Homogenization of random non stationary parabolic operators

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Introduction

We study homogenization problem for a random non stationary parabolic second order equation of the form

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\Big(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla u^\varepsilon(x,t)\Big) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})u^\varepsilon(x,t) + f(x,t), \qquad (1)$$

with a small positive parameter ε. This model equation describes various processes in a medium with spatial microstructure whose characteristics are rapidly changing functions of time.

Throughout this article we assume that the spatial microstructure is periodic and that the characteristics of this microstructure are random stationary rapidly oscillating processes.

The presence in the equation of a large zero order term, linear or nonlinear, leads to rather unusual asymptotic behaviour of a solution of (1), as ε tends to zero. We will show that almost sure (a.s.) homogenization result in general fails to hold and that a weaker averaging result takes place. Namely, under certain mixing conditions, a solution of (1) converges in law in a suitable functional space to a solution of a homogenized stochastic partial differential equation (SPDE).

Our aim is to justify this convergence and to investigate the properties of the limit SPDE.

The presence of a large factor in the lower order terms of the equation is natural when studying long term behaviour of solutions. We illustrate this with the following example. Many applications deal with parabolic operator of the form

$$\frac{\partial}{\partial t}v(y,s) = \operatorname{div}\left(a(y,s)\nabla v(y,s)\right) + \varepsilon g(y,s)v(y,s),$$

with a small potential $\varepsilon g(y,s)$, here ε characterizes the range of oscillation of the potential. In order to study the behaviour of solutions at large time $s \sim \varepsilon^{-2}$, one can make the diffusive change of variables $x = \varepsilon y$, $t = \varepsilon^2 s$. In the new coordinates the equation reads

$$\frac{\partial}{\partial t}v = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla v\right) + \varepsilon^{-1}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v,$$

it is similar to the equation (1).

First rigorous homogenization results for random elliptic and parabolic operators in divergence form were obtained in the works [5], [10]. After that this topic has been studied by many mathematicians, now it is well presented in the existing literature. However, some important problems in the field remain open.

It is known that, in contrast with periodic case, the presence of lower order terms in the equation with random coefficients might change crucially the effective behaviour of solutions.

In this work we consider an intermediate case of equations with lower order terms whose coefficients are periodic in spatial variables and random in time. Averaging problems for these equations with diffusive driving process were studied in [2] in linear case, and in [12] in nonlinear case. The convection diffusion problem of this type with a generic stationary driving process having good mixing properties, have been considered in [6].

1. The setup

This section is devoted to homogenization of equations of the form

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla u^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})u^\varepsilon(x,t) + f(x,t), \qquad (2)$$

with generic random stationary in time and periodic in spatial variables coefficients. For this equation we consider a Cauchy problem in $\mathbb{R}^n \times (0,T)$ with the initial condition

$$u^\varepsilon(x,0) = u_0(x). \qquad (3)$$

Here and later on we assume that $u_0 \in L^2(\mathbb{R}^n)$ and $f \in L^2((0,T)\times\mathbb{R}^n)$.

Remark 1. The Cauchy problem has been chosen for the sake of definiteness. Initial boundary problems with Dirichlet or Neumann conditions can be studied in a similar way.

Problem (2)–(3) will be investigated under the following assumptions on the coefficients.

H1. The coefficients $a_{ij}(y,s)$ and $g(y,s)$ are $[0,1]^n$ periodic in y.

H2. The functions $a_{ij}(y,s)$ and $g(y,s)$ are stationary random functions of s defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, with values in the space of periodic functions of y. We assume that $a_{ij}(y,s) = a_{ij}(y,s,\omega)$ and $g(y,s) = g(y,s,\omega)$ are measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{T}^n)\times\mathcal{B}(-\infty,+\infty)\times\mathcal{F}$, where the symbol $\mathcal{B}$ stands for the Borel σ-algebra.
For simplicity we assume that Ω is equipped with a random dynamical system T_t and that

$$a_{ij}(y,s,\omega) = \breve{a}_{ij}(y,T_s\omega), \quad g(y,s,\omega) = \breve{g}(y,T_s\omega), \tag{4}$$

where $\breve{a}_{ij}(y,\omega)$ and $\breve{g}(y,\omega)$ are given random function with values in $L^\infty(\mathbb{T}^n)$.
Let us recall that T_t is a group of measurable transformations $T_t:\Omega\longrightarrow\Omega$ such that
- $T_{s_1}T_{s_2} = T_{s_1+s_2}, \quad T_0 = \mathrm{Id}$;
- T_s preserves measure $\mathbf{P}$ for any $s\in\mathbb{R}$, i.e. $\mathbf{P}(T_s(\mathcal{G})) = \mathbf{P}(\mathcal{G})$ for any $\mathcal{G}\in\mathcal{F}$;
- $T_s(\omega)$ is a measurable map from $(\Omega\times\mathbb{R},\mathcal{F}\times\mathcal{B})$ to $(\Omega,\mathcal{F})$, where $\mathcal{B}$ stands for a Borel σ-algebra.

H3. Uniform ellipticity:

$$a_{ij}(y,\tau)\zeta_i\zeta_j \ge \lambda|\zeta|^2, \quad \lambda>0,$$
$$|a_{ij}(y,\tau)| \le \lambda^{-1}, \quad |g(y,s)| \le \lambda^{-1}$$

for all y, τ and $\zeta\in\mathbb{R}^n$.

H4. Centering condition. The average of $g(z,s)$ is equal to zero that is

$$\mathbf{E}\int\limits_{[0,1]^n} g(z,s)\,dz = 0 \tag{5}$$

for all $s\in\mathbb{R}$.

In order to formulate one more assumption we first recall the definition of mixing coefficients.

Let ξ_s be a stationary random process defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, and denote $\mathcal{F}_{\le t} = \sigma\{\xi_s,\ s\le t\}$ and $\mathcal{F}_{\ge t} = \sigma\{\xi_s,\ s\ge t\}$.

The function $\kappa(\gamma)$ defined by

$$\kappa(\gamma) = \sup_{E_1\in\mathcal{F}_{\le t},\,E_2\in\mathcal{F}_{\ge(t+\gamma)}} |\mathbf{P}(E_1)\mathbf{P}(E_2) - \mathbf{P}(E_1\cap E_2)|,$$

is called strong mixing coefficient of the process $\xi_\cdot$. Notice that since $\xi_\cdot$ is stationary, $\kappa(\gamma)$ does not depend on t.

The function $\varphi(\gamma)$ defined by

$$\varphi(\gamma) = \sup_{\substack{E_1\in\mathcal{F}_{\le t},\ E_2\in\mathcal{F}_{\ge(t+\gamma)}\\ \mathbf{P}(E_2)\ne 0}} \left|\mathbf{P}(E_1) - \frac{\mathbf{P}(E_1\cap E_2)}{\mathbf{P}(E_2)}\right|,$$

is called the *uniform mixing coefficient* of $\xi_{\cdot}$.

The function $\rho(\gamma)$ defined by

$$\rho(\gamma) = \sup_{\eta_1,\eta_2} \left| \frac{\mathbf{E}((\eta_1 - \mathbf{E}\eta_1)(\eta_2 - \mathbf{E}\eta_2))}{\sqrt{\mathbf{E}\eta_1^2\,\mathbf{E}\eta_2^2}} \right|,$$

$$\eta_1 \in L^2(\Omega, \mathcal{F}_{\leq t}, \mathbf{P}),\ \ \eta_2 \in L^2(\Omega, \mathcal{F}_{\geq (t+\gamma)}, \mathbf{P}).$$

is called the *maximum correlation coefficient* of $\xi_{\cdot}$.

We now consider σ-algebras $\mathcal{F}_{\leq t}$ and $\mathcal{F}_{\geq t}$, generated by the coefficients $a(y,t), g(y,t)$ of operator (2), and impose the following condition on the corresponding mixing coefficients

H5. At least one of the following conditions holds true.

$$\int_0^\infty \sqrt{\kappa(\gamma)}d\gamma < \infty, \qquad \int_0^\infty \varphi(\gamma)d\gamma < \infty, \qquad \int_0^\infty \rho(\gamma)d\gamma < \infty$$

Remark 2. Condition **H4** can be assumed without loss of generality. Indeed, the relation (5) can be achieved by means of the following factorization of unknown function in (2)

$$\tilde{u}^\varepsilon(x,t) = \exp(< \bar{g} > /\varepsilon)u(x,t),$$

with

$$< \bar{g} >= \mathbf{E} \int_{[0,1]^n} g(z,s)dz.$$

If $< \bar{g} >\neq 0$, then the homogenization takes place on the background of exponential growth or decay of the solution.

Under conditions **H1-H3** problem (2)-(3) is well posed for each $\varepsilon > 0$.

Lemma 1. *Let* **H1-H3** *be fulfilled. Then for each $\varepsilon > 0$ problem (2)-(3) has a unique solution $u^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L^2(\mathbb{R}^n))$ for all $\omega \in \Omega$. This solution defines a measurable mapping*

$$u^\varepsilon\ :\ (\Omega,\mathcal{F}) \longrightarrow (L^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L^2(\mathbb{R}^n)), \mathcal{B}).$$

The estimate holds

$$\begin{aligned} &\|u^\varepsilon\|_{C((0,T);L^2(\mathbb{R}^n))} + \|u^\varepsilon\|_{L^2((0,T);H^1(\mathbb{R}^n))} \\ &\leq C(\varepsilon)(\|f\|_{L^2(0,T;H^{-1}(\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)}). \end{aligned} \tag{6}$$

Proof. The existence and the uniqueness of a solution as well as a priori estimate (6) are standard. The measurability is the consequence of the fact that u^ε depends continuously on the data of the problem. □

2. Factorization of the equation

It is convenient to represent $g(y,s)$ as a sum

$$g(y,t) =< g > (t) + \tilde{g}(y,t), \qquad < g > (t) \stackrel{\text{def}}{=} \int\limits_{[0,1]^n} g(z,t)dz,$$

and to introduce a new unknown function $v^\varepsilon = v^\varepsilon(x,t)$ as follows

$$u^\varepsilon(x,t) = v^\varepsilon(x,t)\exp\left(\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right), \tag{7}$$

It is straightforward to check that v^ε satisfies the equation

$$\frac{\partial}{\partial t}v^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla v^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^\varepsilon(x,t)$$

$$+f(x,t)\exp\left(-\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right), \tag{8}$$

$$v^\varepsilon(x,0) = u_0(x).$$

This problem will be studied in the following sections. In the remaining part of this section we deal with the exponential factor in (7).

Lemma 2. *Suppose that at least one of the conditions* **H5** *holds. Then the process*

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon}\int\limits_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds \tag{9}$$

satisfies functional Central Limit Theorem (invariance principle) with zero mean and the diffusion given by

$$\sigma^2 = 2\int\limits_0^\infty \mathbf{E} < g > (0) < g > (s)ds. \tag{10}$$

That is the process $\{\zeta_t^\varepsilon\}$ converges in law in the space $C[0,\infty)$ to the process $\{\sigma W_t\}$, where W_t is a standard Brownian motion.

The **proof** of this statement can be found for instance in [9], Chapter 9. As a consequence of the lemma we obtain the convergence

$$\exp\left(\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right) \xrightarrow{\mathcal{L}} \exp(\sigma W_t) \tag{11}$$

in the space $C[0,\infty)$.

3. A priori estimates for the factorized equation

In this section we derive a priori estimates for a solution of problem (8) and of more general Cauchy problem of the form

$$\frac{\partial}{\partial t}z^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla z^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}\tilde g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})z^\varepsilon(x,t) + h(x,t), \quad (12)$$

$$z^\varepsilon(x,0) = z_0(x),$$

which involves a nontrivial right hand side.

Proposition 1. *A solution z^ε of problem (12) admits an estimate*

$$\|z^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|z^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} \le C(\|z_0\|_{L^2(\mathbb{R}^n)} + \|h\|_{L^2(0,T;H^{-1}(\mathbb{R}^n))}) \quad (13)$$

with a constant C which does not depend on ε.

Proof. By construction the function $\tilde g(y,s)$ has zero average in variable y for all s and ω. Therefore, the equation $\Delta\tilde Q = \tilde g$ is solvable in the space of periodic functions. Denote $\tilde q = \nabla_y\tilde Q$. Since $\|\tilde g\|_{L^\infty} < \infty$, we have $|\tilde q(y,s)| \le C$.

Clearly, $q(y,s)$ satisfies the relation $\operatorname{div_y}\tilde q(y,s) = \tilde g(y,s)$. In coordinates $x = \varepsilon y$, $t = \varepsilon^2 s$ it reads

$$\varepsilon\operatorname{div_x}\tilde q^\varepsilon(x,t) = \tilde g^\varepsilon(x,t); \quad (14)$$

Here and afterwards for a generic function $F(y,s)$ we use the notation

$$F_\varepsilon(x,t) = F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}), \quad \frac{\partial}{\partial y_i}F_\varepsilon(x,t) = \frac{\partial}{\partial y_i}F(y,\frac{t}{\varepsilon^2})\Big|_{y=\frac{x}{\varepsilon}}, \quad (15)$$

$$\frac{\partial}{\partial s}F_\varepsilon(x,t) = \frac{\partial}{\partial s}F(\frac{x}{\varepsilon},s)\Big|_{s=\frac{t}{\varepsilon^2}}$$

Multiplying the equation (12) by z^ε and integrating the resulting relation over the set $\mathbb{R}^n\times(0,T)$ gives

$$\int_{\mathbb{R}^n}(z^\varepsilon(x,t))^2dx - \int_{\mathbb{R}^n}(z_0(x))^2dx$$
$$= -\int_0^t\int_{\mathbb{R}^n}a_{ij,\varepsilon}(x,\tau)\nabla z^\varepsilon(x,\tau)\cdot\nabla z^\varepsilon(x,\tau)dxd\tau +$$
$$+\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^n}\tilde g_\varepsilon(x,\tau)(z^\varepsilon(x,\tau))^2dxd\tau + \int_0^t\int_{\mathbb{R}^n}z^\varepsilon(x,\tau)h(x,\tau)dxd\tau.$$

Considering (14), after multiple integration by parts we get

$$\int_{\mathbb{R}^n} (z^\varepsilon(x,t))^2 dx + \int_0^t \int_{\mathbb{R}^n} a_{ij,\varepsilon}(x,\tau)\nabla z^\varepsilon(x,\tau)\cdot\nabla z^\varepsilon(x,\tau)dxd\tau$$
$$= \int_{\mathbb{R}^n} (z_0(x))^2 dx + \int_0^t \int_{\mathbb{R}^n} z^\varepsilon(x,\tau)\tilde{q}_\varepsilon(x,\tau)\cdot\nabla z^\varepsilon(x,\tau)dxd\tau +$$
$$+ \int_0^t \int_{\mathbb{R}^n} z^\varepsilon(x,\tau)h(x,\tau)dxd\tau. \tag{16}$$

Denote the right hand side here by $R^\varepsilon(t)$. For each $\gamma > 0$ we have

$$|R^\varepsilon(t)| \le \int_{\mathbb{R}^n} (z_0(x))^2 dx + \gamma^{-1}\int_0^t \int_{\mathbb{R}^n} |z^\varepsilon(x,\tau)\tilde{q}_\varepsilon(x,\tau)|^2 dxd\tau$$
$$+\gamma \int_0^t \int_{\mathbb{R}^n} |\nabla z^\varepsilon(x,\tau)|^2 dxd\tau + \int_0^t \|z^\varepsilon(\cdot,\tau)\|_{H^1(\mathbb{R}^n)}\|h(\cdot,\tau)\|_{H^{-1}(\mathbb{R}^n)}d\tau$$
$$\le \int_{\mathbb{R}^n} (z_0(x))^2 dx + C\gamma^{-1}\int_0^t \int_{\mathbb{R}^n} |z^\varepsilon(x,\tau)|^2 dxd\tau + \gamma\int_0^t \int_{\mathbb{R}^n} |\nabla z^\varepsilon(x,\tau)|^2 dxd\tau$$
$$+\gamma^{-1}\int_0^t \|h(\cdot,\tau)\|^2_{H^{-1}(\mathbb{R}^n)}d\tau + \gamma\int_0^t \int_{\mathbb{R}^n} (|z^\varepsilon(\cdot,\tau)|^2 + |\nabla z^\varepsilon(\cdot,\tau)|^2)dxd\tau.$$

It remains to combine this bound with (16). The desired estimate (13) now follows from Gronwall lemma. □

4. Auxiliary problems

Passage to the limit in problem (8) requires introducing a number of auxiliary functions usually called *correctors*. These correctors will be defined as solutions of auxiliary parabolic equations. This section is devoted to those auxiliary problems and their properties.

Denote $A = \frac{\partial}{\partial y_i} a_{ij}(y,s)\frac{\partial}{\partial y_j}$ and consider in a cylinder $\mathbb{T}^n \times (-\infty, +\infty)$ the following two equations:

$$\frac{\partial}{\partial s}\chi^j(y,s) - A\chi^j(y,s) = \frac{\partial}{\partial y_i}a_{ij}(y,s) \tag{17}$$

$$\frac{\partial}{\partial s}G(y,s) - AG(y,s) = \tilde{g}(y,s) \tag{18}$$

Proposition 2. *Equations (17) and (18) have stationary solutions in the space $L^\infty(-\infty,+\infty; C(\mathbb{T}^n)) \cap L^2_{\mathrm{loc}}(-\infty,+\infty; H^1(\mathbb{T}^n))$. Each of these solutions is unique up to a (random) additive constant. The estimates*

$$\|\chi\|_{L^2(N,N+1;H^1(\mathbb{T}^n))} \le C, \qquad \|G\|_{L^2(N,N+1;H^1(\mathbb{T}^n))} \le C, \tag{19}$$

hold uniformly in $N \in \mathbb{R}$. Moreover, the constant C is deterministic and only depends on λ in **H3**.

Proof. Let us show that (17) has a stationary solution. To this end we consider the following Cauchy problem

$$\frac{\partial}{\partial s}\chi^j_N(y,s) - A\chi^j_N(y,s) = \frac{\partial}{\partial y_i}a_{ij}(y,s)\mathbf{1}_{[N,N+1)}(s), \quad (y,s) \in \mathbb{T}^n \times (N,+\infty)$$

$$\chi^j_N(y,N) = 0. \tag{20}$$

For $s < N$ we set $\chi^j_N(y,s) = 0$. By the Nash estimates (see [8]), the function χ^j_N is continuous and satisfies the upper bound

$$\|\chi^j_N\|_{L^\infty((-\infty,N+1]\times\mathbb{T}^n)} \le c_1(\gamma).$$

Since the right hand side in (20) is equal to zero for all $s > N+1$, by the maximum principle the last estimate is valid for all s:

$$\|\chi^j_N\|_{L^\infty((-\infty,+\infty)\times\mathbb{T}^n)} \le c_1(\gamma).$$

We want to show that $\chi^j_N(y,s)$ decays exponentially as $(s-N) \to \infty$.

Lemma 3. *There are nonrandom independent of N constants $c_2(\lambda) > 0$ and $c_3(\lambda) > 0$ such that*

$$|\chi^j_N(y,s)| \le c_2 \exp(-c_3(s-N)) \tag{21}$$

Proof of the lemma. Notice that

$$\int_{\mathbb{T}^n} \chi^j_N(y,s)dy = 0 \tag{22}$$

for each $s \in \mathbb{R}$. Indeed, integrating the equation (17) on the cylinder $(s_1,s_2)\times\mathbb{T}^n$, one has $\int_{\mathbb{T}^n}\chi^j_N(y,s_1)dy = \int_{\mathbb{T}^n}\chi^j_N(y,s_2)dy$. Then (22) follows from the equality $\chi^j_N(y,N) = 0$.

By the Poincaré inequality, considering (22) and **H3**, we get for each s

$$\int_{\mathbb{T}^n} a_{ij}(y,s)\frac{\partial}{\partial y_i}\chi^k_N(y,s)\frac{\partial}{\partial y_j}\chi^k_N(y,s)dy \ge \tag{23}$$

$$\ge \lambda^{-1}\int_{\mathbb{T}^n} |\nabla_y\chi^k_N(y,s)|^2 dy \ge c_4\lambda^{-1}\int_{\mathbb{T}^n}|\chi^k_N(y,s)|^2dy.$$

Now we multiply (17) by $\chi^j_N(y,s)$ and integrate the resulting relation on the cylinder $(\mathbb{T}^n \times (s_1,s_2)$. This gives

$$\|\chi^k_N(\cdot,s_2)\|^2_{L^2(\mathbb{T}^n)} - \|\chi^k_N(\cdot,s_1)\|^2_{L^2(\mathbb{T}^n)} \le -c_4\lambda^{-1}\int_{s_1}^{s_2}\|\chi^k_N(\cdot,s)\|^2_{L^2(\mathbb{T}^n)}ds \tag{24}$$

for all s_1 and s_2 such that $N+1 \le s_1 \le s_2$. This implies the bound

$$\|\chi_N^k(\cdot,s)\|^2_{L^2(\mathbb{T}^n)} \le c_2 \exp(-c_3(s-(N+1))$$

It is also clear that the function $\|\chi_N^k(\cdot,s)\|^2_{L^2(\mathbb{T}^n)}$ is monotone on the interval $(N+1,\infty)$. To complete the proof of the lemma it remains to apply once again the Nash inequality. □

We define a vector function $\chi(y,s)$ by

$$\chi^k(y,s) = \sum_{N=-\infty}^{+\infty} \chi_N^k(y,s) \tag{25}$$

By construction and in view of the last Lemma, $\chi^j(y,s)$ solves the equation (17) and satisfies the estimate (19). We want to show that $\chi^j(y,s)$ is stationary. The fact that any finite dimensional distribution of this random function is invariant with respect to all integer shifts easily follows from the stationarity of $a_{ij}(\cdot,s)$.

Taking in the above procedure an arbitrary rational step size q instead of 1, we construct a solution of equation (17) whose finite dimensional distributions are invariant with respect to any shift of the form kq with integer k. It is easy to check that this new solution coincides with $\chi^j(y,s)$. Thus, by arbitrariness of q, the finite dimensional distributions of $\chi^j(y,s)$ are invariant with respect to any rational shift. Now the stationarity of $\chi^j(y,s)$ follows by continuity arguments.

The uniqueness of a stationary solution up to a (random) additive constant follows from Lemma 3. Indeed, if we assume the existence of two distinct stationary solutions with zero average, then their difference vanishes as $s\to\infty$. This contradicts the stationarity. □

We impose the following normalization conditions for $\chi^k(y,s)$ and $G(y,s)$:

$$\int_{\mathbb{T}^n} \chi^j(y,s)dy = 0, \qquad \int_{\mathbb{T}^n} G(y,s)dy = 0 \tag{26}$$

This makes the choice of the corresponding additive constants unique.

We set

$$\tilde\chi^k(y,\omega) = \chi^k(y,0,\omega), \quad \tilde G(y,\omega) = G(y,0,\omega).$$

Since $\chi^k(y,s)$ and $G(y,s)$ are continuous in s, the functions $\tilde\chi^k$ and G are well defined. By definition $a_{ij}(y,s+\tau,\omega) = a_{ij}(y,s,T_\tau\omega)$. Therefore, considering the uniqueness of solution of problem (17), we have $\chi^k(y,s+\tau,\omega) = \chi^k(y,s,T_\tau\omega)$. In particular,

$$\chi^k(y,s,\omega) = \chi^k(y,0,T_s\omega) = \tilde\chi^k(y,T_s\omega).$$

Similarly,

$$G(y,s,\omega) = G(y,0,T_s\omega) = \tilde G(y,T_s\omega). \qquad \square$$

5. Homogenization of the factorized equation

We begin by considering the equation (2) in the particular case $f = 0$. Our aim is to show that in this case factorized problem (8) admits a.s. homogenization.

Theorem 1. *Let $f = 0$. Then under our standing assumptions a solution v^ε of problem (8) converges a.s., as $\varepsilon \to 0$, in the space $L^\infty(0,T;L^2(\mathbb{R}^n))$ towards a solution of the following Cauchy problem*

$$\begin{aligned}&\frac{\partial}{\partial t}v^0(x,t) = \operatorname{div}\big(\hat{a}\nabla v^0(x,t)\big) + \hat{b}\cdot\nabla v^0(x,t) + \hat{G}v^0(x,t),\\ &v^0(x,0) = u_0(x).\end{aligned} \tag{27}$$

The homogenized equation has constant coefficients defined by

$$\hat{a}_{ij} = \mathbf{E}\int\limits_{\mathbb{T}^n} a_{ik}(y,s)\big(\delta_{kj} + \frac{\partial}{\partial y_k}\chi^j(y,s)\big)dy, \tag{28}$$

$$\hat{b}_i = \mathbf{E}\int\limits_{\mathbb{T}^n}\Big(\tilde{g}(y,s)\chi^i(y,s) + a_{ij}\frac{\partial}{\partial y_j}G(y,s)\big)dy, \tag{29}$$

$$\hat{G} = \mathbf{E}\int\limits_{\mathbb{T}^n}\tilde{g}(y,s)G(y,s)dy. \tag{30}$$

Proof. Assume for a while that $u_0 \in C_0^\infty$. Then a solution v^0 of problem (27) is a C^∞ function which vanishes at infinity, as well as its partial derivatives, faster than any negative power of $(1+|x|)$. We then substitute the following *ansatz*

$$\check{v}^\varepsilon(x,t) = v^0(x,t) + \varepsilon\chi(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\cdot\nabla v^0 + \varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

in the equation (8). Considering (17) and (18), after straightforward rearrangements we get

$$\frac{\partial}{\partial t}(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t)) - \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t))\right)$$

$$-\frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t))$$

$$= \frac{\partial}{\partial t}v^0 + \frac{1}{\varepsilon}\frac{\partial}{\partial s}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0 + \frac{1}{\varepsilon}\frac{\partial}{\partial s}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0 + \varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0$$

$$+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0 - a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0 - \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0 - \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

$$-\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_j}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0 - a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_k\partial x_j}v^0$$

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial^2}{\partial x_k\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0$$

$$-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0-\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_j}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

$$-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial}{\partial x_j}v^0$$

$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

$$=\frac{\partial}{\partial t}v^0-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_k\partial x_j}v^0$$

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial^2}{\partial x_k\partial x_j}v^0-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0$$

$$-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial}{\partial x_j}v^0$$

$$-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0$$

$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Substituting the right hand side of the equation (27) in place of $\frac{\partial}{\partial t}v^0$ gives

$$\frac{\partial}{\partial t}(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))-\operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))\right)$$

$$-\frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))$$

$$=\left\{\hat{a}_{ij}-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-a_{ik}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_k}\chi^j(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-\frac{\partial}{\partial y_k}\left(a_{kj}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^i(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\right\}$$

$$\times\frac{\partial^2}{\partial x_i\partial x_j}v^0+\left\{\hat{b}_i-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^j(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-a_{ji}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right.$$

$$\left.-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\right\}\frac{\partial}{\partial x_j}v^0+\left\{\hat{G}-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right\}v^0$$

$$+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0$$

$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Denote the right hand side of the last formula by R_2^ε. Since the expressions in the figure brackets are periodic in spatial variables and have zero average for any t, they can be represented as in (14):

$$\Big\{\hat a_{ij}-a_{ij}(y,s)-a_{ik}(y,s)\frac{\partial}{\partial y_k}\chi^j(y,s)$$
$$-\frac{\partial}{\partial y_k}\left(a_{kj}(y,s)\chi^i(y,s)\right)\Big\}=\mathrm{div}_y\kappa_{1,ij}(y,s)$$
$$\Big\{\hat b_i-\tilde g(y,s)\chi^j(y,s)-a_{ji}(y,s)\frac{\partial}{\partial y_i}G(y,s)$$
$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,s)G(y,s)\right)\Big\}=\mathrm{div}_y\kappa_{2,i}(y,s)$$
$$\Big\{\hat G-\tilde g(y,s)G(y,s)\Big\}=\mathrm{div}_y\kappa_3(y,s)$$

where the functions $\kappa_1, ij(y,s)$, $\kappa_2, i(y,s)$ and $\kappa_3(y,s)$ are periodic in y and satisfy the estimates

$$\|\kappa_{1,ij}\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,\ \ \|\kappa_{2,i}\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,\ \ \|\kappa_3\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,$$

uniformly in $s\in\mathbb{R}$. Then R_2^ε takes the form

$$R_2^\varepsilon=\varepsilon\mathrm{div}_x\kappa_{1,ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0+\varepsilon\mathrm{div}_x\kappa_{2,i}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_i}v^0+\varepsilon\mathrm{div}_x\kappa_3(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$
$$+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0 \tag{31}$$
$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Due to the properties of v^0 this implies the estimate

$$\|R_2^\varepsilon\|_{L^2((0,T);H^{-1}(\mathbb{R}^n))}\le C\varepsilon.$$

Therefore, by Proposition 1

$$\|\check v^\varepsilon-v^\varepsilon\|_{L^2((0,T);H^1(\mathbb{R}^n))}\le C\varepsilon,\quad \|\check v^\varepsilon-v^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^n))}\le C\varepsilon. \tag{32}$$

Combining the latter estimate with an evident bound

$$\|\check v^\varepsilon-v^0\|_{L^\infty(\mathbb{R}^n\times(0,T))}\le C\varepsilon$$

we obtain the desired statement for all smooth u_0 with compact support.

In order to prove this result for general $u_0\in L^2(\mathbb{R}^n)$ we introduce a family of functions $u_0^\delta\in C_0^\infty(\mathbb{R}^n)$ such that $\|u_0^\delta-u_0\|_{L^2(\mathbb{R}^n)}\le\delta$. If we denote $v^{\delta,\varepsilon}$ a solution of problem (8) with initial condition u_0^δ, then according to Proposition 1 the estimate

$$\|v^{\delta,\varepsilon}-v^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta$$

holds. Evidently, we have $\|v^{\delta,0}-v^0\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta$. As was proved above, $v^{\delta,\varepsilon}$ converges, as $\varepsilon\to0$, to $v^{\delta,0}$ in $L^\infty(0,T;L^2(\mathbb{R}^n))$. Therefore,

$$\limsup_{\varepsilon\to0}\|v^\varepsilon-v^0\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta.$$

Since δ is an arbitrary positive number, the result follows. □

Remark 3. In the proof of Theorem 1 we did not use the mixing conditions **H5**, but only the ergodicity of the dynamical system T_s. The statement of this theorem remains valid for any ergodic dynamical system T_s without mixing assumptions:

In particular, we obtain the following result.

Corollary 1. *Let the coefficients of problem (2)-(3) be periodic in spatial variables and stationary ergodic in time, and suppose the uniform ellipticity conditions* **H3**. *Assume, furthermore, that* $\int_{\mathbb{T}^n} \tilde{g}(y,s)dy = 0$ *for all* $s \in \mathbb{R}$ *a.s. Then problem (2)-(3) admits a.s. homogenization and the limit operator is a non random parabolic operator with constant coefficients given by (28)-(30).*

It should be noted that the methods developed in the proof of Theorem 1 apply to the equation (12) with a right hand side $h(x,t) \in L^2((0,T) \times \mathbb{R}^n)$. The following result holds true.

Theorem 2. *Let* $h(x,t) \in L^2((0,T) \times \mathbb{R}^n)$ *and* $z_0 \in L^2(\mathbb{R}^n)$. *Then a solution of problem (12) converges a.s., as* $\varepsilon \to 0$, *in the space* $L^2(0,T; H^1(\mathbb{R}^n))$ *towards a solution of problem*

$$\frac{\partial}{\partial t} z^0(x,t) = \operatorname{div}\left(\hat{a}\nabla z^0(x,t)\right) + \hat{b}\cdot\nabla z^0(x,t) + \hat{G} z^0(x,t) + h(x,t),$$
$$z^0(x,0) = z_0(x), \tag{33}$$

with constant non random coefficients given by (28)-(30).

We proceed by studying problem (8) with non trivial right hand side. We denote

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right) ds.$$

and introduce $V^\varepsilon(x,t)$ to be a solution to the following Cauchy problem in $\mathbb{R}^n \times (0,T)$

$$\frac{\partial}{\partial t} V^\varepsilon(x,t) = \operatorname{div}\left(\hat{a}\nabla V^\varepsilon(x,t)\right) + \hat{b}\cdot\nabla V^\varepsilon(x,t) + \hat{G} V^\varepsilon(x,t) + f(x,t)\exp(-\zeta_t^\varepsilon),$$
$$V^\varepsilon(x,0) = u_0(x), \tag{34}$$

with coefficients defined in (28)–(30). It is convenient to represent a solution v^ε of problem (8) as a sum $v^\varepsilon = V^\varepsilon + (v^\varepsilon - V^\varepsilon)$. We will show that $(v^\varepsilon - V^\varepsilon)$ tends to zero in probability, as $\varepsilon \to 0$, in the norm of $L^2((0,T)\times\mathbb{R}^n)$, while V^ε converges in law in $L^2((0,T)\times\mathbb{R}^n)$ to a solution of Cauchy problem

$$\frac{\partial}{\partial t} v^0(x,t) = \operatorname{div}\left(\hat{a}\nabla v^0(x,t)\right) + \hat{b}\cdot\nabla v^0(x,t) + \hat{G} v^0(x,t) + f(x,t)\exp(-\sigma W_t),$$
$$v^0(x,0) = u_0(x), \tag{35}$$

with σ given by (10). Notice that this equation has a random right hand side.

Proposition 3. *The $L^2((0,T)\times\mathbb{R}^n)$ norm of the difference $(v^\varepsilon - V^\varepsilon)$ tends to zero in probability as $\varepsilon\to 0$.*

Proof. By Lemma 2 the process ζ_t^ε converges in law in $C(0,T)$, as $\varepsilon\to 0$, towards σW_t. Therefore, a random function $f(x,t)\zeta_t^\varepsilon$ converges in law in $L^2((0,T)\times\mathbb{R}^n)$ to $\sigma f(x,t)W_t$. By the Prokhorov theorem this implies that for any $\delta>0$ there is a compact set $K^\delta\subset L^2((0,T)\times\mathbb{R}^n)$ such that $\mathbf{P}\{f(x,t)\zeta_t^\varepsilon\notin K^\delta\}\le\delta$. Consider a finite δ-net in K^δ, for which we use the notation $\{h_j\}$, $j=1,2,\dots,N(\delta)$, and denote $z_j^\varepsilon(x,t)$ and $z_j^0(x,t)$ respectively solutions of problem (12) and (33) with right hand side $h_j(x,t)$ and initial condition $u_0(x)$.

By Theorem 1 for any $\delta>0$ there exists $\varepsilon_0(\delta)>0$ such that

$$\max \mathbf{P}\{\|z_j^\varepsilon - z_j^0\|_{L^2}>\delta\}<\frac{\delta}{N(\delta)}$$

Let $\mathcal{E}_j$ be the following events

$$\mathcal{E}_j=\{\Omega\,:\,\|f(x,t)\zeta_t^\varepsilon - h_j(x,t)\|_{L^2}<\delta\}.$$

By construction $\mathbf{P}\big(\Omega\setminus\bigcup_{j=1}^{N(\delta)}\mathcal{E}_j\big)<\delta$. Considering the estimate (13) and similar estimate for the homogenized problem, we conclude that for all $\omega\in\mathcal{E}_j$ the inequality holds

$$\|V^\varepsilon - v^\varepsilon\|_{L^2}\le\|V^\varepsilon - z_j^0\|_{L^2}+\|z_j^0-z_j^\varepsilon\|_{L^2}+\|z_j^\varepsilon - v^\varepsilon\|_{L^2}$$

$$\le 2C\delta+\|z_j^0-z_j^\varepsilon\|_{L^2},$$

with a constant C that depends neither on ε nor on ω. Thus

$$\mathbf{P}\{\|V^\varepsilon - v^\varepsilon\|_{L^2}>(2C+1)\delta\}$$

$$\le\mathbf{P}\big(\Omega\setminus\bigcup_{j=1}^{N(\delta)}\mathcal{E}_j\big)+\sum_{j=1}^{N(\delta)}\mathbf{P}\{\mathcal{E}_j\cap\big(\|V^\varepsilon - v^\varepsilon\|_{L^2}>(2C+1)\delta\big)\}$$

$$\delta+\sum_{j=1}^{N(\delta)}\mathbf{P}\{\|z_j^0-z_j^\varepsilon\|_{L^2}>\delta\}\le\delta+\sum_{j=1}^{N(\delta)}\frac{\delta}{N(\delta)}=2\delta.$$

This implies the required convergence in probability. □

Proposition 4. *The function V^ε converges in law, as $\varepsilon\to 0$, in $L^2((0,T)\times\mathbb{R}^n)$ towards a solution v^0 of problem (35).*

Proof. Notice that a solution of problem (33) as a functional of the right hand side defines a continuous mapping from $L^2((0,T)\times\mathbb{R}^n)$ to $L^2((0,T);H^1(\mathbb{R}^n))\cap L^\infty(0,T;L^2(\mathbb{R}^n))$. Then the convergence in law of the right hand side in (33) implies the convergence in law, in the corresponding functional space, of V^ε, and the desired statement follows. □

We summarize the above assertions in the following theorem.

Theorem 3. *Under conditions* **H1**–**H5** *the solution of factorized problem (8) converges in law, as* $\varepsilon \to 0$, *in the strong topology of the space* $L^2((0,T) \times L^2(\mathbb{R}^n))$ *towards a solution of problem (35) whose coefficients are given by (28)–(30).*

Proof. This statement is a consequence of Propositions 3 and 4. □

6. Homogenization of the original equation

We now turn to the homogenization of the original problem (2)–(3).

Notice first that the random process

$$(\exp(\zeta_t^\varepsilon), \exp(-\zeta_t^\varepsilon))$$

converges in law in $(C[0,T])^2$ to the process

$$(\exp(\sigma W_t), \exp(-\sigma W_t)),$$

where ζ_t^ε and σ are defined in (9) and(10) respectively.

The asymptotic behaviour of a solution to problem (2)–(3) is described by the following

Theorem 4. *Let conditions* **H1**–**H5** *be fulfilled. Then, as* $\varepsilon \to 0$, *a solution* u^ε *of problem (2)–(3) converges in law in the strong topology of* $L^2(\mathbb{R}^n \times (0,T))$ *to a solution of the following stochastic partial differential equation*

$$d\hat{u} = \Big(\hat{a}_{ij}\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} + \hat{b}_i\frac{\partial \hat{u}}{\partial x_i} + \hat{g}\hat{u}\Big)dt + \sigma\hat{u}\,dW_t + f(x,t), \tag{36}$$

$$\hat{u}(x,0) = u_0(x),$$

with $\hat{g} = \hat{G} + \frac{1}{2}\sigma^2$ *and* $\hat{a}_{ij}$, $\hat{b}_i$ *and* $\hat{G}$ *given by (28)–(30);* σ *is defined in (10).*

According to [3] problem (36) is well posed and has a unique solution. Hence, the limit law is well defined.

Remark 4. If $\int_{\mathbb{T}^n} g(z,s)dz = 0$ for almost all s then σ is equal to zero and the limit problem (36)is deterministic. As was already mentioned, in this case u^ε converges a.s.

Proof (Theorem 4). The solution u^ε of problem (2)–(3) can be written as a sum

$$u^\varepsilon(x,t) = (v^\varepsilon(x,t) - V^\varepsilon(x,t))\exp(\zeta_t^\varepsilon) + V^\varepsilon(x,t)\exp(\zeta_t^\varepsilon),$$

where v^ε and V^ε satisfies (8) and (34) respectively, and ζ_t^ε is defined in (9). By Proposition 3, the factor $(v^\varepsilon(x,t) - V^\varepsilon(x,t))$ in the first term on the right hand side converges in probability to zero in $L^2(\mathbb{R}^n \times (0,T))$ norm. Since by Lemma 2 the function ζ_t^ε converges in law in the space $C[0,T]$, the product $(v^\varepsilon(x,t) - V^\varepsilon(x,t))\varepsilon(\zeta_t^\varepsilon)$ tends to zero in probability in $L^2(\mathbb{R}^n \times (0,T))$.

The function V^ε as an element of $L^2(\mathbb{R}^n \times (0,T))$, depends continuously on the trajectories of the process $\zeta_\cdot^\varepsilon$ in the topology of $C(0,T)$, so does the product $V^\varepsilon(x,t)\exp(\zeta_t^\varepsilon)$. Therefore, convergence in law of the process $\zeta_\cdot^\varepsilon$ towards $\sigma W_\cdot$ implies convergence in law of the expression $V^\varepsilon \exp(\zeta_\cdot^\varepsilon)$ to the function $v^0 \exp(\sigma W_\cdot)$, where v^0 is a solution to problem (35).

It remain to show that $v^0 \exp(\sigma W_\cdot$ solves the homogenized equation (36). To this end we denote $\hat{u} = v^0 \exp(\sigma W_\cdot)$ and

$$\hat{A} = \hat{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \hat{b}_i \frac{\partial}{\partial x_i} + \hat{G},$$

and consider for arbitrary $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ the inner product $(\hat{u}(t), \varphi)$ taken in $L^2(\mathbb{R}^n)$. This expression defines a diffusion process in $\mathbb{R}$. Applying Ito's formula to this process gives

$$d(\hat{u},\varphi) = \exp(\sigma W_t) d(v^0(t),\varphi) + \sigma(v^0(t),\varphi)\exp(\sigma W_t)dW_t$$

$$+\frac{1}{2}\sigma^2(v^0(t),\varphi)\exp(\sigma W_t)dt = \exp(\sigma W_t)(\hat{A}v^0(t),\varphi)dt$$

$$(f(x,t),\varphi)\exp(\sigma W_t)\exp(-\sigma W_t)dt + \sigma(\hat{u}(t),\varphi)dW_t + \frac{1}{2}\sigma^2(\hat{u}(t),\varphi)dt$$

$$= (\hat{A}\hat{u}(t),\varphi)dt + \frac{1}{2}\sigma^2(\hat{u}(t),\varphi)dt + (f(x,t),\varphi)dt + \sigma(\hat{u}(t),\varphi)dW_t.$$

Considering also an evident relation $\hat{u}(0) = u_0$ we conclude that $\hat{u}$ is a solution of problem (36). According to [3] this problem has a unique solution, thus the limit law is uniquely defined. □

In the end of this section we formulate similar results for initial boundary problems. Given a Lipschitz domain $Q \subset \mathbb{R}^n$, consider in the cylinder $Q \times (0,T)$ a Dirichlet initial boundary problem of the form

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla u^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})u^\varepsilon(x,t) + f(x,t), \quad (37)$$

$$u^\varepsilon(x,t) = 0 \text{ on } \partial Q \times (0,T), \qquad u(x,0) = u_0(x),$$

where $u_0 \in L^2(Q)$ and $f \in L^2(Q \times (0,T))$. Under assumption **H3** for each $\varepsilon > 0$ the existence and the uniqueness of a solution of this problem in the space $L^2((0,T); H_0^1(Q)) \cap C((0,T); L^2(Q))$ are well known, see, for instance, [8].

The statement below can be justified in the same way as that for the case of Cauchy problem. We omit its proof.

Theorem 5. *Let conditions* **H1**–**H5** *be fulfilled. Then, as* $\varepsilon \to 0$, *a solution* u^ε *of problem (37) converges in law in the strong topology of* $L^2(Q \times (0,T))$ *to a solution of the limit (homogenized) stochastic partial differential equation which has the form*

$$d\hat{u} = \Big(\hat{a}_{ij}\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} + \hat{b}_i \frac{\partial \hat{u}}{\partial x_i} + \hat{g}\hat{u}\Big)dt + \sigma \hat{u}\, dW_t + f(x,t), \qquad (38)$$

$$\hat{u}(x,t) = 0 \text{ on } \partial Q \times (0,T), \quad \hat{u}(x,0) = u_0(x).$$

All the coefficients of this equation are the same as in Theorem 4.

Similar results hold true for Neumann and Fourier initial boundary problems.

7. Equations with diffusion driving process

In this section we consider an important particular case of a diffusion finite dimensional driving process in (1). Then problem (2)–(3) reads

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\Big(a(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\nabla u^\varepsilon(x,t)\Big) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})u^\varepsilon(x,t) + f(x,t), \quad (39)$$

$$u^\varepsilon(x,0) = u_0(x);$$

here and afterwards ξ_s is a stationary diffusion process with values in $\mathbb{R}^d$. We denote the generator of this process by $\mathcal{L}$:

$$\mathcal{L} = q_{km}(y)\frac{\partial^2}{\partial y_k \partial y_m} + B(y)\cdot \nabla_y.$$

The advantages of operators with diffusion driving processes are
- the coefficients of homogenized problem can be found in terms of solutions of non random elliptic auxiliary problems;
- sufficient conditions for mixing properties required in **H5** can be formulated explicitly in terms of the coefficients of generator $\mathcal{L}$.

We suppose the following conditions to hold

A1. The coefficients $a(z,y)$, $g(z,y)$ and $q(y)$ are uniformly bounded as well as their derivatives: there exists $C > 0$ such that for all $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$

$$|a_{ij}(z,y)| + |\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le C,$$
$$|g(z,y)| + |\nabla_z g(z,y)| + |\nabla_y g(z,y)| \le C,$$
$$|q_{km}(y)| + |\nabla_y q_{km}(y)| \le C,$$

for all $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$ and for all $1 \le i,j \le n$, $1 \le k,l \le d$; the symbols ∇_z and ∇_y stand for the gradients with respect to z and y respectively.
The vector function B and its derivatives satisfy polynomial growth condition:

$$|B(y)| + |\nabla B(y)| \le C(1 + |y|)^\mu$$

for some $\mu \ge 0$ and $C > 0$.
A2. Matrices a_{ij} and q_{km} are uniformly positive definite: there is $\lambda > 0$ such that

$$\lambda\,|z'|^2 \le a_{ij}(z,y)z'_i z'_j\,,\ \forall z' \in \mathbb{R}^n\,,$$
$$\lambda\,|y'|^2 \le q_{km}(z,y)y'_k y'_m\,,\ \forall y' \in \mathbb{R}^d\,.$$

A3. There exist constants $\alpha > -1$, $R > 0$ and $C > 0$ such that

$$\frac{b(y)\cdot y}{|y|} \le -C|y|^\alpha \qquad \text{for all } y \in \{y\,:\,|y| \ge R\}.$$

A4. Centering condition: $\mathbf{E}\int_{\mathbb{T}}^{n} g(z,\xi_s)dz = 0$.

As was proved in [11], under conditions **A1**–**A3** the process ξ_s has a unique invariant measure in $\mathbb{R}^d$ whose density solves the problem

$$\mathcal{L}^*\rho = 0, \qquad \int\limits_{\mathbb{R}^d} \rho(y)dy = 1;$$

the notation $\mathcal{L}^*$ is used for the adjoint operator. Moreover, for any $N > 0$ there is $C_N > 0$ such that

$$\rho(y) \le C_N(1 + |y|)^{-N}.$$

It was also shown in the same work that the strong mixing coefficient of a stationary version of the diffusion process ξ_s possesses the property **H4**.

Denote $L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ the weighted L^2 space with the norm

$$\|f(z,y)\|^2_\rho = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} f^2(z,y)\rho(y)dydz,$$

and

$$H^1_\rho(\mathbb{T}^n \times \mathbb{R}^d) = \{f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\,:\,|\nabla_z f| + |\nabla_y f| \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\}.$$

Also we will use the notation $\mathcal{A} = \mathrm{div}_z a(z,y)\nabla_z$. The proof of following two technical statements can be found in [2].

Lemma 4. *Let $f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$, and suppose that*

$$|f(z,y)| \le C\,(1 + |y|)^p\ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$$

for some $C > 0$ and $p \in \mathbb{R}$ and that $\int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} f(z,y)\rho(y)dydz = 0.$.Then the equation

$$(\mathcal{A}+\mathcal{L})\, u(z,y) = f(z,y)$$

has a solution in the space of functions of polynomial growth in y:

$$|u(z,y)| \le C\,(1+|y|)^{p+1}\ \ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$$

This solution is unique up to an additive constant.
If, in addition, there is $N>0$ such that for all $n_1,\ n_2 \in \mathbb{N}$ with $n_1+n_2 \le N$ we have

$$\left|\partial_z^{n_1}\, \partial_y^{n_2} f(z,y)\right| \le C\,(1+|y|)^p\ \ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d,$$

then

$$\left|\partial_z^{n_1}\, \partial_y^{n_2} u(z,y)\right| \le C\,(1+|y|^{p+1})\ \ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d.$$

Proposition 5. *For any $t>0,\ \mu>0,\ \gamma>0$ and $\beta>0$ the relation holds*

$$\lim_{\varepsilon\to 0} \mathbf{E}\Big(\sup_{s\le t} \varepsilon^{\beta} |\xi_{s/\varepsilon^{\gamma}}|^{\mu}\Big) = 0.$$

We proceed with averaging procedure. As above we represent $g(z,\xi_s)$ as the sum $g(z,\xi_s) =< g > (\xi_s) + \tilde{g}(z,\xi_s)$ with $< g > (\xi_s) = \int_{\mathbb{T}^n} g(z,\xi_s)dz$. In order to construct the limit operator we need two correctors which are defined as solutions of the following auxiliary equations

$$(\mathcal{A}+\mathcal{L})\chi_j(z,y) = -\frac{\partial}{\partial z_i} a_{ij}(z,y), \tag{40}$$

$$(\mathcal{A}+\mathcal{L})G(z,y) = -\tilde{g}(z,y). \tag{41}$$

By Lemma 4 these equations have solutions in $H^1_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ of polynomial growth in y.

The main result of this section is

Theorem 6. *Under assumptions* **A1**–**A4** *the solution u^ε of problem (39) converges in law, as $\varepsilon \to 0$, in the space $L^2((0,T)\times\mathbb{R}^n)$ to a solution u^0 of the following stochastic PDE*

$$du^0(x,t) = \Big(\mathrm{div}(\hat{a}\nabla u^0(x,t)) - \hat{b}\nabla u^0(x,t)\hat{g}u^0(x,t)\Big)dt + \sigma u^0(x,t)dW_t,$$

$$u^0(x,0) = u_0,$$

where

$$\hat{a} = \int_{\mathbb{T}^n}\int_{\mathbb{R}^d} a(z,y)(\mathbf{Id} + \nabla_z\chi(z,y))\rho(y)dzdy,$$

$$\hat{b} = \int_{\mathbb{T}^n}\int_{\mathbb{R}^d} (g(z,y)\chi(z,y) + a(z,y)\nabla_z G(z,y))\rho(y)dzdy,$$

$$\hat{g} = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} g(z,y)G(z,y)\rho(y)dzdy + \frac{1}{2}\sigma^2, \tag{42}$$

$$\sigma^2 = \int\limits_{\mathbb{R}^d} 2q(y)\Big(\int\limits_{\mathbb{T}^n} \nabla_y G(z,y))dz\Big)\cdot\Big(\int\limits_{\mathbb{T}^n} \nabla_y G(z,y))dz\Big)\rho(y)dy,$$

and W_t is a standard 1D Wiener process.

Proof. Since all the conditions of Theorem 4 are fulfilled, we need not prove the convergence of solutions of problem (39) to a solution of a limit stochastic PDE but only the fact that the effective coefficients given by (42) coincide with those defined in Theorem 4.

First we show that the corresponding diffusion coefficients σ are identical. The validity of the Central Limit Theorem for stationary processes of the form $F(\xi_s)$, $F \in L^q(\mathbb{R}^d)$, with diffusion ξ_s satisfying condition **A3**, have been justified in [11]. In particular, it has been proved that for the process $< g(\xi_s) >$ the corresponding variance σ is given by the formula

$$\sigma^2 = \int\limits_{\mathbb{R}^d} 2q(y)\nabla\bar{G}(y)\cdot\nabla\bar{G}(y)\rho(y)dy,$$

where $\bar{G}$ is a solution to the equation $\mathcal{L}\bar{G}(y) =< g > (y)$. Since operator $\mathcal{L}$ applied to a function $F(z,y)$ commutes with taking the average of F in z, we obtain $\nabla < g > (y) = \nabla\int_{\mathbb{T}^n} G(z,y)dz$, and the desired coincidence follows.

We proceed with the other coefficients. For any $\varphi \in C^\infty(0,T;C_0^\infty(\mathbb{R}^n))$ consider the auxiliary process

$$H^\varepsilon(t) = (\varphi, v^\varepsilon) + \varepsilon(\chi^\varepsilon\cdot\nabla\varphi, v^\varepsilon) + \varepsilon(G^\varepsilon,\varphi);$$

here and afterwards for a generic function $F = F(z,y)$ we use the notation $F^\varepsilon(x,t) = F(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2})$; $(\cdot,\cdot)$ stands for the inner product in $L^2(\mathbb{R}^n)$. Applying Ito's formula to $H^\varepsilon(t)$ gives

$$\begin{aligned}
dH^\varepsilon = {} & \{(a^\varepsilon_{ij}\partial_{x_i}\partial_{x_j}\varphi, v^\varepsilon) + \varepsilon^{-1}(\partial_{z_i}a^\varepsilon_{ij}\partial_{x_i}\varphi, v^\varepsilon)\}dt \\
& +\varepsilon^{-1}(\tilde{g}^\varepsilon\varphi, v^\varepsilon)dt + (\nabla_y G^\varepsilon\varphi + \nabla_y\chi^\varepsilon_i\partial_{x_i}\varphi, v^\varepsilon)\cdot\sqrt{q^\varepsilon}dB_t \\
& +\Big\{\varepsilon^{-1}(\mathcal{L}\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (g^\varepsilon\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (\partial_t\varphi, v^\varepsilon) \\
& +\varepsilon^{-1}(\mathcal{A}\chi^\varepsilon\nabla\varphi, v^\varepsilon) + \varepsilon(\chi^\varepsilon\partial_t\nabla\varphi, v^\varepsilon) \\
& +(\partial_{z_i}(a^\varepsilon_{ij}\chi_k)^\varepsilon\partial_{x_j}\partial_{x_k}\varphi + a^\varepsilon_{ij}\partial_{z_i}\chi^\varepsilon_k\partial_{x_j}\partial_{x_k}\varphi, v^\varepsilon) \\
& +\varepsilon(a^\varepsilon\chi^\varepsilon\nabla\nabla\nabla\varphi, v^\varepsilon) + \varepsilon^{-1}(\mathcal{L}G^\varepsilon\varphi, v^\varepsilon) + (G^\varepsilon g^\varepsilon\varphi, v^\varepsilon) \\
& +\varepsilon^{-1}(\mathcal{A}G^\varepsilon\varphi, v^\varepsilon) + (\partial_{z_i}(a^\varepsilon_{ij}G^\varepsilon)\partial_{x_j}\varphi, v^\varepsilon) + (a^\varepsilon_{ij}\partial_{z_i}G^\varepsilon\partial_{x_j}\varphi, v^\varepsilon) \\
& +\varepsilon(a^\varepsilon G^\varepsilon\nabla\nabla\varphi, v^\varepsilon)\Big\}dt.
\end{aligned}$$

Taking into account the equations (40), (41) we get

$$dH^\varepsilon = \Big\{\{(a^\varepsilon_{ij}\partial_{x_i}\partial_{x_j}\varphi, v^\varepsilon) + (\partial_{z_i}(a^\varepsilon_{ij}\chi_k)^\varepsilon\partial_{x_j}\partial_{x_k}\varphi + a^\varepsilon_{ij}\partial_{z_i}\chi^\varepsilon_k\partial_{x_j}\partial_{x_k}\varphi, v^\varepsilon)$$
$$+(g^\varepsilon\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (G^\varepsilon g^\varepsilon\varphi, v^\varepsilon) + (\partial_t\varphi, v^\varepsilon)$$
$$+(\partial_{z_i}(a^\varepsilon_{ij}G^\varepsilon)\partial_{x_j}\varphi, v^\varepsilon) + (a^\varepsilon_{ij}\partial_{z_i}G^\varepsilon\partial_{x_j}\varphi, v^\varepsilon)\Big\}dt \tag{43}$$
$$(\nabla_y G^\varepsilon\varphi + \nabla_y\chi^\varepsilon_i\partial_{x_i}\varphi, v^\varepsilon)\cdot\sqrt{q^\varepsilon}dB_t + \varepsilon dR^\varepsilon(t),$$

where B_t is a standard n-dimensional Brownian motion, and R^ε satisfies the estimate

$$\mathbf{E}\sup_{t\le T}|R^\varepsilon(t)| \le C.$$

This estimate follows from Proposition 1.

For the stochastic term on the right hand side of (43) we have

Lemma 5. *The bound holds*

$$\lim_{\varepsilon\to 0}\mathbf{E}\sup_{t\le T}\Big|\int_0^t\Big(\nabla_y G^\varepsilon(\cdot,s)\varphi(\cdot,s)$$
$$+\nabla_y\chi_i\varepsilon(\cdot,s)\partial_{x_i}\varphi(\cdot,s), v^\varepsilon(\cdot,s)\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big| = 0.$$

Proof. Since $\int_{\mathbb{T}^n}\nabla_y\chi(z,y)dz = 0$ and $\int_{\mathbb{T}^n}\nabla_y G(z,y)dz = 0$, we have

$$\mathbf{E}\sup_{t\le T}\Big|\int_0^t\Big(\nabla_y G^\varepsilon(\cdot,s)\varphi(\cdot,s) + \nabla_y\chi^\varepsilon_i(\cdot,s)\partial_{x_i}\varphi(\cdot,s), v^\varepsilon(\cdot,s)\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big|$$
$$= \varepsilon\mathbf{E}\sup_{t\le T}\Big|\int_0^t\Big(J^{1,\varepsilon}_i(\cdot,s), \partial_{x_i}(\varphi(\cdot,s)v^\varepsilon(\cdot,s))$$
$$+J^{2,\varepsilon}_{ij}(\cdot,s), \partial_{x_j}(\partial_{x_i}\varphi(\cdot,s)v^\varepsilon(\cdot,s))\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big|$$

where $J^1(z,y)$ and $J^2(z,y)$ are periodic in z functions such that

$$\mathrm{div}_z J^1(z,y) = \nabla_y G(z,y), \quad \mathrm{div}_z J^2(z,y) = \nabla_y\chi(z,y).$$

The statement of the lemma now follows from (1) and the Burkholder-Davis-Gundy inequality. □

To complete the proof of the theorem we notice that by virtue of (1) and the Birkhoff theorem

$$\lim_{\varepsilon\to 0}\mathbf{E}\sup_{t\le T}\Big|(\varphi(\cdot,t), v^\varepsilon(\cdot,t)) - (\varphi(\cdot,0), u_0) - \int_0^t(\partial_s\varphi(\cdot,s), v^\varepsilon(\cdot,s))ds$$
$$-\int_0^t\Big(\Big\{\hat a_{ij}\partial_{x_i}\partial_{x_j}\varphi(\cdot,s) + \hat b\nabla\varphi(\cdot,s) + \bar G\varphi(\cdot,s)\Big\}, v^\varepsilon(\cdot,s)\Big)ds\Big| = 0,$$

where the coefficients $\hat{a}$, $\hat{b}$ have been defined in Theorem 6 and

$$\hat{G} = \int_{\mathbb{T}^n} \int_{\mathbb{R}^d} g(z, y) G(z, y) \rho(y) dz dy. \qquad \square$$

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