$ on $x_d=a,b$ and $P^k=0$ on $x_d=a,b$ for $k\geqslant 1$, which leads to the following boundary conditions for $\bar{P}^k,\ k\geqslant 1$:

$$\bar{P}^k(x)=\sum_{i=0}^{k-1}\sum_{|\alpha|=k-i}v_\alpha^{k-i,\pm}\partial_\alpha^{k-i}\bar{P}^i(x)\quad\text{for } x_d=a,b$$

where $+$ and $-$ are taken on the lower and upper boundary. In view of (13), the terms $P^k(x,x/\varepsilon)$ satisfy the boundary conditions up to an exponentially small discrepancy, which is always neglected in further analysis.

Now with the above notations and definitions, by substituting expansion (11) into (2), we obtain the asymptotic expansion of the Darcy velocity

$$\vec{q}^{\,\varepsilon}\approx\vec{Q}^0+\varepsilon\vec{Q}^1+\varepsilon^2\vec{Q}^2+\cdots\tag{16}$$

and of its divergence

$$\begin{aligned}-\mathrm{div}\left(\mathbf{K}\left(\frac{x}{\varepsilon}\right)\nabla P^\varepsilon\right)\approx{}&\frac{1}{\varepsilon}\mathrm{div}_y\,\vec{Q}^0+\mathrm{div}_x\,\vec{Q}^0+\mathrm{div}_y\,\vec{Q}^1+\varepsilon(\mathrm{div}_x\,\vec{Q}^1+\mathrm{div}_y\,\vec{Q}^2)\\&+\cdots+\varepsilon^{k-1}(\mathrm{div}_x\,\vec{Q}^{k-1}+\mathrm{div}_y\,\vec{Q}^k)+\cdots\end{aligned}$$

This leads to a sequence of the equations in the product domain $\Omega\times Y\times\mathscr{Y}^+\times\mathscr{Y}^-$

$$\mathrm{div}_x\,\vec{Q}^{k-1}+\mathrm{div}_y\,\vec{Q}^k=0\tag{17}$$

$k=0,1,2,\ldots$, where $\vec{Q}^{-1}\equiv 0$. The Darcy velocity terms, for any $k\geqslant 0$, have the form

$$\vec{Q}^k=-\,\mathbf{K}\sum_{i=0}^{k}\sum_{|\alpha|=k+1-i}(\nabla_y\varphi_\alpha^{k+1-i}+\varphi_{\alpha'}^{k-i}\vec{e}_{\alpha_{k+1-i}})\partial_\alpha^{k+1-i}\bar{P}^i\tag{18}$$

and can be further decomposed in periodic and boundary layer parts

$$\vec{Q}^k=\vec{Q}^{k,\#}+\vec{Q}^{k,+}+\vec{Q}^{k,-}\tag{19}$$

all of them having the structure analogous to (18).

Grouping separately the periodic terms and the boundary layer terms, we arrive at a sequences of problems defining $\chi_\alpha^{k,\#}$ and $\chi_\alpha^{k,\pm}$. These problems are studied in the following two sections.

3.2. Local periodic problems

Separation of variables in the periodic equations leads to a sequence of auxiliary periodic problems in Y. Necessary conditions for the existence of the solution of these problems give the equations for $\bar{P}^k$. Since the calculations are standard (see Reference [7] for instance) we omit the details and give the auxiliary periodic problems. For any $\alpha\in\mathscr{I}^k$, $k\geqslant 1$

$$-\mathrm{div}_y(\mathbf{K}\nabla_y\chi_\alpha^{k,\#})=\mathscr{F}_\alpha^{k,\#} \text{ in } Y \tag{20_k}$$

In (20_k) $\mathscr{F}_\alpha^{1,\#}=\mathrm{div}_y(\mathbf{K}\vec{e}_\alpha))$ and for $\alpha\in\mathscr{I}^k$, $k\geqslant 2$:

$$\begin{aligned}\mathscr{F}_\alpha^{k,\#} &= \mathrm{div}_y(\mathbf{K}\vec{e}_{\alpha_k}\chi_{\alpha'}^{k-1,\#})+(\mathbf{K}(\nabla_y\chi_{\alpha'}^{k-1,\#}+\chi_{\alpha''}^{k-2,\#}\vec{e}_{\alpha_{k-1}})\\ &\quad -\langle\mathbf{K}(\nabla_y\chi_{\alpha'}^{k-1,\#}+\chi_{\alpha''}^{k-2,\#}\vec{e}_{\alpha_{k-1}})\rangle)\cdot\vec{e}_{\alpha_k}\end{aligned}$$

By standard arguments we have then:

Lemma 3.1
For any $k\geqslant 1$ and any $\alpha\in\mathscr{I}^k$ Equation (20_k) has a unique periodic solution with zero mean value, $\chi_\alpha^{k,\#}\in C_{\mathrm{per}}^{2+\gamma}(\bar{Y})$.

3.3. Boundary layers

In this section we consider the boundary layer equations. These equations are defined on the product domains $\Omega\times\mathscr{Y}^+$ and $\Omega\times\mathscr{Y}^-$. Since the two boundary layers have completely analogous structure, we only consider that associated to the lower boundary of Ω. After separation of slow and fast variables we arrive at the following auxiliary problems defined in $\mathscr{Y}^+$:

$$\begin{aligned}&-\mathrm{div}_{\bar{y}}(\mathbf{K}\nabla_{\bar{y}}\check{\chi}_\alpha^{k,+})=\mathscr{F}_\alpha^{k,+}, \quad \text{in } \mathscr{Y}^+\\ &\check{\chi}_\alpha^{k,+}(y',0)=-\chi_\alpha^{k,\#}(y',0) \quad \text{for } y'\in Y'\\ &y'\mapsto\check{\chi}_\alpha^{k,+}(y',\bar{\theta}) \quad \text{is } Y'\text{-periodic}\end{aligned} \tag{21_k}$$

where $\mathscr{F}_\alpha^{1,+}=0$ and for $\alpha\in\mathscr{I}^k$, $k\geqslant 2$:

$$\mathscr{F}_\alpha^{k,+}=\mathrm{div}_{\bar{y}}(\mathbf{K}\vec{e}_{\alpha_k}\chi_{\alpha'}^{k-1,+})+\mathbf{K}(\nabla_{\bar{y}}\chi_{\alpha'}^{k-1,+}+\chi_{\alpha''}^{k-2,+}\vec{e}_{\alpha_{k-1}})\cdot\vec{e}_{\alpha_k}$$

The boundary layer function $\chi_\alpha^{k,+}$ is then defined by $\chi_\alpha^{k,+}=\check{\chi}_\alpha^{k,+}-v_\alpha^{k,+}$ where the constants $v_\alpha^{k,+}$ are given by the following lemma.

Lemma 3.2
For any $k\geqslant 1$ problem (21_k) has a unique bounded solution $\check{\chi}_\alpha^{k,+}\in C^{2+\gamma}(\bar{\mathscr{Y}}^+)$. Moreover, there exist constants $v_\alpha^{k,+}$, $C_1>0$ and $c_2>0$ such that

$$\begin{aligned}|\check{\chi}_\alpha^{k,+}(y',\bar{\theta})-v_\alpha^{k,+}| &\leqslant C_1\mathrm{e}^{-c_2\bar{\theta}}, \quad |\nabla\check{\chi}_\alpha^{k,+}(y',\bar{\theta})|\leqslant C_1\mathrm{e}^{-c_2\bar{\theta}}\\ |\nabla^2\check{\chi}_\alpha^{k,+}(y',\bar{\theta})| &\leqslant C_1\mathrm{e}^{-c_2\bar{\theta}}, \quad [\nabla^2\check{\chi}_\alpha^{k,+}(\cdot,\bar{\theta})]_{\gamma;Y'}\leqslant C_1\mathrm{e}^{-c_2\bar{\theta}}\end{aligned}$$

for all $\alpha\in\mathscr{I}^k$ and $y\in\mathscr{Y}^+$.

Proof
The existence, uniqueness and exponential stabilization of the solution $\check{\chi}_\alpha^{k,+}$ is a consequence of Theorem 1 from Reference [13] (see also Reference [11]). Exponential decay of the first two derivatives then follows from the Schauder estimates. □

The boundary layer functions $\chi_\alpha^{k,-}$, $k \geqslant 1$, are constructed in the same way.

3.4. Effective equations

Associated to the local periodic problems (20_k) we define the effective matrix, for all $\alpha \in \mathscr{I}$:

$$\mathbf{K}^h \vec{e}_\alpha = \langle \mathbf{K}(\nabla_y \chi_\alpha^{1,\#} + \vec{e}_\alpha) \rangle$$

and the effective tensors, for all $\alpha \in \mathscr{I}^k$, $k \geqslant 2$:

$$\mathbf{N}_\alpha^k = \langle \mathbf{K}(\nabla_y \chi_\alpha^{k,\#} + \chi_{\alpha'}^{k-1,\#} \vec{e}_{\alpha_k}) \rangle$$

The matrix $\mathbf{K}^h$ is symmetric and positive definite (see Reference [7]).

For $k = 0$ we have

$$\begin{aligned} &\operatorname{div}_x(\mathbf{K}^h \nabla \bar{P}^0) = 0 \quad \text{in } \Omega \\ &\bar{P}^0 = P^+ \quad \text{on } x_d = a \\ &\bar{P}^0 = P^- \quad \text{on } x_d = b \end{aligned} \tag{22_0}$$

and for any $k \geqslant 1$, boundary conditions are set to cancel the non-oscillatory error appearing in the boundary layer correctors:

$$\begin{aligned} -\operatorname{div}_x(\mathbf{K}^h \nabla \bar{P}^k) &= \sum_{i=2}^{k+1} \operatorname{div}_x \left(\sum_{|\alpha|=i} \mathbf{N}_\alpha^i \partial_\alpha^i \bar{P}^{k+1-i} \right) \quad \text{in } \Omega \\ \bar{P}^k(x) &= \sum_{i=0}^{k-1} \sum_{|\alpha|=k-i} v_\alpha^{k-i,+} \partial_\alpha^{k-i} \bar{P}^i(x) \quad \text{on } x_d = a \\ \bar{P}^k(x) &= \sum_{i=0}^{k-1} \sum_{|\alpha|=k-i} v_\alpha^{k-i,-} \partial_\alpha^{k-i} \bar{P}^i(x) \quad \text{on } x_d = b \end{aligned} \tag{22_k}$$

Theorem 3.1
Assume for some $l \geqslant 1$, $P^+, P^- \in C^{l+2+\gamma}(\mathbb{R}^{d-1})$. Then problems (22_k) have unique bounded solutions $\bar{P}^k \in C^{l+2-k+\gamma}(\bar{\Omega})$, for $k = 0, 1, \ldots, l$.

The proof is standard and we omit it.

Corollary 3.1
Let $\vec{Q}^{l,\#}$ and $\vec{Q}^{l,\pm}$ be defined by (19), and assume for some $l \geqslant 0$ and $k \geqslant 1$, $P^{\pm} \in C^{k+l+1+\gamma}(\mathbb{R}^{d-1})$. Then $\vec{Q}^{l,\#} + \vec{Q}^{l,\pm} \in (C^k(\bar{\Omega}; C^{1+\gamma}(\bar{\mathscr{Y}}^{\pm})))^d$ and $\vec{Q}^{l,\pm}$ satisfy the uniform bounds

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial y^\beta} \vec{Q}^{l,\pm}(x, y) \right| \leqslant C_1 \mathrm{e}^{-c_2 |y_d|}, \quad C_1, c_2 > 0$$

for all $|\alpha| \leqslant k$ and $|\beta| \leqslant 1$.

3.5. Error estimates

Let us now assume that the first $k+1$ terms in expansion (11) are constructed. Then we denote

$$p_\varepsilon^{k+1}(x) = P^0(x) + \varepsilon P^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 P^2\left(x, \frac{x}{\varepsilon}\right) + \cdots + \varepsilon^{k+1} P^{k+1}\left(x, \frac{x}{\varepsilon}\right)$$

The velocity field $\vec{Q}^{k+1/2,\varepsilon}(x) = -\mathbf{K}(x/\varepsilon)\nabla p_\varepsilon^{k+1}(x)$ is then given by

$$\vec{Q}^{k+1/2,\varepsilon} = \vec{Q}^0 + \varepsilon \vec{Q}^1 + \varepsilon^2 \vec{Q}^2 + \cdots + \varepsilon^k \vec{Q}^k + \varepsilon^{k+1} \vec{Q}^{k+1/2} \tag{23}$$

where for $j = 0, 1, \ldots, k$

$$\vec{Q}^j = -\mathbf{K}(y)(\nabla_y P^{j+1} + \nabla_x P^j) \quad \text{and } \vec{Q}^{k+1/2} = -\mathbf{K}(y)\nabla_x P^{k+1} \tag{24}$$

The discrepancy $u_\varepsilon^{k+1} = P^\varepsilon - p_\varepsilon^{k+1}$ is the solution of the problem

$$\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon^{k+1}\right) = \varepsilon^k H_\varepsilon(x) \quad \text{in } \Omega \tag{25}$$

$$u_\varepsilon^{k+1}(x) = 0 \quad \text{on } x_d = a, b \tag{26}$$

where

$$H_\varepsilon(x) = \operatorname{div}_x \vec{Q}^k\left(x, \frac{x}{\varepsilon}\right) + \operatorname{div}_y \vec{Q}^{k+1/2}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \operatorname{div}_x \vec{Q}^{k+1/2}\left(x, \frac{x}{\varepsilon}\right)$$

From Lemmas 3.1 and 3.2 and Theorem 3.1 we obtain the following uniform estimate:

$$|H_\varepsilon|_{0;\Omega} + \varepsilon^\gamma [H_\varepsilon]_{\gamma;\Omega} \leqslant C \tag{27}$$

Theorem 3.2
Let u_ε^{k+1} be the $k+1$ order discrepancy term, as defined in (25)–(26), and assume that for some $k \geqslant 1$

$$P^+, P^- \in C^{k+5+\gamma}(\mathbb{R}^{d-1}) \tag{28}$$

Then there are constants $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds

$$|u_\varepsilon^{k+1}|_{0;\Omega} \leqslant C \varepsilon^{k+2} \tag{29}$$

Proof
Using the fact that problem (25), (26) is well posed and that the right-hand side H_ε is uniformly bounded we get immediately the uniform estimate

$$|u_\varepsilon^{k+1}|_{0;\Omega} \leqslant C\varepsilon^k$$

Estimate (29) is a consequence of a special structure of the problem. With the assumed regularity (28) we can construct two further terms in the asymptotic expansion (11) and use p_ε^{k+3} to obtain better uniform norm estimate.

In the rest of the proof we deal with the well posedness of problem (25), (26). First we construct a barrier function as follows:

$$v^\varepsilon\left(x, \frac{x}{\varepsilon}\right) = v_0(x_d) + \varepsilon \sum_{|\alpha|=1} \chi_\alpha^{1,\#}\left(\frac{x}{\varepsilon}\right) \partial_\alpha v_0(x_d) + \varepsilon^2 \sum_{|\alpha|=2} \chi_\alpha^{2,\#}\left(\frac{x}{\varepsilon}\right) \partial_\alpha^2 v_0(x_d)$$

where v_0 is a solution of the problem

$$\operatorname{div}(\mathbf{K}^h \nabla v_0) = -1 \quad \text{in } \Omega$$

$$v_0|_{x_d = a} = v_0|_{x_d = b} = 1$$

Then $v^\varepsilon(x, x/\varepsilon)$ satisfies

$$\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\varepsilon}\right) \nabla v^\varepsilon\right) = -1 + \varepsilon r^\varepsilon\left(x_d, \frac{x}{\varepsilon}\right) \quad \text{in } \Omega$$

$$v^\varepsilon|_{x_d=a,b} = 1 + \varepsilon g^\varepsilon\left(x_d, \frac{x}{\varepsilon}\right)$$

where r^ε and g^ε are bounded functions and therefore, for ε sufficiently small, we have

$$\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\varepsilon}\right) \nabla v^\varepsilon\right) \leqslant -\frac{1}{2} \quad \text{in } \Omega$$

$$v^\varepsilon|_{x_d=a,b} \geqslant \tfrac{1}{2}$$

We will now show that the solution of problem (25), (26) is uniformly bounded in Ω as soon as the right-hand side is uniformly bounded. Since we cannot apply directly the maximum principle in an unbounded domain we will consider problem (25), (26) in a cylinder $\Omega_n = \Omega \cap \{|x'| < n\}$, for large n, with zero boundary condition on the surface $|x'| = n$. The solution of this problem in Ω_n will be denoted by $u_{\varepsilon,n}$. Let C be a bound on the right-hand side in (25). Then

$$\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\varepsilon}\right) \nabla(2Cv^\varepsilon - u_{\varepsilon,n})\right) \leqslant 0 \quad \text{in } \Omega_n$$

$$2Cv^\varepsilon - u_{\varepsilon,n} \geqslant 0 \quad \text{on } x_d = a, b$$

$$2Cv^\varepsilon - u_{\varepsilon,n} = 2Cv^\varepsilon \quad \text{on } |x'| = n$$

and we apply the maximum principle to get

$$2Cv^{\varepsilon}(x) - u_{\varepsilon,n}(x) \geqslant \min\left\{0, 2C\inf_{\Omega} v^{\varepsilon}\right\}$$

and therefore $u_{\varepsilon,n}(x) \leqslant 2C|v^{\varepsilon}|_{0;\Omega}$. Since v^{ε} is uniformly bounded by C_1 for sufficiently small ε, we obtain $u_{\varepsilon,n}(x) \leqslant C_1 C$. By similar construction we obtain uniform lower bound, leading finally to the estimate $|u_{\varepsilon,n}|_{0;\Omega} \leqslant C_1 C$. Applying now the Schauder estimates we can pass to the limit as $n \to \infty$ and obtain the same estimate for u_{ε} in Ω. That proves the well posedness of the problem. □

By means of the change of variable $y = x/\varepsilon$ and the change of function $U_{\varepsilon}^{k+1}(y) = u_{\varepsilon}^{k+1}(\varepsilon y)$ from (25) we obtain the equation in the rescaled domain $\Omega_{\varepsilon} = \mathbb{R}^{d-1} \times (a/\varepsilon, b/\varepsilon)$:

$$\sum_{i,j=1}^{d} k_{i,j}(y) \frac{\partial^2 U_{\varepsilon}^{k+1}}{\partial y_i \partial y_j} + \sum_{i,j=1}^{d} \frac{\partial k_{i,j}}{\partial y_i}(y) \frac{\partial U_{\varepsilon}^{k+1}}{\partial y_j} = \varepsilon^{k+2} H_{\varepsilon}(\varepsilon y) \tag{30}$$

with homogeneous boundary conditions. The rescaling does not change uniform estimate (29) and therefore $|U_{\varepsilon}^{k+1}|_{0;\Omega_{\varepsilon}} \leqslant C\varepsilon^{k+2}$. Applying interior and boundary Schauder estimates (see Reference [14]) to Equation (30) we obtain

$$|U_{\varepsilon}^{k+1}|_{2+\gamma;\Omega_{\varepsilon}} \leqslant C(\varepsilon^{k+2}|H_{\varepsilon}(\varepsilon\cdot)|_{\gamma;\Omega_{\varepsilon}} + |U_{\varepsilon}^{k+1}|_{0;\Omega_{\varepsilon}}) \leqslant C\varepsilon^{k+2}$$

where we used (27). The constant in the Schauder estimates only depends on the differential operator and thus is independent of ε. In particular, we obtain the estimate

$$|\vec{q}^{\varepsilon}(\varepsilon\cdot) - \vec{Q}^{k+1/2,\varepsilon}(\varepsilon\cdot)|_{1+\gamma;\Omega_{\varepsilon}} \leqslant C\varepsilon^{k+1}$$

and since the last term $\vec{Q}^{k+1/2}$ in (23) is uniformly bounded we obtain

Theorem 3.3
Let

$$\vec{Q}^{k,\varepsilon} = \vec{Q}^0 + \varepsilon\vec{Q}^1 + \varepsilon^2\vec{Q}^2 + \cdots + \varepsilon^k\vec{Q}^k$$

where $\vec{Q}^i$ are defined by (18) and assume (28) is fulfilled for some $k \geqslant 1$. Let $\vec{q}^{\varepsilon}$ be Darcy's velocity solution of Equations (1)–(6). Then, for sufficiently small ε, $\vec{q}^{\varepsilon}$ can be approximated by $\vec{Q}^{k,\varepsilon}$ with the following accuracy:

$$|\vec{q}^{\varepsilon}(\varepsilon\cdot) - \vec{Q}^{k,\varepsilon}(\varepsilon\cdot)|_{1+\gamma;\Omega_{\varepsilon}} \leqslant C\varepsilon^{k+1}$$

Moreover, in the domain Ω, the discrepancy for the Darcy velocity is given by

$$|\vec{q}^{\varepsilon} - \vec{Q}^{k,\varepsilon}|_{0;\Omega} + \varepsilon|\nabla(\vec{q}^{\varepsilon} - \vec{Q}^{k,\varepsilon})|_{0;\Omega} + \varepsilon^{1+\gamma}[\nabla(\vec{q}^{\varepsilon} - \vec{Q}^{k,\varepsilon})]_{\gamma;\Omega} \leqslant C\varepsilon^{k+1}$$

4. ASYMPTOTIC EXPANSION FOR CONCENTRATION

We are seeking formal asymptotic expansion for the concentration c^{ε}, knowing that the Darcy velocity $\vec{q}^{\varepsilon}$ has an asymptotic expansion of the form (16), where each term $\vec{Q}^k$, $k = 0, 1, \ldots,$

is decomposed in periodic and boundary-layer parts as in (19). From the structure of problem (1)–(6) and from the asymptotic expansion for the Darcy velocity we then postulate an expansion for the concentration with terms for the inner microscopic periodic behaviour and boundary layer terms for both the effects of the two boundaries a and b of Ω and for the initial time boundary.

4.1. Structure of the expansion

We write down the asymptotic expansion for the concentration in a very general form

$$c^{\varepsilon}(x,t) \approx c^0\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) + \varepsilon c^1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) + \varepsilon^2 c^2\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) + \cdots \tag{31}$$

where each $c^k = c^k(x,t,y,\tau)$, for $k=0,1,2,\ldots$, is decomposed into the sum of a periodic term, two spatial boundary layer terms and a time boundary layer term as follows:

$$\begin{aligned} c^k\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) &= c^k_{\#}\left(x,t,\frac{x}{\varepsilon}\right) + c^k_{+}\left(x',t,\frac{x'}{\varepsilon},\frac{x_d-a}{\varepsilon}\right) \\ &\quad + c^k_{-}\left(x',t,\frac{x'}{\varepsilon},\frac{x_d-b}{\varepsilon}\right) + c^k_{\text{in}}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) \end{aligned} \tag{32}$$

The periodic term $c^k_{\#}(x,t,y)$ and the initial time boundary layer term $c^k_{\text{in}}(x,t,y,\tau)$ are Y-periodic functions in y, the function $\tau \mapsto c^k_{\text{in}}(x,t,y,\tau)$ decays exponentially as $\tau \to +\infty$. The spatial boundary layer terms $c^k_{+}(x',t,\bar{y})$, $\bar{y} \in \mathscr{Y}^+$, and $c^k_{-}(x',t,\underline{y})$, $\underline{y} \in \mathscr{Y}^-$, are Y'-periodic functions in y'. The functions $\bar{\theta} \mapsto c^k_{+}(x',t,y',\bar{\theta})$ and $\underline{\theta} \mapsto c^k_{-}(x',t,y',\underline{\theta})$ decay exponentially as $\bar{\theta}$ and $\underline{\theta}$ tends to $+\infty$ and $-\infty$, respectively. All the details of the structure of the asymptotic expansion terms and corresponding error estimates are studied in the next section.

Now we plug series (16) and (31) in Equation (1) and then collect the terms of the same order of ε. This gives a sequence of equations defining successive terms in expansion (31).

Considering the terms of order $k-1$ we get the equations for the periodic terms $c^k_{\#}$ on the unit cell Y, with periodic boundary conditions, and they read

$$-\text{div}_y(\mathbf{D}\nabla_y c^k_{\#}) + \vec{Q}^{0,\#} \cdot \nabla_y c^k_{\#} = \mathscr{F}^k_{\#} \tag{33}$$

In (33), the right-hand side is $\mathscr{F}^0_{\#} = 0$ and for $k \geqslant 1$, $\mathscr{F}^k_{\#}$ is given by

$$\begin{aligned} \mathscr{F}^k_{\#} &= -\phi \frac{\partial c^{k-1}_{\#}}{\partial t} + \text{div}_y(\mathbf{D}\nabla_x c^{k-1}_{\#}) + \text{div}_x(\mathbf{D}(\nabla_y c^{k-1}_{\#} + \nabla_x c^{k-2}_{\#})) \\ &\quad - \vec{Q}^{0,\#} \cdot \nabla_x c^{k-1}_{\#} - \sum_{j=1}^{k} \vec{Q}^{j,\#} \cdot (\nabla_y c^{k-j}_{\#} + \nabla_x c^{k-j-1}_{\#}) \end{aligned} \tag{34}$$

where we set $c^{-1}_{\#} \equiv 0$. The variables x and t appear as parameters in Equations (33) and (34).

We will now proceed with the spatial and time boundary layer equations. The former equations are defined in the domains $\mathscr{Y}^+$ and $\mathscr{Y}^-$, respectively, while the latter ones in the cylinder $Y \times (0,\infty)$.

First we introduce the following differential operators:

$$\mathscr{A}^{+} = -\mathrm{div}_y(\mathbf{D}(y)\nabla_y) + (\vec{Q}^{0,\#}(x,y) + \vec{Q}^{0,+}(x,y))|_{x_d=a} \cdot \nabla_y \tag{35}$$

$$\mathscr{A}^{-} = -\mathrm{div}_y(\mathbf{D}(y)\nabla_y) + (\vec{Q}^{0,\#}(x,y) + \vec{Q}^{0,-}(x,y))|_{x_d=b} \cdot \nabla_y \tag{36}$$

and state the spatial boundary layer equations for $k \geqslant 0$:

$$\mathscr{A}^{\pm} c_{\pm}^{k} = \mathscr{F}_{\pm}^{k} \tag{37}$$

In (37), $\mathscr{F}_{\pm}^{0} = 0$ for $k = 0$ but to compute $\mathscr{F}_{\pm}^{k}$ for $k \geqslant 1$, we need to develop in the Taylor series the periodic terms and the boundary layer Darcy's velocity terms in the vicinity of $x_d = a$ or b. Then, denoting

$$\vec{Q}^{l,\#}_{(k)} = \frac{\partial^k}{\partial x_d^k}\vec{Q}^{l,\#}, \quad \vec{Q}^{l,\pm}_{(k)} = \frac{\partial^k}{\partial x_d^k}\vec{Q}^{l,\pm}, \quad c^{l}_{\#(k)} = \frac{\partial^k}{\partial x_d^k}c^l_{\#}$$

on the boundary $x_d = a$ we have

$$\vec{Q}^{l,\#} = \vec{Q}^{l,\#}\Big|_{x_d=a} + \frac{\partial}{\partial x_d}\vec{Q}^{l,\#}\Big|_{x_d=a}(x_d - a) + \frac{1}{2!}\frac{\partial^2}{\partial x_d^2}\vec{Q}^{l,\#}\Big|_{x_d=a}(x_d - a)^2 + \cdots$$

$$= \vec{Q}^{l,\#}|_{x_d=a} + \varepsilon\vec{Q}^{l,\#}_{(1)}|_{x_d=a}\bar{\theta} + \varepsilon^2\frac{1}{2!}\vec{Q}^{l,\#}_{(2)}\Big|_{x_d=a}\bar{\theta}^2 + \cdots$$

and after direct calculations, we get

$$\begin{aligned}
\mathscr{F}_{+}^{k} = &-\phi\frac{\partial c_{+}^{k-1}}{\partial t} + \mathrm{div}_y(\mathbf{D}\nabla_x c_{+}^{k-1}) + \mathrm{div}_x(\mathbf{D}(\nabla_y c_{+}^{k-1} + \nabla_x c_{+}^{k-2}))\\
&-(\vec{Q}^{0,\#} + \vec{Q}^{0,+})|_{x_d=a}\cdot\nabla_x c_{+}^{k-1}\\
&-\sum_{j=0}^{k-1}\frac{\bar{\theta}^{k-j}}{(k-j)!}(\vec{Q}^{0,\#}_{(k-j)} + \vec{Q}^{0,+}_{(k-j)})|_{x_d=a}\cdot(\nabla_y c_{+}^{j} + \nabla_x c_{+}^{j-1})\\
&-\sum_{j=1}^{k}\sum_{i=1}^{j}\frac{\bar{\theta}^{k-j}}{(k-j)!}(\vec{Q}^{i,\#}_{(k-j)} + \vec{Q}^{i,+}_{(k-j)})|_{x_d=a}\cdot(\nabla_y c_{+}^{j-i} + \nabla_x c_{+}^{j-i-1})\\
&-\sum_{j=1}^{k}\sum_{i=0}^{j-1}\sum_{l=0}^{k-j}\frac{\bar{\theta}^{k-j-l}}{(k-j-l)!\,l!}\vec{Q}^{i,+}_{(k-j-l)}\cdot(\nabla_y(\bar{\theta}^l c^{j-i}_{\#(l)}) + \nabla_x(\bar{\theta}^l c^{j-i-1}_{\#(l)}))|_{x_d=a}
\end{aligned} \tag{38}$$

with $c_{+}^{-1} \equiv 0$.

For $\mathscr{F}_{-}^{k}$, very similar expressions are obtained in the neighborhood of $x_d = b$, involving c_{-}^{l}, $\vec{Q}^{l,-}$ and $\underline{\theta}$.

Boundary conditions for $c_{\pm}^k$ will be specified later on to compensate periodic oscillations at the boundary of Ω. As in the case of the Darcy velocity, we always neglect exponentially small terms.

Now we proceed with the initial time boundary layer equations. By introducing the differential operator $\mathscr{A}^{\text{in}}$

$$\mathscr{A}^{\text{in}} = \phi(y)\frac{\partial}{\partial\tau} + \vec{Q}^{0,\#}(x,y)\cdot\nabla_y - \operatorname{div}_y(\mathbf{D}(y)\nabla_y) \tag{39}$$

we get the initial time boundary layer equations for $k \geqslant 0$:

$$\mathscr{A}^{\text{in}} c_{\text{in}}^k = \mathscr{F}_{\text{in}}^k \tag{40}$$

The right-hand side in (40) for $k \geqslant 1$ reads

$$\begin{aligned}\mathscr{F}_{\text{in}}^k = &-\phi\frac{\partial c_{\text{in}}^{k-1}}{\partial\tau} + \operatorname{div}_y(\mathbf{D}\nabla_x c_{\text{in}}^{k-1}) + \operatorname{div}_x(\mathbf{D}(\nabla_y c_{\text{in}}^{k-1} + \nabla_x c_{\text{in}}^{k-2}))\\ &-\vec{Q}^{0,\#}\cdot\nabla_x c_{\text{in}}^{k-1} - \sum_{j=1}^{k}\vec{Q}^{j,\#}\cdot(\nabla_y c_{\text{in}}^{k-j} + \nabla_x c_{\text{in}}^{k-j-1})\end{aligned} \tag{41}$$

with $c_{\text{in}}^{-1} \equiv 0$, and $\mathscr{F}_{\text{in}}^0 = 0$.

On the right-hand side of the initial layer equations (40) and (41) we did not take into account the following non-periodic terms

$$-\sum_{j=0}^{k}(\vec{Q}^{j,+} + \vec{Q}^{j,-})\cdot(\nabla_y c_{\text{in}}^{k-j} + \nabla_x c_{\text{in}}^{k-j-1}) \tag{42}$$

arising in the vicinity of the 'corner' sets $\{(x,t)\colon\ x_d = a,b;\ \ t=0\}$. It will be shown that under suitable compatibility conditions (see (54)–(56)) these terms are of order $O(\varepsilon^l)$, with sufficiently large l. Moreover in (40), (41) we will choose later in Section 4.4 the periodic initial conditions in order to compensate spatial periodic oscillations at $t=0$. The existence of the initial time boundary layers c_{in}^k defined in (40) is now given by applying the following standard result:

Lemma 4.1
Let $g \in C(\bar{\Omega}; C^{2+\gamma}(\bar{Y}))$ *and* $f \in C(\bar{\Omega}; C^{\gamma}(\bar{Y}\times[0,+\infty)))$ *be* Y*-periodic functions and let* f *satisfy uniform bound*

$$|f(x,y,\tau)| \leqslant C_1 \mathrm{e}^{-c_2\tau}, \quad C_1, c_2 > 0$$

Then, with $\mathscr{A}^{\text{in}}$ defined in (39), the problem

$$\begin{gathered}\mathscr{A}^{\text{in}} u = f(x,y,\tau) \quad \text{in } Y\times(0,\infty)\\ u(x,y,\tau) = g(x,y) \quad \text{at } \tau = 0\\ y \mapsto u(x,y,\tau) \quad \text{is } Y\text{-periodic}\end{gathered}$$

has a unique solution $u \in C(\bar{\Omega}; C^{2+\gamma}(\bar{Y}\times[0,\infty)))$. Moreover, there exists a function $\mu_{\rm in}(x)$, $\mu_{\rm in}\in C(\bar{\Omega})$, such that

$$|u(x,y,\tau)-\mu_{\rm in}(x)| \leqslant C_3 \mathrm{e}^{-c_4\tau},$$

$$|\nabla_y u(x,y,\tau)| \leqslant C_3 \mathrm{e}^{-c_4\tau}, \qquad |\nabla_y^2 u(x,y,\tau)| \leqslant C_3 \mathrm{e}^{-c_4\tau}$$

The constants $C_3, c_4>0$ are independent of x, y and τ.

By applying Lemma 4.1 recursively to Equations (40) and (41) we get at any order k

$$|c^k_{\rm in}(x,y,\tau)-\mu^k_{\rm in}(x)| \leqslant C_3 \mathrm{e}^{-c_4\tau}$$

$$|\nabla_y c^k_{\rm in}(x,y,\tau)| \leqslant C_3 \mathrm{e}^{-c_4\tau}, \quad |\nabla_y^2 c^k_{\rm in}(x,y,\tau)| \leqslant C_3 \mathrm{e}^{-c_4\tau}$$

4.2. Spatial boundary layers

In order to prove the solvability of the forthcoming effective equations (57) and (71) and boundary layer equations (37) we simplify the hypothesis on the effective behaviour of the Darcy flow by assuming existence of a constant $D>0$, such that

$$\langle \vec{Q}^{0,\#}\rangle \cdot \vec{e}_d \leqslant -D<0 \quad \text{in } \bar{\Omega} \tag{43}$$

With this additional assumption we can now formulate the two following lemmas which state existence and the rate of decay for the boundary layer problems (37) and (38).

Lemma 4.2
Consider the problem

$$\mathscr{A}^+ u = f^+(x',t,\bar{y}) \quad \text{in } \mathscr{Y}^+ \tag{44}$$

$$u(x',t,y',\bar{\theta}) = g(x',t,y') \quad \text{on } \bar{\theta}=0 \tag{45}$$

$$y' \mapsto u(x',t,y',\bar{\theta}) \quad \text{is } Y'\text{-periodic} \tag{46}$$

with $\mathscr{A}^+$ defined in (35), and assume that

$$f^+ \in C(\mathbb{R}^{d-1}\times[0,T]; C^1(\bar{\mathscr{Y}}^+)), \quad g \in C(\mathbb{R}^{d-1}\times[0,T]; C^{2+\gamma}(\bar{\mathscr{Y}}^+))$$

are Y'-periodic functions and that f^+ satisfies uniform bounds

$$|f^+(x',t,\bar{y})|, \quad |\nabla_y f^+(x',t,\bar{y})| \leqslant C_1 \mathrm{e}^{-c_2|\bar{\theta}|}, \quad C_1, c_2>0$$

Then, under condition (43) there exists a unique solution $u\in C(\mathbb{R}^{d-1}\times[0,T]; C^{2+\gamma}(\overline{\mathscr{Y}^+}))$ of problem (44)–(46) in the class of functions that decay exponentially as $\bar{\theta}\to\infty$. More precisely, there are constants $C_3>0$, $c_4>0$, independent of x' and t, such that

$$\begin{aligned} &|u(x',t,\bar{y})| \leqslant C_3 \mathrm{e}^{-c_4|\bar{\theta}|} \\ &|\nabla_y u(x',t,\bar{y})| \leqslant C_3 \mathrm{e}^{-c_4|\bar{\theta}|}, \quad |\nabla_y^2 u(x',t,\bar{y})| \leqslant C_3 \mathrm{e}^{-c_4|\bar{\theta}|} \end{aligned} \tag{47}$$

If in addition

$$f^+ \in C^1(\mathbb{R}^{d-1} \times [0,T];\ C^1(\overline{\mathscr{Y}^+})), \quad g \in C^1(\mathbb{R}^{d-1} \times [0,T];\ C^{2+\gamma}(\overline{\mathscr{Y}^+}))$$

and the derivatives of f^+ with respect to x' and t decay exponentially as $\bar{\theta} \to \infty$, uniformly in x' and t, then $u \in C^1(\mathbb{R}^{d-1} \times [0,T]; C^{2+\gamma}(\overline{\mathscr{Y}^+}))$ and bounds (47) hold for the derivatives of u with respect to x' and t.

Proof
By (43) we have

$$\lim_{N\to\infty} \int_N^{N+1} \int_{Y \times \{\bar{\theta}\}} (Q_d^{0,\#} + Q_d^{0,+})\, \mathrm{d}y' \mathrm{d}\bar{\theta} = \langle Q_d^{0,\#} \rangle \leqslant -D < 0$$

Since the vector field $\vec{Q}^{0,\#}(x,\cdot) + \vec{Q}^{0,+}(x,\cdot)$ is divergence free, by Theorem 2 in Reference [13] problem (44)–(46) has a bounded solution that decays exponentially as $\bar{\theta} \to \infty$, and the solution with this property is unique. Furthermore, by the Schauder estimates we obtain the exponential decay of the first two derivatives (47). The constant c_4 only depends on c_2, D, λ and Λ (see (7)); C_3 also depends on C_1, $|g(x',t,\cdot)|_{0;Y'}$ and they are thus independent of x' and t.

In order to prove the second part of the theorem it is sufficient to show that $\nabla_{x'} u$ and $\partial_t u$ are solutions of the problems obtained by differentiating problem (44)–(46). Then we can again apply Theorem 2 from Reference [13] and obtain the desired bounds. Justification of the formal differentiation relies on Schauder's estimates and the following maximum norm estimate:

$$|u(x',t,\cdot)|_{0;\mathscr{Y}^+} \leqslant |g(x',t,\cdot)|_{0;Y'} + C|\mathrm{e}^{c_2\bar{\theta}} f^+(x',t,\cdot)|_{0;\mathscr{Y}^+} \tag{48}$$

□

Lemma 4.3
Consider the problem

$$\mathscr{A}^- u = f^-(x',t,\underline{y}) \quad \text{in } \mathscr{Y}^- \tag{49}$$

$$u(x',t,y',\underline{\theta}) = g(x',t,y') \quad \text{on } \underline{\theta} = 0 \tag{50}$$

$$y' \mapsto u(x',t,y',\underline{\theta}) \quad \text{is } Y'\text{-periodic} \tag{51}$$

with $\mathscr{A}^-$ defined in (36), and assume that

$$f^- \in C(\mathbb{R}^{d-1} \times [0,T]; C^1(\overline{\mathscr{Y}^-})), \quad g \in C(\mathbb{R}^{d-1} \times [0,T]; C^{2+\gamma}(\overline{\mathscr{Y}^-}))$$

are Y'-periodic functions and that f^- satisfies uniform bounds

$$|f^-(x',t,\underline{y})|, \quad |\nabla_y f^-(x',t,\underline{y})| \leqslant C_1 \mathrm{e}^{-c_2|\underline{\theta}|}, \quad C_1, c_2 > 0$$

Then, under condition (43) there exists a unique bounded solution $u \in C(\mathbb{R}^{d-1} \times [0,T]; C^{2+\gamma}(\overline{\mathscr{Y}^-}))$ of problem (49)–(51) that stabilizes exponentially as $\underline{\theta} \to -\infty$ to a function $\mu(x',t)$,

$\mu \in C(\mathbb{R}^{d-1} \times [0,T])$, that does not depend on $\underline{y}$. More precisely, there are constants $C_3 > 0$, $c_4 > 0$, independent of x' and t, such that

$$\begin{aligned} |u(x',t,\underline{y}) - \mu(x',t)| &\leqslant C_3 \mathrm{e}^{-c_4|\underline{\theta}|} \\ |\nabla_y u(x',t,\underline{y})| \leqslant C_3 \mathrm{e}^{-c_4|\underline{\theta}|}, &\quad |\nabla_y^2 u(x',t,\underline{y})| \leqslant C_3 \mathrm{e}^{-c_4|\underline{\theta}|} \end{aligned} \tag{52}$$

If in addition

$$f^- \in C^1(\mathbb{R}^{d-1} \times [0,T]; C^1(\overline{\mathscr{Y}^-})), \quad g \in C^1(\mathbb{R}^{d-1} \times [0,T]; C^{2+\gamma}(\overline{\mathscr{Y}^-}))$$

and the derivatives of f^- with respect to x' and t decay exponentially as $\underline{\theta} \to -\infty$, uniformly in x' and t, then $u \in C^1(\mathbb{R}^{d-1} \times [0,T]; C^{2+\gamma}(\overline{\bar{Y}^-}))$ and bounds (52) hold for the derivatives of u with respect to x' and t, with μ replaced by its corresponding derivatives.

Proof
The proof relies on Theorem 2 from Reference [13] and is identical to the proof of Lemma 4.2. Boundedness and continuity of the function μ follows from the estimate

$$|\mu(x',t)| \leqslant C|\mathrm{e}^{-c_2\underline{\theta}} f^-(x',t,\cdot)|_{0;\mathscr{Y}^-}$$

and can be proved in the same way as (48). □

By applying recursively Lemmas 4.2 and 4.3 to problems (37), (38) we obtain the exponential stabilization of $c_\pm^k$, for any $k \geqslant 0$.

4.3. Zero-order terms

We consider the first $c^0(x,t,y)$ term in (31), and with assumption (43) we can solve successively the interior periodic problems given by Equations (33), the two spatial boundary layer problems (37) and the initial time boundary layer problem (40). To avoid cumbersome calculations we only construct the first two terms in the asymptotic expansion (31). However, to obtain error estimate for these two terms we have to construct the third term and therefore we will need regularity and compatibility of the initial and boundary value data for the construction of the first three terms in (31). Namely, we assume

$$\begin{aligned} \mathbf{D}, \mathbf{K} \in (C_{\mathrm{per}}^{2+\gamma}(\bar{Y}))^{d^2}, &\quad P^\pm \in C^{7+\gamma}(\mathbb{R}^{d-1}) \\ c_{\mathrm{init}} \in C^5(\bar{\Omega}), &\quad c_\pm \in C^5(\mathbb{R}^{d-1} \times [0,T]) \end{aligned} \tag{53}$$

for some $\gamma > 0$, and

$$c_{\mathrm{init}}|_{x_d=a} = c_+|_{t=0}, \quad c_{\mathrm{init}}|_{x_d=b} = c_-|_{t=0} \tag{54}$$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} c_{\mathrm{init}} = 0, \quad \text{at } x_d = a, b \quad \text{for } 1 \leqslant |\alpha| \leqslant 5 \tag{55}$$

$$\frac{\partial^k}{\partial t^k} \frac{\partial^{|\alpha|}}{\partial x^\alpha} c_\pm = 0 \quad \text{at } t = 0 \quad \text{for } 1 \leqslant |\alpha| + k \leqslant 5 \tag{56}$$

We study now the structure of the zero-order term $c^0(x,t,y)$ by computing the regular part $c^0_\#$ and the corresponding boundary layer term c^0_+. From (33) it follows immediately that $c^0_\#$ does not depend on y, so we write in (32)

$$c^0_\#(x,t,y)=\bar{c}^0(x,t)$$

where the function $\bar{c}^0$ is determined from the condition $\langle \mathscr{F}^1_\# \rangle = 0$. This implies the following problem for $\bar{c}^0$:

$$\mathscr{L}^h \bar{c}^0 = 0 \quad \text{in} \quad \Omega\times(0,T) \tag{57}$$

$$\bar{c}^0 = c_- \quad \text{for } x_d = b \text{ and } t>0 \tag{58}$$

$$\bar{c}^0 = c_{\text{init}} \quad \text{for } x\in\Omega \text{ and } t=0 \tag{59}$$

where the operator $\mathscr{L}^h$ is given by

$$\mathscr{L}^h = \langle \phi\rangle \frac{\partial}{\partial t} + \langle \vec{Q}^{0,\#}\rangle(x)\cdot\nabla_x$$

Since Equation (57) is of the first order, we only keep the boundary condition associated to the inflow i.e. the upper boundary of the layer. We also note that problem (57)–(59) is well posed due to (43), and that $\bar{c}^0$ is at least $C^5(\bar{\Omega}\times[0,T])$ function, due to compatibility conditions (54)–(56).

In order to fit the boundary condition on the lower boundary we use a boundary layer c^0_+, correcting the effects of $\bar{c}^0$:

$$\mathscr{A}^+ c^0_+ = 0 \quad \text{in } \mathscr{Y}^+ \tag{60}$$

$$c^0_+(x',t,\bar{y}) = c_+(x',t) - \bar{c}^0(x',a,t) \quad \text{on } \bar{\theta}=0 \tag{61}$$

$$y'\mapsto c^0_+(x',t,y',\bar{\theta}) \quad \text{is } Y'\text{-periodic} \tag{62}$$

By Lemma 4.2 problems (60)–(62) have a unique bounded solution $c^0_+(x',t,\bar{y})$, decaying exponentially as $\bar{\theta}\to\infty$. We set then

$$c^0_- \equiv 0, \quad c^0_{\text{in}} \equiv 0 \tag{63}$$

4.4. *First-order terms*

The first-order term c^1 in (31) is decomposed in a sum of an oscillating term $c^1_\#$, an initial time boundary layer term c^1_{in} and two spatial boundary layer terms c^1_+ and c^1_-.

The oscillating term $c^1_\#$ is the solution of problem (33) for $k=1$, and it can be written as a sum of an oscillating term

$$\tilde{c}^1_\#(x,t,y) = \sum_{|\alpha|=1} \psi^1_\alpha \phi(x,y)\partial_\alpha \bar{c}^0(x,t) \tag{64}$$

plus a non-oscillating term $\bar{c}^1(x,t)$, where ψ^1_α, $\alpha\in\mathscr{I}$, is a solution of the cell problem on Y:

$$-\mathrm{div}_y(\mathbf{D}(y)(\nabla_y\psi^1_\alpha+\vec{e}_\alpha))+\vec{Q}^{0,\#}\cdot\nabla_y\psi^1_\alpha=\left[\frac{\phi(y)}{\langle\phi\rangle}\langle\vec{Q}^{0,\#}\rangle-\vec{Q}^{0,\#}\right]\cdot\vec{e}_\alpha \tag{65}$$

$$\psi^1_\alpha \quad \text{is } Y\text{-periodic}, \quad \langle\phi\psi^1_\alpha\rangle=0 \tag{66}$$

and where $\bar{c}^1(x,t)$ will be defined later on, in (71)–(73).

To correct the oscillations on the boundary, produced by $\tilde{c}^1_\#$, we introduce a boundary layer term c^1_- on the upper boundary by

$$\begin{aligned}&\mathscr{A}^-\check{c}^1_-=\mathscr{F}^1_-(x',t,\underline{y}) \quad \text{in } \mathscr{Y}^-\\ &y'\mapsto\check{c}^1_-(x',t,y',\underline{\theta}) \quad \text{is } Y'\text{-periodic}\\ &\check{c}^1_-(x',t,\underline{y})=-\sum_{|\alpha|=1}\psi^1_\alpha(x',b,y)\partial_\alpha\bar{c}^0(x',b,t) \quad \text{on } \underline{\theta}=0\end{aligned} \tag{67}$$

This problem, by Lemma 4.3, has a unique bounded solution that stabilizes exponentially as $\underline{\theta}\to-\infty$ towards some function $\mu^1_-(x',t)$, independent of $\underline{y}$. Then we set $c^1_-(x',t,\underline{y})=\check{c}^1_-(x',t,\underline{y})-\mu^1_-(x',t)$. This boundary layer problem (67) was designed to cancel the oscillatory error on the upper boundary, but it produces then a non-oscillating error on this boundary which will be eliminated later on with the help of the function $\bar{c}^1$.

For correcting the trace of $\tilde{c}^1_\#$ at the initial time, we first introduce the function $\check{c}^1_{\mathrm{in}}$ defined by

$$\mathscr{A}^{\mathrm{in}}\check{c}^1_{\mathrm{in}}=0 \quad \text{in } Y\times(0,\infty) \tag{68}$$

$$y\mapsto\check{c}^1_{\mathrm{in}}(x,y,\tau) \text{ is } Y\text{-periodic} \tag{69}$$

$$\check{c}^1_{\mathrm{in}}(x,y,\tau)=-\tilde{c}^1_\#(x,0,y) \quad \text{for } \tau=0 \tag{70}$$

By Lemma 4.1 there exists a function $\mu^1_{\mathrm{in}}(x)$ such that

$$|\check{c}^1_{\mathrm{in}}(x,y,\tau)-\mu^1_{\mathrm{in}}(x)|\leqslant c_1\mathrm{e}^{-c_2\tau}, \quad c_1,c_2>0$$

Then we define $c^1_{\mathrm{in}}(x,y,\tau)=\check{c}^1_{\mathrm{in}}(x,y,\tau)-\mu^1_{\mathrm{in}}(x)$, where in fact, from (66) it is easy to see that $\mu^1_{\mathrm{in}}\equiv 0$.

Remark 4.1
The structure of initial condition (70) suggests the following representation of the solution of problem (68)–(70):

$$c^1_{\mathrm{in}}(x,y,\tau)=\sum_{|\alpha|=1}c^1_{\mathrm{in},\alpha}(x,y,\tau)\partial_\alpha c_{\mathrm{init}}(x)$$

and since, from (55), ∇c_{init} has a fourth-order zero at $x_d=a,b$, the term

$$(\vec{Q}^{0,+}+\vec{Q}^{0,-})\cdot\nabla_y c^1_{\mathrm{in}}$$

is of order $O(\varepsilon^4)$. This justifies a posteriori that we did not need to incorporate these terms in the initial time boundary layer Equation (40). $\square$

We proceed with the term $\bar{c}^1$ that corrects the non-oscillating error arising from the boundary layer corrector c^1_- on the upper boundary. To obtain $\bar{c}^1$ we average Equation (33), for $k=2$. This gives, after simple transformations

$$\mathscr{L}^h \bar{c}^1 = \mathrm{div}_x(\mathbf{D}^h \nabla_x \bar{c}^0) - \langle \vec{Q}^{1,\#} \rangle \cdot \nabla_x \bar{c}^0 \quad \text{in } \Omega \tag{71}$$

$$\bar{c}^1 = \mu^1_- \quad \text{on } x_d = b, \ t > 0 \tag{72}$$

$$\bar{c}^1 = 0 \quad \text{for } x \in \Omega, \ t = 0 \tag{73}$$

where the tensor $\mathbf{D}^h$ is defined, for $\alpha \in \mathscr{I}$, by

$$\mathbf{D}^h(x)\vec{e}_\alpha = \langle \mathbf{D}(\nabla_y \psi^1_\alpha + \vec{e}_\alpha)\rangle - \langle \vec{Q}^{0,\#} \psi^1_\alpha \rangle \tag{74}$$

Then, it is not difficult to verify from (54)–(56) that $\bar{c}^1$ is $C^4(\bar{\Omega} \times [0,T])$ function. We can finally construct boundary layer corrector c^1_+ that will correct on the lower boundary both the errors coming from the oscillatory and non-oscillatory terms. To this end we consider the problem

$$\mathscr{A}^+ c^1_+ = \mathscr{F}^1_+(x', t, \bar{y}) \quad \text{in } \mathscr{Y}^+ \tag{75}$$

$$c^1_+(x', t, \bar{y}) = -c^1_\#(x', a, t, y) \quad \text{on } \bar{\theta} = 0 \tag{76}$$

$$y' \mapsto c^1_+(x', t, y', \bar{\theta}) \quad \text{is } Y'\text{-periodic} \tag{77}$$

By Lemma 4.2 the right-hand side $\mathscr{F}^1_+$, defined by (38), decays exponentially as $\bar{\theta} \to \infty$ and therefore this problem has a unique bounded solution, which decays exponentially as $\bar{\theta} \to \infty$. This completes the construction of the first order terms in expansion (31).

Obviously, if the initial and boundary value data in problem (1)–(6) are regular enough and satisfy compatibility conditions, then this procedure can be continued further and the higher-order terms of the expansion can be constructed.

5. CONVERGENCE RESULTS

In this section we summarize the results from previous sections and state the main convergence result for the approximation given by the first two terms in expansion (16), (31) of the solution $(\vec{q}^\varepsilon, c^\varepsilon)$ of (1).

We set

$$c^{1,\varepsilon} = c^0 + \varepsilon c^1, \quad \vec{Q}^{1,\varepsilon} = \vec{Q}^0 + \varepsilon \vec{Q}^1$$

and we formulate the main theorem:

Theorem 5.1
Let $c^{1,\varepsilon}=c^0+\varepsilon c^1$ where c^0, c^1 are the first two terms in the asymptotic expansion (31), constructed in previous two sections, for the solution c^ε of problem (1)–(6). Assume the regularity hypotheses (43), (53)–(56) fulfilled. Then there is a constant C, independent of ε, such that for sufficiently small ε

$$|c^\varepsilon - c^{1,\varepsilon}|_{0,Q_T} \leqslant C\varepsilon^2, \quad \left|\frac{\partial}{\partial t}c^\varepsilon - \frac{\partial}{\partial t}c^{1,\varepsilon}\right|_{0;Q_T} \leqslant C\varepsilon$$

$$|\nabla c^\varepsilon - \nabla c^{1,\varepsilon}|_{0;Q_T} \leqslant C\varepsilon$$

Proof
We write down the equations verified by $c^{1,\varepsilon}$:

$$\phi\left(\frac{x}{\varepsilon}\right)\frac{\partial c^{1,\varepsilon}}{\partial t} + \vec{Q}^{1,\varepsilon}\cdot\nabla c^{1,\varepsilon} - \varepsilon\,\mathrm{div}(\mathbf{D}\left(\frac{x}{\varepsilon}\right)\nabla c^{1,\varepsilon}) = \mathscr{F}^\varepsilon \quad \text{in } \Omega\times(0,T) \tag{78}$$

$$c^{1,\varepsilon} = c_+ \quad \text{on } x_d = a \tag{79}$$

$$c^{1,\varepsilon} = c_- \quad \text{on } x_d = b \tag{80}$$

$$c^{1,\varepsilon} = c_{\mathrm{init}} \quad \text{at } t=0 \tag{81}$$

In (78) the right-hand side $\mathscr{F}^\varepsilon(x,t,y,\tau)$ includes the first- and second-order terms obtained by plugging the two first terms of the c^ε and $\vec{q}^\varepsilon$ expansions in the original transport equation (1).

Denoting $y=x/\varepsilon$, $\tau=t/\varepsilon$ and introducing $C^{1,\varepsilon}(y,\tau)=c^{1,\varepsilon}(x,t)$ we see that the function $C^{1,\varepsilon}$ is the solution of the following transformed problem:

$$\begin{aligned}&\phi(y)\frac{\partial C^{1,\varepsilon}}{\partial\tau} + \vec{Q}^{1,\varepsilon}(\varepsilon y, y)\cdot\nabla_y C^{1,\varepsilon} - \mathrm{div}_y(\mathbf{D}(y)\nabla_y C^{1,\varepsilon})\\ &= \varepsilon\mathscr{F}^\varepsilon(\varepsilon y, \varepsilon\tau, y, \tau) \quad \text{in } Q_T^\varepsilon\end{aligned} \tag{82}$$

$$C^{1,\varepsilon}(y,\tau) = c_+(\varepsilon y', \varepsilon\tau) \quad \text{on } y_d = a/\varepsilon \tag{83}$$

$$C^{1,\varepsilon}(y,\tau) = c_-(\varepsilon y', \varepsilon\tau) \quad \text{on } y_d = b/\varepsilon \tag{84}$$

$$C^{1,\varepsilon}(y,\tau) = c_{\mathrm{init}}(\varepsilon y) \quad \text{at } \tau=0 \tag{85}$$

in $Q_T^\varepsilon = \Omega_\varepsilon\times(0,T/\varepsilon)$. □

To go further we need the following result.

Lemma 5.1
Let regularity assumptions (43), (53)–(56) be fulfilled. Then the right-hand side in (78) is bounded as follows:

$$|\mathscr{F}^\varepsilon(\varepsilon\,\cdot,\cdot,\varepsilon\,\cdot,\cdot)|_{\gamma,\gamma/2;Q_T^\varepsilon} \leqslant C\varepsilon$$

where the constant C does not depend on ε, and

$$\mathscr{F}^{\varepsilon}\left(x, 0, \frac{x}{\varepsilon}, 0\right) = 0 \quad \text{at } x_d = a, b$$

The proof is straightforward from (43), (53)–(56). We only note that the regularity conditions (43) ensure that all periodic, initial and boundary layer problems have sufficiently regular solutions with respect to the parameters x and t. By the compatibility conditions (54)–(56) then all effective equations have sufficiently regular solutions and all boundary layer functions vanish at $t = 0$, together with their corresponding derivatives with respect to x' and t.

Now by Lemma 5.1 the compatibility conditions of the first order for the transformed problem (82)–(85) are fulfilled and we can apply the Schauder estimates. Using standard maximum norm estimate for parabolic operator in the space of exponentially growing functions we obtain, uniformly with respect to ε, the estimate

$$|\nabla_y C^{2,\varepsilon}|_{\gamma,\gamma/2; Q_T^{\varepsilon}} \leqslant C \tag{86}$$

If we denote $v^{\varepsilon} = c^{\varepsilon} - c^{1,\varepsilon}$ the discrepancy that we want to estimate, then v^{ε} is a solution in $\Omega \times (0, T)$ of

$$\phi\left(\frac{x}{\varepsilon}\right)\frac{\partial v^{\varepsilon}}{\partial t} + \vec{q}^{\varepsilon} \cdot \nabla v^{\varepsilon} - \varepsilon \operatorname{div}\left(\mathbf{D}\left(\frac{x}{\varepsilon}\right)\nabla v^{\varepsilon}\right) = -(\vec{q}^{\varepsilon} - \vec{Q}^{1,\varepsilon}) \cdot \nabla c^{1,\varepsilon} - \mathscr{F}^{\varepsilon} \tag{87}$$

with zero initial and boundary conditions. From Theorem 3.3, Lemma 5.1 and bound (86), it follows immediately that $|v^{\varepsilon}|_{0;Q_T} \leqslant C\varepsilon$. This estimate can be improved like in Theorem 3.2 by constructing the next term in the expansions for the concentration and the Darcy velocity. This leads to the estimate

$$|v^{\varepsilon}|_{0;Q_T} \leqslant C\varepsilon^2 \tag{88}$$

From (87) the normalized discrepancy $V^{\varepsilon}(y, \tau) = v^{\varepsilon}(x, t)$ satisfies

$$\begin{aligned}
\phi(y)\frac{\partial V^{\varepsilon}}{\partial \tau} + \vec{q}^{\varepsilon}(\varepsilon y) \cdot \nabla_y V^{\varepsilon} = {}& \operatorname{div}_y(\mathbf{D}(y)\nabla_y V^{\varepsilon}) \\
& - (\vec{q}^{\varepsilon}(\varepsilon y) - \vec{Q}^{1,\varepsilon}(\varepsilon y, y)) \cdot \nabla_y C^{1,\varepsilon}(y, \tau) \\
& - \varepsilon \mathscr{F}^{\varepsilon}(\varepsilon y, \varepsilon \tau, y, \tau) \quad \text{in } Q_T^{\varepsilon}
\end{aligned}$$

with homogeneous initial and boundary conditions. Applying Schauder's estimates we obtain finally

$$\begin{aligned}
|V^{\varepsilon}|_{2+\gamma, 1+\gamma/2; Q_T^{\varepsilon}} \leqslant {}& C\{|V^{\varepsilon}|_{0;Q_T^{\varepsilon}} + \varepsilon|\mathscr{F}^{\varepsilon}(\varepsilon\,\cdot, \varepsilon\,\cdot, \cdot, \cdot)|_{\gamma,\gamma/2;Q_T^{\varepsilon}} \\
& + |\vec{q}^{\varepsilon}(\varepsilon\,\cdot) - \vec{Q}^{1,\varepsilon}(\varepsilon\cdot, \cdot)|_{\gamma;\Omega_{\varepsilon}} |\nabla_y C^{1,\varepsilon}|_{\gamma,\gamma/2;Q_T^{\varepsilon}}\} \leqslant C\varepsilon^2
\end{aligned}$$

where we used Lemma 5.1, Theorem 3.3, (86) and (88). By returning to the initial variables x and t we obtain the required estimates in Theorem 5.1. □

6. DISPERSION AND EFFECTIVE DIFFUSION

We are now interested in the behaviour of the non-oscillatory part of the asymptotic expansion which describes a homogenized solution of the original problem. Due to singular perturbation in Equation (1) the zero-order concentration approximation $\bar{c}^0(x,t)$ satisfies transport equation (57)–(59) which does not include any diffusion term. The zero-order term $\bar{c}^0(x,t)$ is convected by the zero-order mean Darcy's velocity $\langle \vec{Q}^{0,\#} \rangle$. The transport equation (57)–(59) is a first-order hyperbolic equation which describes the effective transport of the concentration and does not show the diffusion occurring on the background of the effective convection. In order to see the effective diffusion we consider the first two non-oscillating terms of the expansion $\mathscr{C}^1(x,t;\varepsilon) = \bar{c}^0(x,t) + \varepsilon \bar{c}^1(x,t)$, and from the above analysis we see that this function satisfies the equation

$$
\begin{aligned}
\langle \phi \rangle \frac{\partial \mathscr{C}^1}{\partial t} + \langle \vec{Q}^{0,\#} + \varepsilon \vec{Q}^{1,\#} \rangle (x) \cdot \nabla_x \mathscr{C}^1 &= \varepsilon \operatorname{div}(\mathbf{D}^h(x) \nabla \mathscr{C}^1) + O(\varepsilon^2) \\
\mathscr{C}^1 &= g_+ + O(\varepsilon) \quad \text{on } x_d = a \\
\mathscr{C}^1 &= c_- + \varepsilon \mu_-^1 + O(\varepsilon^2) \quad \text{on } x_d = b \\
\mathscr{C}^1 &= c_{\text{init}} \quad \text{at } t = 0
\end{aligned}
\tag{89}
$$

where $g_+(x',t) = \bar{c}^0(x',a,t)$. Dropping all the higher-order terms in problem (89) we obtain a modified problem:

$$
\begin{aligned}
\langle \phi \rangle \frac{\partial \mathscr{C}^*}{\partial t} + \langle \vec{Q}^{0,\#} + \varepsilon \vec{Q}^{1,\#} \rangle (x) \cdot \nabla_x \mathscr{C}^* &= \varepsilon \operatorname{div}(\mathbf{D}^h(x) \nabla \mathscr{C}^*) \\
\mathscr{C}^* &= g_+ \quad \text{on } x_d = a \\
\mathscr{C}^* &= c_- + \varepsilon \mu_-^1 \quad \text{on } x_d = b \\
\mathscr{C}^* &= c_{\text{init}} \quad \text{at } t = 0
\end{aligned}
\tag{90}
$$

It is straightforward to see that $\mathbf{D}^h$ is uniformly positive-definite tensor and therefore problem (90) is well posed. Furthermore, one can easily verify that the difference between the solution of problem (90) and the solution of problem (89) does not exceed $O(\varepsilon^2)$. Moreover, the contribution of the boundary term g_+ decays exponentially in the neighbourhood of the hyperplane $x_d = a$.

From these considerations we see that $(90)_1$ is the first-order effective transport equation for the concentration. The convective part in $(6.90)_1$ is given by the mean value of the first two oscillating terms in the Darcy velocity expansion (16), (19). The effective diffusion tensor $\mathbf{D}^h$ is given by the auxiliary convection type problem (65), (66). Then the effective diffusion matrix $\mathbf{D}^h$ depends on the zero-order Darcy's velocity oscillations $\vec{Q}^{0,\#}$, through the solution of this local problem (65), (66) and is then including the so-called dispersion. Finally, we notice that the first-order effective equation (90) is still parabolic thanks to a small dispersion/diffusion and its solution could be considered as the viscous approximation to $\bar{c}^0(x,t)$, solution of the zero-order effective equation.

Let us finally note that if we substract from the solution $\mathscr{C}^*$ of Equation (90) the purely convective term $\bar{c}^0$, we see that the normalized difference $u = (\mathscr{C}^* - \bar{c}^0)/\varepsilon$ obeys a purely diffusive/dispersive equation with Peclet number of order one.

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