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SYMMETRY-RELATIONS FOR ELASTICALLY DEFORMED PERIODIC ROD-STRUCTURES*

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In this paper we study periodic elastic rod-structures which are locally anisotropic and symmetric with respect to some plane. In order to find the effective behavior and approximate local behavior (so-called corrector-results) of such structures, one has to solve a finite number of boundary-value problems on one period of the rod-structure, the cell problem. For the solution of the cell-problem, it is shown that the components of the displacement satisfy either Neumann or Dirichlet conditions on the sides of the cell of periodicity parallel with the symmetry-plane. This is very useful from a computational point of view since the derived boundary conditions can easily be incorporated into standard numerical schemes. We also study resultant forces and moments and their variations along the rod-structure in several types of cases, even when no symmetry is required.

Keywords: Symmetry relations; rod structures; homogenization.

AMS Subject Classification: 35B27

1. Introduction

Effective stiffness parameters of a periodic rod-structure, such as effective rigidity of extension, flexural stiffness or torsion rigidity, can be found by deforming the structure (e.g. stretching, bending or twisting) in such a way that the stress tensor

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becomes periodic. The corresponding displacement-vector u then takes the form

$$u = v + w,$$

where $w = w(x_1, x_2, x_3)$ is periodic in the x_3 variable (the longitudinal direction) and v is a given function, hereafter referred to as a global function, which depends on the particular effective parameter we want to compute. Then, by comparing the resultant forces or moments with the average elongation, curvature or relative twist angle, we find the corresponding effective stiffness parameters.

Here we give a simple algorithm of determining the effective characteristics of a thin elastic perforated rod. It should be noted that we do not deal here with solutions of any macroscopic boundary value problem. Instead, we determine the effective characteristics in terms of model solutions of the elasticity equations defined in the unbounded rod. Some boundary value problems for thin rods and bars have been studied in Refs. 10 and 11. However, the asymptotics of solutions have been constructed in these works under certain symmetry assumptions. Namely, it was assumed that the rod cross sections possess two symmetry axes. In the presence of these symmetry axes the limit (homogenized) problem is getting decoupled and consists of four independent ordinary differential equations describing respectively the displacements in two transversal directions, tangential displacement and the rod torsion. The corresponding effective elastic moduli coincide with the rod effective moduli obtained (by essentially simpler method) in the present paper.

Our technique does not assume any symmetry of the cross sections. However, in the lack of symmetry the questions on the asymptotic behavior of solutions to boundary value problems remain open. There are arguments in favor of the conjecture that in this case the limit system might consist of four coupled ODE. The derivation of the limit problem in the most general non-symmetric case is still an open problem.

In our model we assume that there are no body forces presented and that the side surface is free from external stresses. Thus the displacement u is uniquely determined (at least within a rigid displacement) by the standard equations of elastic equilibrium defined on one single period, a so-called Y-cell. If the structure is symmetric with respect to some plane, the deformed structure will in some sense inherit this symmetry. In the case shown in Fig. 1, where we have two symmetry planes ($x_3 = 0$ and $x_2 = 0$) and the structure is stretched longitudinally such that

$$v(x_1, x_2, x_3) = (0, 0, \tau x_3)$$

for some constant τ, this fact is physically obvious. As one might also guess from the figure, one consequence of these symmetries is that the resultant torsion moment and the resultant bending moment about the x_1-axis vanish. Another important implication of the symmetry with respect to the plane $x_3 = 0$ is that the periodicity boundary conditions on $w = (w_1, w_2, w_3)$ can be replaced by Neumann conditions for the displacement-components w_1 and w_2 and Dirichlet conditions for w_3 on the two parallel surfaces of the Y-cell normal to the x_3-axes. Thus, since v is known

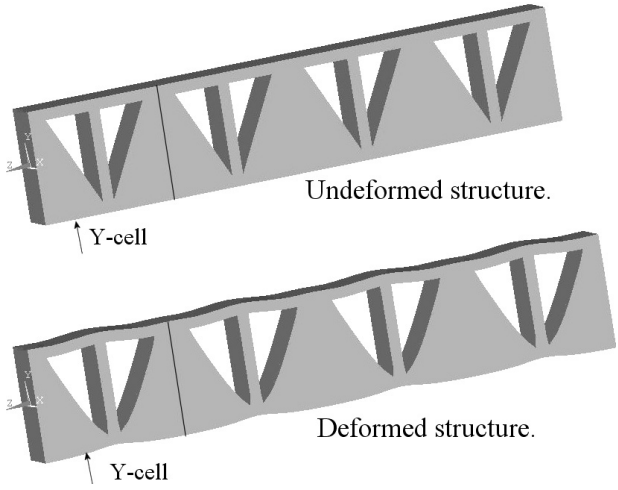

Fig. 1. Periodic rod-structure with isotropic material stretched longitudinally (i.e. along the z-axes in the figure). The Young's modulus is 1 while the Poisson's ratio is 0.3. We observe that the deformed structure inherits the symmetry of the undeformed structure. The figure is generated from a computation performed by the FE-program ANSYS 9.1.

we easily obtain boundary conditions for $u = v + w$. Such boundary conditions are often significantly easier to implement in many FE-codes than the original ones which couples node values for u on these two parallel surfaces. Thus, in computational practice the derived boundary conditions permit us to easily incorporate the periodic boundary value problems into standard numerical schemes.

The measured response of the stretching shown in Fig. 1 can e.g. be used to determine the effective rigidity of extension of the rod-structure. In order to compute other effective parameters of the rod-structure we have to subject the structure to other types of deformations by using suitable adjustments on the global function v. For example, for finding the effective torsion rigidity we may put

$$v(x_1, x_2, x_3) = (-\tau x_3 x_2, \tau x_3 x_1, 0),$$

and for finding the effective flexural stiffness with respect to the x_1-axis we might put

$$v(x_1, x_2, x_3) = (\tau x_3^2, 0, -2\tau x_3 x_1).$$

Even from a physical point of view it is not obvious what resultant forces which vanish (if any) or if we can replace the periodicity condition on w with similar Neumann and Dirichlet boundary conditions in all these cases. However, in this paper we prove precise criteria for the vanishing of resultant forces and moments and prove that a similar change of the boundary conditions, as that described above, is possible for all types of deformations needed in order to calculate the relevant effective parameters. Moreover, our results turn out to be valid even for a large class of multi-component and locally anisotropic rod-structures which possess

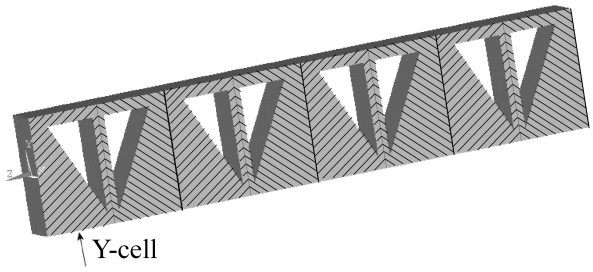

Fig. 2. Rod-structure of monocyclic material with local symmetry plane as indicated in the figure.

a more general type of symmetry than that illustrated in Fig. 1 [see (6.1) and (6.2) below].

This class of problems includes e.g. the rod-structure which is illustrated in Fig. 2. Such structures appear naturally in computational problems related to composites and structural engineering. The analysis of such rod-structures is also important for another reason. It appears namely as an intermediate modelling-step in a homogenization procedure for computing the effective behavior of multiscaled rod-structures similar to that of ordinary multiscaled materials, a topic which has been treated extensively in the literature (see e.g. Refs. 1, 2, 3–6 and the references given therein).

Even if we focus on symmetric rod-structures, we also present results concerning resultant forces and moments and their variations along the rod-structure also in situations where no symmetry is assumed.

The paper is organized as follows. We have collected some preliminaries in Sec. 2. In Sec. 3, we discuss weak formulations and associated classical formulations of the relevant stress problems related to the computation of the effective properties for periodic rod-structures. In Sec. 4, we prove some general results concerning resultant forces and moments. For computational purposes, we describe relations between effective properties and strain energies in Sec. 5. In Sec. 6, we show that the local stress tensor and the corresponding resultant forces and moments belong to certain symmetry classes depending on the symmetry of the rod-structure. These results are used in Sec. 7 to find symmetry properties which ensure that certain resultant forces and moments vanish. Finally we discuss equivalent boundary conditions for rod-structures with longitudinal symmetry in Sec. 8.

2. Preliminaries

We let $\mathbf{S}$ be the space of symmetric 3×3 matrices and let $a \cdot b$ denote the scalar product between two matrices $a = \{a_{ij}\}$ and $b = \{b_{ij}\}$ in $\mathbf{S}$, which is defined by $a \cdot b = \sum_{ij} a_{ij} b_{ij}$. The norm $|a|$ is correspondingly defined by $|a|^2 = \sum_{ij} a_{ij}^2$ (here and in the rest of the paper $\sum_{ij}$ denotes $\sum_{i=1,j=1}^{3}$). If a and b are vectors, $a \cdot b$ will denote the usual scalar product in $\mathbb{R}^3$.

Let $\Omega \subset \mathbb{R}^3$ be the region occupying the rod-structure. We assume that Ω is a connected open set with Lipschitz continuous boundary which is bounded in the x_1 and x_2 variables and periodic in the x_3 variable with respect to some interval $I = (-x_3^0/2, x_3^0/2)$. The set $Y = \{x \in \Omega : x_3 \in I\}$ corresponds to a period of the rod-structure, and is referred to as the Y-cell. The boundary ∂Y of Y with outward unit-normal $n = (n_1, n_2, n_3)$ consists of the two disjoint parts $B = \{x \in \overline{Y} : x_3 = -x_3^0/2 \text{ or } x_3 = x_3^0/2\}$ and $C = \partial Y \backslash B$. We note that $B = B(-x_3^0/2) \cup B(x_3^0/2)$ where $B(t)$ denotes the vertical surface $B(t) = \{x = (x_1, x_2, x_3) \in \overline{Y},\ x_3 = t\}$. Moreover, we let $\mathbf{H}^1_{\text{per},3}(Y)$ denote the closure in the usual Sobolev space $\mathbf{H}^1(Y)$, equipped with the norm

$$\|\varphi\| = \langle |D\varphi|^2 + |\varphi|^2 \rangle^{1/2},$$

of the set $\mathbf{C}^\infty_{\text{per},3}(Y)$ of all smooth vector-valued functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ which is I-periodic in the x_3 variable, i.e.

$$\varphi(x + e_3 x_3^0) = \varphi(x)$$

for all $x \in \mathbb{R}^3$ (e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$). Here, $\langle \cdot \rangle$ denotes as usual the average over the Y-cell, i.e.

$$\langle f \rangle = \frac{1}{|Y|} \int_Y f(x) dx$$

and

$$|D\varphi|^2 = \sum_{ij} \left| \frac{\partial \varphi_i}{\partial x_j} \right|^2 .$$

In accordance with the terminology of mathematical elasticity we let $e(\varphi) = \{e_{ij}(\varphi)\}$ denote the strain

$$e_{ij}(\varphi) = \frac{1}{2} \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right)$$

and let $\sigma = \{\sigma_{ij}\}$ denote the corresponding stress. The Hooke's law is generally expressed by

$$\sigma_{ij}(\varphi) = \sum_{kr} a_{ijkr} e_{kr}(\varphi),$$

where the fourth order elasticity tensor $A = \{a_{ijkr}(x)\}$, $x \in \Omega$ satisfies the following symmetry relations

$$a_{ijkr} = a_{krij}, \quad a_{ijkr} = a_{jikr} = a_{ijrk}, \tag{2.1}$$

together with the inequalities

$$\nu_1 |\xi|^2 \leq \xi \cdot A\xi \leq \nu_2 |\xi|^2 \tag{2.2}$$

for all $\xi \in \mathbf{S}$ and some strictly positive constants ν_1 and ν_2 which are independent of ξ and x. Here, $A\xi$ denote the matrix with elements

$$(A\xi)_{ij} = \sum_{kr} a_{ijkr}\xi_{kr}.$$

The above relations imply that the coefficients a_{ijkr} are bounded.

3. Weak and Classical Formulations of the Stress Problems

In this paper we will also assume that each component a_{ijkr} is Lebesgue-measurable in Ω and I-periodic in the x_3 variable. Moreover, we assume that the global displacement $v \in \mathbf{H}^1(Y)$ is such that $\xi = e(v)$ is I-periodic in the x_3 variable, even though all examples considered in this paper are devoted to situations where ξ is independent of the x_3 variable.

In order to calculate an effective parameter associated with v we have to solve the following problem: Find $w \in \mathbf{H}^1_{\text{per},3}(Y)$ such that

$$\int_Y e(\varphi) \cdot A(\xi + e(w))dx = 0 \quad \text{for all } \varphi \in \mathbf{H}^1_{\text{per},3}(Y).$$

Noting that $\sigma(u) = A(e(u)) = A(\xi + e(w))$, we may rewrite this formulation as follows: Find $u = v + w$, where $w \in \mathbf{H}^1_{\text{per},3}(Y)$ such that

$$\int_Y e(\varphi) \cdot \sigma(u)dx = 0 \quad \text{for all } \varphi \in \mathbf{H}^1_{\text{per},3}(Y). \tag{3.1}$$

This problem is equivalent to finding the corresponding strain energy $W_\xi = F_\xi(w)$ from the variational problem: Find $w \in \mathbf{H}^1_{\text{per},3}(Y)$ such that

$$F_\xi(w) \le F_\xi(\varphi) \quad \text{for all } \varphi \in \mathbf{H}^1_{\text{per},3}(Y), \tag{3.2}$$

where

$$F_\xi(\varphi) = \frac{1}{2}\int_Y (\xi + e(\varphi)) \cdot A(\xi + e(\varphi))dx.$$

Since $e(u) = \xi + e(w)$, we observe that

$$W_\xi = F_\xi(w) = \frac{1}{2}\int_Y e(u) \cdot A(e(u))dx. \tag{3.3}$$

Concerning existence and uniqueness of these problems, see Lemma 8.1 below. We note that (3.1) can be derived from the following classical formulation of the elasticity problem: Find $u = v + w$, such that

$$\begin{cases} \operatorname{div} \sigma(u) = 0 & \text{in } Y, \\ F(u) = 0 & \text{on } C, \\ w \in \mathbf{H}^1_{\text{per},3}(Y), \end{cases} \tag{3.4}$$

where $F(u) = (F_1(u), F_2(u), F_3(u))$ is the stress vector acting on a plane with outward unit normal $n = (n_1, n_2, n_3)$, given by $F_i(u) = \sum_{j=1}^3 \sigma_{ij}(u)n_j$. The first

of the three conditions in (3.4) comes from the assumption that there are no body forces present. The second condition merely tells that the side surface C is free from external stresses. The derivation of (3.1) from (3.4) follows from Green's formula

$$\int_Y \varphi \cdot \operatorname{div} \sigma(u) dx + \int_Y e(\varphi)\sigma(u) dx = \int_{\partial Y} \varphi \cdot F(u) ds. \tag{3.5}$$

Indeed, $F_i(u) = \sigma_{i3}(u)$ and $F_i(u) = -\sigma_{i3}(u)$ on the left and right of B, respectively. Moreover, due to the I-periodicity of $\sigma_{i3}(w)$ and $\sigma_{i3}(v)$, we see that $\sigma_{i3}(u) = \sigma_{i3}(w) + \sigma_{i3}(v)$ is also I-periodic. Thus, since φ is I-periodic, it is clear that $\varphi \cdot F(u)$ takes opposite values on opposite sides of B. Hence, $\int_B \varphi \cdot F(u) ds = 0$. In addition, since $F(u) = 0$ on C, $\int_C \varphi \cdot F(u), ds = 0$. Thus

$$\int_{\partial Y} \varphi \cdot F(u) ds = \int_B \varphi \cdot F(u) ds + \int_C \varphi \cdot F(u) ds = 0,$$

and since $\operatorname{div} \sigma(u) = 0$, we obtain (3.1) from (3.5).

Note that the derivation of (3.1) from (3.4) is only possible if the stress field is sufficiently smooth such that the Green's formula is valid (and makes sense).

4. Resultant Forces and Moments

In connection with the definition of several effective parameters we will use the concept of resultant forces $N_{ij}(x_3)$ and resultant moment $M_i(x_3)$ about the x_i-axis of the stress vector $(\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u))$ applied to the surface $B(x_3)$. These functions are defined by

$$N_{ij}(x_3) = \int_{B(x_3)} \sigma_{ij}(u)(x) dx_1 dx_2,$$

and

$$M_1(x_3) = \int_{B(x_3)} (x_3\sigma_{23}(u) - x_2\sigma_{33}(u)) dx_1 dx_2,$$

$$M_2(x_3) = \int_{B(x_3)} (x_3\sigma_{13}(u) - x_1\sigma_{33}(u)) dx_1 dx_2,$$

$$M_3(x_3) = \int_{B(x_3)} (-x_2\sigma_{13}(u) + x_1\sigma_{23}(u)) dx_1 dx_2.$$

Before discussing this any further, we prove a lemma which will be useful in our study of these quantities. Consider two disjoint intervals $I(p_1, r_1)$ and $I(p_2, r_2)$ in I of lengths $2r_1$ and $2r_2$ and with centers at some fixed points p_1 and p_2, respectively. Moreover, let $g = g(x_3)$ be a continuous periodic function of x_3 defined in I by

$$g'(x_3) = \begin{cases} s_1 & x_3 \in I(p_1, r_1), \\ s_2 & x_3 \in I(p_2, r_2), \\ 0 & \text{elsewhere}, \end{cases} \tag{4.1}$$

where s_1 and s_2 are constants satisfying the condition

$$r_1 s_1 + r_2 s_2 = 0 \tag{4.2}$$

(by this condition g becomes I-periodic). We have the following result.

Lemma 4.1. *Let $f \in L^1(Y)$ and let $k(x_3)$ be defined almost everywhere by*

$$k(x_3) = \int_{B(x_3)} f(x_1, x_2, x_3) dx_1 dx_2.$$

If the identity

$$\int_Y f(x_1, x_2, x_3) g'(x_3) dx = 0 \tag{4.3}$$

holds for all disjoint intervals $I(p_1, r_1)$ and $I(p_2, r_2)$ in I, then there exists a constant k such that $k(x_3) = k$ for almost every x_3.

Proof. From the definition of g we see that (4.3) implies that

$$s_1 \int_{I(p_1,r_1)} k(t) dt + s_2 \int_{I(p_2,r_2)} k(t) dt = 0.$$

Using (4.2) we now find that

$$\frac{1}{2r_1} \int_{I(p_1,r_1)} k(t) dt = \frac{1}{2r_2} \int_{I(p_2,r_2)} k(t) dt,$$

i.e.

$$\frac{1}{|I(p_1, r_1)|} \int_{I(p_1,r_1)} k(t) dt = \frac{1}{|I(p_2, r_2)|} \int_{I(p_2,r_2)} k(t) dt.$$

Since the intervals were chosen arbitrarily, this shows that the average value of $k(t)$ taken over any interval is equal to a constant k. Hence

$$\lim_{r \to 0} \frac{1}{|I(x_3, r)|} \int_{I(x_3,r)} k(t) dt = k,$$

at all points $x_3 \in I$. According to Lebesgue differentiation theorem, almost all points in I are Lebesgue-points, i.e. points x_3 for which the above limit equals $k(x_3)$. Hence, $k(x_3) = k$ a.e. □

We now turn back to our discussion on resultant forces and moments.

Theorem 4.1. *For any global displacement v it holds that the average value $N_{ij} = \langle N_{ij}(\cdot) \rangle = 0$ unless $i = j = 3$. Moreover $N_{i3}(x_3) = N_{i3}$ for almost every x_3.*

Likewise, there exist constants M_i such that $M_i(x_3) = M_i$ for almost every x_3. In addition, the following simplifications are valid:

$$M_1(x_3) = -\int_{B(x_3)} x_2\sigma_{33}(u)dx_1dx_2 \quad \textit{and} \quad M_2(x_3) = -\int_{B(x_3)} x_1\sigma_{33}(u)dx_1dx_2.$$

Proof. The fact that N_{11}, N_{12}, N_{22}, N_{13} and N_{23} vanish follows by inserting $\varphi(x) = (x_1, 0, 0)$, $\varphi(x) = (x_2, x_1, 0)$, $\varphi(x) = (0, x_2, 0)$, $\varphi = (0, 0, x_1)$ and $\varphi = (0, 0, x_2)$, respectively, into (3.1), since we directly obtain that

$$N_{ij} = \frac{1}{x_3^0}\int_Y 1\sigma_{ij}(u)dx = \frac{1}{x_3^0}\int_Y e(\varphi)\cdot\sigma(u)dx = 0,$$

by these choices of test functions.

In order to show that $N_{i3}(x_3)$ is constant we insert

$$\varphi = (\delta_{i1}g(x_3), \delta_{i2}g(x_3), \delta_{i3}g(x_3))$$

into (3.1), where g is defined in (4.1). Observing that $e_{3i}(\varphi) = e_{i3}(\varphi)$ (which equals $g'(x_3)/2$ if $i \neq 3$ and $g'(x_3)$ if $i = 3$) is the only nonvanishing component(s) of $e(\varphi)$, we obtain that

$$\int_Y g'(x_3)\sigma_{i3}(u)dx = \int_Y e(\varphi)\sigma(u)dx = 0.$$

Hence, by Lemma 4.1,

$$N_{i3}(x_3) = \int_{B(x_3)} \sigma_{13}(u)dx_1dx_2 = k$$

for some constant k, which certainly coincide with the average value N_{i3}.

In order to show that $M_1(x_3)$ is constant a.e. we first observe that

$$\begin{aligned} M_1(x_3) &= \int_{B(x_3)} (x_3\sigma_{23}(u) - x_2\sigma_{33}(u))dx_1dx_2 \\ &= x_3N_{23}(x_3) - \int_{B(x_3)} x_2\sigma_{33}(u)dx_1dx_2 \\ &= -\int_{B(x_3)} x_2\sigma_{33}(u)dx_1dx_2, \end{aligned} \tag{4.4}$$

since $N_{23}(x_3) = 0$. Inserting $\varphi = (0, g(x_3), -x_2g(x_3))$ into (3.1) and observing that $e_{32}(\varphi) = e_{23}(\varphi) = (g'(x_3) - g(x_3))/2$ and $e_{33}(\varphi) = -x_2g'(x_3)$ are the only nonvanishing components of $e(\varphi)$, we obtain that

$$\int_Y \frac{g'(x_3) - g(x_3)}{2}\sigma_{23}(u)dx - \int_Y x_2\sigma_{33}(u)g'(x_3)dx = \int_Y e(\varphi)\sigma(u)dx = 0. \tag{4.5}$$

The first term

$$\int_Y \frac{g'(x_3) - g(x_3)}{2}\sigma_{23}(u)dx = \int_{-x_3^0/2}^{x_3^0/2} \frac{g'(x_3) - g(x_3)}{2}N_{23}(x_3)dx_3$$

vanishes since $N_{23}(x_3) = 0$. Thus (4.5) reduces to

$$-\int_Y x_2\sigma_{33}(u)g'(x_3)dx = 0,$$

which by (4.4) implies that

$$\int_Y M_1(x_3)g'(x_3)dx = 0.$$

Hence, according to Lemma 4.1, $M_1(x_3) = M_1$ a.e. for some constant M_1.

The fact that $M_2(x_3) = M_2$ a.e. where the constant

$$M_2 = -\int_{B(x_3)} x_1\sigma_{33}(u)dx_1dx_2,$$

follows by inserting $\varphi = (g(x_3), 0, -x_1g(x_3))$ into (3.1). Similarly as above we observe that $e_{13}(\varphi) = e_{31}(\varphi) = (g'(x_3) - g(x_3))/2$ and $e_{33}(\varphi) = -x_1g'(x_3)$ are the only nonvanishing components of $e(\varphi)$ and obtain that

$$\int_Y (g'(x_3) - g(x_3))\sigma_{23}(u)dx - \int_Y x_2\sigma_{33}(u)g'(x_3)dx = \int_Y e(\varphi)\sigma(u)dx = 0.$$

The rest of the arguments are identical with those used for the moment $M_1(x_3)$.

The fact that $M_3(x_3)$ is constant a.e. is shown as follows. Inserting $\varphi = (-\tau x_2, \tau x_1, 0)g(x_3)$ into (3.1), and observing that

$$e(\varphi) = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\tau x_2 \\ 0 & 0 & \tau x_1 \\ -\tau x_2 & \tau x_1 & 0 \end{pmatrix} g'(x_3),$$

we obtain that (4.3) holds $f(x_1, x_2, x_3) = -\tau x_2\sigma_{13}(u) + \tau x_1\sigma_{23}(u)$. Hence

$$M_3(x_3) = M_3,$$

almost everywhere, for some constant M_3. □

5. Relations Between Effective Properties and Strain Energies

In the case when the global displacement $v = v(x_1, x_2, x_3)$ is of the form,

$$v = (0, 0, \tau x_3) \qquad \text{(extension in } x_3\text{-direction)}, \tag{5.1}$$

$$v = (-\tau x_3x_2, \tau x_3x_1, 0) \qquad \text{(torsion in } x_1x_2\text{-plane)}, \tag{5.2}$$

$$v = (\tau x_3^2, 0, -2\tau x_3x_1) \qquad \text{(pure bending about the } x_2\text{-axis)}, \tag{5.3}$$

$$v = (\tau x_3^2, 0, -2\tau x_3x_2) \qquad \text{(pure bending about the } x_1\text{-axis)}, \tag{5.4}$$

we observe that $\xi = e(v)$ is given by

$$e(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau \end{pmatrix}, \quad e(v) = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\tau x_2 \\ 0 & 0 & \tau x_1 \\ -\tau x_2 & \tau x_1 & 0 \end{pmatrix} \tag{5.5}$$

and

$$e(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\tau x_1 \end{pmatrix}, \quad e(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\tau x_2 \end{pmatrix}, \tag{5.6}$$

respectively.

By (3.1) we generally have that

$$\begin{aligned} F_\xi(w) &= \frac{1}{2}\int_Y (\xi + e(w)) \cdot A(\xi + e(w))dx \\ &= \frac{1}{2}\int_Y \xi \cdot A(\xi + e(w))dx \\ &= \frac{1}{2}\int_Y e(v) \cdot \sigma(u)dx, \end{aligned}$$

i.e.

$$F_\xi(w) = \frac{1}{2}\int_Y e(v) \cdot \sigma(u)dx. \tag{5.7}$$

This identity will be useful for finding the relations between various types of effective properties and the corresponding strain energy $F_\xi(w)$.

5.1. *Effective rigidity of extension*

If v is of the form (5.1), i.e.

$$v(x_1, x_2, x_3) = (0, 0, \tau x_3),$$

the identity (5.7) reduces to

$$F_\xi(w) = \frac{1}{2}\int_Y e_{33}(v)\sigma_{33}(u)dx = \frac{1}{2}\tau\int_Y \sigma_{33}(u)dx. \tag{5.8}$$

Since $\langle\sigma_{33}(u)\rangle$ and $\langle e_{33}(u)\rangle$ are the resultant extending force and the average extension in the x_3-direction, respectively, it is natural to define the *effective rigidity of extension* D_{ex} as the relation

$$D_{ex} = \frac{\langle\sigma_{33}(u)\rangle}{\langle e_{33}(u)\rangle}.$$

Due to the periodicity of w we have that $\langle e_{33}(w)\rangle = 0$, which gives that

$$\langle e_{33}(u)\rangle = \langle e_{33}(v)\rangle + \langle e_{33}(w)\rangle = \langle e_{33}(v)\rangle = \tau.$$

Thus by (5.8) we obtain that

$$D_{ex} = \frac{\langle\sigma_{33}(u)\rangle}{\tau} = \frac{2F_\xi(w)}{|Y|\tau^2}.$$

5.2. *Effective torsion rigidity*

When v takes the form (5.2), i.e.

$$v(x_1, x_2, x_3) = (-\tau x_3 x_2, \tau x_3 x_1, 0)$$

the identity (5.7) reduces to

$$\begin{aligned} F_\xi(w) &= \frac{1}{2}\int_Y e_{13}(v)\sigma_{13}(u) + e_{23}(v)\sigma_{23}(u)\,dx \\ &= \frac{1}{2}\int_Y (-\tau x_2\sigma_{13}(u) + \tau x_1\sigma_{23}(u))dx \\ &= \frac{1}{2}\int_{-x_3^0/2}^{x_3^0/2}\left(\int_{B(t)} (-\tau x_2\sigma_{13}(u) + \tau x_1\sigma_{23}(u))dx_1dx_2\right)dt. \end{aligned}$$

Hence, due to Theorem 4.1,

$$2F_\xi(w) = \tau M_3 x_3^0. \tag{5.9}$$

Due to linearity $u = \tau u_0$, where u_0 is the solution corresponding to the global displacement $v = (-x_3x_2, x_3x_1, 0)$. Thus

$$M_3 = \tau D_3, \tag{5.10}$$

where

$$D_3 = \int_{B(t)} (-x_2\sigma_{13}(u_0) + x_1\sigma_{23}(u_0))dx_1dx_2.$$

This shows that M_3 is proportional to the relative twist τ (the rotation-angle per unit-length in the x_3 direction). In agreement with the terminology of torsion in the two-dimensional case, we call D_3 *the effective torsional rigidity*. By (5.9) and (5.10),

$$D_3 = \frac{2F_\xi(w)}{x_3^0\tau^2}. \tag{5.11}$$

If we assume sufficient regularity on the stress $\sigma(u)$ and the domain Y, it is possible to give a simpler proof of (5.11) (without involving Lemma 4.1) by using Green's formula,

$$\int_Y v\cdot \operatorname{div}\sigma(u)dx + \int_Y e(v)\sigma(u)dx = \int_{\partial Y} v\cdot F(u)ds,$$

which due to (3.4) reduces to

$$\int_Y e(v)\sigma(u)dx = \int_B v\cdot F(u)ds. \tag{5.12}$$

As observed earlier, $F_i(u) = \sigma_{i3}(u)$ and $F_i(u) = -\sigma_{i3}(u)$ on the left and right of B, respectively, and these values are opposite. Moreover, for this particular case v

takes also opposite values on opposite sides of B. Hence,

$$\int_B v \cdot F(u)ds = 2\frac{x_3^0}{2}\int_{B(x_3^0)} (-\tau x_2\sigma_{13}(u) + \tau x_1\sigma_{23}(u))dx_1dx_2 = x_3^0\tau M_3.$$

Thus, by (5.7), (5.10) and (5.12) we obtain (5.11). Note, however, that this proof is less general than that above, which does not rely on the validity of Green's formula and the assumption that the concept of stress vector $F(u)$ is meaningful in classical sense on the entire boundary of Y.

It is natural to ask whether the value of D_3 would be influenced by replacing the x_3-axis by another one, parallel to it. Fortunately, this does not happen. Indeed, let $(a, b, 0)$ be the intersection of the plane $x_3 = 0$ and the new axis. Then we must replace v by the function v' given by

$$v' = v'(x_1, x_2, x_3) = (-\tau x_3(x_2 - b), \tau x_3(x_1 - a), 0).$$

We observe that one corresponding solution $u' = v' + w'$ is obtained by choosing $w' = w + (0, 0, \tau x_2 a - \tau x_1 b)$, where w is the periodic part of the original solution $u = v + w$. Indeed, we easily see that

$$u' = v' + w' = v + w + \psi = u + \psi,$$

where $\psi = (\tau x_3 b, -\tau x_3 a, \tau x_2 a - \tau x_1 b)$, which is a rigid displacement, i.e. $e(\psi) = 0$. Hence $e(u) = e(u')$, so u' is clearly a solution of (3.1), and

$$F_\xi(w) = \frac{1}{2}\int_Y e(u) \cdot A(e(u))dx = \frac{1}{2}\int_Y e(u') \cdot A(e(u'))dx = F_{\xi'}(w'),$$

where $\xi' = e(v')$. Similarly as (5.11) the effective torsional rigidity D_3' corresponding to the new x_3-axis will be given by

$$D_3' = \frac{2F_{\xi'}(w')}{x_3^0\tau^2}.$$

Thus, since $F_\xi(w) = F_{\xi'}(w')$, (5.11) gives that $D_3' = D_3$, i.e. the effective torsional rigidity is unaffected by the choice of the new axis, as long as it is parallel with the old one.

5.3. *Effective flexural stiffness*

In the case when

$$v(x_1, x_2, x_3) = (\tau x_3^2, 0, -2\tau x_3 x_1),$$

i.e. of the form (5.3), the identity (5.7) reduces to

$$F_\xi(w) = \frac{1}{2}\int_Y e_{33}(v) \cdot \sigma_{33}(u)dx = -\int_Y \tau x_1\sigma_{33}(u)dx.$$

Thus, according to Theorem 4.1,

$$F_\xi(w) = \tau x_3^0 M_2. \tag{5.13}$$

Again, due to linearity, $u = \tau u_0$ where u_0 is the solution corresponding to the global displacement

$$v(x_1, x_2, x_3) = (x_3^2, 0, -2x_3x_1).$$

Thus

$$M_2 = \tau D_2, \tag{5.14}$$

where D_2 is the resultant moment corresponding to u_0, i.e.

$$D_2 = -\int_{B(x_3)} x_1 \sigma_{33}(u_0) dx. \tag{5.15}$$

The points on the x_3-axis having before deformation the coordinates $x = (0, 0, x_3)$ will move to points with coordinates

$$\eta(x) + w(x) = (\eta_1(x_3) + w_1(x_3), \eta_2(x_3) + w_2(x_3), \eta_3(x_3) + w_3(x_3)),$$

where $\eta_1(x_3) = \tau x_3^2$, $\eta_2(x_3) = 0$, $\eta_3(x_3) = x_3$ and $w_i(x_3)$ are I-periodic functions. Ignoring these periodic functions, which do not contribute to the global deformation of the rod-structure, we observe that $\tau = d^2\eta_1/dx_3^2$, i.e. τ is the curvature of the global deformation of the x_3-axis. In agreement with the two-dimensional theory of bending we therefore call the relation $D_2 = M_2/\tau$ the *effective flexural stiffness* with respect to bending about the x_2-axis. According to (5.13) and (5.14) we have that

$$D_2 = \frac{F_\xi(w)}{\tau^2 x_3^0}.$$

In the case of pure bending about the x_1-axis (5.4), i.e. when

$$v(x_1, x_2, x_3) = (\tau x_3^2, 0, -2\tau x_3 x_2),$$

we obtain similarly that the *effective flexural stiffness* with respect to bending about the x_1-axis is related to the corresponding strain energy $F_\xi(w)$ by the relation

$$D_1 = \frac{F_\xi(w)}{\tau^2 x_3^0}.$$

In contrast to the case of torsion, the values of effective flexural stiffness turn out to be highly dependent of the intersection between the x_3-axis and the x_1x_2-plane. Indeed, according to (5.15),

$$D_2 = -\int_{B(x_3)} x_1 \sigma_{33}(u_0) dx,$$

where $u = u_0$ is the solution corresponding to the global displacement $v(x_1, x_2, x_3) = (x_3^2, 0, -2x_3x_1)$. As before, let $(a, b, 0)$ be the intersection of the plane $x_3 = 0$ and the new axis. The effective flexural stiffness D_2' with respect to bending about the new x_1-axis, will then be given by

$$D_2' = -\int_{B(x_3)} (x_1 - a)\sigma_{33}(u_0') dx,$$

where $u = u_0'$ is the solution corresponding to the global displacement v' given by

$$v'(x_1, x_2, x_3) = (x_3^2, 0, -2x_3(x_1 - a)).$$

This displacement can be written as the sum $v' = v + v''$, where $v'' = (0, 0, 2ax_3)$, i.e. v' is the sum of a pure extension in the x_3-direction and a pure bending about the original x_2-axis. Letting $u = u_0''$ be the solution corresponding to v'', we obtain by the linearity of the problem that $u_0' = u_0 + u_0''$. Thus

$$\begin{aligned} D_2' &= -\int_{B(x_3)} (x_1 - a)\sigma_{33}(u_0 + u_0'')dx \\ &= -\int_{B(x_3)} x_1\sigma_{33}(u_0)dx + a\int_{B(x_3)} \sigma_{33}(u_0)dx - \int_{B(x_3)} (x_1 - a)\sigma_{33}(u_0'')dx \\ &= D_2 + aN_{33}(u_0) + M_2(u_0''), \end{aligned}$$

where $N_{33}(u_0)$ is the resultant force in the x_3-direction corresponding to the solution u_0 and $M_2(u_0'')$ is the bending moment about the new x_2-axis of the forces in the vertical plane corresponding to the solution u_0''.

6. Symmetric Rod-Structures

We will now consider cases where

$$\Omega \text{ is symmetric with respect to the plane } x_s = 0 \tag{6.1}$$

for some fixed $s \in \{1, 2, 3\}$ and the elasticity tensor A satisfies the following symmetry property with respect to that plane:

$$a_{ijkr}(x) = (-1)^{\delta_{is}+\delta_{js}+\delta_{ks}+\delta_{rs}} a_{ijkr}(y), \tag{6.2}$$

where δ_{ij} is the Kronecker symbol,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is the mirrored coordinate with respect to the plane $x_s = 0$, i.e. $y = (-x_1, x_2, x_3)$, $y = (x_1, -x_2, x_3)$ and $y = (x_1, x_2, -x_3)$ if $s = 1$, $s = 2$ and $s = 3$, respectively. An example of a rod-structure satisfying (6.1) and (6.2) for $s = 3$ is illustrated in Fig. 2.

Note that (6.2) reduces to the simple symmetry condition

$$a_{ijkr}(x) = a_{ijkr}(y), \tag{6.3}$$

in the case when A is monocyclic with respect to the symmetry plane $x_s = 0$ at each point x. For the proof of this fact we refer to Ref. 7. In particular, (6.3) holds if the material is locally orthotropic with respect to the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, which certainly includes the class of isotropic materials.

Let $\mathcal{S}$ be the space of all functions $f : \Omega \to \mathbf{S}$ and consider the subspaces $\mathcal{S}_{|s}$ and $\mathcal{S}_{\perp s}$ defined by:

$$\mathcal{S}_{|s} = \{\xi \in \mathcal{S} : \xi_{ij}(x) = (-1)^{\delta_{is}+\delta_{js}}\xi_{ij}(y)\}, \tag{6.4}$$

$$\mathcal{S}_{\perp s} = \{\xi \in \mathcal{S} : -\xi_{ij}(x) = (-1)^{\delta_{is}+\delta_{js}}\xi_{ij}(y)\}. \tag{6.5}$$

In the case when ξ is constant these spaces reduce to

$$\mathbf{S}_{s|} = \{\xi \in \mathbf{S} : \xi_{is} = 0, i \neq s\},$$

$$\mathbf{S}_{s\perp} = \{\xi \in \mathbf{S} : \xi_{ss} = 0, \xi_{ij} = 0, i \neq s, j \neq s\},$$

respectively.

Remark 6.1. We observe from (5.5) and (5.6) that the strain $e(v)$ corresponding to the global displacement v belongs to the following spaces:

- If v is a pure extension (5.1) then $e(v) \in \mathcal{S}_{|1}$, $\mathcal{S}_{|2}$, $\mathcal{S}_{|3}$,
- If v is a torsion (5.2) then $e(v) \in \mathcal{S}_{\perp 1}$, $\mathcal{S}_{\perp 2}$, $\mathcal{S}_{\perp 3}$,
- If v is a pure bending about the x_2-axis (5.3) then $e(v) \in \mathcal{S}_{\perp 1}$, $\mathcal{S}_{|2}$, $\mathcal{S}_{|3}$,
- If v is a pure bending about the x_1-axis (5.4) then $e(v) \in \mathcal{S}_{|1}$, $\mathcal{S}_{\perp 2}$, $\mathcal{S}_{|3}$.

Theorem 6.1. *Let A be a Y-periodic tensor which is I-periodic in the x_3 variable and satisfying* (2.1), (2.2), (6.1) *and* (6.2) *for some fixed $s \in \{1, 2, 3\}$. In addition, let $v \in \mathbf{H}^1(Y)$ be such that $\xi = e(v)$ is I-periodic in the x_3 variable. Then, the stress tensor $\sigma(u)$ belongs to $\mathcal{S}_{|s}$ if $\xi \in \mathcal{S}_{|s}$ and $\mathcal{S}_{\perp s}$ if $\xi \in \mathcal{S}_{\perp s}$. Moreover, if $f : Y \to \mathbb{R}$ is symmetric with respect to the plane $x_s = 0$, i.e. $f(x) = f(y)$, then the symmetric matrix $K = \{K_{ij}\}$, where*

$$K_{ij} = \int_Y f(x)\sigma_{ij}(u)(x)dx,$$

belongs to $\mathbf{S}_{s|}$ if $\xi \in \mathcal{S}_{|s}$ and $\mathbf{S}_{s\perp}$ if $\xi \in \mathcal{S}_{\perp s}$. Conversely, if $f : Y \to \mathbb{R}$ is anti-symmetric with respect to the plane $x_s = 0$, i.e. $f(x) = -f(y)$, then $K = \{K_{ij}\}$ belongs to $\mathbf{S}_{s\perp}$ if $\xi \in \mathcal{S}_{|s}$ and $\mathbf{S}_{s|}$ if $\xi \in \mathcal{S}_{\perp s}$.

Proof. Let $\varphi \in \mathbf{H}^1_{\text{per},3}(Y)$. If $\xi \in \mathbf{S}_{s|}$, we define the function $\widetilde{\varphi}$ as follows:

$$\begin{aligned} &\widetilde{\varphi}_s(x) = -\varphi_s(y), \\ &\widetilde{\varphi}_i(x) = \varphi_i(y) \quad \text{for all } i \neq s. \end{aligned} \tag{6.6}$$

We obtain the relations

$$\frac{\partial \widetilde{\varphi}_s(x)}{\partial x_s} = \frac{\partial \varphi_s(y)}{\partial y_s},$$

$$\frac{\partial \widetilde{\varphi}_s(x)}{\partial x_i} = -\frac{\partial \varphi_s(y)}{\partial y_i}, \frac{\partial \widetilde{\varphi}_i(x)}{\partial x_s} = -\frac{\partial \varphi_i(y)}{\partial y_s} \quad \text{for } i \neq s$$

and

$$\frac{\partial \widetilde{\varphi}_i(x)}{\partial x_j} = \frac{\partial \varphi_i(y)}{\partial y_j} \quad \text{for } i \neq s,\ j \neq s.$$

Thus,

$$e_{ij}(\widetilde{\varphi})(x) = (-1)^{\delta_{is}+\delta_{js}} e_{ij}(\varphi)(y). \tag{6.7}$$

This identity holds in particularly for the solution w of (3.1), i.e.

$$e_{kr}(\widetilde{w})(x) = (-1)^{\delta_{ks}+\delta_{rs}} e_{kr}(w)(y),$$

and by adding this to the identity (6.4) (with i and j replaced by k and r, respectively),

$$\xi_{kr}(x) = (-1)^{\delta_{ks}+\delta_{rs}} \xi_{kr}(y),$$

and multiplying with (6.2), we obtain

$$a_{ijkr}(e_{kr}(\widetilde{w}) + \xi_{kr})(x) = (-1)^{\delta_{is}+\delta_{js}} a_{ijkr} e_{ij}(\varphi)(e_{kr}(w) + \xi_{kr})(y). \tag{6.8}$$

Thus, multiplying with (6.7) gives

$$e_{ij}(\widetilde{\varphi}) a_{ijkr}(e_{kr}(\widetilde{w}) + \xi_{kr})(x) = e_{ij}(\varphi) a_{ijkr}(e_{kr}(w) + \xi_{kr})(y),$$

which together with (3.1) implies

$$\int_Y e(\widetilde{\varphi}) \cdot A(e(\widetilde{w}) + \xi)(x) dx = \int_Y e(\varphi) \cdot A(e(w) + \xi)(y) dy = 0. \tag{6.9}$$

Noting that every function $\varsigma \in \mathbf{H}^1_{\mathrm{per},3}(Y)$ can be represented by $\varsigma = \widetilde{\varphi}$ where $\varphi = \widetilde{\varsigma}$, we obtain that

$$\int_Y e(\varsigma) \cdot A(e(\widetilde{w}) + \xi)(x) dx = 0 \quad \text{for all } \varsigma \in \mathbf{H}^1_{\mathrm{per},3}(Y),$$

i.e. $\widetilde{w}$ is also a solution of (3.1). Hence, $\widetilde{w} - w$ is a rigid displacement, i.e. $e(\widetilde{w}) = e(w)$. By summation over k and r in (6.8) we therefore find that the stress component $\sigma_{ij}(u)$ satisfies the condition

$$\sigma_{ij}(u)(x) = (-1)^{\delta_{is}+\delta_{js}} \sigma_{ij}(u)(y).$$

Thus, by (6.4) $\sigma(u) \in \mathcal{S}_{|s}$. Hence, if $f : Y \to \mathbb{R}$ is symmetric with respect to the plane $x_s = 0$, i.e. $f(x) = f(y)$, then

$$\int_Y f(x) \sigma_{ij}(u)(x) dx = (-1)^{\delta_{is}+\delta_{js}} \int_Y f(y) \sigma_{ij}(u)(y) dy,$$

and we obtain that

$$K_{ij} = (-1)^{\delta_{is}+\delta_{js}} K_{ij}, \tag{6.10}$$

which is equivalent with saying that K belongs to $\mathbf{S}_{s|}$. Similarly, we obtain that $-K_{ij} = (-1)^{\delta_{is}+\delta_{js}} K_{ij}$ if f is anti-symmetric with respect to the plane $x_s = 0$, which implies that K belongs to $\mathbf{S}_{s\perp}$.

In the case when $\xi \in \mathbf{S}_{s\perp}$ we replace (6.6) by

$$\begin{aligned} &\widetilde{\varphi}_s(x) = \varphi_s(y), \\ &\widetilde{\varphi}_i(x) = -\varphi_i(y) \quad \text{for all } i \neq s. \end{aligned} \tag{6.11}$$

Similarly as above we obtain that

$$-e_{ij}(\widetilde{\varphi})(x) = (-1)^{\delta_{is}+\delta_{js}} e_{ij}(\varphi)(y).$$

This identity holds in particularly for the solution w of (3.1), i.e.

$$-e_{kr}(\widetilde{w})(x) = (-1)^{\delta_{ks}+\delta_{rs}} e_{kr}(w)(y),$$

and adding this to the identity (6.5) (with i and j replaced by k and r, respectively),

$$-\xi_{kr}(x) = (-1)^{\delta_{ks}+\delta_{rs}} \xi_{kr}(y),$$

and multiplying with (6.2) gives

$$-a_{ijkr}(e_{kr}(\widetilde{w}) + \xi_{kr})(x) = (-1)^{\delta_{is}+\delta_{js}} a_{ijkr} e_{ij}(\varphi)(e_{kr}(w) + \xi_{kr})(y).$$

Using similar arguments as above, the rest of the theorem follows directly. □

We close this section with a corollary, which will be useful in the analysis of resultant forces and resultant moments.

Corollary 6.1. *If*

$$K_{ij} = \int_Y \sigma_{ij}(u)(x)dx \tag{6.12}$$

or

$$K_{ij} = \int_Y x_t \sigma_{ij}(u)(x)dx, \tag{6.13}$$

where $t \neq s$, then $K = \{K_{ij}\}$ belongs to $\mathbf{S}_{s|}$ if $\xi = e(v) \in \mathcal{S}_{|s}$ and $\mathbf{S}_{s\perp}$ if $\xi \in \mathcal{S}_{\perp s}$. Moreover, if

$$K_{ij} = \int_Y x_s \sigma_{ij}(u)(x)dx, \tag{6.14}$$

then $K = \{K_{ij}\}$ belongs to $\mathbf{S}_{s\perp}$ if $\xi \in \mathcal{S}_{|s}$ and $\mathbf{S}_{s|}$ if $\xi \in \mathcal{S}_{\perp s}$.

Proof. The result follows directly from Theorem 6.1 by letting f be the symmetric functions $f(x) = 1$ and $f(x) = x_t$, and next letting f be the anti-symmetric function $f(x) = x_s$. □

7. The Vanishing of Resultant Forces and Moments

From Theorem 4.1 we know that generally all the resultant forces N_{ij} vanish except for N_{33}. Several of the remaining quantities N_{33}, M_1, M_2 and M_3 may also vanish. For example in the special case of symmetric homogeneous isotropic bar, bounded by cylindrical (prismatic) surface, only one of these four quantities are different from zero for a given global displacement v. More precisely, when v is of the forms (5.1)–(5.4) the only nonvanishing resultant forces and moments are $\langle\sigma_{33}(u)\rangle$, M_3, M_2 and M_1, respectively. Even though the presentation of the theory of the periodic case presented here is completely different from that presented for the classical theory of deformation of bars, we still recommend that the reader be familiar with the theory presented e.g. in Ref. 9.

This property of vanishing resultant forces and moments is not directly inherited in our more general situation. However, by Theorem 6.1 we are able to find symmetry properties which imply this property in each of the four principle cases. Before we draw this conclusion, let us first use Corollary 6.1 to obtain the following remarks for $s = 1, 2, 3$ separately:

The case $s = 1$

If $\xi = e(v) \in \mathcal{S}_{|1}$, then

$$\int_Y x_2\sigma_{13}(u)(x)dx = 0,$$

i.e. one term of M_3 vanishes, and

$$\int_Y x_1\sigma_{33}(u)(x)dx = 0,$$

i.e. $M_2 = 0$. If $\xi \in \mathcal{S}_{\perp 1}$, then

$$\int_Y x_2\sigma_{33}(u)(x)dx = 0,$$

i.e. $M_1 = 0$, and

$$\int_Y \sigma_{33}(u)(x)dx = 0,$$

i.e. $N_{33} = 0$.

The case $s = 2$

If $\xi = e(v) \in \mathcal{S}_{|2}$, then

$$\int_Y x_1\sigma_{23}(u)(x)dx = 0,$$

i.e. one part of M_3 vanishes, and

$$\int_Y x_2\sigma_{33}(u)(x)dx = 0,$$

i.e. $M_1 = 0$. If $\xi \in \mathcal{S}_{\perp 2}$, then

$$\int_Y x_1 \sigma_{33}(u)(x)dx = 0,$$

hence $M_2 = 0$, and

$$\int_Y \sigma_{33}(u)(x)dx = 0,$$

i.e. $N_{33} = 0$.

The case $s = 3$

If $\xi = e(v) \in \mathcal{S}_{|3}$, then

$$\int_Y x_1 \sigma_{13}(u)(x)dx = \int_Y x_2 \sigma_{23}(u)(x)dx = 0,$$

i.e. $M_3 = 0$. If $\xi \in \mathcal{S}_{\perp 3}$, then

$$\int_Y x_1 \sigma_{33}(u)(x)dx = \int_Y x_2 \sigma_{33}(u)(x)dx = 0,$$

i.e. $M_1 = M_2 = 0$ and

$$\int_Y \sigma_{33}(u)(x)dx = 0,$$

i.e. $N_{33} = 0$.

By Remark 6.1 we are now able to determine which of the resultant forces and moment that vanish in the four principal cases. The conclusions are presented in the following four tables.

Extension in x_3-direction

		Symmetry-planes			
	Generally	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	$x_1 = 0$ and $x_2 = 0$
N_{33}	—	—	—	—	—
M_1	—	—	0	—	0
M_2	—	0	—	—	0
M_3	—	—	—	0	0

Torsion in x_1x_2-plane

		Symmetry-planes			
	Generally	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	$x_1 = 0$ and $x_2 = 0$
N_{33}	—	0	0	0	0
M_1	—	0	—	0	0
M_2	—	—	0	0	0
M_3	—	—	—	—	—

Pure bending about the x_2-axis					
		Symmetry-planes			
	Generally	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	$x_1 = 0$ and $x_2 = 0$
N_{33}	—	0	—	—	—
M_1	—	0	0	—	0
M_2	—	—	—	—	—
M_3	—	—	—	0	—

Pure bending about the x_1-axis					
		Symmetry-planes			
	Generally	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	$x_1 = 0$ and $x_2 = 0$
N_{33}	—	—	0	—	—
M_1	—	—	—	—	—
M_2	—	0	0	—	0
M_3	—	—	—	0	—

8. Equivalent Boundary Conditions

Let $\mathbf{C}^\infty_{|'}(Y)$ be the space of smooth vector valued functions $u \in \mathbf{C}^\infty(\overline{Y})$ satisfying

$$u_3\left(x_1, x_2, -\frac{x_3^0}{2}\right) = u_3\left(x_1, x_2, \frac{x_3^0}{2}\right) \quad \text{on } B, \tag{8.1}$$

and let $\mathbf{C}^\infty_{\perp'}(Y)$ denote the space of smooth vector-valued functions u satisfying

$$u_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) = u_i\left(x_1, x_2, \frac{x_3^0}{2}\right) \quad \text{on } B \quad \text{for } i = 2 \text{ and } i = 3. \tag{8.2}$$

In addition, let $\mathbf{C}^\infty_{|}(Y)$ denote the space of smooth vector-valued functions u satisfying

$$u_3 = 0 \quad \text{on } B \tag{8.3}$$

and let $\mathbf{C}^\infty_{\perp}(Y)$ denote the space of smooth vector-valued functions u satisfying

$$u_i = 0 \quad \text{on } B \quad \text{for } i = 2 \text{ and } i = 3. \tag{8.4}$$

Moreover, let $\mathbf{H}_{|'}(Y)$, $\mathbf{H}_{\perp'}(Y)$, $\mathbf{H}_{|}(Y)$ and $\mathbf{H}_{\perp}(Y)$ denote the closures of these spaces in $\mathbf{H}^1(Y)$, respectively.

Lemma 8.1. *Let X be a closed subspace of $\mathbf{H}^1(Y)$. Then the problem*

$$\int_Y e(\varphi) \cdot A(\xi + e(w))dx = 0 \quad \textit{for all } \varphi \in X, \tag{8.5}$$

has a solution $w \in X$ which is unique up to a rigid body displacement. More precisely, if $w = w_1$ and $w = w_2$ are solutions of (8.5)*, then $r = w_1 - w_2 \in \mathcal{R}$, where*

$$\mathcal{R} = \{r \in X : e(r) = 0\}.$$

In the following cases $r = (r_1, r_2, r_3)$ *takes the following form*:

(1) $X = \mathbf{H}^1_{\mathrm{per},3}(Y)$:

$$r = (b_1, b_2, b_3) + m_{12}(x_2, -x_1, 0),$$

i.e. a translation in $\mathbb{R}^3$ *and a rotation in the* x_1x_2*-plane of angle* m_{12} (*if* m_{12} *is small*).

(2) $X = \mathbf{H}_{|'}(Y)$:

$$r = (b_1, b_2, b_3) + m_{12}(x_2, -x_1, 0) + m_{13}(x_3, 0, -x_1) + m_{23}(0, x_3, -x_2),$$

(3) $X = \mathbf{H}_{\perp'}(Y)$:

$$r = (b_1, b_2, b_3) + m_{12}(x_2, -x_1, 0),$$

(4) $X = \mathbf{H}_{|}(Y)$:

$$r = (b_1, b_2, 0) + m_{12}(x_2, -x_1, 0),$$

(5) $X = \mathbf{H}_{\perp}(Y)$:

$$r = (0, 0, b_3).$$

Proof. Let $\mathcal{R}$ denote the set of all rigid displacements in X, i.e.

$$\mathcal{R} = \{r \in X : e(r) = 0\}.$$

Denote $X' = X/\mathcal{R}$ the factor space which is well-defined since $\mathcal{R}$ is finite dimensional and X is closed. The linear functional $(e(\varphi), A\xi)$ and the bilinear form $(e(\varphi), Ae(w))$ are well defined on X' by construction. By the Korn inequality this form $(e(\varphi), Ae(\varphi))$ is coercive on X', and the desired statement on the existence and the uniqueness of a solution follows.

It is possible to show that for all rigid displacements $r \in \mathbf{H}^1(Y)$ there exists a constant matrix m with $m_{ij} = -m_{ji}$ and a constant vector b such that $r(x) = mx + b$, i.e.

$$r = (b_1, b_2, b_3) + m_{12}(x_2, -x_1, 0) + m_{13}(x_3, 0, -x_1) + m_{23}(0, x_3, -x_2). \tag{8.6}$$

For the proof of this fact we refer to Ref. 8. If $u = (u_1, u_2, u_3) \in \mathcal{R}$ and u_i is I-periodic in the x_3 variable, we obtain that $m_{i3} = 0$ in (8.6). Moreover, if $u_3 = 0$ on B, then by (8.6) we find that $m_{13} = m_{23} = b_3 = 0$, and if $u_1 = 0$ or $u_2 = 0$ we obtain that $m_{12} = b_1 = b_2 = 0$. Items (1)–(5) follow directly by this. □

Theorem 8.1. *Let* A *be a* Y*-periodic tensor which is* I*-periodic in the* x_3*-variable and satisfying* (2.1), (2.2), (6.1) *and* (6.2) *for* $s = 3$. *Moreover, let* $v \in \mathbf{H}^1(Y)$ *such that* $\xi = e(v)$ *is* I*-periodic in the* x_3*-variable. Then, if* $\xi \in \mathcal{S}_{|3}$, *any solution* w *of* (8.5) *for* $X = \mathbf{H}_{|}(Y)$ *is also a solution of* (3.1), (*or equivalently* (3.2)). *Similarly, if* $\xi \in \mathcal{S}_{\perp 3}$, *any solution* w *of* (8.5) *for* $X = \mathbf{H}_{\perp}(Y)$ *is also a solution of* (3.1).

Using similar arguments as we used for deriving (3.1) from (3.4) via Green's formula (3.5), we obtain from Theorem 8.1 the following two pairs of weak versus. classical formulations.

(1) The case $e(v) \in \mathcal{S}_{|3}$. The weak formulation takes the form: Find $u = v + w$, such that $w \in \mathbf{H}_{|}(Y)$ and

$$\int_Y e(\varphi) \cdot A(e(u))dx = 0 \quad \text{for all } \varphi \in \mathbf{H}_{|}(Y). \tag{8.7}$$

The classical formulation takes the form: Find $u \in \mathbf{H}^1(Y)$, such that

$$\begin{cases} \operatorname{div} \sigma(u) = 0 & \text{in } Y, \\ F_i(u) = 0 & \text{on } B, \quad i = 1, 2, \\ F(u) = 0 & \text{on } C, \\ u_3 = v_3 & \text{on } B. \end{cases} \tag{8.8}$$

(2) The case $e(v) \in \mathcal{S}_{\perp 3}$. The weak formulation takes the form: Find $u = v + w$, such that $w \in \mathbf{H}_{\perp}(Y)$ and

$$\int_Y e(\varphi) \cdot A(e(u))dx = 0 \quad \text{for all } \varphi \in \mathbf{H}_{\perp}(Y). \tag{8.9}$$

The classical formulation takes the form: Find $u = v + w$, such that

$$\begin{cases} \operatorname{div} \sigma(u) = 0 & \text{in } Y, \\ F_3(u) = 0 & \text{on } B, \\ F(u) = 0 & \text{on } C, \\ u_i = v_i & \text{on } B, \quad i = 1, 2. \end{cases} \tag{8.10}$$

Remark 8.1. Approximate solutions on both of the above types of problems can easily be found in finite dimensional spaces of continuous functions by using commercially available FE-programs. For example, in the program ANSYS these problems are solved by using "structural problem" with no body forces and specifying the Dirichlet boundary conditions $u_3 = v_3$ (or $u_1 = v_1$ and $u_2 = v_2$) on the two parallel surfaces constituting the set B. This is certainly the same as putting $w_3 = 0$ (or $w_1 = 0$ and $w_2 = 0$) on B. The above Neumann boundary condition are automatically imposed by leaving the corresponding displacements on these surfaces unspecified. This gives us a numerical solution u which according to Lemma 8.1 is unique only within a rigid displacement of the form $r = (b_1, b_2, 0) + m_{12}(x_2, -x_1, 0)$ (or $r = (0, 0, b_3)$). In order to obtain a unique solution we may specify this rigid displacement as follows. By imposing a Dirichlet boundary condition $w_i = 0$ (i.e. $u_i = v_i$) at some point, $x^k = (x_1^k, x_2^k, x_3^k) \in Y$ we obtain the condition $b_i = 0$. Similarly, the constant m_{12} is removed by defining e.g. $w_1 = 0$ (i.e. $u_1 = v_1$) or $w_2 = 0$ (i.e. $u_2 = v_2$) at some other point $x^l = (x_1^l, x_2^l, x_3^l) \in Y$, where $(x_1^l, x_2^l) \neq (x_1^k, x_2^k)$.

Remark 8.2. From (5.5) and (5.6) we observe that $\xi = e(v) \in \mathcal{S}_{|3}$, $\mathcal{S}_{\perp 3}$, $\mathcal{S}_{|3}$ and $\mathcal{S}_{|3}$ for the cases (5.1)–(5.4), respectively. Hence from the above results it is clear that the problems corresponding to these cases can be solved by using the following

Dirichlet boundary conditions on the left (right) of B and the points (x_1^k, x_2^k, x_3^k) and (x_1^l, x_2^l, x_3^l), where $(x_1^l, x_2^l) \neq (x_1^k, x_2^k)$. Note that the corresponding effective parameters are independent of the constant τ. This parameter may therefore be chosen arbitrarily when the only purpose of the computation is to calculate effective properties.

(1) Extension in x_3-direction (5.1):

$$u_3 = \frac{-\tau x_3^0}{2} \quad \left(u_3 = \frac{\tau x_3^0}{2}\right) \quad \text{on } B$$

$$u_1(x_1^k, x_2^k, x_3^k) = u_2(x_1^k, x_2^k, x_3^k) = 0 \quad \text{and} \quad u_1(x_1^l, x_2^l, x_3^l) = 0.$$

(2) Torsion in x_1x_2-plane (5.2):

$$u_1 = \frac{\tau x_3^0 x_2}{2}, \quad u_2 = -\frac{\tau x_3^0 x_1}{2} \quad \left(u_1 = -\frac{\tau x_3^0 x_2}{2}, \quad u_2 = \frac{\tau x_3^0 x_1}{2}\right) \quad \text{on } B$$

$$u_3(x_1^k, x_2^k, x_3^k) = 0.$$

(3) Pure bending about the x_2-axis (5.3):

$$u_3 = \tau x_1 x_3^0 \quad (u_3 = -\tau x_1 x_3^0) \quad \text{on } B$$

$$u_1(x_1^k, x_2^k, x_3^k) = \tau (x_3^k)^2, \quad u_2(x_1^k, x_2^k, x_3^k) = 0 \quad \text{and} \quad u_2(x_1^l, x_2^l, x_3^l) = 0.$$

(4) Pure bending about the x_1-axis (5.4):

$$u_3 = \tau x_2 x_3^0 \quad (u_3 = -\tau x_2 x_3^0) \quad \text{on } B$$

$$u_1(x_1^k, x_2^k, x_3^k) = 0, \quad u_2(x_1^k, x_2^k, x_3^k) = \tau (x_3^k)^2 \quad \text{and} \quad u_1(x_1^l, x_2^l, x_3^l) = 0.$$

Proof of Theorem 8.1. Assume that $\xi \in \mathcal{S}_{|3}$. By repeating the proof of Theorem 6.1 with $\mathbf{H}^1_{\text{per},3}(Y)$ replaced by the larger space $\mathbf{H}_{|'}(Y)$, we find that if w is a solution, so is the function $\widetilde{w}$ given by

$$\begin{aligned} &\widetilde{w}_3(x) = -w_3(y), \\ &\widetilde{w}_i(x) = w_i(y) \quad \text{for all } i \neq 3. \end{aligned} \tag{8.11}$$

Hence, it is easy to see that the convex combination

$$\psi = \frac{1}{2} w + \frac{1}{2} \widetilde{w}$$

is also a solution. By the I-periodicity in the x_3-variable of w_i for $i = 3$, we obtain from (8.11) that

$$\begin{aligned} \psi_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) &= \frac{1}{2} w_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) + \frac{1}{2} \widetilde{w}_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) \\ &= \frac{1}{2} w_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) - \frac{1}{2} w_i\left(x_1, x_2, \frac{x_3^0}{2}\right) = 0, \end{aligned}$$

and similarly that $\psi_i(x_1, x_2, x_3^0/2) = 0$ for $i = 3$. This shows that $\psi \in \mathbf{H}_|(Y)$. Moreover, for $i \neq 3$ we obtain by using (8.11) twice that

$$\begin{aligned}\psi_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) &= \frac{1}{2}w_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) + \frac{1}{2}\widetilde{w}_i\left(x_1, x_2, -\frac{x_3^0}{2}\right)\\ &= \frac{1}{2}w_i\left(x_1, x_2, -\frac{x_3^0}{2}\right) + \frac{1}{2}w_i\left(x_1, x_2, \frac{x_3^0}{2}\right)\\ &= \frac{1}{2}\widetilde{w}_i\left(x_1, x_2, \frac{x_3^0}{2}\right) + \frac{1}{2}w_i\left(x_1, x_2, \frac{x_3^0}{2}\right) = \psi_i\left(x_1, x_2, \frac{x_3^0}{2}\right).\end{aligned}$$

Hence, we also have that $\psi \in \mathbf{H}^1_{\mathrm{per},3}(Y)$. Thus, since ψ is solution of (8.5) for $X = \mathbf{H}_{|'}(Y)$ and at the same time belongs to the smaller function spaces $\mathbf{H}_|(Y)$ and $\mathbf{H}^1_{\mathrm{per},3}(Y)$, it follows directly that ψ is a solution of (8.5) for $X = \mathbf{H}_|(Y)$ and $X = \mathbf{H}^1_{\mathrm{per},3}(Y)$. If $\xi \in \mathcal{S}_{\perp 3}$, we argue exactly as above by replacing $\mathbf{H}_{|'}(Y)$ and $\mathbf{H}_|(Y)$ with $\mathbf{H}_{\perp'}(Y)$ and $\mathbf{H}_\perp(Y)$, respectively, and using

$$\begin{aligned}&\widetilde{w}_3(x) = w_3(y),\\ &\widetilde{w}_i(x) = -w_i(y) \quad \text{for all } i \neq 3,\end{aligned} \tag{8.12}$$

instead of (8.11), and the rest of the theorem follows.

□

References

1. A. Braides and D. Lukkassen, Reiterated homogenization of integral functionals, *Math. Mod. Meth. Appl. Sci.* **10** (2000) 47–71.
2. J. Bystrøm, J. Helsing and A. Meidell, Some computational aspects of iterated structures, *Composites Part B* **32** (2001) 485–490.
3. J.-L. Lions, D. Lukkassen, L.-E. Persson and P. Wall, Reiterated homogenization of nonlinear monotone operators, *Chin. Ann. Math., Ser. B* **22** (2001) 1–12.
4. D. Lukkassen, Reiterated homogenization of non-standard Lagrangians, *C. R. Acad. Sci., Paris, Ser. I, Math.* **332** (2001) 999–1004.
5. D. Lukkassen and G. W. Milton, On hierarchical structures and reiterated homogenization., *Proc. of the Conf. on Function Spaces, Interpolation Theory and Related Topics,* in Honour of Jaak Peetre on His 65th Birthday, August 17–22, 2000 (Walter de Gruyter, 2002), pp. 311–324.
6. D. Lukkassen, G. Nguetseng and P. Wall, Two scale convergence. *Int. J. Pure Appl. Math.* **2** (2002) 35–86.
7. D. Lukkassen, A. Meidell and S. Vigdergauz, On the elastic deformation of symmetric periodic structures, *Quart. J. Mech. Appl. Math.* **56** (2003) 441–454.
8. O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization* (North-Holland, 1992).
9. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd ed. (Noordhoff 1975).
10. G. P. Panasenko, Asymptotic analysis of bar systems. I, *Russ. J. Math. Phys.* **2** (1994) 325–352.
11. G. P. Panasenko, Asymptotic analysis of bar systems. II, *Russ. J. Math. Phys.* **4** (1996) 87–116.