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Variational Problems with Percolation: Dilute Spin Systems at Zero Temperature

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Abstract We study the asymptotic behavior of dilute spin lattice energies by exhibiting a continuous interfacial limit energy computed using the notion of Γ-convergence and techniques mixing Geometric Measure Theory and Percolation while scaling to zero the lattice spacing. The limit is not trivial above a percolation threshold. Since the lattice energies are not equi-coercive, a suitable notion of limit magnetization must be defined, which can be characterized by two phases separated by an interface. The macroscopic surface tension at this interface is characterized through a first-passage percolation formula, which highlights interesting connections between variational problems and percolation issues. A companion result on the asymptotic description on energies defined on paths in a dilute environment is also given.

Keywords Dilute spins · Lattice energies · First-passage percolation · Variational problems · Gamma-convergence

1 Introduction

Variational theories in Materials Science (and elsewhere) often require the modeling of media with fine microstructure and their description via averaged effective energies. This process is usually referred to as *homogenization*; see e.g. [8, 14, 21]. In some cases, the analysis of *periodic* microstructure is sufficient to all modeling purposes (e.g., in Optimal Design [1], or in the study of composite materials [27]), but in other cases random media

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have to be considered. In the case of surface energies, or more in general of energies with competing surface and volume terms, the introduction of a random dependence of coefficients is particularly delicate, since the value of these coefficients on sets of zero measure may dramatically influence the averaged description of the media. Recently, homogenization methods have been applied to treat discrete models that are "equivalent" to classical models on the continuum such as mesoscopic phase-transition energies or models in Fracture Mechanics by a rigorous passage "from-discrete-to-continuous". In this discrete context, the introduction of a random dependence is particularly simple and clear, as it amounts to describe the behavior of energy densities depending e.g. on bond or site location. To this end the methods and results of Statistical Physics, and Percolation Theory *in primis*, are particularly relevant and useful. We will see that variational problems translate into interesting questions in Percolation Theory, for which an answer often requires non-trivial elaborations of existing results.

This paper is about variational problems for whose solution we can use results for dilute spins at zero temperature. The model problem that we have in mind is that of a crystalline solid with randomly distributed voids subject to fracture. We suppose that the thermal fluctuations are negligible (zero-temperature case) and that the relevant scale is that of the surface (fracture) energy so that we may neglect the elastic energy of the lattice (this can be taken separately into account as in the paper [15]). In this case, depending on the applied forces or boundary displacements of the sample, a fracture may appear, separating two regions where the displacement is constant. In the Griffith theory of Fracture the energy necessary for the creation of a crack is proportional to its area; in a discrete setting this is translated in the number of atomic bonds that are broken. If voids are already present then the corresponding bonds are not accounted for in the computation of the energy. This model translates into a dilute spin problem, where the two values of the spin parameterize the two regions of constant displacement of the crystal. We note that in this problem the random distribution of defects is considered as fixed and as a characteristic of the crystalline material, so that we are interested in almost sure properties of the overall energies when the measure of the sample is large with respect to the atomic distance.

The way we will describe the overall behavior of those systems is by scaling the domain lattice by a small parameter ε and introducing the corresponding scaled energies, and then compute the variational limit (Γ-limit) of such energies, which is defined on the continuum. The Γ-limit is nothing but an effective energy which describes the behavior of minimum problems for the discrete energies, so that minimizers and minimal energies in the discrete are approximated by minimizers and minimal energies on the continuum. A description of the use of Γ-convergence in the context of Statistical Mechanics (but not directly on a discrete setting) can be found in the book of Presutti starting from the Lebowitz-Penrose functional (see [29], Chap. 7). We will obtain the same type of surface energies, which shows phase segregation through the description of their Γ-limit as a surface energy between two phases and the identification of a surface tension between such phases. As clearly described in [29], a key point in the computation of the surface tension of the Γ-limit is the possibility of reducing to minimum problems that (in the limit) have a minimizer with a flat interface. In our case the asymptotic behavior of such problems can be rephrased in terms of percolation issues.

The microscopic energy under examination is of the form

$$-\sum_{ij}\sigma_{ij}^{\omega}u_i u_j,$$

where $u_i \in \{\pm 1\}$ is a spin variable indexed on the lattice $\mathbb{Z}^d$, the sum runs on nearest neighbors in a given portion $\mathbf{D} \cap \mathbb{Z}^d$ of $\mathbb{Z}^d$, the coefficients σ_{ij}^{ω} depend on the realization ω of an i.i.d. random variable, and

$$\sigma_{ij}^{\omega} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p, \end{cases}$$

with $p \in [0,1]$ fixed. In order to describe the behavior as the size of $\mathbf{D}$ diverges we introduce a scaled problem, as is customary in the passage from lattice systems to continuous variational problems, in which, on the contrary, $\mathbf{D}$ is kept fixed, but scaled energies are defined as follows. A small parameter $\varepsilon > 0$ is introduced, the lattice is scaled accordingly to $\varepsilon\mathbb{Z}^d$, and the energies are scaled to

$$E_{\varepsilon}^{\omega}(u) := \sum_{ij} \varepsilon^{d-1} \sigma_{ij}^{\omega} (u_i - u_j)^2$$

(this amounts to adding proper normalization constants depending on ω and multiplying by 2). Note that considering $(u_i - u_j)^2$ in place of $-u_i u_j$ is merely technical and amounts to the translation of the energies so that uniform states (which are pointwise minimizers of the 'integrand') have zero energy; moreover, the 'surface scaling' ε^{d-1} is driven by the knowledge that for $p = 1$ (i.e., for ferromagnetic interactions) the Γ-limit with that scaling is not trivial (as shown e.g. by Alicandro, Braides and Cicalese [3]). For other scalings the limit energy will be trivial (either identically 0 or finite only on trivial uniform states), so that it will give very little information on the overall behavior of the system at finite ε. After this scaling, the sum is taken on nearest neighbors in $\mathbf{D} \cap \varepsilon\mathbb{Z}^d$, and the normalization allows also to consider $\mathbf{D} = \mathbb{R}^d$ (in this case the domain of the energy is composed of all u which are constant outside a bounded set).

The coarse graining of these energies corresponds to a general approach in the theory of Γ-convergence for lattice system where the discrete functions $u = \{u_i\}$ are identified with their piecewise-constant extensions, and the scaled lattice energies with energies on the continuum whose asymptotic behavior is described by taking L^1-limits in the u variable and applying a mesoscopic homogenization process to the energies. The comparison with the case $p = 1$ ensures that the limit is finite (but possibly trivial) on u with $\partial\{u = 1\}$ of finite area in $\mathbf{D}$. A general theory for interfacial energies by Ambrosio and Braides [4] suggests the identification with functionals of the form

$$\int_{\mathbf{D} \cap \partial\{u=1\}} \varphi(x, \nu)\, d\mathcal{H}^{d-1},$$

with ν the normal to $\partial\{u = 1\}$. In the dilute case, however, neither the existence of an average macroscopic magnetization (the L^1 limit of the u) nor the definition of a limit surface tension follow from this general theory. They can instead be translated in almost-sure properties of the corresponding Bernoulli bond percolation model. Below the (lower) percolation threshold the energy is indeed trivial (the Γ-limit identically vanishing on its domain), since interfaces with zero energy are asymptotically L^1-dense. Above the percolation threshold instead the coarse graining leads first to showing that indeed we may define a limit magnetization u taking values in $\{\pm 1\}$. This u is obtained as a L^1-limit on the scaled infinite strong cluster, thus neglecting the values u_i on nodes i isolated from that cluster. It should be noted that for p far from the critical value this limit variable can be alternatively thought as a renormalization of the 'effective magnetization' (the one obtained by local averages; i.e., as the weak L^1 limit of the spins on the scaled lattices). This effective magnetization does not take only the values ± 1 but may take all values u with $|u| \in [m_{\text{eff}}, 1]$, where m_{eff}

is the limit (almost sure) deterministic average (depending only on p) of the function taking the value 1 on points connected to the strong cluster, and -1 elsewhere. The surface tension is obtained by optimizing the almost sure contribution of the interfaces, and showing that it can be expressed as a first-passage percolation problem, so that the limit is of the form

$$\int_{\mathbf{D}\cap\partial\{u=1\}} \varphi_p(\nu)\, d\mathcal{H}^{d-1}.$$

It should also be noted that the possibility of the definition of the limit magnetization is translated in variational terms in a equi-coerciveness result; i.e., that if we have a sequence (u_ε) such that $\sup_\varepsilon E_\varepsilon^\omega(u_\varepsilon) < +\infty$ then we may deduce that such a sequence is pre-compact. In the case that $\sigma_{ij}^\omega = 1$ for all ij strong L^1-compactness is ensured by the theory of the functions of bounded variation. For periodic homogenization problems the compactness is essentially equivalent to the existence of a positive surface tension. In the random case, the compactness does not follow immediately from the existence of a positive surface tension, but requires an "extension principle" from the strong cluster to the whole $\mathbb{Z}^d$. This type of variational percolation results can be linked to an earlier paper by the authors [15] where discrete fracture is studied and linked to large deviations for the chemical distance in supercritical Bernoulli percolation, thus showing a stimulating interaction between Variational Calculus and Percolation theory.

The Γ-convergence of spin systems is linked to recent progress in the understanding of phase segregation and the validity of the Wulff construction for Ising-type models through a L^1 approach (see, e.g., [2, 5–7, 9, 10, 17–20, 31]). Establishing a link between the Wulff construction, large deviations and Γ-convergence seems to be a promising field of research.

The paper is organized as follows. After briefly setting notation in Sect. 2, in Sect. 3.1 we prove some asymptotic properties of connected subsets of points in the underlying percolation model, and deduce the coerciveness of energies in the supercritical case $p > p_c$. The convergence theorem is then proved in Sect. 3.2 by a blow-up argument, which corresponds to a coarse graining at the interfaces, using geometric measure theoretical properties and the description of (optimal) interfaces through a first-passage percolation formula. Section 4 deals briefly with the subcritical and critical regimes. Finally, in Sect. 5 we give a 'dual' result to the one above. Again, we may think of a random crystalline medium whose metric properties are described by an energy on paths counting the number of connections in the path with the same weights σ_{ij}^ω as introduced above. As a model we may think of a network of resistors, with a random distribution of perfect conductors. The overall metric properties of the medium are described by the Γ-limit of these path energies. We give an almost-sure representation for the limit as an integral on continuous paths of the form

$$\int_0^L \psi_p(\gamma')\, dt$$

for all $0 < p < 1$. The shape of ψ_p is linked to properties of first-passage percolation in the supercritical case and of the chemical distance in supercritical Bernoulli percolation in the subcritical regime.

2 Setting of the Problem

We use the notation for bond-percolation problems in $\mathbb{Z}^d$ in [15], Sect. 2.4, and introduce coefficients

$$\sigma_{\hat{z}}^{\omega} = \begin{cases} 1 & \text{if } \xi_{\hat{z}}(\omega) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\xi_{\hat{z}} = \begin{cases} 0 \text{ ('weak')} & \text{with probability } 1-p, \\ 1 \text{ ('strong')} & \text{with probability } p. \end{cases} \tag{1}$$

We also write $\sigma_{\hat{z}}^{\omega} = \sigma_{ij}^{\omega}$, after identifying each $\hat{z}$ with a pair of nearest neighbors in $\mathbb{Z}^d$.

A *path* of points in $\mathbb{Z}^d$ is a finite or infinite sequence $\{i_k : k = 0, 1, \dots\}$ such that $|i_k - i_{k+1}| = 1$ for all $k = 0, 1, \dots$. A path is *weak* (*resp., strong*) if $\sigma_{i_k i_{k-1}}^{\omega} = 0$ (resp., 1) for all $k = 1, \dots$. Two points i, j are said to be *weakly* (*resp., strongly*) *connected* if there exists a weak (resp., strong) path containing them. It is known that there exist a critical *percolation threshold* $p_c \le \frac{1}{2}$ depending on the dimension d (equal to $1/2$ only in dimension 2) such that if $p > p_c$ a.s. there exists a unique infinite *strong cluster* denoted by $\mathcal{S}^{\omega}$; i.e. an infinite set such that all its sets are strongly connected. Symmetrically, if $p < 1 - p_c$ there exists a unique infinite weak cluster.

For each ω we consider the energies

$$E_{\varepsilon}^{\omega}(u) = \frac{1}{8} \sum_{i,j \in \mathbf{D}_{\varepsilon}} \varepsilon^{d-1} \sigma_{ij}^{\omega} (u_i - u_j)^2 \tag{2}$$

defined on $u : \mathbf{D}_{\varepsilon} \to \{\pm 1\}$, where we use the notation $\mathbf{D}_{\varepsilon} = \frac{1}{\varepsilon}\mathbf{D} \cap \mathbb{Z}^d$, and $\mathbf{D}$ is a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary. The factor 8 is a normalization factor due to the fact that each bond is accounted for twice and $(u_i - u_j)^2 \in \{0, 4\}$. Each function $u : \mathbf{D}_{\varepsilon} \to \{\pm 1\}$ is identified with the piecewise-constant function such that $u(x) = u_i$ on each coordinate d-cube of centre εi and side length ε contained in $\mathbf{D}$ and e.g. 1 otherwise. In this way all u can be considered as functions in $L^1(\mathbf{D})$, and more precisely in $BV(\mathbf{D}; \{\pm 1\})$. With this identification in mind, we will describe the Γ-limit of E_{ε} with respect to the strong convergence in $L^1(\mathbf{D})$ for $p > p_c$ and with respect to the weak convergence in $L^1(\mathbf{D})$ for $p < p_c$. Note that L^1-compactness for $p > p_c$ does not follow from the boundedness of the energy only, but can nevertheless be assumed to hold on optimal sequences, as shown in Sect. 3.

The case $p = 1$ corresponds to a *ferromagnetic spin system*, which can be described approximately as $\varepsilon \to 0$ by the anisotropic perimeter energy (see [3])

$$F^1(u) = \int_{\partial^*\{u=1\} \cap \mathbf{D}} \|\nu_u\|_1 \, d\mathcal{H}^{d-1}$$

defined on $u \in BV(\mathbf{D}; \{\pm 1\})$, here $\partial^*\{u = 1\}$ denotes the measure-theoretical reduced boundary of the set of finite perimeter $\{u = 1\}$ and ν_u its inner normal; see e.g. [11]. We denote

$$\|x\|_1 = \sum_{k=1}^{d} |x_k|.$$

In this approximation we identify each function $u : \mathbf{D}_{\varepsilon} \to \{\pm 1\}$ with the set $A = \bigcup \{\varepsilon i + \varepsilon Q : i \in \mathbf{D}_{\varepsilon} : u_i = 1\}$ or the function $u \in BV(\mathbf{D}; \{\pm 1\})$ given by $u = 2\chi_A - 1$, where Q denotes the coordinate (semi-open) unit square in $\mathbb{R}^d$ centered at 0.

3 The Supercritical Regime: $p > p_c$

We define

$$\mathcal{Z}^\omega = \left\{ i \in \mathbb{Z}^d\colon \exists j \in \mathbb{Z}^d \text{ such that } \hat{z}(i,j) \in \mathcal{S}^\omega \right\},$$

where $\mathcal{S}^\omega$ denotes the strong cluster, and

$$\mathcal{W} = \mathcal{W}^\omega = \bigcup \left\{ i + Q\colon\ i \in \mathcal{Z}^\omega \right\}.$$

We will use the following terminology:

- the *boundary* of a set $I \subset \mathbb{Z}^d$ is $\{i \in I\colon \exists j \in \mathbb{Z}^d \setminus I : |i - j| = 1\}$;
- the *interior* of a bounded set $I \subset \mathbb{Z}^d$ is the set of points i such that there is no unbounded path with starting point i (i.e., such that $i_0 = i$) not intersecting the boundary of I. Note that the interior of I may contain also points not in I;
- the *size* of a bounded subset $I \subset \mathbb{Z}^d$ is the cardinality of its interior.

Note that the definition of "interior" of a discrete set I given here (the reader will excuse the abuse of notation with the topological notion) corresponds to the complement of the infinite connected component of $\mathbb{Z}^d$ not containing I. Loosely speaking, it is the portion of lattice enclosed by the "external boundary" of I.

3.1 Coerciveness

We will show that our energies are equicoercive with respect to the strong L^1-convergence, which will derive from their equicoerciveness with respect to the weak*-convergence in $BV(\mathbf{D}; \{\pm 1\})$. More precisely, we will show that, a.s., given a sequence $\{u_\varepsilon\}$ with bounded energy we may construct another sequence $\{\tilde{u}_\varepsilon\}$, close to the first one on the strong cluster (i.e., such that the corresponding piece-wise constant extensions are close in the L^1-norm) such that $E^\omega_\varepsilon(\tilde{u}_\varepsilon) \le E^\omega_\varepsilon(u_\varepsilon)$ and $\{\tilde{u}_\varepsilon\}$ admits a converging subsequence in the weak*-convergence in $BV(\mathbf{D}; \{\pm 1\})$. This will be done for $p > 1 - p_c$. In that case we can adapt a method similar to the one used for surface energies on periodically perforated domains (see [30]): we may define a "relevant phase" $+1$ or -1 on the strong cluster; on each isolated component of the complement of the strong cluster we extend the function by a constant defined as the relevant phase having a majority on the boundary of the corresponding component. The delicate issue is here to show that the $\mathcal{H}^{d-1}$-measure of the additional part of $\partial\{\tilde{u}_\varepsilon = 1\}$ introduced by this process outside the strong cluster can be estimated by the energy.

We now consider the case $d \ge 2$, and suppose that $p > 1 - p_c$. Notice that in the 2D case $p > 1 - p_c$ coincides with $p > p_c$.

Lemma 3.1 *Let* $\mathbf{D}$ *be a bounded Lipschitz open set. For a set of* ω *of full probability, if* (u_ε) *is a sequence such that* $\sup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) < +\infty$, *then there exists a sequence* $(\tilde{u}_\varepsilon)$ *such that* $E^\omega_\varepsilon(\tilde{u}_\varepsilon) \le E^\omega_\varepsilon(u_\varepsilon)$,

$$\left\| (u_\varepsilon - \tilde{u}_\varepsilon)\chi_{\mathbf{D}\cap\varepsilon\mathcal{W}} \right\|_{L^1(\mathbf{D})} = o(1) \tag{3}$$

as $\varepsilon \to 0$, *and there are no connected components of the sets* $\{i : \tilde{u}_\varepsilon = 1\}$ *and* $\{i : \tilde{u}_\varepsilon = -1\}$ *with size not exceeding* $1/\varepsilon$.

Proof We extend each function as $u_\varepsilon = 1$ on $\mathbb{Z}^d \setminus \mathbf{D}_\varepsilon$.

We first consider all the connected components of the complement of $\mathcal{Z}^\omega$. If $u_\varepsilon = 1$ identically on the boundary of one such component we set $\tilde{u}_\varepsilon = 1$ on its interior. In the remaining cases, we set $\tilde{u}_\varepsilon = -1$. With this operation we do not change the value of u_ε on $\mathcal{Z}^\omega$ and we have $E^\omega_\varepsilon(\tilde{u}_\varepsilon) \leq E^\omega_\varepsilon(u_\varepsilon)$. We can therefore assume from the beginning that u_ε is constant on each such connected component. So, in what follows we identify $\tilde{u}_\varepsilon$ and u_ε.

We can now subdivide $\mathbb{Z}^d$ into connected components $(I^{\varepsilon,+}_m)_{m\in M^+_\varepsilon}$ and $(I^{\varepsilon,-}_m)_{m\in M^-_\varepsilon}$ defined as the maximal connected components where $u_\varepsilon = 1$ and $u_\varepsilon = -1$, respectively. Note that here we do not use the geometry of the cluster.

Note that we have

$$\sum_{i,j: i\in I^{\varepsilon,+}_m} \sigma^\omega_{ij}(u_{\varepsilon,i} - u_{\varepsilon,j})^2 \geq 1, \qquad \sum_{i,j: i\in I^{\varepsilon,-}_m} \sigma^\omega_{ij}(u_{\varepsilon,i} - u_{\varepsilon,j})^2 \geq 1 \tag{4}$$

for all m since otherwise we would have $\sigma^\omega_{ij} = 0$ identically on the boundary of such connected components; i.e., that the boundary is contained in the complement of $\mathcal{Z}^\omega$, which contradicts the construction above. We then have

$$\#M^+_\varepsilon \leq \frac{C}{\varepsilon^{d-1}}, \qquad \#M^-_\varepsilon \leq \frac{C}{\varepsilon^{d-1}}, \tag{5}$$

where $\#M^+_\varepsilon$ and $\#M^-_\varepsilon$ are the number of maximal connected components of the set where $u_\varepsilon = 1$ and $u_\varepsilon = -1$, respectively.

We fix $\delta > 0$ and consider a component $I^{\varepsilon,-}_m$ with interior of size not more than $\varepsilon^{-1+\delta}$. We denote by $M^-_\varepsilon(\delta) \subset M^-_\varepsilon$ the set of the corresponding indices m. If we identify each $I^{\varepsilon,-}_m$ with a subset of $\mathbb{R}^d$, as usual taking the union of the corresponding ε-squares, we estimate the measure of $I^{\varepsilon,-}_m$ by

$$\left|I^{\varepsilon,-}_m\right| \leq \varepsilon^d \cdot \varepsilon^{-1+\delta} = \varepsilon^{d-1+\delta}$$

The total volume of such components is then

$$\left|\bigcup\left\{I^{\varepsilon,-}_m : m \in M^-_\varepsilon(\delta)\right\}\right| \leq \frac{C}{\varepsilon^{d-1}}\varepsilon^{d-1+\delta} = C\varepsilon^\delta$$

by (5). We can then set $\tilde{u}_\varepsilon = 1$ on the interior of this sets. This change is compatible with (3) and decreases the energy. We may repeat the corresponding process with the components $I^{\varepsilon,+}_m$ with interior of size not exceeding $\varepsilon^{-1+\delta}$. □

By what just proved, up to substituting u_ε for $\tilde{u}_\varepsilon$ we then may suppose that there is no connected components of the sets $\{i : u_\varepsilon = 1\}$ and $\{i : u_\varepsilon = -1\}$ with size not exceeding $\varepsilon^{-1+\delta}$.

We consider now the components $I^{\varepsilon,-}_m$ with interior of size in the interval $(\varepsilon^{-1+\delta}, \varepsilon^{-1}]$. We denote by $N^-_\varepsilon(\delta) \subset M^-_\varepsilon$ the set of the corresponding indices m. In particular each their measure is greater than $\varepsilon^{d-1+\delta}$, so that their perimeter is then estimated as

$$\mathcal{H}^{d-1}\big(\partial I^{\varepsilon,-}_m\big) \geq C\varepsilon^{(d-1+\delta)(d-1)/d}.$$

Since the maximum size of a connected component with $\sigma^\omega_{\hat{z}} = -1$ is of order $|\log\varepsilon|$ (see e.g. [25]) then the number of $\hat{z}$ along the boundary of $I^{\varepsilon,-}_m$ such that $\sigma^\omega_{\hat{z}} = 1$ is at least

$$C\frac{1}{|\log\varepsilon|^2}\frac{1}{\varepsilon^{d-1}}\varepsilon^{(d-1+\delta)(d-1)/d} = C\frac{\varepsilon^{(-1+\delta)(d-1)/d}}{|\log\varepsilon|^2}.$$

We then deduce that the energy contribution of each such component is at least

$$\sum_{i,j:i\in I_m^{\varepsilon,-}} \varepsilon^{d-1}\sigma_{ij}^{\omega}(u_{\varepsilon,i}-u_{\varepsilon,j})^2 \geq C\varepsilon^{d-1}\frac{\varepsilon^{(-1+\delta)(d-1)/d}}{|\log\varepsilon|^2} = C\frac{\varepsilon^{(d-1+\delta)(d-1)/d}}{|\log\varepsilon|^2}. \tag{6}$$

In particular, by the boundedness of the energy, we deduce that

$$\# N_\varepsilon^-(\delta) \leq C|\log\varepsilon|^2\varepsilon^{-(d-1+\delta)(d-1)/d}.$$

The measure of each such $I_m^{\varepsilon,-}$ is at most ε^{d-1}, so that the total measure of the union of these components is

$$\left|\bigcup\left\{I_m^{\varepsilon,-} : m \in N_\varepsilon^-(\delta)\right\}\right| \leq C|\log\varepsilon|^2\varepsilon^{(d-1)(1-\delta)/d} = o(1).$$

We can therefore again change the value setting $\tilde{u}_\varepsilon = 1$ in each $I_m^{\varepsilon,-}$, and reason similarly for the analogous $I_m^{\varepsilon,+}$.

At the end of the process above we obtain a sequence $(\tilde{u}_\varepsilon) \in BV(\mathbf{D};\{\pm1\})$ with the desired properties.

Theorem 3.2 (Percolation animal) *Let $p > 1 - p_c$. Then almost surely there exist a deterministic positive constant α and $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that for all connected sets contained in a cube $\{\|x\|_1 \leq M/\varepsilon\}$ and of size larger than $\varepsilon^{-1/d}$ with $\varepsilon < \varepsilon_0$, the proportion of strong links (such that $\sigma_{ij}^{\omega} = 1$) in each such a set is at least α.*

Proof Denote $n = \lfloor\varepsilon^{-1/d}\rfloor$, and let $\mathcal{Z}^d$ be the lattice dual to $\mathbb{Z}^d$. Our aim is to prove that almost surely, for sufficiently large n, any connected subset of $[-Mn^d, Mn^d]^d \cap Z^d$ of size n contains at least μn strong edges with a non-random $\mu = \mu(p, M) > 0$.

We begin by proving the result for probabilities p close enough to 1. To this end we modify the notion of adjacent points. We say that two points $i, j \in \mathbb{Z}^d$ are L^∞-adjacent, if $\|i - j\|_{L^\infty} = 1$. The notion of L^∞-connectedness is introduced accordingly.

First we recall the estimate for the number of L^∞-connected sets of size n in $\mathbb{Z}^d$ which contain the origin. It reads (see [24], pp. 81–82)

$$\#\left\{A \subset \mathbb{Z}^d : |A| = n, 0 \in A\right\} \leq C_d^n, \tag{7}$$

where $|\cdot|$ stands for the number of vertices in a subset of $\mathbb{Z}^d$.

Consider the site-percolation model where the probability that any vertex of $\mathbb{Z}^d$ be open is equal to $\tilde{p}$. For this model the range of dependence in the L^∞ norm is equal to 1, that is, for any set $j_1, \dots j_N$ in $\mathbb{Z}^d$ such that $\|j_i - j_k\|_{L^\infty} > 1$, $i, k = 1, \dots, N$, $i \neq k$, the random variables characterizing the state of j_i are independent.

For a fixed L^∞-connected set A with $|A| = n$ and any $\mu \in (0, 1)$, the probability that A contains less than μn strong vertices admits the upper bound

$$\begin{aligned}\mathbf{P}\{A \text{ contains less than } \mu n \text{ strong edges}\} &\leq (1-\tilde{p})^{(1-\mu)3^{-d}n}\sum_{k=\lfloor(1-\mu)n\rfloor}^{n}\binom{k}{n}\\ &\leq (1-\tilde{p})^{(1-\mu)3^{-d}n}2^n.\end{aligned}$$

Denote by $\mathcal{G}_\mu(n)$ the event

$$\begin{aligned}\mathcal{G}_\mu(n) = \big\{&\text{there is an } L^\infty\text{-connected set } A \subset \left[-Mn^d, Mn^d\right]^d\\ &\text{of size } n \text{ that has at most } \mu n \text{ strong edges}\big\}.\end{aligned}$$

From the last two estimates we deduce the inequality

$$\mathbf{P}\{\mathcal{G}_\mu(n)\} \le \left(Mn^d\right)^d C_d^n 2^n (1-\tilde{p})^{(1-\mu)3^{-d}n}. \tag{8}$$

Therefore, there is $\tilde{p}_0 = \tilde{p}_0(\mu) < 1$ such that for all $\tilde{p} \in (\tilde{p}_0, 1)$ the inequality

$$\mathbf{P}\{\mathcal{G}_\mu(n)\} \le C(M)(1/2)^n \tag{9}$$

holds. With the help of the Borel-Cantelli lemma this yields

$$\mathbf{P}\left\{\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\mathcal{G}_\mu(i)\right\} = 0.$$

In particular, the desired statement of Theorem 3.2 follows for $\tilde{p} \in (\tilde{p}_0, 1)$. □

We are now going to use the renormalization technique. Let $p > 1 - p_c$. Consider the set of cubes $Q_y^N = Ny + [0, N]^d$ and $\mathcal{Q}_{y,\delta}^N = Ny + [-\delta N, (1+\delta)N]^d$ with $y \in \mathbb{Z}^d$, $\delta > 0$ and integer $N > 1$.

Proposition 3.3 *For any $p > 1 - p_c$, $\tilde{\alpha} > 0$, $\delta > 0$, and any $p_1 < 1$ there is $N_0 = N_0(p, p_1, \tilde{\alpha}) > 0$ such that for each $N > N_0$ and $y \in \mathbb{Z}^d$ it holds*

$$\mathbf{P}\{\textit{any connected subset } A \textit{ of } \mathcal{Q}_{y,\delta}^N \textit{ with } |A| \ge \tilde{\alpha}N \textit{ contains at least one strong edge}\} > p_1. \tag{10}$$

Proof The statement of proposition is a straightforward consequence of the exponential estimates for the size of a weak cluster in the case $1 - p < p_c$.

We proceed by applying the renormalization arguments and set $\delta = 1/4$ and $\tilde{\alpha} = \delta/2$. Given $p > 1 - p_c$, we choose $p_1 \in (\tilde{p}_0, 1)$ and $\mu > 0$ so that (9) holds, and then choose N such that (10) holds true. Assuming that n is an integer multiplier of N, we then partition the big cube $[-Mn^d, Mn^d]^d$ into the cubes Q_y^N, $y \in [-Mn^d/N, Mn^d/N]^d \cap \mathbb{Z}^n$, and consider the cubes $\mathcal{Q}_{y,\delta}^N$ for the same set of y.

Given a connected set $A \subset \mathbb{Z}^d$, we will say that a cube $\mathcal{Q}_{y,\delta}^N$ is good if it contains a connected subset of A of size at least N.

Let $A \subset [-Mn^d, Mn^d]^d$ be a connected set with $|A| \ge n$. Our aim is to build an L^∞-connected set $\tilde{\mathcal{A}}$ of y such that

$$|\tilde{\mathcal{A}}| \ge \nu n/N^d, \quad \nu > 0, \tag{11}$$

and for each $y \in \tilde{\mathcal{A}}$ the cube $\mathcal{Q}_{y,\delta}^N$ is good. To this end we choose first an arbitrary cube $Q_{y^0}^N$ having a nontrivial intersection with A and denote by x_0 a point of $\mathbb{Z}^d$ that belongs to this intersection. We set $\bar{Y}^0 = \{y \in \mathbb{Z}^d\colon \|y - y^0\|_{L^\infty} \le 1\}$ and $\mathcal{X}_0 = \bigcup_{y \in \bar{Y}^0} Q_y^N$.

Since A is connected, there is a path that belongs to $A \cap \mathcal{X}_0$ and connects x_0 with some point at the boundary of $\mathcal{X}_0$. Denote this point by x_1 and the index of the corresponding cube by y^1 so that $x_1 \in Q_{y^1}^N$. By construction the cube $\mathcal{Q}_{y^0,\delta}^N$ contains a connected component of A that has at least δN elements. Thus, $\mathcal{Q}_{y^0,\delta}^N$ is good.

At the second step we define the following sets: $\bar{Y}^1 = \{y \in \mathbb{Z}^d\colon \min_{i=0,1} \|y - y^i\|_{L^\infty} \le 1\}$ and $\mathcal{X}_1 = \bigcup_{y \in \bar{Y}^1} Q_y^N$. Due to the connectedness of A, there is a path that belongs to $A \cap \mathcal{X}_1$ and connects x_1 with some point at the boundary of $\mathcal{X}_1$. This point is denoted by x_2 and the index of the corresponding cube by y^2 so that $x_2 \in Q_{y^2}^N$. Again, by construction $\mathcal{Q}_{y^1,\delta}^N \setminus \mathcal{Q}_{y^0,\delta}^N$ contains a connected subset of A whose size is at least δN.

At the $(k+1)$-th step we define $\bar{Y}^k = \{y \in \mathbb{Z}^d\colon \min_{i=0,\dots,k} |y - y^i\|_{L^\infty} \le 1\}$ and $\mathcal{X}_k = \bigcup_{y \in \bar{Y}^k} Q_y^N$. Due to the connectedness of A, there is a path that belongs to $A \cap \mathcal{X}_k$ and connects x_k with some point at the boundary of $\mathcal{X}_k$. This point is denoted by x_{k+1} and the index of the corresponding cube by y^{k+1} so that $x_{k+1} \in Q^N_{y^{k+1}}$. Again, by construction $\mathcal{Q}^N_{y^k,\delta} \setminus \bigcup_{i=0,\dots,k-1} \mathcal{Q}^N_{y^i,\delta}$ contains a connected subset of A whose size is at least δN.

Since the cardinality of $\mathcal{X}_k$ does not exceed $k3^d N^d$, we can make at least $3^{-d}N^{-d}n$ steps. This yields an L^∞-connected set $\tilde{A}$ of good cubes, which contains at least $3^{-d}N^{-d}n$ cubes; here we identify a cube $\mathcal{Q}^N_{y,\delta}$ with its index y.

Let ζ^N_y be a random variable equal to 1 if any connected subset of $\mathcal{Q}^N_y$ whose size is at least δN contains a strong edge, and 0 otherwise. It is straightforward to check that for any set $y_1, \dots, y_K \in \mathbb{Z}^d$, such that $\|y_i - y_j\|_{L^\infty} \geq 2$, $i \neq j$, the random variables $\zeta^N_{y_1}, \dots, \zeta^N_{y_K}$ are independent.

Since every good cube contains a strong edge with probability greater than p_1, then for sufficiently large n the number of strong edges in the set A is at least $\frac{\mu n}{3^d N^d}$, as desired. □

Lemma 3.4 *For a set of ω of full probability, if* $\sup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) < +\infty$ *and all connected components of the sets* $\{i : u_\varepsilon = 1\}$ *and* $\{i : u_\varepsilon = -1\}$ *have size greater than* $1/\varepsilon$, *then* $\{u_\varepsilon = 1\}$ *has equi-bounded perimeter in* $\mathbf{D}$, *and in particular* (u_ε) *is pre-compact in the weak topology of* $BV(\mathbf{D}; \{\pm 1\})$.

Proof Each connected component of $\{i : u_\varepsilon = 1\}$ and $\{i : u_\varepsilon = -1\}$ has perimeter at least $\varepsilon^{-(d-1)/d}$. By Theorem 3.2 we then have

$$\mathcal{H}^{d-1}\big(\partial\{u_\varepsilon = 1\}\big) \leq \frac{C}{\alpha} E_\varepsilon(u_\varepsilon) + C\mathcal{H}^{d-1}(\partial \mathbf{D}),$$

which proves the desired statement. □

We can collect the previous lemmas in the following one.

Lemma 3.5 *Let* $\mathbf{D}$ *be a bounded Lipschitz open set. For a set of ω of full probability, if* (u_{ε_j}) *is a sequence such that* $\sup_j E^\omega_{\varepsilon_j}(u_{\varepsilon_j}) < +\infty$, *then there exists a function* $u \in BV(\mathbf{D}, \{\pm 1\})$ *and a subsequence, still denoted by* (u_{ε_j}), *such that*

$$\lim_j \big\| (u_{\varepsilon_j} - u)\chi_{\mathbf{D}\cap\varepsilon_j\mathcal{W}} \big\|_1 = 0. \tag{12}$$

Proof It suffices to apply Lemma 3.4 to the sequence $(\tilde{u}_{\varepsilon_j})$ obtained from Lemma 3.1. In this way we have $u \in BV(\mathbf{D}, \{\pm 1\})$ such that, up to subsequences, $\tilde{u}_{\varepsilon_j} \to u$ in $L^1(\mathbf{D})$. We then get

$$\lim_j \big\| (u_{\varepsilon_j} - u)\chi_{\mathbf{D}\cap(\varepsilon_j\mathcal{W})} \big\|_{L^1} \leq \lim_j \big\| (u_{\varepsilon_j} - \tilde{u}_{\varepsilon_j})\chi_{\mathbf{D}\cap(\varepsilon_j\mathcal{W})} \big\|_{L^1} + \lim_j \|\tilde{u}_{\varepsilon_j} - u\|_{L^1} = 0,$$

as desired. □

Note that by this last lemma, the functionals E_ε are equicoercive with respect to the convergence

$$u_\varepsilon \to u \quad \Longleftrightarrow \quad \lim_{\varepsilon\to 0} \big\| (u_\varepsilon - u)\chi_{\mathbf{D}\cap(\varepsilon\mathcal{W})} \big\|_{L^1} = 0, \tag{13}$$

which highlights that the relevant values for u_ε are those in $\varepsilon\mathcal{W}$. This convergence can be used equivalently in the statements of our results.

3.2 Definition of Surface Tension and Convergence of the Energies

For any vector $\nu \in \mathbb{R}^d$ and $s > 0$ we define R_s^ν, the *unit-basis parallelepiped with height* s, as follows: we choose a orthonormal basis $\nu^1, \dots, \nu^d$ with first vector $\nu^1 = \nu$, and set

$$R_s^\nu = \big\{x \in \mathbb{R}^d\colon\ \big|\langle x, \nu\rangle\big| \le s,\ \big|\langle x, \nu^k\rangle\big| \le 1 \text{ for } k = 2, \dots, d\big\}.$$

For all $T > 0$, $\delta > 0$, $x \in \mathbb{R}^d$ we define

$$\psi_s^\omega(x, T, \nu) = \min\left\{\frac{1}{8}\sum_{i,j} \varepsilon^{d-1}\sigma_{ij}^\omega(u_i - u_j)^2\colon\ i \text{ or } j \in x_T + T R_s^\nu\right\}, \tag{14}$$

where the infimum is taken over all $u : \mathbb{Z}^d \to \{\pm 1\}$ such that $u_i = \pm 1$ if $\pm\langle i - x, \nu\rangle \ge sT$ (this is the discrete analog of the condition that $u = \pm 1$ on the two opposite faces of $x_T + T R_s^\nu$ orthogonal to ν). Note that we take into account both interactions interior to the parallelepiped, when both i and $j \in x_T + T R_s^\nu$, and through its boundary, when only one of the two indices belongs to $x_T + T R_s^\nu$.

We then have the following result.

Lemma 3.6 (Cerf-Théret [20]) *Let $p > p_c$, $s > 0$ and let $\{x_T\}_{T>1}$ satisfy $\sup_T |x_T|/T < +\infty$. Then there exists a.s. and is deterministic the limit*

$$\psi_p(\nu) = \lim_{T\to+\infty} \frac{1}{T^{d-1}} \psi_s^\omega(x_T, T, \nu). \tag{15}$$

The limit is independent of $\{x_T\}$, s and ω. Moreover ψ_p defines a norm on $\mathbb{R}^d$.

Remark 3.7 (Two-dimensional statement) In two dimensions we may take into account that the minimal-interface problem defining $\psi_s^\omega(x, T, \nu)$ can be reduced to a "minimal length" problem since it is not restrictive to suppose that the set of bonds with $u_i \ne u_j$ is a single path, and hence the function ψ_p can be defined alternatively in a dual way as follows.

For $x, y \in \mathbb{Z}^2$ and $\omega \in \Sigma$ we denote

$$\psi^\omega(x, y) = \min\left\{\sum_{n=1}^K \sigma_{i_n i_{n-1}}^\omega\colon\ i_0 = x,\ i_K = y,\ K \in \mathbb{N}\right\}, \tag{16}$$

where the minimum is taken over all paths joining x and $y \in \mathbb{Z}^2$.

The following is the analog of Lemma 3.6.

Lemma 3.8 (See, for instance, [26]) *For any $\tau \in \mathbb{R}^2$ the following limit exists almost surely and does not depend on ω*

$$\psi_p(\tau) = \lim_m \frac{1}{m} \psi^\omega\big(0, \lfloor m\tau\rfloor\big), \tag{17}$$

where $\lfloor m\tau\rfloor_k = \lfloor m\tau_k\rfloor$ is the integer part of the k-th component of $m\tau$. Moreover, ψ_p defines a norm in $\mathbb{R}^2$ and, given any sequence of points $i_m \in \mathbb{Z}^2$ with $\sup_m |i_m|/m < +\infty$ we have

$$\psi_p(\tau) = \lim_m \frac{1}{m} \psi^\omega\big(i_m, i_m + \lfloor m\tau\rfloor\big).$$

Our main result is the following. For the definition and properties of Γ-convergence we refer to [12–14, 22].

Theorem 3.9 *Let* $\mathbf{D}$ *be a bounded Lipschitz open set and* $p > p_c$, *then* $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* E^ω_ε *with respect to the* $L^1(\mathbf{D})$*-convergence on* $BV(\mathbf{D}; \{\pm 1\})$, *it is deterministic, and is given by*

$$F_p(u) = \int_{\partial^*\{u=1\}\cap\mathbf{D}} \psi_p(\nu)\, d\mathcal{H}^{d-1} \tag{18}$$

for $u \in BV(\mathbf{D}; \{\pm 1\})$.

Proof We begin with the proof of the lower bound (liminf inequality), and fix a family $u_\varepsilon \to u$ in $L^1(\mathbf{D})$ with $u \in BV(\mathbf{D}; \{\pm 1\})$. We pass to a subsequence (not relabeled) such that

$$\lim_{\varepsilon\to 0} E^\omega_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} E^\omega_\varepsilon(u_\varepsilon).$$

For all ε we consider the set in the dual lattice $\varepsilon\mathcal{Z}$ of $\varepsilon\mathbb{Z}^d$ defined by

$$S_\varepsilon = \left\{\varepsilon k\colon\ k = \frac{i+j}{2},\ \varepsilon i, \varepsilon j \in \mathbf{D}_\varepsilon, |i-j| = 1, \tilde{u}_\varepsilon(\varepsilon i) = 1, \tilde{u}_\varepsilon(\varepsilon j) = -1\right\}$$

and the measure

$$\mu_\varepsilon = \sum_{\varepsilon k \in S_\varepsilon} \varepsilon \sigma^\omega_k \delta_{\varepsilon k}.$$

Note that $E^\omega_\varepsilon(u_\varepsilon) = \mu_\varepsilon(\mathbf{D})$ so that the family of measures μ_ε is equibounded. Hence, up to further subsequences we can assume that μ_ε converges weakly* to a finite measure μ.

Let $A = \{u = 1\}$ and $A_\varepsilon = \{u_\varepsilon = 1\}$. With fixed $h \in \mathbb{N}$ we can consider the collection $\mathcal{Q}_h$ of squares $Q^\nu_\rho(x)$ such that the following conditions are satisfied:

(i) $x \in \partial^* A$ and $\nu = \nu(x)$;
(ii) $|(Q^\nu_\rho(x) \cap A)\triangle \Pi^\nu(x)| \le \frac{1}{h}\rho^d$, where $\Pi^\nu(x) = \{y \in \mathbb{R}^d\colon\ \langle y - x, \nu\rangle \ge 0\}$;
(iii) $|\frac{\mu(Q^\nu_\rho(x))}{\rho^{d-1}} - \frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x)| \le \frac{1}{h}$;
(iv) $|\frac{1}{\rho^{d-1}} \int_{Q^\nu_\rho(x)\cap\partial^* A} \psi_p(\nu(y)) d\mathcal{H}^{d-1}(y) - \psi_p(\nu(x))| \le \frac{1}{h}$;
(v) $\mu(Q^\nu_\rho(x)) = \mu(\overline{Q^\nu_\rho(x)})$.

Note that for fixed $x \in \partial^* A$ and for ρ small enough (ii) is satisfied by the definition of reduced boundary (see [11]), (iii) follows from the Besicovitch Derivation Theorem provided that

$$\frac{d\mu}{d\mathcal{H}^{d-1}\llcorner\partial^* A}(x) < +\infty;$$

(iv) holds by the same reason, and (v) is satisfied for almost all $\rho > 0$ since μ is a finite measure. We deduce that $\mathcal{Q}_h$ is a fine covering of the set

$$\partial^* A_\mu = \left\{x \in \partial^* A\colon\ \frac{d\mu}{d\mathcal{H}^{d-1}\llcorner\partial^* A}(x) < +\infty\right\},$$

so that (by Morse's lemma, see [28]) there exists a countable family of disjoint closed cubes $\{\overline{Q^{\nu_j}_{\rho_j}(x_j)}\}$ still covering $\partial^* A_\mu$. Note that we have

$$\mathcal{H}^{d-1}\big(\partial^* A \setminus \partial^* A_\mu\big) = 0$$

since $\mu(\partial^* A) < +\infty$.

We now fix one of such cubes $Q^\nu_\rho(x)$. Since $|A_\varepsilon \triangle A| \to 0$, for ε small enough we have

$$\big|\big(Q^\nu_\rho(x)\cap A_\varepsilon\big)\triangle \Pi^\nu(x)\big| \le \frac{2}{h}\rho^d \tag{19}$$

by (ii) above.

For simplicity of notation we can suppose that $\nu = e_2$ and $x = 0$. With fixed $\delta < 1/2$, from (19) we have in particular

$$\left|\big(\big(Q^\nu_\rho(x)\cap A_\varepsilon\big)\triangle \Pi^\nu(x)\big)\cap\left\{y\colon\, \rho\frac{\delta}{2}\le \operatorname{dist}\big(y,\partial Q^\nu_\rho(x)\big)\le\rho\delta\right\}\right| \le \frac{2}{h}\rho^d. \tag{20}$$

We deduce that there exists

$$t\in\left[\frac{\rho\delta}{2},\rho\delta\right]$$

such that

$$\mathcal{H}^{d-1}\big(\big(\big(Q^\nu_\rho(x)\cap A_\varepsilon\big)\triangle \Pi^\nu(x)\big)\cap\big\{y\colon\, \operatorname{dist}\big(y,\partial Q^\nu_\rho(x)\big)=t\big\}\big)\le\frac{4}{h\delta}\rho^{d-1}. \tag{21}$$

We can then define the subset $A^1_\varepsilon\subset Q^\nu_\rho(x)$ by

$$A^1_\varepsilon=\begin{cases} A_\varepsilon & \text{on } Q^\nu_{\rho-t}(x)\\ \Pi^\nu(x) & \text{otherwise.}\end{cases} \tag{22}$$

In this way the set A^1_ε has the same trace as $\Pi^\nu(x)$ on $\partial Q^\nu_\rho(x)$ and

$$\mathcal{H}^{d-1}\big(\big(\partial A^1_\varepsilon\setminus\partial A_\varepsilon\big)\cap Q^\nu_\rho(x)\big)\le\frac{4}{h\delta}\rho^{d-1}+\frac{\delta}{2}\rho^{d-1}. \tag{23}$$

Since $Q^\nu_\rho(x)=x+\rho R^\nu_1$ in the notation of Lemma 3.6 we can use as test function (the discretization of) $-1+2\chi_{A^1_\varepsilon}$ to estimate function $\psi^\omega_s(x_T,T,\nu)$ in (14) with $T=\rho/\varepsilon$, $s=1$ and $x_T=x/\varepsilon$. By estimate (23) we have

$$\mu_\varepsilon\big(Q^\nu_\rho(x)\big)\ge\rho^{d-1}\psi^\omega_s(x_T,T,\nu)-\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\rho^{d-1}$$

Taking the limit as $T\to+\infty$, by Lemma 3.6 we get

$$\liminf_{\varepsilon\to0}\mu_\varepsilon\big(Q^\nu_\rho(x)\big)\ge\rho^{d-1}\psi_p(\nu)-\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\rho^{d-1}$$

By (iv) above we then have

$$\liminf_{\varepsilon\to0}\mu_\varepsilon\big(Q^\nu_\rho(x)\big)\ge\int_{Q^\nu_\rho(x)\cap\partial^*A}\psi_p\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-\frac{1}{h}\rho^{d-1}-\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\rho^{d-1},$$

and we finally deduce that

$$\begin{aligned}\liminf_{\varepsilon\to0}\mu_\varepsilon(\mathbf{D})&\ge\sum_j\liminf_{\varepsilon\to0}\mu_\varepsilon\big(Q^{\nu_j}_{\rho_j}(x_j)\big)\\ &\ge\sum_j\int_{Q^{\nu_j}_{\rho_j}(x_j)\cap\partial^*A}\psi_p\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-C\left(\frac{1}{h}+\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\right)\\ &=\int_{\mathbf{D}\cap\partial^*A}\psi_p\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-C\left(\frac{1}{h}+\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\right),\end{aligned}$$

which gives the liminf inequality by the arbitrariness of h and δ.

The construction of a recovery sequence giving the upper bound can be performed just for polyhedral sets, since they are dense in energy in the class of sets of finite perimeter. We only give the construction when the set is of the form $\Pi^\nu(x)\cap\mathbf{D}$ since it is easily generalized to each face of a polyhedral boundary. It is no restriction to suppose that

$$\mathcal{H}^{d-1}\big(\partial\mathbf{D}\cap\partial\Pi^\nu\big)=0. \tag{24}$$

Note that, by Lemma 14, for fixed M and s the limit

$$\lim_{T\to+\infty}\frac{1}{T^{d-1}}\psi_s^\omega(x_T,T,\nu)$$

is uniform on families such that $|x_T|\le MT$. We fix $\eta>0$ and $\delta>0$. We set

$$M=\frac{1}{\delta}\sup\big\{|y|:\ y\in\mathbf{D}\big\},$$

and consider T_M^η large enough so that

$$\left|\frac{1}{T^{d-1}}\psi_\eta^\omega(y,T,\nu)-\psi_p(\nu)\right|\le\eta \tag{25}$$

for all $T\ge T_M^\eta$ and $|y|\le\frac{1}{\delta}TM$.

We fix $s>0$ and consider a family $\{x_k^\eta\}$ of points in $\Pi^\nu(x)$ such that $x_k^\eta+\delta R_s^\nu$ are mutually disjoint, are contained in $\mathbf{D}$ and

$$\mathcal{H}^{d-1}\left(\left(\Pi^\nu(x)\setminus\bigcup_k\big(x_k^\eta+\delta R_s^\nu\big)\right)\cap\mathbf{D}\right)=O(\delta)$$

For each k we choose a function u_k minimizing $\psi_s^\omega(\frac{1}{\varepsilon}x_k^\eta,\frac{\delta}{\varepsilon},\nu)$. Taking $T=\frac{\delta}{\varepsilon}$, note that

$$\left|\frac{1}{\varepsilon}x_k^\eta\right|\le\frac{1}{\delta}TM,$$

so by (25) we have

$$\left|\frac{\varepsilon^{d-1}}{\delta^{d-1}}\psi_s^\omega\left(\frac{1}{\varepsilon}x_k^\eta,\frac{\delta}{\varepsilon},\nu\right)-\psi_p(\nu)\right|\le\eta \tag{26}$$

for ε small enough.

We then define the function u_ε by setting

$$(u_\varepsilon)_i=\begin{cases}(u_k)_i & \text{if } \varepsilon i\in x_k^\eta+\delta R_s^\nu\\ 2\chi_{\Pi^\nu(x)}(\varepsilon i)-1 & \text{otherwise.}\end{cases}$$

We can estimate the energy of u_ε by summing up the contributions due to the rectangles $x_k^\eta+\delta R_s^\nu$ (except possibly on their lateral boundaries, i.e., those with normal orthogonal to ν), the lateral boundaries themselves (outside which u_ε can be not optimal for the problems defining ψ_s^ω), and the interface corresponding to $\Pi^\nu(x)$. We then have

$$\begin{aligned}\limsup_{\varepsilon\to0}E_\varepsilon^\omega(u_\varepsilon)&\le\lim_{\varepsilon\to0}\sum_k\varepsilon^{d-1}\psi_s^\omega\left(\frac{1}{\varepsilon}x_k^\eta,\frac{\delta}{\varepsilon},\nu\right)+C\delta s+CO(\delta)\\&\le\lim_{\varepsilon\to0}\sum_k\delta^{d-1}\big(\psi_p(\nu)+\eta\big)+C\delta s+CO(\delta)\\&=\psi_p(\nu)\mathcal{H}^{d-1}\big(\mathbf{D}\cap\partial\Pi^\nu(x)\big)+C\big(\eta+\delta s+O(\delta)\big).\end{aligned}$$

Note that by Lemma 3.4 we can suppose that u_ε converge to some u^δ weakly* in $BV(\mathbf{D})$ and strongly in $L^1(\mathbf{D})$. This u^δ coincides with $2\chi_{\Pi^\nu(x)} - 1$ at a distance larger than δs from $\partial\Pi^\nu(x)$.

We have

$$
\begin{aligned}
F''_\omega\big(u^\delta\big) := \Gamma\text{-}\limsup_{\varepsilon\to 0^+} E^\omega_\varepsilon\big(u^\delta\big) &\le \limsup_{\varepsilon\to 0^+} E^\omega_\varepsilon(u_\varepsilon) \\
&\le \psi_p(\nu)\mathcal{H}^{d-1}\big(\mathbf{D}\cap\partial\Pi^\nu(x)\big) + O(\delta)
\end{aligned}
$$

by the arbitrariness of η and s. Since $u^\delta \to u := 2\chi_{\Pi^\nu(x)} - 1$, by the lower semicontinuity of the functional F''_ω we deduce then that

$$
F''_\omega(u) \le \liminf_{\delta\to 0^+} F''_\omega\big(u^\delta\big) \le \psi_p(\nu)\mathcal{H}^{d-1}\big(\partial\Pi^\nu \cap \overline{\mathbf{D}}\big)
$$

Eventually, we obtain the desired inequality recalling that $\mathcal{H}^{d-1}(\overline{\mathbf{D}} \cap \partial\Pi^\nu) = \mathcal{H}^{d-1}(\mathbf{D} \cap \partial\Pi^\nu)$ by (24). □

4 The Subcritical Regime: $p \le p_c$

In this regime the surface tension is 0 as a consequence of the absence of an infinite strong cluster. Therefore the Γ-limit is trivial, and the topology with respect to which it can be computed is that of the weak $L^1(\mathbf{D})$-convergence.

Theorem 4.1 *Let* $\mathbf{D}$ *be a bounded Lipschitz set and* $p < p_c$. *Then* $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* E^ω_ε *with respect to the weak* L^1 *convergence and it coincides with the functional*

$$
F_0(u) = \begin{cases} 0 & \text{if } \|u\|_\infty \le 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{27}
$$

Proof Since the constraint $\|u\|_\infty \le 1$ is weakly closed in $L^1(\mathbf{D})$ and the domain of the energies E^ω_ε is composed of functions with $\|u\|_\infty = 1$ then we immediately get that $F(u) = +\infty$ if $\|u\|_\infty > 1$.

It suffices then to prove the limsup inequality if $\|u\|_\infty \le 1$. The proof follows exactly as in Theorem 3.9 since $BV(\mathbf{D}; \{\pm 1\})$ is weakly dense in the set $\{u \in L^1(\mathbf{D}) : \|u\|_\infty \le 1\}$. □

In the case $d = 2$ we can state a more precise result, highlighting that if $p < 1/2$ the Γ-limit degenerates at all orders.

Theorem 4.2 *Let* $\mathbf{D}$ *be a bounded Lipschitz set.*

(i) (critical regime) *if* $p = 1/2$ *then* $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* E^ω_ε *with respect to the weak* L^1 *convergence. The limit functional is given by* F_0 *above.*
(ii) (subcritical regime) *if* $p < 1/2$, *then for all choices of scaling factors* $C_\varepsilon > 0$ $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* $C_\varepsilon E^\omega_\varepsilon$ *with respect to the weak* L^1 *convergence and it coincides with the functional* F_0 *above.*

Proof (i) By the lower-semicontinuity of the Γ-limsup it suffices to check that

$$
F''(u) := \Gamma\text{-}\limsup_{\varepsilon\to 0} E^\omega_\varepsilon(u) = 0
$$

for an L^1-strongly dense set of functions in $BV(\mathbf{D};\{\pm 1\})$ since the latter is weakly dense in the unit ball of L^∞. This immediately follows by the construction of the limsup inequality in the previous section, after remarking that $\psi_{1/2}=0$ (see [26]);

(ii) In this case, by the arbitrariness of C_ε we have to show that for all u in a dense set of functions in $BV(\mathbf{D};\{\pm 1\})$ there exists a sequence $u_\varepsilon \rightharpoonup u$ in $L^1(\mathbf{D})$ such that $E^\omega_\varepsilon(u_\varepsilon)=0$ for all ε. To this end we can use arguments similar to those used for the proof of the Γ-limsup inequality in the previous section.

As remarked therein it is enough to compute the Γ-limsup for $u=2\chi_{\Pi^\nu}-1$, where $\Pi^\nu=\Pi^\nu(x)$ in the notation. We fix $\eta>0$ and set $K^\eta_\varepsilon=\lfloor\frac{\eta}{S\varepsilon}\rfloor$. We also consider $M>0$ large enough so that $\mathbf{D}\subset\subset Q^\nu_M(0)$. We consider a path γ_ε in the weak cluster of the dual lattice $\mathcal{Z}$ contained in the strip $\{x:|\langle x,\nu\rangle|\le\eta/\varepsilon\}$ and with the two endpoints lying at distance at most 2ε from the two sides $\{x:\langle x,\nu^\perp\rangle=\pm M/2\}$. The existence of such a path in the subcritical regime is well known (see [25]). Note that, after identifying it with a curve in $\mathbb{R}^2$, for ε small enough γ_ε disconnects $\frac{1}{\varepsilon}\mathbf{D}$. We can therefore consider $\mathbf{D}^+_\varepsilon$ the maximal connected component of $\frac{1}{\varepsilon}\mathbf{D}\setminus\gamma_\varepsilon$ containing $\mathbf{D}\cup\{\langle x,\nu\rangle\ge\eta/\varepsilon\}$, and define

$$u^\eta_\varepsilon(\varepsilon i)=\begin{cases}1 & \text{if } i\in\mathbb{Z}^2\cap\mathbf{D}^+_\varepsilon\\ -1 & \text{otherwise.}\end{cases}$$

Note that up to subsequences u^η_ε converges weakly in $L^1(\mathbf{D})$ to some u^η with $\|u_\eta-u\|_{L^1\mathbf{D}}=O(\eta)$. Since $E^\omega_\varepsilon(u^\eta_\varepsilon)=0$ we obtain the desired sequence by a diagonal argument. □

5 Curves with 'Dilute' Length

We define a path γ in $\mathbf{D}_\varepsilon$ as an array of points

$$\varepsilon i_0,\varepsilon i_1,\dots,\varepsilon i_{N-1},\varepsilon i_N\in\mathbf{D}_\varepsilon,\qquad N\in\mathbb{N},$$

such that

$$|i_n-i_{n-1}|=1.$$

Note that self-intersections are allowed by this definition. Each such path can be identified by the piecewise-affine continuous curve $\gamma:[0,\varepsilon N]\to\mathbb{R}^d$ satisfying $\gamma(\varepsilon n)=\varepsilon i_n$ for $n=0,1,\dots,N$, parameterized by arc length. We say that a path γ joins x to y if $\gamma(0)=\varepsilon i_0=x$ and $\gamma(\varepsilon N)=\varepsilon i_N=y$.

The energy of a path γ in $\mathbf{D}_\varepsilon$ is

$$F^\omega_\varepsilon(\gamma)=\sum_{n=1}^N\varepsilon\sigma^\omega_{i_ni_{n-1}},\tag{28}$$

with the same σ^ω_{ij} as in the previous sections. In order to study the behavior of such energies we extend each path to $\gamma(t)=\gamma(0)$ if $t<0$ and $\gamma(t)=\gamma(\varepsilon N)$ if $t>\varepsilon N$, so that we may define the convergence $\gamma_\varepsilon\to\gamma$ as the L^∞_{loc}-convergence of such extended curves.

When σ^ω_{ij} satisfy

$$0<\alpha\le\sigma^\omega_{ij}\le\beta<+\infty\tag{29}$$

the homogenization of such energies has been studied in [16], remarking first that F^ω_ε are L^∞-equi-coercive, in the sense that if $F^\omega_\varepsilon(\gamma_\varepsilon)\le C<+\infty$, and γ_ε are parameterized on $\{0,\dots,N_\varepsilon\}$ then $\varepsilon N_\varepsilon$ is bounded, so that $(\gamma_\varepsilon-\gamma_\varepsilon(0))$ is bounded in L^∞. In particular, up to subsequences $\varepsilon N_\varepsilon\to L$; the Γ-limit is almost surely given by

$$F(\gamma)=\int_0^L \psi_p(\gamma')\,dt, \tag{30}$$

where for $\|\tau\|_1<1$ the energy density $\psi_p(\tau)=\psi_p^\omega(\tau)$ is a.s. independent of ω and defined as the first-passage percolation time constant defined by

$$\psi_p^\omega(\tau)=\lim_m \frac{1}{m}\inf\left\{\sum_{n=1}^m \sigma^\omega_{i_n i_{n-1}}:\ i_0=0,\ i_m=\lfloor m\tau\rfloor\right\}, \tag{31}$$

where $\lfloor m\tau\rfloor$ denotes the vector each component of which is the integer part of the corresponding component of $m\tau$, extended by continuity to $\|\tau\|_1=1$, while we set $\psi_p^\omega(\tau)=+\infty$ if $\|\tau\|_1>1$.

In the dilute case the system is not elliptic and the energies E_ε^ω are not a priori L^∞ equicoercive; i.e., we may have $L=+\infty$. The E_ε^ω are trivially equicoercive with respect to the $W^{1,\infty}_{\rm loc}(0,+\infty)$ topology, and their limit can be described from the results in [16].

Remark 5.1 For all $0\le p<1$ we have a.s. $\psi_p^\omega(0)=0$. Indeed it suffices to remark that for fixed ω we can choose i_ω, i'_ω with $\|i_\omega-i'_\omega\|=1$ and $\sigma^\omega_{i_\omega i'_\omega}=0$, and for m large enough choose a path in the definition of $\psi_p(0)$ with only a finite number (independent of m) of pairs $\{i_n,i_{n-1}\}$ not equal to $\{i_\omega,i'_\omega\}$.

After this remark, we can state the convergence theorem, remarking that even though the Γ-limit is written as an integral on $(0,+\infty)$, it also comprises the case when $\varepsilon N_\varepsilon\to L$ after extending functions as constant for $t\ge L$.

Theorem 5.2 *Let* $0<p<1$; *then almost surely the energies* E_ε^ω *Γ-converge to the energy*

$$F_p(\gamma)=\int_0^{+\infty}\psi_p(\gamma')\,dt$$

defined on $W^{1,\infty}((0,+\infty);\mathbb{R}^d)$.

Proof (i) We first check the liminf inequality. We will reduce to the Γ-convergence result of [16] with $\tilde\sigma^\omega_{ij}=\sigma^\omega_{ij}+1$, which then is an elliptic model. Note that correspondingly, we have energies $\tilde E_\varepsilon^\omega$ whose limit is described by $\tilde\psi_p(\tau)=\psi_p(\tau)+1$.

Let $\gamma_\varepsilon\to\gamma$ be given, with γ_ε parameterized on $[0,\varepsilon N_\varepsilon]$. It suffices to consider the case $\varepsilon N_\varepsilon\to+\infty$. We fix $L>0$ and $\tilde N_\varepsilon$ such that $\varepsilon\tilde N_\varepsilon\to L$, and we consider the paths γ_ε^L being the restriction of γ_ε to $[0,\varepsilon N_\varepsilon]$. By [16] we then have

$$\begin{aligned}\liminf_{\varepsilon\to0}E_\varepsilon^\omega(\gamma_\varepsilon)&\ge\liminf_{\varepsilon\to0}\tilde E_\varepsilon^\omega(\gamma_\varepsilon^L)-L\\&\ge\int_0^L\tilde\psi_p(\gamma')\,dt-L=\int_0^L\psi_p(\gamma')\,dt.\end{aligned}$$

By letting $L\to+\infty$ we then obtain the desired lower bound. Note that if $\varepsilon N_\varepsilon$ is bounded then it is not restrictive to suppose that $\varepsilon N_\varepsilon\to L$ and the argument above keeps working without the passage to the limit as $L\to+\infty$.

(ii) We now prove the limsup inequality. Again we can use the elliptic result in [16].

Given γ such that $F_p(\gamma)<+\infty$, and given $L>0$, we can find a recovery sequence γ_ε^L for $\tilde F_p(\gamma;L)=\int_0^L\tilde\psi_p(\gamma')\,dt$. After extending such γ_ε^L by a constant, we have $\gamma_\varepsilon^L\to\gamma^L$ where $\gamma^L=\gamma$ on $[0,L]$ and $\gamma^L(t)=\gamma(L)$ for $t>L$. Again, by [16] we have

$$\lim_{\varepsilon\to 0} E_\varepsilon^\omega\big(\gamma_\varepsilon^L\big) = \lim_{\varepsilon\to 0}\big(\tilde{E}_\varepsilon^\omega\big(\gamma_\varepsilon^L\big) - L\big) \le \int_0^L \tilde{\psi}_p\big(\gamma'\big)\,dt - L = \int_0^L \psi_p\big(\gamma'\big)\,dt,$$

so that

$$F_p''\big(\gamma^L\big) := \Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon^\omega\big(\gamma^L\big) \le \int_0^L \psi_p\big(\gamma'\big)\,dt.$$

Note that $\gamma^L \to \gamma$ in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$ and then by the lower semicontinuity of the Γ-limsup

$$F_p''(\gamma) \le \liminf_{L\to+\infty} F_p''\big(\gamma^L\big) \le \lim_{L\to+\infty}\int_0^L \psi_p\big(\gamma'\big)\,dt = \int_0^{+\infty} \psi_p\big(\gamma'\big)\,dt,$$

as desired. □

The statements below describe the structure of the limit functional F_p. As was shown in Theorem 5.2, the function $\psi_p^\omega(\tau)$ is deterministic: $\psi_p^\omega(\tau) = \psi_p(\tau)$. If the probability of a weak bond is such that an infinite weak cluster does not exist a.s. then we have the two propositions below that follow from Theorem 2.3 in [26].

Proposition 5.3 *Let $p > 1 - p_c$; then almost surely the function ψ_p defined above is deterministic and $\psi_p(\tau) \ge C_p|\tau|$ for some positive constant C_p.*

Proof To check the lower bound $\psi_p(\tau) \ge C_p|\tau|$ it suffices to remark that given a path $\gamma_m = \{i_0,\dots,i_m\}$ minimizing the functional $\{\sum_{n=1}^m \sigma^\omega_{i_n i_{n-1}} : i_0 = 0, i_m = \lfloor m\tau\rfloor\}$ we can find a non intersecting path in $\mathbb{Z}^d$ joining 0 and $\lfloor m\tau\rfloor$ contained in the image of γ_m, which then consists of at least $\|\lfloor m\tau\rfloor\|_1$ edges. For sufficiently large m then the number of edges k of this path such that $\sigma_k^\omega = 1$ is at least $C_p\|\lfloor m\tau\rfloor\|_1$, which implies the desired estimate. □

Proposition 5.4 *Let $p > 1 - p_c$, and let $\sup_\varepsilon E_\varepsilon^\omega(\gamma_\varepsilon) < +\infty$ with $\gamma_\varepsilon(0)$ equibounded. Then almost surely the sequence (γ_ε) is bounded in L^∞.*

Proof This is also a straightforward consequence of Theorem 2.3 in [26]. □

Finally, if the infinite weak cluster exists a.s. then the function ψ_p satisfies the following property.

Proposition 5.5 *Let $p < 1 - p_c$; then we have $\psi_p(\tau) = 0$ if $|\tau| \le \varphi_{1-p}(\tau/|\tau|)$, where φ_s is the asymptotic chemical distance as defined in* [15].

Proof It suffices to remark that by the properties of the chemical distance (see [23]) for such τ there exists a.s. a path from 0 to $\lfloor m\tau\rfloor$ contained in the weak cluster (up to a $o(m)$ number of nodes). □

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