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Homogenization

In Memory of Serguei Kozlov

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CENTRAL LIMIT THEOREM AND SPECTRAL ASYMPTOTICS FOR NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATION WITH WEAK NONLINEARITY

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We prove the central limit theorem for one nonlinear stochastic partial differential equation with weak nonlinearity and study the spectral asymptotics of the infinitesimal generator of the corresponding Markov process.

1 Introduction

In 1994 S.M. Kozlov in the preprint [1] posed two interrelated problems: the one about the limiting behavior of normalized integrals of solutions to one nonlinear stochastic partial differential equation (SPDE) with weak nonlinearity (see equation (2.1) below), and the other one about the spectral asymptotics for the corresponding infinitesimal generator. He formulated the conjecture that the central limit theorem (CLT) holds for the solution of SPDE mentioned above, obtained the conditional result, found the covariance operator of the limit distribution and proposed a nonrigorous method of calculating the spectral asymptotics. He hypothized, in particular, the exponential growth of the counting function. It turns out, however, that this method does not yield the correct asymptotics.

In the present paper we propose another approach to the spectral asymptotics problem and, also, justify some results which has not been proved in [1]. In particular, we prove the relative compactness of the operator that corresponds to the nonlinear term.

Recent years are marked by the growing interest of mathematicians and physicists in SPDEs and in stochastic equations on Hilbert and Banach spaces. Many modern investigations in hydrodynamics, material sciences, and other applied fields, rely on the equations of this type. Also, many asymptotic methods used in random media description result in infinite dimensional stochastic problems.

Earliest mathematical works on the subject were mainly focused on the existence and uniqueness results for the simplest SPDEs and on the existence of an invariant measure. Further investigations were devoted to more complex equations and such qualitative properties as Markovity, irreducibility, ergodicity, the Feller property, and many others.

We mention here the books [2], [3], [4], [5], [6] that comprise main aspects of the theory and where further bibliography can be found.

One of the natural and important questions is the long-term behavior of solutions, in particular, the limits of applicability of the CLT. As shown in [1], the CLT holds for SPDEs with weak nonlinearity. On the other hand, there is a conjecture that, in contrast to the finite-dimensional case, the CLT fails to hold for SPDEs with strong nonlinearity.

2 Setting of the problem

Consider the SPDE

$$\begin{gathered} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \alpha \sin u(x,t) + \dot{w}_t, \quad x \in (0,\pi); \\ u(0,t) = u(\pi,t) = 0, \qquad u|_{t=0} = u_0(x); \end{gathered} \tag{2.1}$$

here w_t is functional Brownian motion in $L^2(0,\pi)$. It is well known (see, for example, [7], [8]) that a weak solution of (2.1) does exist for all $t > 0$, is unique, and belongs to the functional space $V = L^2(0,T;H^1(0,\pi)) \cap C(0,T;L^2_w(0,\pi))$, where the symbol w indicates that the corresponding space is endowed with the weak topology.

Consider, also, the linear equation

$$\begin{gathered} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \dot{w}_t, \quad x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0, \qquad u|_{t=0} = u_0(x); \end{gathered} \tag{2.2}$$

which, in fact, is an infinite-dimensional version of Ornstein–Ulenbeck equation. The invariant measure μ_0 of the latter equation is unique (see [7]) and can be found explicitly: it is the Gaussian measure in $L^2(0,\pi)$ with a zero mean value and with a correlation operator $A = \left(\frac{d^2}{dx^2}\right)^{-1}$, which is diagonal in the basis $\left\{\sqrt{\frac{2}{\pi}}\sin kx\right\}\big|_{k=1}^{\infty}$ with eigenvalues k^{-2}

$$(Au,u)=\sum_{k=1}^{\infty}k^{-2}u_k^2;$$

here u_k are the corresponding Fourier coefficients of $u(x)$. Define the functional space $\mathcal{L}^2(L^2(0,\pi),\mu_0)$ of square–integrable functions over $L^2(0,\pi)$ with respect to the measure μ_0 with the norm

$$\left\|F(\cdot)\right\|_{\mathcal{L}^2}=\int\limits_{L^2(0,\pi)}F^2(u)\mu_0(du).$$

The solution of (2.2) defines a Markov process in $L^2(0,\pi)$, and the corresponding infinitesimal generator has the form

$$(\mathcal{A}_0F(\cdot),G(\cdot))_{\mu_0}=\frac{1}{2}\int\limits_{L^2(0,\pi)}\nabla_uF(u)\cdot\nabla_uG(u)\mu_0(du),$$

where we denote $\nabla_uF(u)=\dfrac{\delta F}{\delta u}(u)$. In the coordinates $u_1,\ u_2,\dots,u_k,\dots,$ the operator $\mathcal{A}_0$ can be formally rewritten as follows:

$$\mathcal{A}_0F(u)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\partial^2F}{\partial u_k^2}(u)+\nabla F(u)\cdot\frac{d^2}{dx^2}u.$$

Now, let us define a new measure $\mu(du)$ absolutely continuous with respect to μ_0, by the following relation

$$\frac{d\mu}{d\mu_0}(u)=\exp\left\{-\alpha\int_0^{\pi}\cos u(x)dx\right\}.$$

Taking into account the fact that the function $\alpha\sin u(x)$ is a variational derivative of $-\alpha\int\limits_0^{\pi}\cos u(x)dx$, one can verify that $\mu(du)$ is an invariant measure of equation (2.1), and that the solution $u(x,t)$ of (2.1) is a Markov process (see [7] for more detail) which infinitesimal generator is defined as follows

$$(\mathcal{A}F(\cdot),G(\cdot))_{\mu}=\frac{1}{2}\int\limits_{L^2(0,\pi)}\nabla_uF(u)\cdot\nabla_uG(u)\mu(du).$$

The purpose of this paper is to investigate the main properties of the operator $\mathcal{A}$.

3 Spectral asymptotics for $\mathcal{A}_0$

In the proper basis $\{e_j\} = \{\sqrt{\frac{2}{\pi}}\sin jx\}$ the operator $\mathcal{A}_0$ can be decomposed into direct sum of one–dimensional operators

$$\mathcal{A}_0 = \sum_{k=1}^{\infty} A_k,$$

here $A_k = \frac{1}{2}\frac{d^2}{du_k^2} + k^2 u_k \frac{d}{du_k}$ acts in the space $L^2(R, \exp(-k^2 u_k^2)dx)$. The spectrum of A_k is well-known: $\lambda_n^k = nk^2$, $n = 0, 1, 2, \ldots$; $\varphi_n^k(u_k) = H_n(ku_k)$ where $h_n(x)$ are Chebysheff-Hermite polynomials.

Now we can form the eigenbasis of $\mathcal{A}_0$ in $\mathcal{L}^2(L^2(0,\pi), \mu_0)$ as follows (see [9], Ch.VIII)

$$\nu_{k_1,\ldots,k_s}^{n_1,\ldots,n_s}(u) = \varphi_{n_1}^{k_1}(u_{k_1})\varphi_{n_2}^{k_2}(u_{k_2})\ldots\varphi_{n_s}^{k_s}(u_{k_s});$$

here $s \geq 0$, n_j and k_j are arbitrary natural numbers, $k_j \neq k_i$ for $j \neq i$. The corresponding eigenvalue $\Lambda_J = \Lambda_{k_1,\ldots,k_s}^{n_1,\ldots,n_s}$ is equal to $\sum_{j=1}^{s} k_j^2 n_j$. For $s = 0$ we set $\nu_0(u) = 1$.

The above considerations show that the counting function $N_{\mathcal{A}_0}(\Lambda)$ coincides with the number of integer nonnegative solutions $(n_1, n_2, \ldots)$ of the following inequality

$$\sum_{k=1}^{\infty} n_k k^2 \leq \Lambda.$$

Theorem 3.1 *The following limiting relation holds:*

$$\lim_{\Lambda\to\infty} \frac{\ln N_{\mathcal{A}_0}(\Lambda)}{\Lambda^{1/3}} = c_0,$$

c_0 *being the maximum in the variational problem*

$$\begin{gathered} c_0 = \max_{p\in\mathbf{B}} \int_0^{\infty} [(p(x)+1)\ln(p(x)+1) - p(x)\ln p(x)]dx, \\ \mathbf{B} = \left\{ p(x) \geq 0 \Big| \int_0^{\infty} x^2 p(x)dx = 1 \right\} \end{gathered} \tag{3.1}$$

Proof. First of all let us note that $\Theta(p) = ((p+1)\ln(p+1) - p\ln p)$ is a monotone convex function, $\Theta(0) = 0$, and $\Theta(p) \sim \ln p$ as $p \to \infty$. Thus, by the standard arguments, the problem (3.1) is well-posed and, using of the Lagrange multipliers method, we obtain

$$p(x) = \frac{1}{\exp(\lambda x^2) - 1}, \qquad \int\limits_0^\infty \frac{x^2 dx}{\exp(\lambda x^2) - 1} = 1$$

and

$$c_0 =$$

$$\int\limits_0^\infty \left(\frac{\exp(\lambda x^2)}{\exp(\lambda x^2) - 1} \ln \frac{\exp(\lambda x^2)}{\exp(\lambda x^2) - 1} - \frac{1}{\exp(\lambda x^2) - 1} \ln \frac{1}{\exp(\lambda x^2) - 1} \right) dx$$

In order to estimate $N_{\mathcal{A}_0}(\Lambda)$ from below we fix an interval (δ, δ^{-1}) and divide this interval into $L = \big((\delta^{-1} - \delta)/\varepsilon\big)$ equal parts. In what follows we assume the relation $\varepsilon \ll \delta \ll 1$ holds. Denote $(n_1, n_2, \ldots)$ by $\bar{n}$. We say that $\mathrm{supp}(\bar{n}) \in (j_1, j_2)$ if $n_i = 0$ for all $i \notin (j_1, j_2)$. Clearly, for any continuous $p(x)$ such that $\int\limits_\delta^{\delta^{-1}} x^2 p(x) dx \le 1$ we have

$$\begin{aligned} \#\Big\{\bar{n} \,\Big|\, \sum_{i=1}^\infty n_i i^2 \le \Lambda\Big\} \ge \prod_{k=0}^{L} \#\Big\{\bar{n} \,\Big|\, &\mathrm{supp}(\bar{n}) \in \Lambda^{1/3} I_k, \\ &\sum n_i i^2 \le \Lambda\varepsilon \min_{x\in I_k} x^2 p(x)\Big\}, \end{aligned} \tag{3.2}$$

where the notation $I_k = (k\varepsilon + \delta, (k+1)\varepsilon + \delta)$ has been used and the symbol $\#$ stands for the number of elements in the set. Let us estimate each factor in the right hand side of (3.2)

$$\#\Big\{\bar{n} \,|\, \mathrm{supp}(\bar{n}) \in \Lambda^{1/3} I_k, \sum_i n_i i^2 \le \varepsilon\Lambda q_k\Big\} >$$

$$> \#\Big\{\bar{n} \,|\, \mathrm{supp}(\bar{n}) \in \Lambda^{1/3} I_k, \sum_i n_i \le \frac{\varepsilon\Lambda q_k}{\Lambda^{2/3}((k+1)\varepsilon + \delta)^2}\Big\} =$$

$$= \left(\Lambda^{1/3}\frac{\varepsilon q_k}{((k+1)\varepsilon + \delta)^2} + \Lambda^{1/3}\varepsilon\right)! \Big/ \left(\Lambda^{1/3}\frac{\varepsilon q_k}{((k+1)\varepsilon + \delta)^2}\right)! \left(\Lambda^{1/3}\varepsilon\right)!;$$

here we have also used the standard combinatorial formula. By the Stirling formula

$$\ln\left\{\left(\Lambda^{1/3}\Big(\frac{\varepsilon q_k}{((k+1)\varepsilon + \delta)^2} + \varepsilon\Big)\right)! \Big/ \left(\Lambda^{1/3}\Big(\frac{\varepsilon q_k}{((k+1)\varepsilon + \delta)^2}\Big)\right)! (\varepsilon\Lambda^{1/3})!\right\} =$$

$$= \ln\Big[\Big(\Lambda^{1/3}\big(\frac{\varepsilon q_k}{((k+1)\varepsilon+\delta)^2}+\varepsilon\big)\Big)^{\Lambda^{1/3}\left(\frac{\varepsilon q_k}{((k+1)\varepsilon+\delta)^2}+\varepsilon\right)}\times$$

$$\Big(\Lambda^{1/3}\big(\frac{\varepsilon q_k}{((k+1)\varepsilon+\delta)^2}\big)^{-\Lambda^{1/3}\left(\frac{\varepsilon q_k}{((k+1)\varepsilon+\delta)^2}\right)}(\varepsilon\Lambda^{1/3})^{-\varepsilon\Lambda^{1/3}}\Big](1+o(1)) =$$

$$= \Lambda^{1/3}\varepsilon\Big[\big(\frac{q_k}{((k+1)\varepsilon+\delta)^2}+1\big)\ln\big(\frac{q_k}{((k+1)\varepsilon+\delta)^2}+1\big)-$$

$$\frac{q_k}{((k+1)\varepsilon+\delta)^2}\ln\frac{q_k}{((k+1)\varepsilon+\delta)^2}\Big](1+o(1))$$

as $\Lambda\to\infty$. It remains to put $q_k=\min\limits_{I_k} x^2p(x)$ and substitute the last relation into (3.2):

$$\ln\#\Big\{\bar{n}\mid\sum_{i=1}^{\infty}n_ii^2\le\Lambda\Big\}\ge\varepsilon\Lambda^{1/3}\sum_{k=0}^{L}\Big[\Big(\frac{q_k}{((k+1)\varepsilon+\delta)^2}+1\Big)\times$$

$$\ln\Big(\frac{q_k}{((k+1)\varepsilon+\delta)^2}+1\Big)-\frac{q_k}{((k+1)\varepsilon+\delta)^2}\ln\frac{q_k}{((k+1)\varepsilon+\delta)^2}\Big]\ge$$

$$\ge\Lambda^{1/3}\Big(\int_{\delta}^{\delta^{-1}}(p(x)+1)\ln(p(x)+1)-p(x)\ln p(x))dx-\delta_1(\varepsilon)\Big)\ge$$

$$\ge\Lambda^{1/3}\Big(\int_{0}^{\infty}(p(x)+1)\ln(p(x)+1)-p(x)\ln p(x))dx-\delta_2(\varepsilon,\delta)\Big),$$

where $\delta_1\to0$ as $\varepsilon\to0$ and $\delta_2\to0$ as ε and $\delta\to0$. Thus,

$$\liminf_{\Lambda\to\infty}\frac{\ln\#\Big\{\bar{n}\mid\sum\limits_{i=1}^{\infty}n_ii^2\le\Lambda\Big\}}{\Lambda^{1/3}}\ge c_0. \tag{3.3}$$

To establish the upper bound let us consider an arbitrary sequence $\{q_k\}$ such that $\sum\limits_{k=1}^{L}\varepsilon q_k\le1$, and estimate $\#\{\mathrm{supp}(\bar{n})\in I_k\mid\sum\limits_i n_ii^2\le\varepsilon q_k\Lambda\}$. In the same way as above one can prove that

$$\begin{aligned}&\ln\#\{\mathrm{supp}(\bar{n})\in I_k\mid\sum_i n_ii^2\le\varepsilon q_k\Lambda\}\le\\ &\le\varepsilon\Lambda^{1/3}\Big[\Big(\frac{q_k}{(k\varepsilon+\delta)^2}+1\Big)\ln\Big(\frac{q_k}{(k\varepsilon+\delta)^2}+1\Big)-\\ &-\frac{q_k}{(k\varepsilon+\delta)^2}\ln\frac{q_k}{(k\varepsilon+\delta)^2}\Big](1+o(1))\end{aligned} \tag{3.4}$$

.

as $\Lambda \to \infty$. Similarly,

$$\ln \#\{\mathrm{supp}(\bar{n}) \in (\frac{\delta\Lambda^{1/3}}{2^{j+1}}, \frac{\delta\Lambda^{1/3}}{2^j}) \mid \sum_i n_i i^2 \leq \Lambda\} \leq$$

$$\leq \ln\Big[\big(\frac{2^{2j+2}\Lambda^{1/3}}{\delta^2} + \frac{\delta\Lambda^{1/3}}{2^j}\big)!/\big(\frac{2^{2j+2}\Lambda^{1/3}}{\delta^2}\big)!/\big(\frac{\delta\Lambda^{1/3}}{2^j}\big)!\Big] \leq$$

$$\leq \Lambda^{1/3}\big(\frac{2^{2j+2}}{\delta^2}+\frac{\delta}{2^j}\big)\ln\big(\frac{2^{2j+2}}{\delta^2}+\frac{\delta}{2^j}\big)-\Lambda^{1/3}\frac{2^{2j+2}}{\delta^2}\ln\frac{2^{2j+2}}{\delta^2}-\Lambda^{1/3}\frac{\delta}{2^j}\ln\frac{\delta}{2^j} =$$

$$= \Lambda^{1/3}\frac{2^{2j+2}}{\delta^2}\ln\big(1+\frac{\delta^3}{2^{3j+2}}\big)+\Lambda^{1/3}\frac{\delta}{2^j}\ln\big(1+\frac{2^{3j+2}}{\delta^3}\big) \leq$$

$$\leq \Lambda^{1/3}\frac{\delta}{2^{j-1}}((3j+2)\ln 2 - 3\ln\delta)$$

and summing the leftmost and the rightmost parts of the latter inequality over j, we find

$$\begin{gathered}\ln \#\{\mathrm{supp}(\bar{n}) \in (1, \delta\Lambda^{1/3}) \mid \sum_i n_i i^2 \leq \Lambda\} \leq \\ \sum_{j=1}^{\infty} \Lambda^{1/3}\tfrac{\delta}{2^{j-1}}((3j+2)\ln 2 - 3\ln\delta) \leq \Lambda^{1/3}(4\delta\ln 2 - 3\delta\ln\delta)\end{gathered} \tag{3.5}$$

Similar estimates can be obtained for the interval $\mathrm{supp}(\bar{n}) \in (\Lambda^{1/3}/\delta, \Lambda^{1/2})$:

$$\begin{gathered}\ln \#\{\mathrm{supp}(\bar{n}) \in (\Lambda^{1/3}/\delta, \Lambda^{1/2}) \mid \sum_i n_i i^2 \leq \Lambda\} \leq \\ \sum_{j=1}^{\infty} \ln\Big[\big(\frac{2^j}{\delta}\Lambda^{1/3} + \frac{\delta^2}{2^{2j-2}}\Lambda^{1/3}\big)!/\big(\frac{2^j}{\delta}\Lambda^{1/3}\big)!/\big(\frac{\delta^2}{2^{2j-2}}\Lambda^{1/3}\big)!\Big] \leq \\ \leq \Lambda^{1/3}(8\delta^2\ln 2 - 12\delta^2\ln\delta)\end{gathered} \tag{3.6}$$

In view of (3.4)-(3.6) the number of solutions to the problem

$$\Big\{\bar{n} \mid \sum_{i=1}^{\infty} i^2 n_i \leq \Lambda, \sum_{i=\Lambda^{1/3}(\delta+\varepsilon k)}^{\Lambda^{1/3}(\delta+\varepsilon(k+1))} i^2 n_i \leq \varepsilon\Lambda q_k\Big\}$$

does not exceed

$$\exp\Big[\Lambda^{1/3}\big\{(4\delta\ln 2 - 3\delta\ln\delta) + (8\delta^2\ln 2 - 12\delta^2\ln\delta) +$$

$$+\varepsilon\sum_{k=1}^{L}\Big(\frac{q_k}{(\delta+\varepsilon k)^2}+1\Big)\ln\Big(\frac{q_k}{(\delta+\varepsilon k)^2}+1\Big)$$
$$-\frac{q_k}{(\delta+\varepsilon k)^2}\ln\frac{q_k}{(\delta+\varepsilon k)^2}\Big\}(1+o(1))\Big]$$

as $\Lambda \to \infty$. Denoting $\dfrac{q_k}{(\delta+k\varepsilon)^2}$ by p_k and recalling the definition of $p(x)$ we derive

$$\ln\#\Big\{\bar{n}\,|\,\sum_{i=1}^{\infty}i^2n_i\le\Lambda,\ \sum_{i=\Lambda^{1/3}(\delta+\varepsilon k)}^{\Lambda^{1/3}(\delta+\varepsilon(k+1))}i^2n_i\le\varepsilon\Lambda q_k\Big\}\le$$
$$\Lambda^{1/3}\Big(\int_0^\infty((p(x)+1)\ln(p(x)+1)-p(x)\ln p(x))dx\Big)(1+\delta_1(\delta,\varepsilon)) \tag{3.7}$$

where $\delta_1(\delta,\varepsilon)\to 0$ as ε and $\delta\to 0$ and $\Lambda\to\infty$. It remains to estimate the number of integer vectors $\{\varepsilon\Lambda q_k\}$, $0\le k\le L$, such that $\sum_{k=1}^{L}\varepsilon q_k\le 1$:

$$\#\Big\{\{\varepsilon\Lambda q_k\}\,|\,\sum_{k=1}^{L}\varepsilon q_k\le 1\Big\}\le\Big(\frac{\delta^{-1}}{\varepsilon}+\Lambda\Big)!/\Big(\frac{\delta^{-1}}{\varepsilon}\Big)!/\Lambda!\le$$
$$\le\exp\Big\{\Big(\frac{\delta^{-1}}{\varepsilon}+\Lambda\Big)\ln\Big(\frac{\delta^{-1}}{\varepsilon}+\Lambda\Big)-\frac{\delta^{-1}}{\varepsilon}\ln\frac{\delta^{-1}}{\varepsilon}-\Lambda\ln\Lambda\Big\}\le$$
$$\le\exp\Big\{\frac{\ln\Lambda}{\delta\varepsilon}+\frac{1}{\Lambda(\delta\varepsilon)^2}+\frac{1}{\delta\varepsilon}\Big\}.$$

Finally, (3.7) and the last inequality together imply

$$\#\{\bar{n}\,|\,\sum_i n_ii^2\le\Lambda\}\le\sum_{\{q_k\}}\#\{\bar{n}\,|\,\sum_i n_ii^2\le\Lambda,$$
$$\sum_{j=\Lambda^{1/3}(\delta+k\varepsilon)}^{\Lambda^{1/3}(\delta+(k+1)\varepsilon)}n_ii^2\le\Lambda\varepsilon q_k\}\le$$
$$\le\exp\Big[\Lambda^{1/3}\Big(\int_0^\infty((p(x)+1)\ln(p(x)+1)-p(x)\ln p(x))dx+\delta_1(\delta,\varepsilon)+$$
$$+\frac{\ln\Lambda}{\Lambda^{1/3}\delta\varepsilon}+\frac{1}{\lambda^{4/3}(\delta\varepsilon)^2}\Big)\Big].$$

Passing to the limit in Λ and choosing δ and ε sufficiently small, we have

$$\limsup_{\Lambda\to\infty} \Lambda^{-1/3} \ln \#\{\bar{n} \mid \sum_{i=1}^{\infty} n_i i^2\} \leq c_0.$$

The theorem is proved.

4 Compactness results

In this section we prove that the operator $\mathcal{A}$ is a relatively compact perturbation of operator the $\mathcal{A}_0$. This allows us to study the spectral properties of $\mathcal{A}$.

We start with proving the following simple statement.

Proposition 1 *There exists a constant* $c > 0$ *such that*

$$c(\mathcal{A}_0 F(\cdot), F(\cdot))_{\mu_0} \leq (\mathcal{A}F(\cdot), F(\cdot))_{\mu} \leq c^{-1}(\mathcal{A}_0 F(\cdot), F(\cdot))_{\mu_0}.$$

Proof.

This assertion is a direct consequence of the following evident relation

$$c \leq \frac{d\mu}{d\mu_0}(u) \leq c_{-1}.$$

By the Courant minimax principle, we derive from the last proposition that

$$N_{\mathcal{A}_0}(c\Lambda) \leq N_{\mathcal{A}}(\Lambda) \leq N_{\mathcal{A}_0}(c^{-1}\Lambda).$$

In particular, the resolvent of $\mathcal{A}$ is a compact operator. In fact, the operator $(\mathcal{A} - \mathcal{A}_0)$ is relatively compact with respect to $\mathcal{A}_0$.

Lemma 4.1 *The operator* $(\mathcal{A} - \mathcal{A}_0)(\mathcal{A}_0 + I)^{-1}$ *is compact in* $\mathcal{L}^2(L^2(0,\pi), \mu_0)$.

Proof.

Let $\mathcal{H}$ be a dense subspace of $\mathcal{L}^2(L^2(0,\pi), \mu_0)$ endowed with the inner product

$$(F,G)_{\mathcal{H}} = (F,G)_{\mathcal{L}^2} + \int_{\mathcal{L}^2} (\nabla_u F, \nabla_u G)\mu_0(du) = ((\mathcal{A}_0 + I)F, G)_{\mathcal{L}^2}.$$

$\mathcal{H}$ is embedded compactly in $\mathcal{L}^2$. Denote by $\mathcal{H}^{-1}$ the adjoint space of $\mathcal{H}$. Then $(\mathcal{A}_0 + I)$ is a bounded operator from $\mathcal{H}$ to $\mathcal{H}^{-1}$. For any $F \in \mathcal{H}$ by the Schwartz inequality

$$\| \sin u \cdot \nabla_u F(u)\|^2_{\mathcal{L}^2} = \int_{\mathcal{L}^2} (\sin u \cdot \nabla_u F(u))^2 \mu_0(du) \leq$$

$$\leq \int_{\mathcal{L}^2} \| \sin u\|^2_{L^2(0,\pi)} |\nabla_u F(u)\|^2_{L^2(0,\pi)} \mu_0(du) \leq \|F\|^2_{\mathcal{H}}$$

Thus, $(\mathcal{A}-\mathcal{A}_0)$ is a bounded operator from $\mathcal{H}$ to $\mathcal{L}^2$. Clearly, $(\mathcal{A}_0+I)^{-1}$ is a compact operator from $\mathcal{L}^2$ to $\mathcal{H}$. Consequently, $(\mathcal{A}-\mathcal{A}_0)(\mathcal{A}_0+I)^{-1}$ is compact in $\mathcal{L}^2$. The lemma is proved.

Combining the last lemma with ([10], Theorem 8.2), we derive

Corollary 4.1 *The following limiting relation takes place*

$$\lim_{\Lambda\to\infty} \frac{\ln N_{\mathcal{A}}(\Lambda)}{\Lambda^{1/3}} = c_0.$$

5 Central Limit Theorem

The main result of the section is the following.

Theorem 5.1 *Let the initial distribution $u_0(\cdot)$ be absolutely continuous with respect to $\mu(du)$ (or $\mu_0(du)$) and let its density $U_0(u)$ belong to the space $\mathcal{L}^2(L^2(0,\pi),\mu(du))$. Then the family of normalized integrals $\frac{1}{\sqrt{t}}\int_0^t u(s,x)ds$ converges in distribution in the space $L^2(0,\pi)$ as $t \to \infty$ to a Gaussian measure with a zero mean value and the covariance operator*

$$(B\varphi,\varphi) = C \int_{L^2(0,\pi)} |\nabla_u n_\varphi(u)|^2 \mu(du) = C(\mathcal{A}n_\varphi, n_\varphi)_{\mathcal{L}^2(L^2,\mu)} =$$

$$= C\Big(\mathcal{A}^{-1}(u,\varphi)_{L^2(0,\pi)}, (u,\varphi)_{L^2(0,\pi)}\Big)_{\mathcal{L}^2(L^2,\mu)};$$

here n_φ is a solution of $\mathcal{A}n_\varphi = (u,\varphi)$ and $C = \int_{L^2(0,\pi)} U_0(u)\mu(du)$.

Proof.

First of all let us note that $\int_{L^2(0,\pi)} (u,\varphi)_{L^2}\mu(du) = 0$ for any $\varphi \in L^2$. Hence, a solution $n_\varphi(u)$ indeed exists. We fix a solution by the following condition $\int_{L^2(0,\pi)} n_\varphi(u)\mu(du) = 0$. Using Ito's formula one can verify the identity

$$\int_0^t (u(s,\cdot),\varphi(\cdot))ds + n_\varphi(u(t,\cdot)) - n_\varphi(u_0(\cdot)) = \int_0^t \nabla_u n_\varphi(u(s,\cdot))dw_s$$

which implies, in particular, that the process in the left hand side is a martingale. Obviously, the limiting distribution of

$$\left(\frac{1}{\sqrt{t}}\left[\int_0^t (u(s,x),\varphi(x))ds + n_\varphi(u(t,\cdot)) - n_\varphi(u_0(\cdot))\right]\right)$$

in V coincides with the limiting distribution of $\frac{1}{\sqrt{t}}\left[\int_0^t (u(s,x),\varphi(x))ds\right]$. Indeed, due to the properties of $\mathcal{A}$, the function $\int_{L^2(0,\pi)} n_\varphi^2(u(t,\cdot))\mu(du)$ decreases monotonically in t. The limiting behavior of

$$\frac{1}{\sqrt{t}}\int_0^t \nabla_u n_\varphi(u(s,\cdot))dw_s$$

can, in turn, be studied by the methods developed in [11], [12], and the theorem follows.

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