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Homogenization of immiscible compressible two-phase flow in highly heterogeneous porous media with discontinuous capillary pressures

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This paper presents a study of immiscible compressible two-phase, such as water and gas, flow through highly heterogeneous porous media with periodic microstructure. Such models appear in gas migration through engineered and geological barriers for a deep repository for radioactive waste. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. Consequently, the model involves highly oscillatory characteristics and internal nonlinear interface conditions. The microscopic model is written in terms of the phase formulation, i.e. where the wetting (water) saturation phase and the non-wetting (gas) pressure phase are primary unknowns. This formulation leads to a coupled system consisting of a nonlinear parabolic equation for the gas pressure and a nonlinear degenerate parabolic diffusion-convection equation for the water saturation, subject to appropriate transmission, boundary and initial conditions. The major difficulties related to this model are in the nonlinear degenerate structure of the equations, as well as in the coupling in the system. Moreover, the transmission conditions are nonlinear and the saturation is discontinuous at interfaces separating different media. Under some realistic assumptions on the data, we obtain a nonlinear homogenized coupled system with effective coefficients which are computed via a cell problem and give a rigorous mathematical derivation of the upscaled model by means of the two-scale convergence.

Keywords: Homogenization; immiscible compressible; interface conditions; nonlinear degenerate system; two-phase flow; porous media; nuclear waste, water–gas.

AMS Subject Classification: 35B27, 35K65, 76S05, 76T05, 76T10

1. Introduction

The displacement process of immiscible fluids in porous media occurs in a wide variety of applications. The most concentrated research in the field of multiphase flow over the past four decades has focused on unsaturated groundwater flow, and flow in underground petroleum reservoirs. Most recently, multiphase flow have generated serious interest among engineers concerned with deep geological repository for radioactive waste, and for CO_2 capture and storage sequestration.

The long-term safety of the disposal of nuclear waste is an important issue in all countries with a significant nuclear program. Repositories for the disposal of high-level and long-lived radioactive waste generally rely on a multi-barrier system to isolate the waste from the biosphere. The multi-barrier system typically comprises the natural geological barrier provided by the repository host rock and its surroundings and an engineered barrier system, i.e. engineered materials placed within a repository, including the waste form, waste canisters, buffer materials, backfill and seals.

In this paper, we focus our attention on the modeling of immiscible compressible two-phase flow in heterogeneous porous media, in the framework of the geological disposal of radioactive waste. As a matter of fact, one of the solutions envisaged for managing waste produced by nuclear industry is to dispose it in deep geological formations chosen for their ability to prevent and attenuate possible releases of radionuclides in the geosphere. In the frame of designing nuclear waste geological repositories, a problem of possible two-phase flow of water and gas appears, for more details see for instance Refs. 27 and 28. Multiple recent studies have established that in such installations important amounts of gases, mainly hydrogen, are expected to be produced in particular due to the corrosion of metallic components used in the repository design. The creation and transport of a gas phase is an issue of concern with regard to the capability of the engineered and natural barriers to evacuate the gas phase and avoid overpressure, thus preventing mechanical damages. It has become necessary to carefully evaluate those issues while assessing the performance of a geological repository, see e.g. Refs. 19 and 29 and the references therein.

The upscaling or homogenization of multiphase flow through heterogeneous porous media has been a problem of interest for many years and many methods have been developed. There is an extensive literature on this subject. We will not attempt a literature review here, but merely mention a few references. Here we restrict ourselves to the mathematical homogenization method as described in Ref. 25 for flow and transport in porous media. A recent review of the methods developed for incompressible immiscible two-phase flow in porous media and compressible miscible flow in porous media can be viewed in Refs. 3 and 24. Let us also mention that few homogenization results were obtained in the case of fields with different rock types: porosity, absolute permeability, relative permeabilities and capillary pressure curves being different in each type of porous media, see, e.g., Refs. 11, 14, 24 and 30. We refer for instance to Refs. 12, 13, 15–17 and 23 for more information on the homogenization of one-phase flow in the framework of the geological disposal of radioactive waste.

However, as reported in Ref. 6, the situation is quite different for immiscible compressible two-phase flow in porous media, where, only recently few results have been obtained in the case of a single rock type model. In this model the capillary pressure and relative permeabilities depend solely on the saturation. In Ref. 3 homogenization results were obtained for water–gas flow in porous media using the phase formulation, i.e. where the phase pressures and the phase saturations are primary unknowns. In Ref. 6 homogenization results were obtained for immiscible compressible two-phase flow in porous media, the wetting and non-wetting phases are treated to be compressible, the problem is formulated in terms of a new global pressure[5,7] and it is fully equivalent to the original equations. To our knowledge, convergence results on the homogenization of immiscible compressible two-phase flow in porous media with different rock types are still missing.

This paper is concerned with a nonlinear degenerate system of diffusion–convection equations modeling the flow and transport of immiscible compressible fluids through highly heterogeneous porous media, capillary and gravity effects being taken into account. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. In the literature, this may be rephrased by saying that we consider a field containing several rock types, see, e.g., Ref. 18. We restrict our attention to water (incompressible) and gas such as hydrogen (compressible) in the context of gas migration through engineered and geological barriers for a deep repository for radioactive waste, however the methodology and the analysis can be extended to problems where both fluids are assumed to be compressible. For more details on the formulation of such problems, we refer for instance to Ref. 8 and the references therein.

For notational convenience we only consider a field which contains two different rock types. But it is easy to see that all the results are valid in a domain with several rock types. The model to be presented herein is formulated in terms of the wetting (water) saturation phase and the non-wetting (gas) pressure phase, and the feature of the global pressure as introduced in Refs. 9 and 18 for incompressible immiscible flows is used to establish *a priori* estimates. The governing equations are derived from the mass conservation laws of both fluids, along with constitutive relations relating the velocities to the pressures gradients and gravitational effects.

Traditionally, the standard Muskat–Darcy law provides this relationship. This formulation leads to a coupled system consisting of a nonlinear parabolic equation for the gas pressure and a nonlinear degenerate parabolic diffusion–convection equation for the water saturation, subject to appropriate transmission, boundary and initial conditions. Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling, the degeneracy of the diffusion term in the saturation equation and the degeneracy of the temporal term in the pressure equation. Moreover the transmission conditions are nonlinear and the saturation is discontinuous at the interface separating the two media.

We start with a microscopic model defined on a domain with periodic microstructure. Here we consider a field which contains two different rock types. Namely, we consider that the porosity, the absolute permeability, the relative permeabilities and the capillary pressure are rapidly oscillating functions of the microscopic scale $y = x/\varepsilon$, where x is the macroscopic scale and ε is a small parameter which characterizes the microscopic length scale. Our aim is to study the macroscopic behavior of solutions of this system of equations as ε tends to zero and give a rigorous mathematical derivation of an upscaled model by means of the two-scale convergence. Thus, we extend the results of Ref. 3 to the case of fields with different rock types.

The paper is organized as follows. At the beginning of Sec. 2 we, first, consider the microscopic model in terms of the phase formulation. Then in Sec. 2.1 we introduce an additional variable called the global pressure as introduced in Refs. 9 and 18 for incompressible immiscible flows and rewrite the initial system of equations in terms of the global pressure and the water saturation. In Sec. 2.2 we formulate the main assumptions on the data. The definition of a weak solution to this system is then given in Sec. 2.3. Section 3 is devoted to the derivation of the uniform estimates of the solutions and is essentially based on the energy equality by using test functions introduced in Ref. 22. Section 4 is devoted to the convergence results for the sequences $\{\mathsf{P}_\ell^\varepsilon\}_{\varepsilon>0}$, $\{S_\ell^\varepsilon\}_{\varepsilon>0}$, $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0}$. First, in Sec. 4.1 we construct the extensions of the functions $\mathsf{P}_\ell^\varepsilon$, S_ℓ^ε, Θ_ℓ^ε. Then in Sec. 4.2 we prove the compactness results for the sequences $\{\widetilde{S}_\ell^\varepsilon\}_{\varepsilon>0}$, $\{\widetilde{\Theta}_\ell^\varepsilon\}_{\varepsilon>0}$. Theses compactness results are rather delicate due to the discontinuity of the functions of interest at the microscopic interface. To obtain these results we elaborated a new approach based on the ideas from Ref. 3 and monotonicity arguments. The main result of the paper, i.e. Theorem 5.1, is given in Sec. 5 in terms of the homogenized phases formulation. The theorem is then proved in Sec. 6. Since the original system is fully nonlinear and degenerates, the homogenization procedure is getting nontrivial. Our approach relies on cutoff techniques. Section 7 is devoted to the study of disperse porous media. Finally, we conclude in Sec. 8.

2. Microscopic Model and Main Assumptions

We consider a reservoir $\Omega \subset \mathbb{R}^3$ which is assumed to be a bounded, connected Lipschitz domain with a periodic microstructure. More precisely, we will scale this periodic structure by a parameter ε which represents the ratio of the cell size to the whole region and we assume that $0 < \varepsilon \ll 1$ is a small parameter tending to zero. Let $Y = (0,1)^3$ be a cell consisting of two subdomains (corresponding to two types of rocks) called Y_1 and Y_2. We denote by $\Gamma_{1,2}$ the interface between two subdomains in Y. Let $\mathbf{1}_2(y)$ be the characteristic function of Y_2 extended Y-periodically to the whole $\mathbb{R}^3$. Thus, the medium Ω contains two subdomains Ω_1^ε and Ω_2^ε, representing two different types of rocks and satisfying:

$$\Omega_2^\varepsilon \subset \left\{x \in \Omega : \mathbf{1}_2\left(\frac{x}{\varepsilon}\right) = 1\right\} \quad \text{and} \quad \Omega_1^\varepsilon = \Omega \backslash \overline{\Omega_2^\varepsilon}. \tag{2.1}$$

We assume that both subdomains are connected sets. We also introduce the notation:

$$\Gamma_{1,2}^\varepsilon \stackrel{\text{def}}{=} \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon \cap \Omega, \tag{2.2}$$

$$\Omega_T \stackrel{\text{def}}{=} \Omega \times (0,T), \quad \Omega_{\ell,T}^\varepsilon \stackrel{\text{def}}{=} \Omega_\ell^\varepsilon \times (0,T), \quad \Sigma_{1,2,T}^\varepsilon \stackrel{\text{def}}{=} \Gamma_{1,2}^\varepsilon \times (0,T), \tag{2.3}$$

where $T > 0$ is fixed, and, from now on, $\ell = 1, 2$.

The water–gas flow in porous reservoirs can be described in terms of the following characteristics:

— $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ is the porosity of the medium Ω;
— $K^\varepsilon(x) = K(\frac{x}{\varepsilon})$ is the absolute permeability tensor of Ω;
— ϱ_w, ϱ_g are the densities of water and gas, respectively.
— $S_{\ell,w}^\varepsilon = S_{\ell,w}^\varepsilon(x,t)$, $S_{\ell,g}^\varepsilon = S_{\ell,g}^\varepsilon(x,t)$ are the saturations of water and gas in Ω_ℓ, respectively;
— $k_{r,w}^{(\ell)} = k_{r,w}^{(\ell)}(S_{\ell,w}^\varepsilon)$, $k_{r,g}^{(\ell)} = k_{r,g}^{(\ell)}(S_{\ell,g}^\varepsilon)$ are the relative permeabilities of water and gas in the medium Ω_ℓ, respectively;
— $p_{\ell,w}^\varepsilon = p_{\ell,w}^\varepsilon(x,t)$, $p_{\ell,g}^\varepsilon = p_{\ell,g}^\varepsilon(x,t)$ are the pressures of water and gas in $\Omega_{\ell,T}^\varepsilon$, respectively.

The conservation of mass in each phase can be written as (see, e.g., Refs. 10, 18 and 26)

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S_{\ell,w}^\varepsilon \varrho_w(p_{\ell,w}^\varepsilon)) + \operatorname{div}(\varrho_w(p_{\ell,w}^\varepsilon)\mathbf{q}_{\ell,w}^\varepsilon) = 0 & \text{in } \Omega_{\ell,T}^\varepsilon; \\ \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S_{\ell,g}^\varepsilon \varrho_g(p_{\ell,g}^\varepsilon)) + \operatorname{div}(\varrho_g(p_{\ell,g}^\varepsilon)\mathbf{q}_{\ell,g}^\varepsilon) = 0 & \text{in } \Omega_{\ell,T}^\varepsilon, \end{cases} \tag{2.4}$$

where the velocities of the water and gas $\mathbf{q}_{\ell,w}^\varepsilon$, $\mathbf{q}_{\ell,g}^\varepsilon$ are defined by the Darcy–Muskat law:

$$\mathbf{q}_{\ell,w}^\varepsilon = -K^\varepsilon(x)\lambda_{\ell,w}(S_{\ell,w}^\varepsilon)(\nabla p_{\ell,w}^\varepsilon - \varrho_w(p_{\ell,w}^\varepsilon)\mathbf{g}),$$
$$\text{with} \quad \lambda_{\ell,w}(S_{\ell,w}^\varepsilon) = \frac{k_{r,w}^{(\ell)}}{\mu_w}(S_{\ell,w}^\varepsilon); \tag{2.5}$$

$$\mathbf{q}^\varepsilon_{\ell,g} = -K^\varepsilon(x)\widetilde{\lambda}_{\ell,g}(S^\varepsilon_{\ell,g})(\nabla p^\varepsilon_{\ell,g} - \varrho_g(p^\varepsilon_{\ell,g})\mathbf{g}),$$
$$\text{with} \quad \widetilde{\lambda}_{\ell,g}(S^\varepsilon_{\ell,g}) = \frac{k^{(\ell)}_{r,g}}{\mu_g}(S^\varepsilon_{\ell,g}). \tag{2.6}$$

Here $\mathbf{g}$, μ_w, μ_g are the gravity vector and the viscosities of the water and gas, respectively. From now on we assume that the density of the water is constant, which for the sake of simplicity will be taken equal to one, i.e. $\varrho_w(p^\varepsilon_{\ell,w}) = \text{const.} = 1$, and the gas density ϱ_g is a smooth monotone function such that

$$\begin{gathered} \varrho_g(p) = \varrho_{\min} \quad \text{for } p \le p_{\min}; \quad \varrho_g(p) = \varrho_{\max} \quad \text{for } p \ge p_{\max}; \\ \varrho_{\min} \le \varrho_g(p) \le \varrho_{\max} \quad \varrho'_g(p) > 0 \quad \text{for } p_{\min} < p < p_{\max}. \end{gathered} \tag{2.7}$$

Here the pair of constants $\varrho_{\min}, \varrho_{\max}$ and $p_{\min}, p_{\max}$ satisfies the bounds:

$$0 < \varrho_{\min} < \varrho_{\max} < +\infty \quad \text{and} \quad 0 < p_{\min} < p_{\max} < +\infty. \tag{2.8}$$

To close the system, we need two additional supplementary equations. The first is the saturation balance,

$$S^\varepsilon_{\ell,w} + S^\varepsilon_{\ell,g} = 1 \quad \text{with } S^\varepsilon_{\ell,w}, S^\varepsilon_{\ell,g} \ge 0. \tag{2.9}$$

and the second describes the relation between the pressures,

$$P_{\ell,c}(S^\varepsilon_\ell) = p^\varepsilon_{\ell,g} - p^\varepsilon_{\ell,w} \quad \text{with } P'_{\ell,c}(S^\varepsilon_\ell) < 0 \text{ for all } S^\varepsilon_\ell \in [0,1] \text{ and } P_{\ell,c}(1) = 0, \tag{2.10}$$

where we use the notation:

$$S^\varepsilon_\ell \stackrel{\text{def}}{=} S^\varepsilon_{\ell,w} \tag{2.11}$$

and where $P_{\ell,c}$ is a given capillary pressure–saturation function, $P'_{\ell,c}(s)$ denotes the derivative of the function $P_{\ell,c}(s)$ with respect to the variable s.

Now due to (2.5)–(2.7), (2.11), and the assumption on the density of the water phase we rewrite system (2.4) as follows:

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial S^\varepsilon_\ell}{\partial t} - \operatorname{div}(K^\varepsilon(x)\lambda_{\ell,w}(S^\varepsilon_\ell)(\nabla p^\varepsilon_{\ell,w} - \mathbf{g})) = 0 & \text{in } \Omega^\varepsilon_{\ell,T}; \\ \Phi^\varepsilon(x)\dfrac{\partial(\varrho_g(p^\varepsilon_{\ell,g})(1 - S^\varepsilon_\ell))}{\partial t} & \\ \quad -\operatorname{div}(K^\varepsilon(x)\lambda_{\ell,g}(S_\ell)\varrho_g(p^\varepsilon_{\ell,g})(\nabla p^\varepsilon_{\ell,g} - \varrho_g(p^\varepsilon_{\ell,g})\mathbf{g})) = 0 & \text{in } \Omega^\varepsilon_{\ell,T}; \\ P_{\ell,c}(S^\varepsilon_\ell) = p^\varepsilon_{\ell,g} - p^\varepsilon_{\ell,w} & \text{in } \Omega^\varepsilon_{\ell,T}, \end{cases} \tag{2.12}$$

where

$$\lambda_{\ell,g}(S^\varepsilon_\ell) = \widetilde{\lambda}_{\ell,g}(1 - S^\varepsilon_\ell). \tag{2.13}$$

The phase fluxes and pressures are required to be continuous on the interface $\Gamma^\varepsilon_{1,2}$. Namely,

$$\begin{cases} \mathbf{q}^\varepsilon_{1,w} \cdot \boldsymbol{\nu} = \mathbf{q}^\varepsilon_{2,w} \cdot \boldsymbol{\nu} \quad \text{and} \quad \mathbf{q}^\varepsilon_{1,g} \cdot \boldsymbol{\nu} = \mathbf{q}^\varepsilon_{2,g} \cdot \boldsymbol{\nu} & \text{on } \Sigma^\varepsilon_{1,2,T}; \\ p^\varepsilon_{1,w} = p^\varepsilon_{2,w} \quad \text{and} \quad p^\varepsilon_{1,g} = p^\varepsilon_{2,g} & \text{on } \Sigma^\varepsilon_{1,2,T}, \end{cases} \tag{2.14}$$

where $\Sigma^\varepsilon_{1,2,T}$ is defined in (2.3), $\boldsymbol{\nu}$ is the unit outer normal to $\Gamma^\varepsilon_{1,2}$, and the fluxes $\mathbf{q}^\varepsilon_{\ell,w}, \mathbf{q}^\varepsilon_{\ell,g}$, in the notation (2.11), (2.13) are given by:

$$\begin{aligned} \mathbf{q}^\varepsilon_{\ell,w} &= -K^\varepsilon(x)\lambda_{\ell,w}(S^\varepsilon_\ell)(\nabla p^\varepsilon_{\ell,w} - \mathbf{g}), \\ \mathbf{q}^\varepsilon_{\ell,g} &= -K^\varepsilon(x)\lambda_{\ell,g}(S^\varepsilon_\ell)(\nabla p^\varepsilon_{\ell,g} - \varrho_g(p^\varepsilon_{\ell,g})\mathbf{g}). \end{aligned}$$

Now we specify the boundary and initial conditions. We suppose that the boundary $\partial\Omega$ consists of two parts Γ_{inj} and Γ_{imp} such that $\Gamma_{inj} \cap \Gamma_{imp} = \emptyset$, $\partial\Omega = \overline{\Gamma}_{inj} \cup \overline{\Gamma}_{imp}$. The boundary conditions are given by:

$$\begin{cases} p^\varepsilon_{\ell,g}(x,t) = p^\varepsilon_{\ell,w}(x,t) = 0 & \text{on } \Gamma_{inj} \times (0,T); \\ \mathbf{q}^\varepsilon_{\ell,w} \cdot \boldsymbol{\nu} = \mathbf{q}^\varepsilon_{\ell,g} \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_{imp} \times (0,T). \end{cases} \tag{2.15}$$

Finally, the initial conditions read:

$$\mathsf{p}^\varepsilon_w(x,0) = \mathsf{p}^{\mathbf{0}}_w(x) \quad \text{and} \quad \mathsf{p}^\varepsilon_g(x,0) = \mathsf{p}^{\mathbf{0}}_g(x) \quad \text{in } \Omega, \tag{2.16}$$

where

$$\begin{aligned} \mathsf{p}^\varepsilon_g(x,t) &\stackrel{\text{def}}{=} p^\varepsilon_{1,g}(x,t)\mathbf{1}^\varepsilon_1(x) + p^\varepsilon_{2,g}(x,t)\mathbf{1}^\varepsilon_2(x); \\ \mathsf{p}^\varepsilon_w(x,t) &\stackrel{\text{def}}{=} p^\varepsilon_{1,w}(x,t)\mathbf{1}^\varepsilon_1(x) + p^\varepsilon_{2,w}(x,t)\mathbf{1}^\varepsilon_2(x), \end{aligned} \tag{2.17}$$

with $\mathbf{1}^\varepsilon_\ell(x) = \mathbf{1}_\ell(\frac{x}{\varepsilon})$ being the characteristic function of the subdomain Ω^ε_ℓ.

2.1. *Global pressure and useful relations*

In what follows we will make use of the so-called *global pressure* as introduced in Refs. 9 and 18 for incompressible immiscible two-phase flow. It plays a crucial mathematical role, in particular, for compactness results. The idea of introducing the global pressure for each subdomain Ω^ε_ℓ is as follows. We replace the water–gas flow in the corresponding medium by a flow of a fictive fluid obeying Darcy's law with a non-degenerating coefficient. Namely, we are looking for a pressure $\mathsf{P}^\varepsilon_\ell$ and the coefficient $\gamma_\ell(S^\varepsilon_\ell)$ such that $\gamma_\ell(S^\varepsilon_\ell) > 0$ in $[0,1]$ and

$$\lambda_{\ell,w}(S^\varepsilon_\ell)\nabla p^\varepsilon_{\ell,w} + \lambda_{\ell,g}(S^\varepsilon_\ell)\nabla p^\varepsilon_{\ell,g} = \gamma_\ell(S^\varepsilon_\ell)\nabla \mathsf{P}^\varepsilon_\ell. \tag{2.18}$$

Now, for each subdomain Ω^ε_ℓ, we define the global pressure $\mathsf{P}^\varepsilon_\ell$ as follows:

$$p^\varepsilon_{\ell,w} \stackrel{\text{def}}{=} \mathsf{P}^\varepsilon_\ell + \mathsf{G}_{\ell,w}(S^\varepsilon_\ell) \quad \text{and} \quad p^\varepsilon_{\ell,g} \stackrel{\text{def}}{=} \mathsf{P}^\varepsilon_\ell + \mathsf{G}_{\ell,g}(S^\varepsilon_\ell), \tag{2.19}$$

where the functions $\mathsf{G}_{\ell,w}(s)$, $\mathsf{G}_{\ell,g}(s)$ are defined by

$$\mathsf{G}_{\ell,g}(s) \stackrel{\text{def}}{=} \mathsf{G}_{\ell,g}(0) + \int_0^s \frac{\lambda_{\ell,w}(\tau)}{\lambda_\ell(\tau)} P'_{\ell,c}(\tau) d\tau \tag{2.20}$$

and

$$\mathsf{G}_{\ell,w}(s) \stackrel{\text{def}}{=} \mathsf{G}_{\ell,g}(s) - P_{\ell,c}(s); \tag{2.21}$$

here

$$\lambda_\ell(s) \stackrel{\text{def}}{=} \lambda_{\ell,w}(s) + \lambda_{\ell,g}(s). \tag{2.22}$$

Then it is easy to see that (2.18) is satisfied with $\gamma_\ell = \lambda_\ell$. Moreover, it is straightforward to check that

$$\nabla \mathsf{G}_{\ell,w}(S_\ell^\varepsilon) = -\frac{\lambda_{\ell,g}(S_\ell^\varepsilon)}{\lambda_\ell(S_\ell^\varepsilon)} P'_{\ell,c}(S_\ell^\varepsilon) \nabla S_\ell^\varepsilon. \tag{2.23}$$

From (2.20) and (2.23) we get:

$$\begin{aligned} \lambda_{\ell,w}(S_\ell^\varepsilon) \nabla \mathsf{G}_{\ell,w}(S_\ell^\varepsilon) &= \alpha_\ell(S_\ell^\varepsilon) \nabla S_\ell^\varepsilon, \\ \lambda_{\ell,g}(S_\ell^\varepsilon) \nabla \mathsf{G}_{\ell,g}(S_\ell^\varepsilon) &= -\alpha_\ell(S_\ell^\varepsilon) \nabla S_\ell^\varepsilon, \end{aligned} \tag{2.24}$$

where

$$\alpha_\ell(s) \stackrel{\text{def}}{=} \frac{\lambda_{\ell,g}(s)\lambda_{\ell,w}(s)}{\lambda_\ell(s)} |P'_{\ell,c}(s)|. \tag{2.25}$$

Performing some simple calculations, the following relation can be obtained as in Refs. 9 and 18:

$$\begin{aligned} &\lambda_{\ell,g}(S_\ell^\varepsilon)|\nabla p_{\ell,g}^\varepsilon|^2 + \lambda_{\ell,w}(S_\ell^\varepsilon)|\nabla p_{\ell,w}^\varepsilon|^2 \\ &\quad = \lambda_\ell(S_\ell^\varepsilon)|\nabla \mathsf{P}_\ell|^2 + \frac{\lambda_{\ell,w}(S_\ell^\varepsilon)\lambda_{\ell,g}(S_\ell^\varepsilon)}{\lambda_\ell(S_\ell^\varepsilon)} |\nabla P_{\ell,c}(S_\ell^\varepsilon)|^2. \end{aligned} \tag{2.26}$$

If we use the global pressure and the saturation as new unknown functions, then (2.12) reads:

$$\begin{cases} \Phi^\varepsilon(x) \dfrac{\partial S_\ell^\varepsilon}{\partial t} - \operatorname{div}(K^\varepsilon(x)[\lambda_{\ell,w}(S_\ell^\varepsilon) \nabla \mathsf{P}_\ell^\varepsilon + \nabla \beta_\ell(S_\ell^\varepsilon) \\ \quad - \lambda_{\ell,w}(S_\ell^\varepsilon)\mathbf{g}]) = 0 & \text{in } \Omega_{\ell,T}^\varepsilon; \\ \Phi^\varepsilon(x) \dfrac{\partial \Theta_\ell^\varepsilon}{\partial t} - \operatorname{div}(K^\varepsilon(x) \varrho_{\ell,g}^\varepsilon [\lambda_{\ell,g}(S_\ell^\varepsilon) \nabla \mathsf{P}_\ell^\varepsilon \\ \quad - \nabla \beta_\ell(S_\ell^\varepsilon) - \lambda_{\ell,g}(S_\ell^\varepsilon) \varrho_{\ell,g}^\varepsilon \mathbf{g}]) = 0 & \text{in } \Omega_{\ell,T}^\varepsilon, \end{cases} \tag{2.27}$$

where for brevity we introduced the notation

$$\Theta_\ell^\varepsilon = \Theta_\ell^\varepsilon(S_\ell^\varepsilon, \mathsf{P}_\ell^\varepsilon) \stackrel{\text{def}}{=} \varrho_g(\mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,g}(S_\ell^\varepsilon))(1 - S_\ell^\varepsilon) = (1 - S_\ell^\varepsilon)\varrho_{\ell,g}^\varepsilon; \tag{2.28}$$

$$\varrho_{\ell,g}^\varepsilon = \varrho_g(\mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,g}(S_\ell^\varepsilon)) \tag{2.29}$$

and

$$\beta_\ell(s) \stackrel{\text{def}}{=} \int_0^s \alpha_\ell(\xi) d\xi. \tag{2.30}$$

We have to complete system (2.27) by the corresponding interface, boundary, and initial conditions. We start by considering the interface conditions (2.14). The corresponding conditions for the fluxes read:

$$\mathbf{q}_{1,w}^\varepsilon \cdot \boldsymbol{\nu} = \mathbf{q}_{2,w}^\varepsilon \cdot \boldsymbol{\nu} \quad \text{and} \quad \mathbf{q}_{1,g}^\varepsilon \cdot \boldsymbol{\nu} = \mathbf{q}_{2,g}^\varepsilon \cdot \boldsymbol{\nu} \quad \text{on } \Sigma_{1,2,T}^\varepsilon, \tag{2.31}$$

where the fluxes $\mathbf{q}_{\ell,w}^\varepsilon, \mathbf{q}_{\ell,g}^\varepsilon$, expressed in terms of global pressure and saturation are given by:

$$\mathbf{q}_{\ell,w}^\varepsilon \stackrel{\text{def}}{=} -K^\varepsilon(x)[\lambda_{\ell,w}(S_\ell^\varepsilon)\nabla \mathsf{P}_\ell^\varepsilon + \nabla\beta_\ell(S_\ell^\varepsilon) - \lambda_{\ell,w}(S_\ell^\varepsilon)\mathbf{g}]; \tag{2.32}$$

$$\mathbf{q}_{\ell,g}^\varepsilon \stackrel{\text{def}}{=} -K^\varepsilon(x)[\lambda_{\ell,g}(S_\ell^\varepsilon)\nabla \mathsf{P}_\ell^\varepsilon - \nabla\beta_\ell(S_\ell^\varepsilon) - \lambda_{\ell,g}(S_\ell^\varepsilon)\widetilde{\varrho}_{\ell,g}^\varepsilon\mathbf{g}]. \tag{2.33}$$

We turn to the continuity of the phase pressures. We recall that

$$p_{1,w}^\varepsilon = p_{2,w}^\varepsilon \quad \text{and} \quad p_{1,g}^\varepsilon = p_{2,g}^\varepsilon \quad \text{on } \Sigma_{1,2,T}^\varepsilon. \tag{2.34}$$

As an immediate consequence of the definition of the global pressure and (2.34), we have

$$\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^\varepsilon) = \mathsf{P}_2^\varepsilon + \mathsf{G}_{2,w}(S_2^\varepsilon) \quad \text{and} \quad \mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(S_1^\varepsilon) = \mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(S_2^\varepsilon) \quad \text{on } \Sigma_{1,2,T}^\varepsilon. \tag{2.35}$$

Remark 2.1. Notice that P_1^ε need not be equal to P_2^ε on $\Sigma_{1,2,T}^\varepsilon$. Thus, the global pressure function might be discontinuous at the interface. Also, $\mathsf{G}_{1,w}(S_1^\varepsilon)$ need not coincide with $\mathsf{G}_{2,w}(S_2^\varepsilon)$ on $\Sigma_{1,2,T}^\varepsilon$. This makes the compactness result in Sec. 4 nontrivial.

On the other hand, it follows from the definition of the capillary pressure (2.10) and (2.34) that

$$P_{1,c}(S_1^\varepsilon) = P_{2,c}(S_2^\varepsilon) \quad \text{on } \Sigma_{1,2,T}^\varepsilon. \tag{2.36}$$

Thus, the new interface conditions read:

$$\begin{cases} \mathbf{q}_{1,w}^\varepsilon \cdot \boldsymbol{\nu} = \mathbf{q}_{2,w}^\varepsilon \cdot \boldsymbol{\nu} \quad \text{and} \quad \mathbf{q}_{1,g}^\varepsilon \cdot \boldsymbol{\nu} = \mathbf{q}_{2,g}^\varepsilon \cdot \boldsymbol{\nu} & \text{on } \Sigma_{1,2,T}^\varepsilon; \\ \mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^\varepsilon) = \mathsf{P}_2^\varepsilon + \mathsf{G}_{2,w}(S_2^\varepsilon) & \text{on } \Sigma_{1,2,T}^\varepsilon; \\ \mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(S_1^\varepsilon) = \mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(S_2^\varepsilon) & \text{on } \Sigma_{1,2,T}^\varepsilon; \\ P_{1,c}(S_1^\varepsilon) = P_{2,c}(S_2^\varepsilon) & \text{on } \Sigma_{1,2,T}^\varepsilon. \end{cases} \tag{2.37}$$

Consider now the boundary conditions. Since

$$p_{\ell,g}^\varepsilon(x,t) - p_{\ell,w}^\varepsilon(x,t) = P_{\ell,c}(S_\ell^\varepsilon) = 0 \quad \text{on } \Gamma_{inj} \times (0,T),$$

then it follows from (2.10) that $S_\ell^\varepsilon = 1$ on $\Gamma_{inj} \times (0,T)$. This boundary condition along with (2.19) imply that $\mathsf{P}_\ell^\varepsilon = \text{const.}$ on $\Gamma_{inj} \times (0,T)$. For the sake of simplicity, we will assume that this constant is equal to zero. Thus, the boundary conditions for system (2.27) read:

$$\begin{cases} S_\ell^\varepsilon(x,t) = 1 \quad \text{and} \quad \mathsf{P}_\ell^\varepsilon(x,t) = 0 & \text{on } \Gamma_{inj} \times (0,T); \\ \mathbf{q}_{\ell,w}^\varepsilon \cdot \boldsymbol{\nu} = \mathbf{q}_{\ell,g}^\varepsilon \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_{imp} \times (0,T), \end{cases} \tag{2.38}$$

where the fluxes $\mathbf{q}_{\ell,w}^\varepsilon, \mathbf{q}_{\ell,g}^\varepsilon$, written in terms of the global pressure and the saturation, are given by (2.32)–(2.33).

Finally, the initial conditions read:

$$S_\ell^\varepsilon(x,0) = S_\ell^{\mathbf{0}}(x) \quad \text{and} \quad \mathsf{P}_\ell^\varepsilon(x,0) = \mathsf{P}_\ell^{\mathbf{0}}(x) \quad \text{in } \Omega_\ell^\varepsilon, \tag{2.39}$$

where the initial datum for the global pressure, i.e. $\mathsf{P}_\ell^{\mathbf{0}}$, can be calculated from the corresponding data for $p_{\ell,g}^\varepsilon$ using relations (2.19) and (2.20).

2.2. *Main assumptions*

The main assumptions on the data are as follows:

(A.1) The function $\Phi = \Phi(y)$ is a Y-periodic function given by:

$$\Phi(y) \stackrel{\text{def}}{=} \Phi_1 \mathbf{1}_1(y) + \Phi_2 \mathbf{1}_2(y), \tag{2.40}$$

where Φ_1, Φ_2 are constants such that $0 < \Phi_1, \Phi_2 < 1$.

(A.2) The tensor $K = K(y)$ is a Y-periodic function, it belongs to $(L^\infty(Y))^{3\times 3}$. Moreover, there exist positive constants $K_\pm$ such that

$$K_-|\xi|^2 \le (K(y)\xi,\xi) \le K_+|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3, \text{ a.e. in } Y. \tag{2.41}$$

(A.3) The function $\varrho_g = \varrho_g(p)$ given by (2.7) is a monotone C^1-function in $\mathbb{R}$.

(A.4) The capillary pressure function $P_{\ell,c}(s) \in C^1([0,1];\mathbb{R}^+)$. Moreover, $P'_{\ell,c}(s) < 0$ in $[0,1]$ and $P_{\ell,c}(1) = 0$. We also assume that $P_{1,c}(0) = P_{2,c}(0)$.

(A.5) The functions $\lambda_{\ell,w}, \lambda_{\ell,g}$ belong to the space $C([0,1];\mathbb{R}^+)$ and satisfy the following properties:

- (i) $0 < \lambda_{\ell,w}, \lambda_{\ell,g} < 1$ in $(0,1)$;
- (ii) $\lambda_{\ell,w}(0) = 0$ and $\lambda_{\ell,g}(1) = 0$;
- (iii) there is a positive constant L_0 such that $\lambda_\ell(s) = \lambda_{\ell,w}(s) + \lambda_{\ell,g}(s) \ge L_0 > 0$ in $[0,1]$.

(A.6) The function $\alpha_\ell \in C^1([0,1];\mathbb{R}^+)$. Moreover, $\alpha_\ell(0) = \alpha_\ell(1) = 0$ and $\alpha_\ell > 0$ in $(0,1)$.

(A.7) The function β_ℓ^{-1}, inverse of β_ℓ defined in (2.30) is a Hölder continuous function of order θ with $\theta \in (0,1)$ on the interval $[0,\beta_\ell(1)]$. That is, there exists a positive constant C_β such that for all $s_1, s_2 \in [0,\beta_\ell(1)]$ the following inequality holds:

$$|\beta_\ell^{-1}(s_1) - \beta_\ell^{-1}(s_2)| \le C_\beta |s_1 - s_2|^\theta. \tag{2.42}$$

(A.8) The initial data for the phase pressures defined in (2.16) are such that $\mathsf{p}_g^{\mathbf{0}}, \mathsf{p}_w^{\mathbf{0}} \in L^2(\Omega)$.

(A.9) The initial data for the saturation is such that $\mathsf{S}_{\mathbf{0}}^\varepsilon \in L^\infty(\Omega)$ and $0 \le \mathsf{S}_{\mathbf{0}}^\varepsilon \le 1$ a.e. in Ω.

Remark 2.2. The assumptions (A.1)–(A.9) are classical and physically meaningful for two-phase flow in porous media. They are similar to the assumptions made in our previous work[8] that dealt with the existence of a weak solution of the studied problem.

Remark 2.3. Some of the assumptions (A.1)–(A.9) can be weakened at the price of additional technical steps in the proofs. In particular, in (A.1) we can assume that Φ_1 and Φ_2 are not constant but periodic functions of y. Also, in (A.4) it suffices to assume that

$$P_{l,c} \in C([0,1];\mathbb{R}^+), \quad P'_{l,c} < 0 \quad \text{on } (0,1), \quad P_{l,c}(1) = 0 \quad \text{and} \quad P_{l,c}(0) = P_{2,c}(0).$$

In this case the derivative of $P'_{l,c}$ can have a singularity at the end points of the interval $(0,1)$. Under these assumptions the compactness results required for the existence of a solution and for the homogenization remain valid. The case of unbounded capillary pressure is an open problem.

2.3. *Definition of a weak solution*

In order to formulate the main result of the paper, we introduce the following Sobolev space:

$$H^1_{\Gamma_{inj}}(\Omega) \stackrel{\text{def}}{=} \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_{inj}\}.$$

The space $H^1_{\Gamma_{inj}}(\Omega)$ is a Hilbert space. The norm in this space is given by $\|u\|_{H^1_{\Gamma_{inj}}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^3}$.

In what follows we make use of two equivalent definitions of the weak solutions of our problem.

First, we introduce the notation:

$$\begin{aligned} \mathsf{P}^\varepsilon(x,t) &\stackrel{\text{def}}{=} \mathsf{P}_1^\varepsilon(x,t)\mathbf{1}_1^\varepsilon(x) + \mathsf{P}_2^\varepsilon(x,t)\mathbf{1}_2^\varepsilon(x); \\ \mathsf{S}^\varepsilon(x,t) &\stackrel{\text{def}}{=} S_1^\varepsilon(x,t)\mathbf{1}_1^\varepsilon(x) + S_2^\varepsilon(x,t)\mathbf{1}_2^\varepsilon(x). \end{aligned} \tag{2.43}$$

We start by the definition in terms of the phase pressures.

Definition 2.1. We say that the pair of functions $\langle \mathsf{p}_w^\varepsilon, \mathsf{p}_g^\varepsilon \rangle$ is a weak solution to problem (2.12)–(2.16) if:

(i) $0 \le \mathsf{S}^\varepsilon \le 1$ a.e. in Ω_T.

(ii) $\mathsf{p}_w^\varepsilon, \mathsf{p}_g^\varepsilon \in L^2(\Omega_T)$, $\lambda_w(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon)\nabla p_w^\varepsilon \in L^2(\Omega_T)$, $\lambda_g(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon)\nabla p_g^\varepsilon \in L^2(\Omega_T)$.
(iii) The interface and boundary conditions (2.14), (2.15) are satisfied.
(iv) For any $\varphi_w, \varphi_g \in C^1([0,T]; H^1_{\Gamma_{inj}}(\Omega))$ satisfying $\varphi_w(T) = \varphi_g(T) = 0$, we have:

Saturation equation:

$$
\begin{aligned}
&-\int_{\Omega_T} \Phi^\varepsilon(x)\mathsf{S}^\varepsilon \frac{\partial \varphi_w}{\partial t} dxdt + \int_\Omega \Phi^\varepsilon(x)\mathsf{S}^\varepsilon_{\mathbf{0}}(x)\varphi_w(x,0)dx \\
&\quad + \int_{\Omega_T} K^\varepsilon(x)\lambda_w\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)\nabla \mathsf{p}_w^\varepsilon \cdot \nabla\varphi_w dxdt \\
&\quad - \int_{\Omega_T} K^\varepsilon(x)\lambda_w\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)\mathbf{g}\cdot\nabla\varphi_w dxdt = 0. \qquad (2.44)
\end{aligned}
$$

Pressure equation:

$$
\begin{aligned}
&-\int_{\Omega_T} \Phi^\varepsilon(x)\Theta^\varepsilon \frac{\partial \varphi_g}{\partial t} dxdt + \int_\Omega \Phi^\varepsilon(x)\Theta^\varepsilon_{\mathbf{0}}(x)\varphi_g(x,0)dx \\
&\quad + \int_{\Omega_T} K^\varepsilon(x)\lambda_g\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)\varrho_g(\mathsf{p}_g^\varepsilon)\nabla \mathsf{p}_g \cdot \nabla\varphi_w dxdt \\
&\quad - \int_{\Omega_T} K^\varepsilon(x)\lambda_g\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)[\varrho_g(\mathsf{p}_g^\varepsilon)]^2\mathbf{g}\cdot\nabla\varphi_w dxdt = 0, \qquad (2.45)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_w\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right) &\stackrel{\text{def}}{=} \lambda_{1,w}(S_1^\varepsilon)\mathbf{1}_1^\varepsilon(x) + \lambda_{2,w}(S_2^\varepsilon)\mathbf{1}_2^\varepsilon(x), \\
\lambda_g\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right) &\stackrel{\text{def}}{=} \lambda_{1,g}(S_1^\varepsilon)\mathbf{1}_1^\varepsilon(x) + \lambda_{2,g}(S_2^\varepsilon)\mathbf{1}_2^\varepsilon(x).
\end{aligned} \qquad (2.46)
$$

$$
\Theta^\varepsilon(x,t) \stackrel{\text{def}}{=} \Theta_1^\varepsilon(x,t)\mathbf{1}_1^\varepsilon(x) + \Theta_2^\varepsilon(x,t)\mathbf{1}_2^\varepsilon(x,t), \qquad (2.47)
$$

$$
\mathsf{S}^\varepsilon_{\mathbf{0}}(x) \stackrel{\text{def}}{=} S_1^{\mathbf{0}}(x)\mathbf{1}_1^\varepsilon(x) + S_2^{\mathbf{0}}(x)\mathbf{1}_2^\varepsilon(x), \qquad (2.48)
$$

$$
\Theta^\varepsilon_{\mathbf{0}}(x) \stackrel{\text{def}}{=} \Theta_1^{\mathbf{0}}(x)\mathbf{1}_1^\varepsilon(x) + \Theta_2^{\mathbf{0}}(x)\mathbf{1}_2^\varepsilon(x). \qquad (2.49)
$$

Definition 2.2. We say that the pair of functions $\langle \mathsf{P}^\varepsilon, \mathsf{S}^\varepsilon\rangle$ is a weak solution to problem (2.27)–(2.29), (2.37)–(2.39) if:

(i) $0 \le \mathsf{S}^\varepsilon \le 1$ a.e. in Ω_T.
(ii) $\mathsf{P}_1^\varepsilon \in L^2(0,T; H^1_{\Gamma_{inj}}(\Omega_1^\varepsilon))$ and $\mathsf{P}_2^\varepsilon \in L^2(0,T; H^1(\Omega_2^\varepsilon))$.
(iii) The interface and boundary conditions (2.37), (2.38) are satisfied.
(iv) For any $\varphi_w, \varphi_g \in C^1([0,T]; H^1_{\Gamma_{inj}}(\Omega))$ satisfying $\varphi_w(T) = \varphi_g(T) = 0$, we have:

Saturation equation:

$$
\begin{aligned}
-\int_{\Omega_T} \Phi^\varepsilon(x)\mathsf{S}^\varepsilon \frac{\partial \varphi_w}{\partial t} dxdt + \int_\Omega \Phi^\varepsilon(x)\mathsf{S}^\varepsilon_{\mathbf{0}}(x)\varphi_w(0,x)dx \\
+\sum_{\ell=1}^{2}\int_{\Omega^\varepsilon_{\ell,T}} K^\varepsilon(x)\nabla\beta_\ell(S^\varepsilon_\ell)\cdot\nabla\varphi_w dxdt \\
+\sum_{\ell=1}^{2}\int_{\Omega^\varepsilon_{\ell,T}} K^\varepsilon(x)\lambda_{\ell,w}(S^\varepsilon_\ell)(\nabla\mathsf{P}^\varepsilon_\ell - \mathbf{g})\cdot\nabla\varphi_w dxdt = 0.
\end{aligned}
\tag{2.50}
$$

Pressure equation:

$$
\begin{aligned}
-\int_{\Omega_T} \Phi^\varepsilon(x)\Theta^\varepsilon \frac{\partial \varphi_g}{\partial t} dxdt + \int_\Omega \Phi^\varepsilon(x)\Theta^\varepsilon_{\mathbf{0}}(x)\varphi_g(0,x)dx \\
-\sum_{\ell=1}^{2}\int_{\Omega^\varepsilon_{\ell,T}} K^\varepsilon(x)\varrho_g(p^\varepsilon_{\ell,g})\nabla\beta_\ell(S^\varepsilon_\ell)\cdot\nabla\varphi_g dxdt \\
+\sum_{\ell=1}^{2}\int_{\Omega^\varepsilon_{\ell,T}} K^\varepsilon(x)\lambda_{\ell,g}(S^\varepsilon_\ell)\varrho_g(p^\varepsilon_{\ell,g})(\nabla\mathsf{P}^\varepsilon_\ell - \varrho_g(p^\varepsilon_{\ell,g})\mathbf{g})\cdot\nabla\varphi_g dxdt = 0.
\end{aligned}
\tag{2.51}
$$

According to Ref. 8, for each $\varepsilon > 0$, problem (2.12)–(2.16) (or equivalent problem (2.27)–(2.29), (2.37)–(2.39)) has at least one weak solution.

The rest of the paper is organized as follows. In Sec. 3, we establish the *a priori* uniform estimates for solutions of problem (2.27), (2.37)–(2.39). Then in Sec. 4 we obtain extension and compactness results for the sequences $\{\mathsf{P}^\varepsilon_\ell\}_{\varepsilon>0}$, $\{S^\varepsilon_\ell\}_{\varepsilon>0}$, $\{\Theta^\varepsilon_\ell\}_{\varepsilon>0}$. In Sec. 5, we formulate the main result of the paper which is then proved in Sec. 6.

Notational convention. In what follows $C, C_1, \ldots$ denote generic constants that do not depend on ε.

3. Uniform Estimates for Solutions of Problem (2.27)–(2.29), (2.37)–(2.39)

In this section we establish *a priori* estimates for solution of (2.27)–(2.29), (2.37)–(2.39). These estimates rely on a special choice of test functions in the integral identities (2.50)–(2.51). We use suitable test functions introduced in Ref. 22 in the case of a single rock type model.

To obtain the *energy equality* in the case of a porous medium made of two types of rocks, we introduce the functions:

$$
\mathsf{R}_{\ell,w}(p^\varepsilon_{\ell,w}) \stackrel{\text{def}}{=} \int_0^{p^\varepsilon_{\ell,w}} d\xi = p^\varepsilon_{\ell,w} \quad \text{and} \quad \mathsf{R}_{\ell,g}(p^\varepsilon_{\ell,g}) \stackrel{\text{def}}{=} \int_0^{p^\varepsilon_{\ell,g}} \frac{d\xi}{\varrho_g(\xi)}. \tag{3.1}
$$

It is clear that

$$
\nabla\mathsf{R}_{\ell,w}(p^\varepsilon_{\ell,w}) = \nabla p^\varepsilon_{\ell,w} \quad \text{and} \quad \nabla\mathsf{R}_{\ell,g}(p^\varepsilon_{\ell,g}) = \frac{1}{\varrho^\varepsilon_{\ell,g}(p^\varepsilon_{\ell,g})}\nabla p^\varepsilon_{\ell,g}.
$$

Then, following the lines of Refs. 22 and 3, one can prove the following statement.

Lemma 3.1. (Energy equality) *Let* $\langle \mathsf{p}_g^\varepsilon, \mathsf{p}_w^\varepsilon \rangle$ *be a solution to problem* (2.12)–(2.16). *Then*

$$\frac{d}{dt}\int_\Omega \Phi^\varepsilon(x)\mathcal{E}^\varepsilon(x,t)dx + \int_\Omega K^\varepsilon(x)\left\{\lambda_w\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)\nabla \mathsf{p}_w^\varepsilon \cdot (\nabla \mathsf{p}_w^\varepsilon - \mathbf{g}) \right.$$
$$\left. + \lambda_g\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)\varrho_g(\mathsf{p}_g^\varepsilon)\nabla \mathsf{p}_g^\varepsilon \cdot (\nabla \mathsf{p}_g^\varepsilon - \varrho_g(\mathsf{p}_g^\varepsilon)\mathbf{g})\right\} dx = 0, \tag{3.2}$$

in the sense of distributions. Here S^ε *is defined in* (2.43),

$$\mathcal{E}^\varepsilon(x,t) \stackrel{\text{def}}{=} \mathcal{E}_1^\varepsilon(x,t)\mathbf{1}_1^\varepsilon(x) + \mathcal{E}_2^\varepsilon(x,t)\mathbf{1}_2(x),$$

with

$$\mathcal{E}_\ell^\varepsilon \stackrel{\text{def}}{=} (1 - S_\ell^\varepsilon)\mathcal{R}_\ell(p_{\ell,g}^\varepsilon) - \digamma(S_\ell^\varepsilon), \tag{3.3}$$

where

$$\digamma_\ell(s) \stackrel{\text{def}}{=} \int_0^s P_{\ell,c}(\xi)d\xi \quad \textit{and} \quad \mathcal{R}_\ell(p) \stackrel{\text{def}}{=} \varrho_g(p)\mathsf{R}_{\ell,g}(p) - p. \tag{3.4}$$

Furthermore, $\mathcal{R}_\ell(p) \geq 0$ *for all* $p \in \mathbb{R}$.

As a consequence of the inequality $\mathcal{R}_\ell(p) \geq 0$ and condition (A.4), we have

$$\mathcal{E}_\ell^\varepsilon = (1 - S_\ell^\varepsilon)\mathcal{R}_\ell(p_{\ell,g}^\varepsilon) - \digamma(S_\ell^\varepsilon) \geq -\digamma(1) \geq -\max_{S_\ell^\varepsilon \in [0,1]} P_{\ell,c}(S_\ell^\varepsilon). \tag{3.5}$$

Our next goal is to obtain *a priori* estimates for $\sqrt{\lambda_w(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon)}\nabla \mathsf{p}_w^\varepsilon$ and $\sqrt{\lambda_g(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon)}\nabla \mathsf{p}_g^\varepsilon$.

Lemma 3.2. *Let* $\langle \mathsf{p}_g^\varepsilon, \mathsf{p}_w^\varepsilon \rangle$ *be a solution to problem* (2.12)–(2.16). *Then*

$$\int_{\Omega_T}\left\{\lambda_w\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)|\nabla \mathsf{p}_w^\varepsilon|^2 + \lambda_g\left(\frac{x}{\varepsilon}, \mathsf{S}^\varepsilon\right)|\nabla \mathsf{p}_g^\varepsilon|^2\right\} dx \leq C. \tag{3.6}$$

Proof of Lemma 3.2. Consider the energy equality (3.2). We integrate (3.2) with respect to $t \in (0,T)$. Then the statement of the lemma immediately follows from the boundedness of the function $\mathcal{E}_\ell^\varepsilon$, (3.5), and the Cauchy inequality. Lemma 3.2 is proved. □

Now we are in a position to obtain the *a priori* estimates for $\mathsf{P}_\ell^\varepsilon$ and $\beta_\ell(S_\ell^\varepsilon)$. We have.

Lemma 3.3. *Let* $\langle \mathsf{p}_g^\varepsilon, \mathsf{p}_w^\varepsilon\rangle$ *be a solution of problem* (2.12)–(2.16) *and let* $\{\mathsf{P}_\ell^\varepsilon\}_{\varepsilon>0}$ *and* $\{\beta_\ell(S_\ell^\varepsilon)\}_{\varepsilon>0}$ *be the sequences of functions defined in* (2.19) *and* (2.30), *respectively. Then*

$$\{\mathsf{P}_1^\varepsilon\}_{\varepsilon>0} \textit{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_{inj}}(\Omega_1^\varepsilon)); \tag{3.7}$$

$$\{\mathsf{P}_2^\varepsilon\}_{\varepsilon>0} \textit{ is uniformly bounded in } L^2(0,T;H^1(\Omega_2^\varepsilon)); \tag{3.8}$$

$$\{\beta_\ell(S_\ell^\varepsilon)\}_{\varepsilon>0} \textit{ is uniformly bounded in } L^2(0,T;H^1(\Omega_\ell^\varepsilon)). \tag{3.9}$$

Proof of Lemma 3.3. It is easy to see that equality (2.26) along with Lemma 3.2 and the definition of the function β_ℓ immediately imply the following bound:

$$\int_{\Omega_{\ell,T}^\varepsilon} \{|\nabla \mathsf{P}_\ell^\varepsilon|^2 + |\nabla\beta_\ell(S_\ell^\varepsilon)|^2\}dx \le C, \quad \ell = 1,2. \tag{3.10}$$

Now the uniform boundedness of the sequence $\{\mathsf{P}_1^\varepsilon\}_{\varepsilon>0}$ in the space $L^2(0,T; H^1_{\Gamma_{inj}}(\Omega_1))$ follows from (3.10), the boundary condition (2.38), and the Friedrich inequality.

Consider now the sequence $\{\mathsf{P}_2^\varepsilon\}_{\varepsilon>0}$. As opposed to the function P_1^ε, the Friedrich inequality does not apply to P_2^ε. Therefore, we proceed in another way. We make use of the ideas from Ref. 20. As $\mathsf{P}_2^\varepsilon \in L^2(0,T;H^1(\Omega_2^\varepsilon))$ and $\mathsf{P}_1^\varepsilon \in L^2(0,T;H^1_{\Gamma_{inj}}(\Omega_1^\varepsilon))$, with techniques proposed in Ref. 20 p. 1055, we have:

$$\|\mathsf{P}_2^\varepsilon\|_{L^2(\Omega_{2,T}^\varepsilon)} \le C(\varepsilon\|\nabla \mathsf{P}_2^\varepsilon\|_{L^2(\Omega_{2,T}^\varepsilon)} + \sqrt{\varepsilon}\|\mathsf{P}_2^\varepsilon\|_{L^2(\Sigma_{1,2,T}^\varepsilon)}). \tag{3.11}$$

Then due to the definition of the global pressure P_2^ε, (2.19) and the interface condition (2.35) one can obtain the following estimate:

$$\begin{aligned}\|\mathsf{P}_2^\varepsilon\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} &\le \|\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,w}(S_2^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} + \|\mathsf{G}_{2,w}(S_2^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)}\\ &= \|\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} + \|\mathsf{G}_{2,w}(S_2^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)}\\ &\le \|\mathsf{P}_1^\varepsilon\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} + \|\mathsf{G}_{1,w}(S_2^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} + \|\mathsf{G}_{2,w}(S_2^\varepsilon)\|_{L^2(\Sigma_{1,2,T}^\varepsilon)}.\end{aligned} \tag{3.12}$$

Now, taking into account the boundedness of the functions $\mathsf{G}_{\ell,w}(S_\ell^\varepsilon)$, the geometry of the subdomain $\Omega_{2,T}^\varepsilon$, (3.12), and the estimate

$$\sqrt{\varepsilon}\|\mathsf{P}_1^\varepsilon\|_{L^2(\Sigma_{1,2,T}^\varepsilon)} \le C(\varepsilon\|\nabla \mathsf{P}_1^\varepsilon\|_{L^2(\Omega_{1,T}^\varepsilon)} + \|\mathsf{P}_1^\varepsilon\|_{L^2(\Omega_{1,T}^\varepsilon)}), \tag{3.13}$$

we, finally, obtain that

$$\|\mathsf{P}_2^\varepsilon\|_{L^2(\Omega_{2,T}^\varepsilon)} \le C. \tag{3.14}$$

Finally, consider the sequence $\{\beta_\ell(S_\ell^\varepsilon)\}_{\varepsilon>0}$. The desired upper bound (3.9) follows from condition (A.6) and (3.10). This completes the proof of Lemma 3.3. □

Let Θ^ε and S^ε be the functions defined in (2.47) and (2.43), respectively. The uniform bounds of the time derivatives of Θ^ε and S^ε are given by the following:

Lemma 3.4. *Let* $\langle \mathsf{p}_g^\varepsilon, \mathsf{p}_w^\varepsilon\rangle$ *be a solution to problem* (2.12)–(2.16). *Then*

$$\{\partial_t(\Phi^\varepsilon\Theta^\varepsilon)\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega)); \tag{3.15}$$

$$\{\partial_t(\Phi^\varepsilon\mathsf{S}^\varepsilon)\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{3.16}$$

Proof of Lemma 3.4. First, we rewrite system (2.12) as follows:

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial \mathsf{S}^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\lambda_w\left(\dfrac{x}{\varepsilon},\mathsf{S}^\varepsilon\right)(\nabla \mathsf{p}_w^\varepsilon - \mathbf{g})\right) = 0 & \text{in } \Omega_T; \\ \Phi^\varepsilon(x)\dfrac{\partial \Theta^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\lambda_g\left(\dfrac{x}{\varepsilon},\mathsf{S}^\varepsilon\right)\varrho_g(\mathsf{p}_g^\varepsilon)(\nabla \mathsf{p}_g^\varepsilon \right. & \\ \qquad \left. - \varrho_g(\mathsf{p}_g^\varepsilon)\mathbf{g})\right) = 0 & \text{in } \Omega_T; \\ P_c\left(\dfrac{x}{\varepsilon},\mathsf{S}^\varepsilon\right) = \mathsf{p}_g^\varepsilon - \mathsf{p}_w^\varepsilon & \text{in } \Omega_T. \end{cases} \tag{3.17}$$

Consider the first statement of the lemma. Multiplying the second equation in (3.17) by $\varphi_g \in \mathcal{D}(\Omega_T)$ and integrating by parts, we get:

$$\begin{aligned} &-\int_{\Omega_T} \Phi^\varepsilon(x)\Theta^\varepsilon \frac{\partial \varphi_g}{\partial t}dxdt \\ &\quad = \int_{\Omega_T} K^\varepsilon(x)\varrho_g(\mathsf{p}_g^\varepsilon)\left\{\lambda_g\left(\frac{x}{\varepsilon},\mathsf{S}^\varepsilon\right)(\nabla \mathsf{p}_g^\varepsilon - \varrho_g(\mathsf{p}_g^\varepsilon)\mathbf{g})\right\}\cdot \nabla\varphi_g dxdt. \end{aligned} \tag{3.18}$$

Then it follows from Cauchy's inequality and the definition of the function ϱ_g that

$$\left|\int_{\Omega_T} \Phi^\varepsilon(x)\Theta^\varepsilon \frac{\partial \varphi_g}{\partial t}dxdt\right| \le C\left(1 + \left\|\sqrt{\lambda_w\left(\frac{x}{\varepsilon},\mathsf{S}^\varepsilon\right)}|\nabla \mathsf{p}_w^\varepsilon|\right\|_{L^2(\Omega_T)}\right)\|\nabla\varphi_g\|_{L^2(\Omega_T)}. \tag{3.19}$$

Inequality (3.19) along with (3.6) implies the desired bound of the sequence $\{\partial_t(\Phi^\varepsilon\Theta^\varepsilon)\}_{\varepsilon>0}$ in the space $L^2(0,T;H^{-1}(\Omega))$.

The uniform boundedness of $\{\partial_t(\Phi^\varepsilon\mathsf{S}^\varepsilon)\}_{\varepsilon>0}$ in the space $L^2(0,T;H^{-1}(\Omega))$ can be obtained in a similar way. Lemma 3.4 is proved. □

4. Extensions and Compactness Results for $\{\mathsf{P}_\ell^\varepsilon\}_{\varepsilon>0}$, $\{S_\ell^\varepsilon\}_{\varepsilon>0}$, $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0}$

The outline of the section is as follows. First, in Sec. 4.1 we extend the functions S_ℓ^ε and $\mathsf{P}_\ell^\varepsilon$ from the subdomain Ω_ℓ^ε to the whole Ω and obtain the uniform estimates for the extended functions $\widetilde{S}_\ell^\varepsilon$, $\widetilde{\mathsf{P}}_\ell^\varepsilon$. Then in Sec. 4.2, using these estimates, we prove the compactness result for $\widetilde{S}_\ell^\varepsilon$ and $\widetilde{\Theta}_\ell^\varepsilon$, where $\widetilde{\Theta}_\ell^\varepsilon$ is an extension of the function Θ_ℓ^ε from the subdomain Ω_ℓ^ε to the whole Ω.

4.1. *Extensions and additional uniform estimates*

To obtain some additional *a priori* estimates and to homogenize the microscopic problem, we will extend the functions $\mathsf{P}_\ell^\varepsilon$, S_ℓ^ε, Θ_ℓ^ε to the whole Ω. It follows from Ref. 1 that there exists a linear continuous extension operator $\Pi_\ell^\varepsilon : H^1(\Omega_\ell^\varepsilon) \to H^1(\Omega)$ such that:

(i) $\Pi_\ell^\varepsilon u = u$ in Ω_ℓ^ε,
(ii) for any $u \in H^1(\Omega_\ell^\varepsilon)$,

$$\|\Pi_\ell^\varepsilon u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega_\ell^\varepsilon)} \quad \text{and} \quad \|\nabla(\Pi_\ell^\varepsilon u)\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega_\ell^\varepsilon)}, \tag{4.1}$$

where C is a constant that does not depend on u, ε.

Then it follows from Lemma 3.3 that there is an extension $\Pi_\ell^\varepsilon \mathsf{P}_\ell^\varepsilon = \widetilde{\mathsf{P}}_\ell^\varepsilon$ such that $\widetilde{\mathsf{P}}_\ell^\varepsilon = \mathsf{P}_\ell^\varepsilon$ in Ω_ℓ^ε and

$$\int_{\Omega_T} (|\widetilde{\mathsf{P}}_\ell^\varepsilon|^2 + |\nabla \widetilde{\mathsf{P}}_\ell^\varepsilon|^2) dx dt \leq C. \tag{4.2}$$

Now we turn to the extension of the functions S_ℓ^ε. To this end we introduce the function $\mathcal{C}$ defined by:

$$\mathcal{C}(s) \stackrel{\text{def}}{=} P_{2,c}^{-1}(P_{1,c}(s)) \quad \text{for } s \in [0,1]. \tag{4.3}$$

Notice that due to the properties of the capillary pressure functions, $\mathcal{C}$ is a smooth increasing function such that $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = 1$.

According to the relation (2.36), we define the extensions of the functions $S_1^\varepsilon, S_2^\varepsilon$ from the subdomains $\Omega_1^\varepsilon, \Omega_2^\varepsilon$ to the whole Ω by the following formula:

$$\widetilde{S}_1^\varepsilon(x,t) \stackrel{\text{def}}{=} \begin{cases} S_1^\varepsilon & \text{in } \Omega_1^\varepsilon \times (0,T); \\ \mathcal{C}^{-1}(S_2^\varepsilon) & \text{in } \Omega_2^\varepsilon \times (0,T); \end{cases} \quad \widetilde{S}_2^\varepsilon(x,t) \stackrel{\text{def}}{=} \begin{cases} S_2^\varepsilon & \text{in } \Omega_2^\varepsilon \times (0,T); \\ \mathcal{C}(S_1^\varepsilon) & \text{in } \Omega_1^\varepsilon \times (0,T). \end{cases} \tag{4.4}$$

In order to obtain the regularity properties of the functions $\widetilde{S}_1^\varepsilon, \widetilde{S}_2^\varepsilon$, we introduce the following functions:

$$\hat{\alpha}(s) = \min\{\alpha_1(s), \alpha_2(\mathcal{C}(s))\}, \quad \hat{\beta}(s) = \int_0^s \hat{\alpha}(\tau) d\tau, \quad 0 \leq s \leq 1. \tag{4.5}$$

Now we are in a position to formulate the regularity properties of the extended saturations. We have.

Lemma 4.1. *Under our standing conditions, the following bound holds true*:

$$\|\nabla \hat{\beta}(\widetilde{S}_\ell^\varepsilon)\|_{L^2(\Omega_T)} \leq C. \tag{4.6}$$

Proof of Lemma 4.1. Consider, for example, the function $\widetilde{S}_1^\varepsilon$. We have:

$$\begin{aligned}|\nabla\hat{\beta}(\widetilde{S}_1^\varepsilon)| &= |\nabla\hat{\beta}(S_1^\varepsilon)\mathbf{1}_1^\varepsilon(x)| + |\nabla\hat{\beta}(\mathcal{C}^{-1}(S_2^\varepsilon))\mathbf{1}_2^\varepsilon(x)| \\ &= |\hat{\alpha}(S_1^\varepsilon)||\nabla S_1^\varepsilon|\mathbf{1}_1^\varepsilon(x) + |\hat{\alpha}(\mathcal{C}^{-1}(S_2^\varepsilon))|\left|\frac{d}{ds}\mathcal{C}^{-1}(S_2^\varepsilon)\right||\nabla S_2^\varepsilon|\mathbf{1}_2^\varepsilon(x) \\ &\le |\alpha_1(S_1^\varepsilon)||\nabla S_1^\varepsilon|\mathbf{1}_1^\varepsilon(x) + C|\alpha_2(S_2^\varepsilon)||\nabla S_2^\varepsilon|\mathbf{1}_2^\varepsilon(x), \end{aligned} \tag{4.7}$$

with $C = \|\frac{d}{ds}\mathcal{C}^{-1}(S_2^\varepsilon)\|_{L^\infty}$. Due to (3.9), this yields

$$\|\nabla\hat{\beta}(\widetilde{S}_1^\varepsilon)\|_{L^2(\Omega_T^\varepsilon)} \le C. \tag{4.8}$$

This completes the proof. □

In order to define the extension of the functions Θ_ℓ^ε, we introduce the function:

$$P_g^\varepsilon \stackrel{\text{def}}{=} \begin{cases} \widetilde{\mathsf{P}}_1^\varepsilon + \mathsf{G}_{1,g}(\widetilde{S}_1^\varepsilon) & \text{in } \Omega_1^\varepsilon \times (0,T); \\ \widetilde{\mathsf{P}}_2^\varepsilon + \mathsf{G}_{1,g}(\widetilde{S}_2^\varepsilon) & \text{in } \Omega_2^\varepsilon \times (0,T). \end{cases} \tag{4.9}$$

Recall that

$$\Theta_\ell^\varepsilon = \varrho_g(\mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,g}(S_\ell^\varepsilon))(1 - S_\ell^\varepsilon).$$

Then we define the extension of the functions Θ_ℓ^ε to the whole Ω_T by

$$\widetilde{\Theta}_\ell^\varepsilon \stackrel{\text{def}}{=} \varrho_g(P_g^\varepsilon)(1 - \widetilde{S}_\ell^\varepsilon). \tag{4.10}$$

4.2. *Compactness results for the sequences* $\{\widetilde{S}_\ell^\varepsilon\}_{\varepsilon>0}$, $\{\widetilde{\Theta}_\ell^\varepsilon\}_{\varepsilon>0}$

We start this section by obtaining the following compactness result for the family $\{\widetilde{S}_\ell^\varepsilon\}_{\varepsilon>0}$. It is assured by the following statement.

Proposition 4.1. *Under our standing assumptions the family $\{\widetilde{S}_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^2(\Omega_T)$.*

Proof of Proposition 4.1. Consider, for example, the family $\{\widetilde{S}_2^\varepsilon\}$, the proof of the compactness result for the family $\{\widetilde{S}_1^\varepsilon\}$ can be done in a similar way. Without loss of generality, we assume that $\Omega \Subset \mathbf{B}$, where $\mathbf{B} = (0, 2\pi)^d$.

Let us introduce the function:

$$\mathbb{S}^\varepsilon \stackrel{\text{def}}{=} \begin{cases} \Phi_2 S_2^\varepsilon & \text{in } \Omega_2^\varepsilon; \\ \Phi_1\mathcal{C}^{-1}(\widetilde{S}_2^\varepsilon) & \text{in } \Omega_1^\varepsilon, \end{cases} \tag{4.11}$$

where the constants Φ_1, Φ_2 are defined in condition (A.1), and $\widetilde{S}_2^\varepsilon$ is introduced in (4.4). It is important to notice that it follows from Lemma 3.4 that

$$\frac{\partial\mathbb{S}^\varepsilon}{\partial t} \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{4.12}$$

It also follows from Lemma 4.1 that

$$\hat{\beta}(\widetilde{S}_2^\varepsilon) \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)). \tag{4.13}$$

Then in the same way as in the proof of Lemma 4.2 in Ref. 3, it can be shown that, for any $\delta > 0$, there is $N = N(\delta)$ such that $\widetilde{S}_2^\varepsilon$ can be represented as follows:

$$\widetilde{S}_2^\varepsilon(x,t) = \sum_{|j|\le N(\delta)} \eta_j^\varepsilon(t)\psi_j(x) + r_N^\varepsilon(x,t), \tag{4.14}$$

where $\{\psi_j\}$ is an orthonormal basis in $H^1_{\mathrm{per}}(\mathbf{B})$ and the function r_N^ε is such that

$$\|r_N^\varepsilon\|^2_{L^2(\Omega_T)} < \delta. \tag{4.15}$$

We also assume that the functions $\{\psi_j\}_{1\le j\le N}$ are orthogonal in $L^2(\mathbf{B})$.

Now let us introduce the following mapping from $L^2(\Omega)$ to $L^2(\Omega)$:

$$\mathcal{U}^\varepsilon(u) = \begin{cases} \Phi_2 u & \text{in } \Omega_2^\varepsilon; \\ \Phi_1 \mathcal{C}^{-1}(u) & \text{in } \Omega_1^\varepsilon. \end{cases} \tag{4.16}$$

Notice that the mapping $\mathcal{U}^\varepsilon$ is continuous in $L^2(\Omega)$ and that the continuity is uniform with respect to $\varepsilon > 0$. In fact, taking into account the regularity properties of the capillary pressure functions, we have:

$$\begin{aligned} &\int_\Omega |\mathcal{U}^\varepsilon(u+v) - \mathcal{U}^\varepsilon(u)|^2 dx \\ &\quad = \Phi_2 \int_{\Omega_2^\varepsilon} |v|^2 dx + \Phi_1 \int_{\Omega_1^\varepsilon} |\mathcal{C}^{-1}(u+v) - \mathcal{C}^{-1}(u)|^2 dx \\ &\quad \le \Phi_2 \int_{\Omega_2^\varepsilon} |v|^2 dx + C \max_{s\in[0,1]} (\mathcal{C}^{-1}(s))' \int_{\Omega_1^\varepsilon} |v|^2 dx \le C \int_\Omega |v|^2 dx. \end{aligned} \tag{4.17}$$

Consider now the function $\mathcal{U}^\varepsilon(\widetilde{S}_2^\varepsilon)$. It can be represented as follows:

$$\mathbb{S}^\varepsilon = \mathcal{U}^\varepsilon(\widetilde{S}_2^\varepsilon) = \sum_{|j|\le N(\delta)} \xi_j^\varepsilon(t)\psi_j(x) + D_N^\varepsilon(x,t) \stackrel{\text{def}}{=} B^\varepsilon(x,t) + D_N^\varepsilon(x,t), \tag{4.18}$$

where for each t, $D_N^\varepsilon(x,t)$ is orthogonal to $\mathrm{span}(\psi_{j=1,\dots,N})$. Notice that the norm of $D_N^\varepsilon(x,t)$ need not be small. Clearly, $\xi_j^\varepsilon(t) = (\mathcal{U}^\varepsilon(\widetilde{S}_2^\varepsilon(t)), \psi_j)_{L^2(\Omega)}$. From (4.12) it now follows in the standard way (see Ref. 3) that

$$|\xi_j^\varepsilon(t+r) - \xi_j^\varepsilon(t)| \le C\sqrt{r}, \quad 0 \le t, t+r \le T, \tag{4.19}$$

where the constant C depends on the upper bounds in (4.12) and on N. Indeed,

$$\begin{aligned} \left\| \frac{\partial \xi_j^\varepsilon}{\partial t} \right\|_{L^2(0,T)} &= \left\| \left(\frac{\partial \mathbb{S}^\varepsilon}{\partial t}, \psi_j \right) \right\|_{L^2(0,T)} \\ &\le C \left\| \frac{\partial \mathbb{S}^\varepsilon}{\partial t} \right\|_{L^2(0,T;H^{-1}(\Omega))} \|\psi_j\|_{H^1(\Omega)} \le C(N). \end{aligned}$$

By the Schwartz inequality this implies (4.19).

Let us define a map $F^\varepsilon : \mathbb{R}^N \mapsto \mathbb{R}^N$ by

$$\{\eta_j\}_{1\le j\le N} \to u = \sum \eta_j \psi_j \to \{(\mathcal{U}^\varepsilon(u), \psi_k)_{L^2}\}_{1\le k\le N}.$$

It is easy to check that this map is smooth uniformly in ε. Moreover, due to positiveness of $\mathcal{C}$, we have

$$\begin{aligned}(F^\varepsilon(\eta') - F^\varepsilon(\eta'')) \cdot (\eta' - \eta'') &= (\mathcal{U}^\varepsilon(u') - \mathcal{U}^\varepsilon(u''), u' - u'')_{L^2(\mathbf{B})} \\ &\ge C\|u' - u''\|^2_{L^2(\mathbf{B})} \ge C|\eta' - \eta''|^2,\end{aligned}$$

with a constant C that does not depend on ε; the second inequality here readily follows from the definition of $\mathcal{U}$ and the fact that $\mathcal{C}$ is a positive and increasing function.

Thus, the maps F^ε are uniformly in ε smooth and monotone. This implies that the inverse maps $(F^\varepsilon)^{-1}$ are also uniformly in ε smooth. Therefore, from (4.19) we deduce the estimate

$$|\eta_j^\varepsilon(t + r) - \eta_j^\varepsilon(t)| \le C\sqrt{r}, \quad 0 \le t, t + r \le T, \tag{4.20}$$

with a constant C that does not depend on ε. This yields the desired compactness in $L^2(\Omega_T)$ (see Ref. 3). □

As a consequence of Proposition 4.1 and the uniform L^∞-bound for $\widetilde{S}_\ell^\varepsilon$ we have the following:

Corollary 4.1. *The family $\{\widetilde{S}_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_T)$ for all $q \in [1, \infty)$.*

Finally, we formulate the compactness result for the sequence $\{\widetilde{\Theta}_\ell^\varepsilon\}_{\varepsilon>0}$, where the function $\widetilde{\Theta}_\ell^\varepsilon$ is defined in (4.10). We have:

Proposition 4.2. *Under our standing assumptions the family $\{\widetilde{\Theta}_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_T)$ for all $q \in [1, \infty)$.*

The proof of Proposition 4.2 is done by the arguments similar to those used in the proof of Proposition 4.1.

5. Formulation of the Main Result

We study the asymptotic behavior of the solution to problem (2.12)–(2.16) (or equivalent problem (2.27)–(2.39)) as $\varepsilon \to 0$.

In order to introduce the effective problem, for any $s_1, s_2 \in [0, 1]$, we first define the following auxiliary functions

$$\lambda_w(y, s_1, s_2) \stackrel{\text{def}}{=} \begin{cases}\lambda_{1,w}(s_1) & \text{if } y \in Y_1;\\ \lambda_{2,w}(s_2) & \text{if } y \in Y_2;\end{cases} \qquad \lambda_g(y, s_1, s_2) \stackrel{\text{def}}{=} \begin{cases}\lambda_{1,g}(s_1) & \text{if } y \in Y_1;\\ \lambda_{2,g}(s_2) & \text{if } y \in Y_2\end{cases}$$

and consider auxiliary cell problems:

$$\operatorname{div}_y\{K(y)\lambda_w(y,s_1,s_2)(\mathbb{I}+\nabla_y\chi_w(y))\}=0, \quad \chi_w\in H^1_{\rm per}(Y); \tag{5.1}$$

$$\operatorname{div}_y\{K(y)\lambda_g(y,s_1,s_2)(\mathbb{I}+\nabla_y\chi_g(y))\}=0, \quad \chi_g\in H^1_{\rm per}(Y). \tag{5.2}$$

Here $\mathbb{I}$ stands for the unit matrix. For any $s_1, s_2 \in (0,1)$, problems (5.1)–(5.2) are solvable and have a unique up to an additive constant solution. We set

$$\Lambda_w(s_1,s_2)=\int_Y K(y)\lambda_w(y,s_1,s_2)\{\mathbb{I}+\nabla_y\chi_w(s_1,s_2,y)\}dy,$$

$$\Lambda_g(s_1,s_2)=\int_Y K(y)\lambda_g(y,s_1,s_2)\{\mathbb{I}+\nabla_y\chi_g(s_1,s_2,y)\}dy.$$

We are going to show that the effective problem reads:

$$\langle\Phi\rangle\frac{\partial S^\star}{\partial t}-\operatorname{div}_x\{\Lambda_w(S_1,S_2)(\nabla P_w-\mathbf{g})\}=0 \quad \text{in } \Omega_T; \tag{5.3}$$

$$\langle\Phi\rangle\frac{\partial \Theta^\star}{\partial t}-\operatorname{div}_x\{\Lambda_g(S_1,S_2)\varrho_g(P_g)(\nabla P_g-\varrho_g(P_g)\mathbf{g})\}=0 \quad \text{in } \Omega_T, \tag{5.4}$$

where

$$P_w(x,t)\stackrel{\rm def}{=}\mathsf{P}_1(x,t)+\mathsf{G}_{1,w}(S_1(x,t))\stackrel{\rm def}{=}\mathsf{P}_2(x,t)+\mathsf{G}_{2,w}(S_2(x,t)); \tag{5.5}$$

$$P_g(x,t)\stackrel{\rm def}{=}\mathsf{P}_1(x,t)+\mathsf{G}_{1,g}(S_1(x,t))\stackrel{\rm def}{=}\mathsf{P}_2(x,t)+\mathsf{G}_{2,g}(S_2(x,t)) \tag{5.6}$$

and

$$S_2=\mathcal{C}(S_1) \quad \text{a.e. in } \Omega_T, \text{ where } \quad \mathcal{C}(s)\stackrel{\rm def}{=}P_{2,c}^{-1}(P_{1,c}(s)). \tag{5.7}$$

Here the following notation has been used:

- $\langle\Phi\rangle$ denotes the mean value of the function Φ over the cell Y.
- The functions $S^\star = S^\star(S_1, S_2)$ and $\Theta^\star = \Theta^\star(S_1, S_2; \mathsf{P}_1, \mathsf{P}_2)$ are defined by:

$$S^\star\stackrel{\rm def}{=}\sum_{\ell=1}^{2}\frac{|Y_\ell|}{\langle\Phi\rangle}\overline{\Phi}_\ell S_\ell \quad \text{and} \quad \Theta^\star\stackrel{\rm def}{=}(1-S^\star)\varrho_g(P_g), \tag{5.8}$$

where $|Y_\ell|$ is the measure of the set Y_ℓ, $\ell = 1, 2$,

$$\overline{\Phi}_\ell\stackrel{\rm def}{=}\frac{1}{|Y_\ell|}\int_{Y_\ell}\Phi(y)dy. \tag{5.9}$$

Equations (5.3)–(5.4) are equipped with the following boundary and initial conditions:

$$P_w(x,t)=P_g(x,t)=0 \quad \text{on } \Gamma_{inj}\times(0,T); \tag{5.10}$$

$$\mathbf{q}_w\cdot\boldsymbol{\nu}=\mathbf{q}_g\cdot\boldsymbol{\nu}=0 \quad \text{on } \Gamma_{imp}\times(0,T), \tag{5.11}$$

where the vectors $\mathbf{q}_w, \mathbf{q}_g$ are given by:

$$\mathbf{q}_g \stackrel{\text{def}}{=} -\Lambda_g(S_1, S_2)\varrho_g(P_g)(\nabla P_g - \varrho_g(P_g)\mathbf{g}); \tag{5.12}$$

$$\mathbf{q}_w \stackrel{\text{def}}{=} -\Lambda_w(S_1, S_2)(\nabla P_w - \mathbf{g}). \tag{5.13}$$

Finally, the initial conditions read:

$$P_w(x,0) = \mathsf{p}_w^{\mathbf{0}}(x) \quad \text{and} \quad P_g(x,0) = \mathsf{p}_g^{\mathbf{0}}(x) \quad \text{in } \Omega, \tag{5.14}$$

where the functions $\mathsf{p}_w^{\mathbf{0}}, \mathsf{p}_g^{\mathbf{0}}$ are defined in (2.16).

The homogenized model described above could be obtained formally by the technique of two-scale asymptotic expansions. Here the homogenization process for the problem is rigorously obtained by using the two-scale approach, see, e.g., Ref. 2. For the reader's convenience, let us recall the definition of the two-scale convergence.

Definition 5.1. A function, $\varphi \in L^2(\Omega_T; C^2_{\text{per}}(Y))$, which is Y-periodic in y and which satisfies

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \left| \varphi\left(x, \frac{x}{\varepsilon}, t\right) \right|^2 dx dt = \int_{\Omega_T \times Y} |\varphi(x,y,t)|^2 dy dx dt$$

is called an *admissible test function.*

Here $L^2(\Omega_T; C^2_{\text{per}}(Y))$ is the space of functions $\phi = \phi(x,y,t)$ periodic and two times continuously differentiable in y for a.e. $(x,t) \in \Omega_T$ with the norm

$$\|\phi\|^2_{L^2(\Omega_T; C^2_{\text{per}}(Y))} = \int_{\Omega_T} \|\phi(x,\cdot,t)\|^2_{C^2(Y)} dx dt.$$

Definition 5.2. A sequence of functions $v^\varepsilon \in L^2(\Omega_T)$ *two-scale converges* to $v \in L^2(\Omega_T \times Y)$ if for any admissible test function $\varphi(x,y,t)$,

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} v^\varepsilon(x,t)\varphi\left(x, \frac{x}{\varepsilon}, t\right) dx dt = \int_{\Omega_T \times Y} v(x,y,t)\varphi(x,y,t) dy dx dt.$$

This convergence is denoted by $v^\varepsilon(x,t) \xrightarrow{2s} v(x,y,t)$. In what follows for the sake of brevity we use the notation

$$\varrho_g^\star \stackrel{\text{def}}{=} \varrho_g(P_g). \tag{5.15}$$

We are also reminded of the following notation (see (2.17), (2.19)):

$$\mathsf{p}_w^\varepsilon(x,t) \stackrel{\text{def}}{=} \mathbf{1}_1^\varepsilon(x)(\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^\varepsilon)) + \mathbf{1}_2^\varepsilon(x)(\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,w}(S_2^\varepsilon)), \tag{5.16}$$

$$\mathsf{p}_g^\varepsilon(x,t) \stackrel{\text{def}}{=} \mathbf{1}_1^\varepsilon(x)(\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(S_1^\varepsilon)) + \mathbf{1}_2^\varepsilon(x)(\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(S_2^\varepsilon)). \tag{5.17}$$

The main result of the paper is the following theorem.

Theorem 5.1. *Let assumptions* (A.1)–(A.9) *be fulfilled. Then the pair of functions* $\langle \mathsf{p}_w^\varepsilon, \mathsf{p}_g^\varepsilon \rangle$ *solution of problem* (2.12)–(2.16) *converges in the two-scale sense to the pair of functions* $\langle P_w, P_g \rangle$ *solution to* (5.3)–(5.14).

6. Proof of Theorem 5.1

The proof of Theorem 5.1 consists of the following steps. In Sec. 6.1, we obtain a number of convergence results that rely on our energy estimates and compactness lemmata. Section 6.2 deals with the passage to the limit in Eqs. (2.50) and (2.51). Finally, we establish the relations between the limit functions S_1, S_2 and $\mathsf{P}_1, \mathsf{P}_2$.

6.1. *Convergence results*

The following statement holds.

Lemma 6.1. *There exist functions* $S_\ell \in L^\infty(\Omega_T)$, $0 \le S_\ell \le 1$, $\mathsf{P}_\ell \in L^2(0,T; H^1(\Omega))$, $\mathsf{w}_{\ell,p} \in L^2(\Omega_T; H^1_{\rm per}(Y))$ *such that up to a subsequence*:

$$\widetilde{\mathsf{P}}_\ell^\varepsilon(x,t) \rightharpoonup \mathsf{P}_\ell(x,t) \quad \textit{weakly in } L^2(0,T;H^1(\Omega)); \tag{6.1}$$

$$\mathbf{1}_\ell^\varepsilon(x)\mathsf{P}_\ell^\varepsilon(x,t) \xrightharpoonup{2s} \mathbf{1}_\ell(y)\mathsf{P}_\ell(x,t); \tag{6.2}$$

$$\widetilde{S}_\ell^\varepsilon(x,t) \to S_\ell(x,t) \quad \textit{strongly in } L^q(\Omega_T)\,\forall\, q \in [1,+\infty); \tag{6.3}$$

$$\mathbf{1}_\ell^\varepsilon(x)S_\ell^\varepsilon(x,t) \xrightarrow{2s} \mathbf{1}_\ell(y)S_\ell(x,t); \tag{6.4}$$

$$\widetilde{\beta}_\ell^\varepsilon \to \beta_\ell(S_\ell) \quad \textit{strongly in } L^q(\Omega_T) \textit{ for all } q \in [1,+\infty); \tag{6.5}$$

$$\widetilde{\Theta}_\ell^\varepsilon(x,t) \to \Theta_\ell \stackrel{\rm def}{=} (1 - S_\ell(x,t))\varrho_g(P_g) \quad \textit{strongly in } L^q(\Omega_T)\,\forall\, q \in [1,+\infty). \tag{6.6}$$

Proof of Lemma 6.1. Relations (6.1)–(6.5) follow from the estimates of Sec. 3 in the standard way. Relation (6.6) can be justified by the arguments used in the proof of Lemma 4.8 from Ref. 3. □

6.2. *Passage to the limit in system* (2.12)

It is easy to justify the passage to the two-scale limit in the temporal terms, see for instance Ref. 3.

We begin the section by obtaining an auxiliary convergence result. First, we recall that we extend the functions S_ℓ^ε as follows:

$$\widetilde{S}_1^\varepsilon(x,t) \stackrel{\rm def}{=} \begin{cases} S_1^\varepsilon & \text{in } \Omega_{1,T}^\varepsilon; \\ \mathcal{C}^{-1}(S_2^\varepsilon) & \text{in } \Omega_{2,T}^\varepsilon \end{cases} \quad \text{and} \quad \widetilde{S}_2^\varepsilon(x,t) \stackrel{\rm def}{=} \begin{cases} S_2^\varepsilon & \text{in } \Omega_{2,T}^\varepsilon; \\ \mathcal{C}(S_1^\varepsilon) & \text{in } \Omega_{1,T}^\varepsilon. \end{cases} \tag{6.7}$$

Now, for any $\delta > 0$, we introduce the functions

$$\tilde{S}_\ell^{\varepsilon,\delta} = \min(1-\delta, \max(\delta, \tilde{S}_\ell^\varepsilon)).$$

These functions satisfy the estimates

$$\|\tilde{S}_\ell^{\varepsilon,\delta}\|_{L^2(0,T;H^1(\Omega))} \le C(\delta), \quad \ell = 1,2.$$

Therefore, $S_\ell^\delta = \min(1-\delta, \max(\delta, S_\ell)) \in L^2(0,T;H^1(\Omega))$ for any $\delta > 0$. Thus, for a subsequence,

$$\nabla(\mathbf{1}_1(x)(\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^{\varepsilon,\delta})) + \mathbf{1}_2(x)(\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,w}(S_2^{\varepsilon,\mathcal{C}(\delta)})))$$
$$\xrightarrow{2s} \nabla_x(\mathsf{P}_1 + \mathsf{G}_{1,w}(S_1^\delta)) + \nabla_y \mathsf{V}_w^\delta$$

with $\mathsf{V}_w^\delta \in L^2(\Omega_T; H^1(Y))$.

We set:

$$\varphi_w^\varepsilon(x,t) \stackrel{\text{def}}{=} \varepsilon\varphi(x,t)R(\widetilde{S}_1^\varepsilon)\zeta\left(\frac{x}{\varepsilon}\right), \tag{6.8}$$

with $R(s)$ being a smooth function equal to zero for $s \notin (\delta, 1-\delta)$; $\zeta(y)$ is smooth periodic, and φ is a smooth function with a compact support in Ω_T. Using φ_w^ε as a test function in the first equation in (2.12) yields

$$\int_{\Omega_T} K^\varepsilon(x)\lambda_w\left(\frac{x}{\varepsilon}, S_1^\varepsilon, S_2^\varepsilon\right)(\nabla \mathsf{p}_w^\varepsilon - \mathbf{g})\nabla\zeta\left(\frac{x}{\varepsilon}\right)\varphi R(\widetilde{S}_1^\varepsilon)dxdt = O(\varepsilon).$$

Passing here to the two-scale limit, we obtain

$$\int_{\Omega_T}\int_Y (K(y)\lambda_w(y, S_1, S_2)(\nabla \mathsf{P}_1 + \nabla G_{1,w}(S_1)$$
$$+ \nabla_y \mathsf{V}_w^\delta(t,x,y) - \mathbf{g})\nabla\zeta(y)R(S_1)\varphi)dydxdt = 0.$$

Therefore,

$$\mathsf{V}_w^\delta = \chi_w(y)(\nabla_x \mathsf{P}_1 + \nabla_x \mathsf{G}_{1,w}(S_1) - \mathbf{g}), \tag{6.9}$$

for all $(x,t) \in \Omega_T$ such that $S^1 \in (\delta, 1-\delta)$, where χ_w has been defined in (5.1). Since δ is an arbitrary positive number, representation (6.9) is valid for all (x,t) such that $S_1 \in (0,1)$. In particular, V_w^δ does not depend on δ: $\mathsf{V}_w^\delta = \mathsf{V}_w$. Notice that χ_w also depends on S_1 and S_2.

With the help of our *a priori* estimates we deduce in the standard way that

$$K^\varepsilon\lambda_w\left(\frac{x}{\varepsilon}, S_1^\varepsilon, S_2^\varepsilon\right)(\nabla \mathsf{p}_w^\varepsilon - \vec{g})$$
$$\xrightarrow{2s} K(y)\lambda_w(y, S_1, S_2)(\mathbb{I} + \nabla_y\chi_w(y))(\nabla_x \mathsf{P}_1 + \nabla_x \mathsf{G}_{1,w}(S_1) - \mathbf{g}).$$

Choosing now a smooth test function $\varphi = \varphi(x,t)$ in the first equation in (2.12) we arrive at (5.3).

We proceed with the second equation in (2.12). In view of (6.6), the passage to the two-scale limit in the temporal term is standard.

In order to pass to the limit in the spatial term we follow the same strategy as above. We have, for a subsequence,

$$\nabla(\mathbf{1}_1(x)(\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(S_1^{\varepsilon,\delta})) + \mathbf{1}_2(x)(\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(S_2^{\varepsilon,\mathcal{C}(\delta)})))$$
$$\xrightarrow{2s} \nabla_x(\mathsf{P}_1 + \mathsf{G}_{1,g}(S_1^\delta)) + \nabla_y \mathsf{V}_g^\delta,$$

with $\mathsf{V}_g^\delta \in L^2(\Omega_T; H^1(Y))$. Letting

$$\varphi_g^\varepsilon(x,t) \stackrel{\text{def}}{=} \varepsilon\varphi(x,t)R(\widetilde{S}_1^\varepsilon)\zeta\left(\frac{x}{\varepsilon}\right), \tag{6.10}$$

with the same $R(s)$ as above, we get

$$\int_{\Omega_T} K^\varepsilon(x)\lambda_g\left(\frac{x}{\varepsilon}, S_1^\varepsilon, S_2^\varepsilon\right)\varrho_g(\mathsf{p}_g^\varepsilon)(\nabla\mathsf{p}_g^\varepsilon - \mathbf{g})\nabla\zeta\left(\frac{x}{\varepsilon}\right)\varphi R(\widetilde{S}_1^\varepsilon)dxdt = O(\varepsilon). \tag{6.11}$$

It is shown below in Lemma 6.2 that $\mathsf{P}_1 + \mathsf{G}_{1,g}(S_1) = \mathsf{P}_2 + \mathsf{G}_{2,g}(S_2)$. Since, according to (6.6), $\varrho_g(\mathsf{p}_g^\varepsilon)(1-\widetilde{S}_1^\varepsilon)$ converge to $\varrho_g(\mathsf{P}_1 + \mathsf{G}_{1,w}(S_1))(1-S_1)$ a.e., then

$$R(\widetilde{S}_1^\varepsilon)\varrho_g(\mathsf{p}_g^\varepsilon) \to R(S_1)\varrho_g(\mathsf{P}_1 + \mathsf{G}_{1,g}(S_1)) \quad \text{in } L^2(\Omega_T).$$

Passing now to the two-scale limit in (6.10) yields

$$\int_{\Omega_T}\int_Y K(y)\varrho_g(\mathcal{P}_g)\lambda_g(y, S_1, S_2)(\nabla\mathsf{P}_1 + \nabla\mathsf{G}_{1,g}(S_1) + \nabla_y\mathsf{V}_g^\delta(t,x,y) - \varrho_g(\mathcal{P}_g)\mathbf{g})\nabla\zeta(y)R(S_1)\varphi dydxdt = 0.$$

From this relation, Eq. (5.4) can be derived by means of the same arguments as were used above.

It remains to justify (5.5)–(5.7). The following lemma holds.

Lemma 6.2. *Let* S_1, S_2 *and* $\mathsf{P}_1, \mathsf{P}_2$ *be the functions defined in* (6.1) *and* (6.3). *Then*

$$S_2 = \mathcal{C}(S_1) \quad \text{a.e. in } \Omega_T, \text{ where} \quad \mathcal{C}(s) \stackrel{\text{def}}{=} P_{2,c}^{-1}(P_{1,c}(s)) \tag{6.12}$$

and

$$\mathsf{P}_1 + \mathsf{G}_{1,w}(S_1) = \mathsf{P}_2 + \mathsf{G}_{2,w}(S_2) \quad \text{and} \quad \mathsf{P}_1 + \mathsf{G}_{1,g}(S_1) = \mathsf{P}_2 + \mathsf{G}_{2,g}(S_2) \quad \text{a.e. in } \Omega_T, \tag{6.13}$$

where the functions $\mathsf{G}_{\ell,w}, \mathsf{G}_{\ell,g}$ *are defined in* (2.20) *and* (2.21).

Proof of Lemma 6.2. Let us prove relation (6.12). Since $\mathcal{C}$ is a smooth bounded function, it follows from (6.3) that $\mathcal{C}(\widetilde{S}_1^\varepsilon(x,t)) \to \mathcal{C}(S_1(x,t))$ in $L^2(\Omega_T)$. Therefore, passing to the limit in the relation $(\widetilde{S}_2^\varepsilon(x,t)) = \mathcal{C}(\widetilde{S}_1^\varepsilon(x,t))$ in Ω_T we obtain the desired formula (6.12).

Justification of (6.13) requires more delicate arguments because the functions $\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)$ and $\mathsf{G}_{\ell,w}(\widetilde{S}_\ell^\varepsilon)$ are not bounded in $L^2(0,T;H^1(\Omega))$. In order to prove (6.13), we are going to exploit the fact that $\mathsf{G}_{\ell,w}(\widetilde{S}_\ell^\varepsilon)$ admits a better estimate on the set $\{(x,t) : \widetilde{S}_\ell^\varepsilon \geq \delta > 0\}$, and $\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)$ admits a better estimates on the set $\{(x,t) : \widetilde{S}_\ell^\varepsilon \leq 1-\delta\}$. To this end we introduce a function $\chi \in C^\infty([0,1])$ such that $0 \leq \chi \leq 1$,

$\chi(s) = 1$ if $s \leq 1/3$ and $\chi(s) = 0$ if $s \geq 2/3$, and notice that the functions $\chi(\widetilde{S}_1^\varepsilon)\mathsf{G}_{1,g}(\widetilde{S}_1^\varepsilon)$ and $\chi(\widetilde{S}_1^\varepsilon)\mathsf{G}_{2,g}(\widetilde{S}_2^\varepsilon)$ satisfy the estimate

$$\|\chi(\widetilde{S}_1^\varepsilon)\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Indeed,

$$|\nabla(\chi(\widetilde{S}_1^\varepsilon)\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)| \leq \left|\frac{d}{ds}\chi(\widetilde{S}_1^\varepsilon)\right| |\nabla \widetilde{S}_\ell^\varepsilon||\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)| + \chi(\widetilde{S}_1^\varepsilon)\left|\frac{\lambda_{\ell,w}(\widetilde{S}_\ell^\varepsilon)}{\lambda_\ell(\widetilde{S}_\ell^\varepsilon)}\right| |P'_{\ell,c}(\widetilde{S}_\ell^\varepsilon)||\nabla \widetilde{S}_\ell^\varepsilon|. \tag{6.14}$$

Since the support of $\frac{d}{ds}\chi(s)$ belongs to the interval $[1/3, 2/3]$, the first term on the right-hand side here satisfies the estimate

$$\left\|\frac{d}{ds}\chi(\widetilde{S}_1^\varepsilon)\nabla \widetilde{S}_\ell^\varepsilon \mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)\right\|_{L^2(\Omega_T)} \leq C. \tag{6.15}$$

For the second one we have

$$\chi(\widetilde{S}_1^\varepsilon)\left|\frac{\lambda_{\ell,w}(\widetilde{S}_\ell^\varepsilon)}{\lambda_\ell(\widetilde{S}_\ell^\varepsilon)}\right| |P'_{\ell,c}(\widetilde{S}_\ell^\varepsilon)||\nabla \widetilde{S}_\ell^\varepsilon| \leq C\chi(\widetilde{S}_1^\varepsilon)\left|\frac{\lambda_{\ell,w}(\widetilde{S}_\ell^\varepsilon)\lambda_{\ell,g}(\widetilde{S}_\ell^\varepsilon)}{\lambda_\ell(\widetilde{S}_\ell^\varepsilon)}\right| |P'_{\ell,c}(\widetilde{S}_\ell^\varepsilon)||\nabla \widetilde{S}_\ell^\varepsilon| = C\chi(\widetilde{S}_1^\varepsilon)|\nabla\beta_\ell(\widetilde{S}_\ell^\varepsilon)|; \tag{6.16}$$

here we have used the fact that $\chi(s) = 0$ on $[2/3, 1]$. Combining (6.14)–(6.16) we conclude that

$$\|\nabla\chi(\widetilde{S}_1^\varepsilon)\mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon)\|_{L^2(\Omega_T)} \leq C$$

and thus

$$\|\chi(\widetilde{S}_1^\varepsilon)(\mathsf{P}_\ell^\varepsilon + \mathsf{G}_{\ell,g}(\widetilde{S}_\ell^\varepsilon))\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Since $\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(\widetilde{S}_1^\varepsilon) = \mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(\widetilde{S}_2^\varepsilon)$ on $\Gamma_{1,2,T}^\varepsilon$, we deduce from the last estimate, by means of the Poincaré–Friedrich inequality, that

$$\chi(\widetilde{S}_1^\varepsilon)(\mathsf{P}_1^\varepsilon + \mathsf{G}_{1,g}(\widetilde{S}_1^\varepsilon)) = \chi(\widetilde{S}_1^\varepsilon)(\mathsf{P}_2^\varepsilon + \mathsf{G}_{2,g}(\widetilde{S}_2^\varepsilon)) + R^\varepsilon, \quad \text{in } \Omega_T, \tag{6.17}$$

with $\|R^\varepsilon\|_{L^2}(\Omega_T) \leq C\varepsilon$. Passing to the limit $\varepsilon \to 0$ in (6.17) yields

$$\chi(S_1)(\mathsf{P}_1 + \mathsf{G}_{1,g}(S_1)) = \chi(S_1)(\mathsf{P}_2 + \mathsf{G}_{2,g}(S_2)), \quad \text{in } \Omega_T. \tag{6.18}$$

In the same way one can show that

$$(1 - \chi(S_1))(\mathsf{P}_1 + \mathsf{G}_{1,w}(S_1)) = (1 - \chi(S_1))(\mathsf{P}_2 + \mathsf{G}_{2,w}(S_2)), \quad \text{in } \Omega_T. \tag{6.19}$$

Summing up (6.18) and (6.19), and considering (2.21), we finally obtain

$$\mathsf{P}_1 + \mathsf{G}_{1,w}(S_1) = \mathsf{P}_2 + \mathsf{G}_{2,w}(S_2),$$

as desired. The second relation in (6.13) can be obtained in a similar way. □

Relation (6.13) allows us to derive the expression for the function $\Theta^\star$ given by the second relation in (5.8). This concludes the proof of Theorem 5.1. □

Remark 6.1. We conclude this section with the following remark about the homogenized model obtained. It could be written in a form which is more suitable for numerical simulations. We have shown that the homogenized system has the following form:

$$\begin{cases} \langle\Phi\rangle \dfrac{\partial S^\star}{\partial t} - \mathrm{div}_x\{\Lambda_w(S_1,S_2)[\nabla P_w - \mathbf{g}]\} = 0 & \text{in } \Omega_T; \\ \langle\Phi\rangle \dfrac{\partial \Theta^\star}{\partial t} - \mathrm{div}_x\{\Lambda_g(S_1,S_2)\varrho_g(P_g)[\nabla P_g - \varrho_g(P_g)\mathbf{g}]\} = 0 & \text{in } \Omega_T; \\ P_c(y,S_1,S_2) = P_g - P_w & \text{in } \Omega_T \times Y; \end{cases} \tag{6.20}$$

subject to appropriate boundary and initial conditions.

In Eq. (6.20) the microscopic variable y appears, at least formally, in the capillary pressure law. In order to write the system (6.20) exclusively in macroscopic variables, we will eliminate

$$S \stackrel{\text{def}}{=} S_1(x,t)\mathbf{1}_1(y) + S_2(x,t)\mathbf{1}_2(y) \tag{6.21}$$

and replace it by the macro-scale saturation $S^\star$. To do this it is sufficient to construct *an effective capillary pressure function* $P_c^\star$ such that

$$P_c^\star(S^\star) \stackrel{\text{def}}{=} P_g - P_w \quad \text{in } \Omega_T. \tag{6.22}$$

Then, for any given $S^\star$ the capillary pressure is known, and therefore S is also known. We can then express all homogenized tensors as functions of the macro-scale saturation $S^\star$.

It follows from condition (A.4) that

$$\min_{s\in[0,1]} P_{\ell,c}(s) = P_{\ell,c}(1) = 0, \quad \max_{s\in[0,1]} P_{1,c}(s) = \max_{s\in[0,1]} P_{2,c}(s) = P_{\ell,c}(0), \tag{6.23}$$

where $\ell = 1,2$. We denote $P_{\ell,c}(0) = \sigma$. Then, for any $u \in [0,\sigma]$ we can find S given by (6.21) by solving the equations:

$$u = P_{1,c}(S_1) = P_{2,c}(S_2). \tag{6.24}$$

After the values S_1 and S_2 are obtained, we can compute $S^\star$ by the formula (5.8), that is

$$S^\star \stackrel{\text{def}}{=} \sum_{\ell=1}^{2} \frac{|Y_\ell|}{\langle\Phi\rangle}\overline{\Phi}_\ell S_\ell \quad \text{and} \quad \Theta^\star \stackrel{\text{def}}{=} (1-S^\star)\varrho_g(P_g).$$

So we have defined a function $f : [0,\sigma] \to [0,1]$ such that $S^\star = f(u)$. It is easy to see that, due to strict monotonicity of both capillary pressure functions, the function $u \mapsto S^\star$ is strictly decreasing, and thus have strictly decreasing inverse function: $u = f^{-1}(S^\star)$, $f^{-1} : [0,1] \to [0,\sigma]$. This function defines the effective capillary pressure function and it will be denoted $P_c^\star$.

When the effective capillary pressure function is computed, we can calculate for any value of macro-scale saturation $S^\star \in [0,1]$ the corresponding micro-scale repartition of the saturation $S = S_1\mathbf{1}_1(y) + S_2\mathbf{1}_2(y)$, where the values S_1 and S_2 are solutions to the equations $P_{1,c}(S_1) = P_{1,c}(S_2) = P_c^\star(S^\star)$. For the repartition S we solve the cell problems (5.1), (5.1), and calculate effective phase mobility tensors:

$$\Lambda_w(S^\star) \stackrel{\text{def}}{=} \Lambda_w(S_1, S_2) \quad \text{and} \quad \Lambda_g(S^\star) \stackrel{\text{def}}{=} \Lambda_g(S_1, S_2). \tag{6.25}$$

These tensors depend only on the macroscopic saturation $S^\star$. Finally we have the macroscopic conservation laws in the form:

$$\begin{cases} 0 \le S^\star \le 1 & \text{in } \Omega_T; \\ \langle\Phi\rangle \dfrac{\partial S^\star}{\partial t} - \operatorname{div}_x\{\Lambda_w(S^\star)[\nabla P_w - \mathbf{g}]\} = 0 & \text{in } \Omega_T; \\ \langle\Phi\rangle \dfrac{\partial((1-S^\star)\varrho_g(P_g))}{\partial t} - \operatorname{div}_x\{\Lambda_g(S^\star)\varrho_g(P_g)[\nabla P_g - \varrho_g(P_g)\mathbf{g}]\} = 0 & \text{in } \Omega_T; \\ P_c^\star(S^\star) = P_g - P_w & \text{in } \Omega_T, \end{cases}$$

subject to appropriate boundary and initial conditions.

Notice that the structure of the macroscopic two-phase flow equations is the same as the structure of micro-scale equations. The only difference is in the effective phase mobilities $\Lambda_w(S^\star)$ and $\Lambda_g(S^\star)$, which are now generally full symmetric tensors, calculated by the resolution of the local cell problems, and are not naturally factorized into absolute (intrinsic) permeability and relative (phase) mobilities. Let us also mention that this homogenization result has been used successfully in Ref. 4 to simulate numerically a benchmark test proposed in the framework of the European Project FORGE: Fate Of Repository Gases Ref. 21.

7. Disperse Media

In this section we relax the geometric assumptions and consider disperse porous media. More precisely, we assume that the medium Ω_1^ε is connected while the medium Ω_2^ε consists of disjoint periodically situated inclusions. The rigorous definition is as follows.

We recall that Y stands for a unit cell $(0,1)^n$, $n = 2$ or 3 and that Y_1 and Y_2 are two subdomains of Y. From now on we assume that Y_2 is a smooth connected set such that $\overline{Y_2}$ is a compact subset of Y, and Y_1 is a connected set.

As above, we introduce the characteristic functions of Y_1 and Y_2 and denote their periodic extensions by $\mathbf{1}_1(y)$ and $\mathbf{1}_2(y)$, respectively. Then, we set

$$\Omega_2^\varepsilon \subset \left\{x \in \Omega : \mathbf{1}_2\left(\frac{x}{\varepsilon}\right) = 1\right\} \quad \text{and} \quad \Omega_1^\varepsilon = \Omega \backslash \overline{\Omega_2^\varepsilon}. \tag{7.1}$$

We then consider system (2.4)–(2.11). The uniform estimates of Sec. 3 remain unchanged.

The extensions $\widetilde{S}_1^\varepsilon$ and $\widetilde{S}_1^\varepsilon$ of functions S_1^ε and S_2^ε are given by formula (4.4).

Under our new assumptions on the geometry, the function P_1^ε can be extended to the whole domain Ω so that

$$\|\widetilde{\mathsf{P}}_1^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C\|\mathsf{P}_1^\varepsilon\|_{L^2(0,T;H^1(\Omega_1^\varepsilon))}.$$

We then set

$$\widetilde{\mathsf{P}}_2^\varepsilon = \mathsf{P}_1^\varepsilon + \mathsf{G}_{1,w}(S_1^\varepsilon) - \mathsf{G}_{2,w}(\widetilde{S}_2^\varepsilon) \quad \text{in } \Omega_{1,T}^\varepsilon.$$

From the properties of functions P_1^ε, $\widetilde{S}_1^\varepsilon$ and $\widetilde{S}_2^\varepsilon$ it easily follows that

$$\lim_{\delta\to 0}\sup_{|y|\leq\delta}\int_{\Omega_T}|\mathsf{P}_2^\varepsilon(x+y,t)-\mathsf{P}_2^\varepsilon(x,t)|^2dxdt = 0$$

and, consequently,

$$\lim_{\delta\to 0}\sup_{|y|\leq\delta}\int_{\Omega_T}|\widetilde{\Theta}_2^\varepsilon(x+y,t)-\widetilde{\Theta}_2^\varepsilon(x,t)|^2dxdt = 0.$$

With these definitions the compactness results for $\widetilde{S}_\ell^\varepsilon$ and $\widetilde{\Theta}_\ell^\varepsilon$ can be proved exactly in the same way as in Sec. 4.

Finally, the convergence result stated in Theorem 5.1 also remains valid.

8. Concluding Remarks

We have presented a homogenization result for a degenerate system modeling immiscible compressible two-phase flow through a porous medium made of several types of rocks. We have assumed that the porosity, the absolute permeability, the capillary and relative permeabilities curves are different in each type of porous media. This leads to nonlinear transmission conditions representing the continuity of some physical characteristics such as water and gas pressures, at the interfaces that separate different media. Then the saturation and some other characteristics are getting discontinuous at the interfaces.

In this work we assumed that the capillary pressure is bounded, and $\varrho_{\min} > 0$. However, these conditions are violated in several important applications. Rigorous mathematical study of the case when these conditions do not hold is an interesting issue. It is also interesting to consider a similar model in asymptotically high contrast media.

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