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Γ-convergence and homogenization of functionals in Sobolev spaces with variable exponents

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Abstract

This paper is devoted to homogenization and minimization problems for variational functionals in the framework of Sobolev spaces with continuous variable exponents. We assume that the sequence of exponents converges in the uniform metric and that the Lagrangian has a periodic microstructure. Then under natural coerciveness assumptions we prove a Γ-convergence result and, as a consequence, the convergence of minimizers (solutions to the corresponding Euler equations).

Keywords: Γ-convergence; Homogenization; Variable exponent

1. Introduction

This paper is devoted to homogenization and Γ-convergence problems for variational functionals in the framework of Sobolev spaces with variable exponents. More precisely, we study the asymptotic behaviour of functionals

$$J^\varepsilon[u] = \int_\Omega \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right) \left|\nabla u(x)\right|^{p_\varepsilon(x)} dx + \int_\Omega \left(\frac{1}{p_\varepsilon(x)} b\left(\frac{x}{\varepsilon}\right) \left|u(x)\right|^{p_\varepsilon(x)} - f(x)u(x)\right) dx, \tag{1.1}$$

as $\varepsilon \to 0$; here $p_\varepsilon(x) > 1$ is a continuous function whose modulo of continuity satisfies some technical conditions described below in Section 2, $a(y)$ and $b(y)$ are periodic positive functions. Under the assumptions that $p_\varepsilon(x)$ converges in the uniform metric to a function $p_0(x)$, we prove that the family J^ε Γ-converges to a limit functional with a convex Lagrangian of $p_0(x)$-growth. This convex Lagrangian coincides with the effective Lagrangian of the periodic functional with frozen exponent $p_0(x)$.

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This Γ-convergence result allows us to study the corresponding variational problem and to prove the convergence of the minimums and of the minimizers.

In recent years functionals with variable exponents and the corresponding Sobolev spaces were widely studied in the mathematical literature, see [1–3,6,7,9,11,13,19]. In particular, the conditions under which C_0^∞ functions are dense in $W^{1,p(\cdot)}$ have been found. Also, Meyers estimates which are used a lot in the present work, have been obtained in [1,19]. Let us mention that such equations arise in many engineering disciplines, such as electrorheological fluids, non-Newtonian fluids with thermo-convective effects, and nonlinear Darcy flow of compressible fluids in heterogeneous porous media (see for instance [2]).

Γ-convergence and minimization problems for functionals with periodic and locally periodic rapidly oscillating Lagrangians of p-growth with a constant p are well understood by now, see for instance [4,5] and the bibliography therein.

The works [10,15–18] (see also [14]) focus on variational functionals with non-standard growth conditions. In particular, the homogenization and Γ-convergence problems for Lagrangians with variable rapidly oscillating exponents $p(x)$ were considered in [16,17]. It was shown that the energy minimums and the homogenized Lagrangians in the spaces $W^{1,r}$ might depend on the value of r (so-called Lavrentiev phenomenon). For example, such a behaviour can be observed for the Lagrangian $|\nabla u|^{p(x/\varepsilon)}$ with a periodic "chess-board" exponent $p(y)$ and a small parameter $\varepsilon > 0$.

Another interesting example of Lagrangian with rapidly oscillating exponent was considered in [10]. Namely, for functionals

$$\mathcal{J}_\varepsilon[u] = \int |\nabla u|^{p(x/\varepsilon)}\, dx$$

with a smooth periodic exponent $p(y)$ such that $p(x) > 1$, it was shown that, contrary to our case, the limit functional is bounded on a Sobolev–Orlicz space of functions with gradient in an L^α log-space where α is the fiber percolation level of $p(x)$.

Variational functionals with non-standard growth conditions have also been considered in [4]. Chapter 21 of this book focuses on the Γ-convergence of such functionals in L^p spaces, but with different conditions on p_ε than ours.

Regarding the technique we have utilized, it is worth to mention one trick used in the paper. In order to prove the Γ-liminf inequality we first replace the original exponent $p_\varepsilon(x)$ by a new one $\pi_\varepsilon(x) = \min(p_\varepsilon(x), p_0(x))$, and consider the corresponding family of auxiliary variational functionals. This family is equi-continuous in the space $W^{1,p_0(x)}$ and thus it is sufficient to prove the Γ-liminf inequality for the dense set of piecewise affine functions. This is done in Section 3.1. Then it is not difficult to show that the Γ-liminf functional for the auxiliary family estimates from below the Γ-liminf functional for the original problem.

The Γ-convergence results of the paper admit the following generalization. Let $\Psi(y, u, \vec{\xi})$ be a function defined for $y \in Y$, $u \in \mathbb{R}$ and $\vec{\xi} \in \mathbb{R}^n$ periodic in y, convex in $\vec{\xi}$ and satisfies the following conditions:

$$c_1|\vec{\xi}| \leqslant \Psi(y, u, \vec{\xi}) \leqslant c_2|\vec{\xi}| + |u| + C, \qquad \left|\Psi(y, u_1, \vec{\xi}) - \Psi(y, u_2, \vec{\xi})\right| \leqslant C|u_1 - u_2|,$$

with $0 < c_1 \leqslant c_2 < +\infty$, $C > 0$, and suppose that $p_\varepsilon(x)$ and $p_0(x)$ are the same as in (1.1). Then the above mentioned Γ-convergence results are also valid for functionals

$$\mathcal{J}^\varepsilon[u] = \int_\Omega \left(\Psi\left(\frac{x}{\varepsilon}, u(x), \nabla u(x)\right)\right)^{p_\varepsilon(x)} dx.$$

The paper is organized as follows. In Section 2 we state the problem and formulate the main result. This result is proved in three steps in Section 3. On the first step (Section 3.1) we derive the "lim inf"-inequality. Section 3.2 is devoted to obtaining the "lim sup"-inequality. Finally, in Section 3.3 we prove the convergence of minimizers.

2. Problem setup and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$) with a sufficiently smooth boundary, and denote $Y = (0,1)^n$. We assume that a family of continuous functions $p_\varepsilon = p_\varepsilon(x)$, $\varepsilon > 0$, is defined in $\overline{\Omega}$ and satisfies the following conditions:

(A1) *the functions p_ε are bounded from below and from above*:

$$1 < \mathsf{p}^- \leqslant p_\varepsilon^- \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^+ \leqslant \mathsf{p}^+ \quad \text{in } \overline{\Omega}; \tag{2.1}$$

(A2) *for any $x, y \in \Omega$ and any $\varepsilon > 0$, the inequality holds*

$$\left|p_\varepsilon(x) - p_\varepsilon(y)\right| \leqslant \omega_\varepsilon\left(|x-y|\right) \quad \textit{with} \ \varlimsup_{\tau\to 0} \omega_\varepsilon(\tau)\ln\left(\frac{1}{\tau}\right) < +\infty; \tag{2.2}$$

(A3) *the functions p_ε converge to a function p_0 uniformly in Ω, i.e.,*

$$\lim_{\varepsilon\to 0} \|p_\varepsilon - p_0\|_{C(\overline{\Omega})} = 0, \tag{2.3}$$

where the limit function p_0 satisfies the condition:

$$\left|p_0(x) - p_0(y)\right| \leqslant \omega_0\left(|x-y|\right) \quad \textit{with} \ \varlimsup_{\tau\to 0} \omega_0(\tau)\ln\left(\frac{1}{\tau}\right) < +\infty. \tag{2.4}$$

We also suppose that

(A4) *$a = a(y)$, $b = b(y)$ are Y-periodic measurable functions such that*

$$0 < a_0 \leqslant a(y) \leqslant a_1, \qquad 0 < b_0 \leqslant b(y) \leqslant b_1; \tag{2.5}$$

(A5) $f \in C(\Omega)$.

For notational convenience we set

$$a^\varepsilon_{p_\varepsilon}(x) = \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right), \qquad b^\varepsilon_{p_\varepsilon}(x) = \frac{1}{p_\varepsilon(x)} b\left(\frac{x}{\varepsilon}\right). \tag{2.6}$$

In what follows we refer to [7,9,11,13] for the properties of Sobolev spaces with variable exponents. Following [7,9,11,13], for any $\varepsilon > 0$, we introduce the Sobolev space $W^{1,p_\varepsilon(\cdot)}(\Omega)$ with a variable exponent p_ε defined by

$$W^{1,p_\varepsilon(\cdot)}(\Omega) = \left\{\phi \in L^{p_\varepsilon(\cdot)}(\Omega) \colon |\nabla\phi| \in L^{p_\varepsilon(\cdot)}(\Omega)\right\}. \tag{2.7}$$

Here by $L^{p_\varepsilon(\cdot)}(\Omega)$ we denote the space of measurable functions ϕ in Ω such that

$$A_{p_\varepsilon(\cdot)}(\phi) = \int_\Omega \left|\phi(x)\right|^{p_\varepsilon(x)}\,dx < +\infty. \tag{2.8}$$

This space equipped with the norm

$$\|\phi\|_{L^{p_\varepsilon(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \colon A_{p_\varepsilon(\cdot)}\left(\frac{\phi}{\lambda}\right) \leqslant 1\right\} \tag{2.9}$$

becomes a Banach space.

On the space $L^{p_\varepsilon(\cdot)}(\Omega)$ we define the functionals $J^\varepsilon : L^{p_\varepsilon(\cdot)}(\Omega) \to \mathbb{R}$:

$$J^\varepsilon[u] = \begin{cases} \int_\Omega \{a^\varepsilon_{p_\varepsilon}(x)|\nabla u|^{p_\varepsilon(x)} + b^\varepsilon_{p_\varepsilon}(x)|u|^{p_\varepsilon(x)} - f(x)u\}\,dx, & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega); \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.10}$$

We study the asymptotic behaviour of J^ε and their minimizers as $\varepsilon \to 0$. Our analysis relies on the Γ-convergence approach in Sobolev spaces with variable exponent. Notice that under assumptions (2.2), (2.4), the spaces $W^{1,p_0(\cdot)}(\Omega)$ and $W^{1,p_\varepsilon(\cdot)}(\Omega)$ are separable and reflexive. Moreover, $C_0^\infty(\Omega)$ is dense in these spaces. We also recall the definition of Γ-convergence (see, e.g., [4,5,8] and the bibliography therein). In our case this definition takes the following form.

Definition 2.1 *($\Gamma_{p_0(\cdot)}$-convergence).* The functionals $I^\varepsilon : L^{p_\varepsilon(\cdot)}(\Omega) \to \mathbb{R} \cup \{\infty\}$ are said to $\Gamma_{p_0(\cdot)}$-converge to a functional $I : L^{p_0(\cdot)}(\Omega) \to \mathbb{R} \cup \{\infty\}$ if

(a) (“*lim inf*”-*inequality*) for any $u \in L^{p_0(\cdot)}(\Omega)$ and any sequence $\{u^\varepsilon\} \subset L^{p_0(\cdot)}(\Omega)$ which converges to the function u strongly in the space $L^{p_0(\cdot)}(\Omega)$ we have:

$$\varliminf_{\varepsilon\to 0} I^\varepsilon\left[u^\varepsilon\right] \geqslant I[u];$$

(b) ("*lim sup*"-*inequality*) for any $u \in L^{p_0(\cdot)}(\Omega)$ there exists a sequence $\{w^\varepsilon\} \subset L^{p_0(\cdot)}(\Omega)$ such that $\{w^\varepsilon\}$ converges to the function $u(\cdot)$ strongly in the space $L^{p_0(\cdot)}(\Omega)$, and

$$\overline{\lim_{\varepsilon\to 0}} I^\varepsilon\big[w^\varepsilon\big] \leqslant I[u].$$

The main result of the paper is the following:

Theorem 2.2. *Let assumptions* (A1)–(A5) *be fulfilled. Then*

(C.1) *The functionals J^ε defined in* (2.10), *$\Gamma_{p_0(\cdot)}$-converge to the functional $J_{\text{hom}}: L^{p_0(\cdot)}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by*

$$J_{\text{hom}}[u] = \begin{cases} \int_\Omega \{\mathsf{T}(x, \nabla u) + \beta_p(x)|u|^{p_0(x)} - f(x)u\}\,dx, & \textit{if } u \in W^{1,p_0}(\Omega), \\ +\infty, & \textit{otherwise}, \end{cases} \tag{2.11}$$

where

$$\mathsf{T}(x, \vec{\xi}) = \inf\left\{ \frac{1}{p_0(x)} \int_Y a(y)\big|\nabla v(y) + \vec{\xi}\,\big|^{p_0(x)}\,dy\text{: } v \in W_\#^{1,p_0(x)}(Y)\right\} \tag{2.12}$$

and

$$\beta_p(x) = \frac{1}{p_0(x)} \int_Y b(y)\,dy. \tag{2.13}$$

(C.2) *Minimizers u^ε of the functionals J^ε converge to a minimizer u of the functional J_{hom} strongly in the space $L^{p_0(\cdot)}(\Omega)$.*

Throughout the paper C, C_1, C_2, etc. are generic constants independent of ε.

3. Proof of Theorem 2.2

The proof of the statement (C.1) of Theorem 2.2 is given below in Sections 3.1 and 3.2.

The statement (C.2) of Theorem 2.2 is then a consequence of the statement (C.1) of this theorem, as proved in Section 3.3.

3.1. Proof of the "lim inf"-inequality

The proof of the "*lim inf*"-*inequality* is performed in two main steps. First we introduce auxiliary functionals $\widetilde{J}^\varepsilon$ and prove the "lim inf"-inequality for these functionals. Then, at the second step, we show that the "lim inf"-inequality for the auxiliary functionals $\widetilde{J}^\varepsilon$ implies the "lim inf"-inequality for J^ε.

Step 1. *An auxiliary inequality.*

We denote

$$\pi_\varepsilon(x) = \min\big\{p_\varepsilon(x), p_0(x)\big\}$$

and on the space $L^{\pi_\varepsilon(\cdot)}(\Omega)$ define the functional $\widetilde{J}^\varepsilon : L^{\pi_\varepsilon(\cdot)}(\Omega) \to \mathbb{R}$:

$$\widetilde{J}^\varepsilon[u] = \begin{cases} \int_\Omega \{a^\varepsilon_{p_\varepsilon}(x)|\nabla u|^{\pi_\varepsilon(x)} + b^\varepsilon_{p_\varepsilon}(x)|u|^{p_\varepsilon(x)} - f(x)u\}\,dx, & \text{if } u \in W^{1,\pi_\varepsilon(\cdot)}(\Omega); \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.1}$$

In what follows we make use of Hölder's inequality for Sobolev spaces with variable exponents. Let $\phi \in L^{\mathsf{p}(\cdot)}(\Omega)$, $\psi \in L^{\mathsf{q}(\cdot)}(\Omega)$ with

$$\frac{1}{\mathsf{p}(x)} + \frac{1}{\mathsf{q}(x)} = 1, \quad 1 < \mathsf{p}^- \leqslant \mathsf{p}(x) \leqslant \mathsf{p}^+ < +\infty, \quad 1 < \mathsf{q}^- \leqslant \mathsf{q}(x) \leqslant \mathsf{q}^+ < +\infty,$$

then

$$\int_\Omega |\phi\,\psi|\,dx \leqslant 2\|\phi\|_{L^{\mathsf{p}(\cdot)}(\Omega)}\|\psi\|_{L^{\mathsf{q}(\cdot)}(\Omega)}. \tag{3.2}$$

We also make use of the following results from the theory of Sobolev spaces with a variable exponent $\mathsf{p} = \mathsf{p}(x)$. Let $\mathsf{p}(x)$ satisfy the inequality $1 < \mathsf{p}^- = \inf_\Omega \mathsf{p}(x) \leqslant \mathsf{p}(x) \leqslant \sup_\Omega \mathsf{p}(x) = \mathsf{p}^+ < +\infty$, and suppose that $|\mathsf{p}(x) - \mathsf{p}(y)| \leqslant \omega(|x-y|)$ for all $x, y \in \Omega$, with $\overline{\lim}_{\tau\to 0}\omega(\tau)\ln(\frac{1}{\tau}) < +\infty$. Then the following inequalities hold:

$$\begin{cases} \min\big(\|\phi\|^{\mathsf{p}^-}_{L^{\mathsf{p}(\cdot)}(\Omega)}, \|\phi\|^{\mathsf{p}^+}_{L^{\mathsf{p}(\cdot)}(\Omega)}\big) \leqslant A_{\mathsf{p}(\cdot)}(\phi) \leqslant \max\big(\|\phi\|^{\mathsf{p}^-}_{L^{\mathsf{p}(\cdot)}(\Omega)}, \|\phi\|^{\mathsf{p}^+}_{L^{p(\cdot)}(\Omega)}\big), \\ \min\big(A^{1/\mathsf{p}^-}_{\mathsf{p}(\cdot)}(\phi), A^{1/\mathsf{p}^+}_{\mathsf{p}(\cdot)}(\phi)\big) \leqslant \|\phi\|_{L^{\mathsf{p}(\cdot)}(\Omega)} \leqslant \max\big(A^{1/\mathsf{p}^-}_{\mathsf{p}(\cdot)}(\phi), A^{1/\mathsf{p}^+}_{\mathsf{p}(\cdot)}(\phi)\big), \end{cases} \tag{3.3}$$

where $A_{\mathsf{p}(\cdot)}(\phi)$ is defined in (2.8).

Without loss of generality, we assume that meas $\Omega > 1$. Then using (3.2), (3.3) and the definition of the functions $a^\varepsilon_{p_\varepsilon}, b^\varepsilon_{p_\varepsilon}$, one can obtain the following inequality:

$$\big|\widetilde{J}^\varepsilon[u] - \widetilde{J}^\varepsilon[v]\big| \leqslant \|u-v\|_{W^{1,\pi_\varepsilon(\cdot)}(\Omega)}\left(\int_\Omega \big(1+|u|+|\nabla u|+|v|+|\nabla v|\big)^{\pi_\varepsilon(x)}\,dx\right)^{1/q_0^-}, \tag{3.4}$$

which implies the uniform in ε continuity of the functional $\widetilde{J}^\varepsilon$ in the space $W^{1,\pi_\varepsilon(\cdot)}(\Omega)$.

Denote by $\mathcal{A}(\Omega)$ the class of piecewise affine functions defined in the domain Ω. First, we prove the inequality

$$\underline{\lim}_{\varepsilon\to 0}\ \widetilde{J}^\varepsilon\big[u^\varepsilon\big] \geqslant J_{\mathrm{hom}}[u] \quad \forall u \in \mathcal{A}(\Omega). \tag{3.5}$$

Consider an arbitrary function $u \in \mathcal{A}(\Omega)$. It can be represented as follows:

$$u(x) = A_j x + B_j \quad \text{in } \Omega_j,\ j = 1, 2, \ldots, M, \tag{3.6}$$

with Ω_j such that $\Omega = \bigcup_{j=1}^M \Omega_j$. Let $\{u^\varepsilon\}$ be a sequence which converges to u strongly in $L^{p_0(\cdot)}(\Omega)$ and such that $\widetilde{J}^\varepsilon[u^\varepsilon] \leqslant C$. In order to prove inequality (3.5) we introduce a covering of the domain Ω by cubes with edge length h ($0 < \varepsilon \ll h \ll 1$). Denote

$$K_h = \left[-\frac{h}{2}, \frac{h}{2}\right]^n, \qquad K_h^\alpha = K_h + h\alpha, \quad \alpha \in \mathbb{Z}^n,$$

and $\mathcal{K}_h = \{K_h^\alpha,\ \alpha \in \mathbb{Z}^n\}$. Without loss of generality, we assume that h is an integer multiplier of ε. Then we denote:

$$\Pi_h = \mathbb{R}^n \setminus \bigcup_{\alpha\in\mathbb{Z}^n}\left(h\alpha + \left[-\frac{h-h^2}{2}, \frac{h-h^2}{2}\right]^n\right). \tag{3.7}$$

Making a translation of the cubic structure $\mathcal{K}_h$ with a vector $\widetilde{y}$ and varying this vector over K_h, one can show that there exists $\widetilde{y}$ such that

$$\int_{(\Pi_h+\widetilde{y})\cap\Omega} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \leqslant C\,h \tag{3.8}$$

and

$$\int_{(\Pi_h+\widetilde{y})\cap\Omega} \big\{\big|\nabla u^\varepsilon\big|^{\pi_\varepsilon(x)} + \big|u^\varepsilon\big|^{\pi_\varepsilon(x)}\big\}\,dx \leqslant Ch. \tag{3.9}$$

We keep for the shifted cubes the same notation K_h^α and denote their centers by x^α, that is $x^\alpha = h\alpha + \widetilde{y}$. Notice that $\widetilde{y}$ might depend on ε. However, this dependence is of no importance for us, and we will not indicate it explicitly.

Using the notation

$$\mathsf{F}_{\pi_\varepsilon}(x, u, \nabla u) = a^\varepsilon_{p_\varepsilon}(x)|\nabla u|^{\pi_\varepsilon(x)} + b^\varepsilon_{p_\varepsilon}(x)|u|^{p_\varepsilon(x)} - f(x)u, \tag{3.10}$$

$\Omega_h = \{\bigcup_\alpha K_h^\alpha; K_h^\alpha \Subset \Omega\}$ and $\widetilde{\Omega}_h = \Omega \setminus \Omega_h$, one can represent $\widetilde{J}^\varepsilon[u^\varepsilon]$ as follows

$$\widetilde{J}^\varepsilon\big[u^\varepsilon\big] = \int\limits_{\Omega_h} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx + \int\limits_{\widetilde{\Omega}_h} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx. \tag{3.11}$$

It is clear that

$$\operatorname{meas} \widetilde{\Omega}_h = O(h) \quad \text{as } h \to 0. \tag{3.12}$$

Using (3.12) and the fact that $\{u^\varepsilon\}$ converges strongly to $u \in \mathcal{A}(\Omega)$ in $L^{p_0(\cdot)}(\Omega)$, one can easily show that the second term on the right-hand side of (3.11) admits the estimate

$$\lim_{h\to 0} \varliminf_{\varepsilon\to 0} \int\limits_{\widetilde{\Omega}_h} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant 0. \tag{3.13}$$

We proceed with the first term on the right-hand side of (3.11). Denoting

$$\Gamma_h = \left(\bigcup_{j=1}^{M} \partial\Omega_j\right) \cap \Omega$$

and letting Ω_Γ be the set of the cubes $K_h^\alpha \subset \Omega_h$ such that $K_h^\alpha \cap \Gamma_h \neq \emptyset$, we obtain

$$\operatorname{meas} \Omega_\Gamma = O(h), \quad \text{as } h \to 0, \tag{3.14}$$

and

$$\lim_{h\to 0} \varliminf_{\varepsilon\to 0} \int\limits_{\Omega_h} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant \lim_{h\to 0} \varliminf_{\varepsilon\to 0} \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx, \tag{3.15}$$

where

$$\Omega_{h,\Gamma} = \Omega_h \setminus \Omega_\Gamma. \tag{3.16}$$

Then from (3.13) and (3.15) we get:

$$\lim_{h\to 0} \varliminf_{\varepsilon\to 0} \int\limits_{\Omega} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant \lim_{h\to 0} \varliminf_{\varepsilon\to 0} \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx. \tag{3.17}$$

Consider now an arbitrary cube $K_h^\alpha \subset \Omega_{h,\Gamma}$. Denote by φ_α^h the cut-off function in K_h^α such that: $0 \leqslant \varphi_\alpha^h(x) \leqslant 1$; $\varphi_\alpha^h(x) = 0$ for $x \notin K_h^\alpha$; $\varphi_\alpha(x) = 1$ for $x \in K_h^\alpha \setminus (\Pi_h \cap K_h^\alpha)$; $|\nabla\varphi_\alpha(x)| \leqslant Ch^{-2}$. We extend φ_α^h to the whole space h-periodically, the notation φ^h stands for the extended function. Denote for brevity $\zeta^\varepsilon(x) = u^\varepsilon(x) - u(x)$ and rewrite u^ε as follows:

$$u^\varepsilon(x) = u(x) + \varphi^h(x)\zeta^\varepsilon(x) + \big(1 - \varphi^h(x)\big)\zeta^\varepsilon(x) \equiv u_h^\varepsilon(x) + \big(1 - \varphi^h(x)\big)\zeta^\varepsilon(x).$$

Then we have

$$\varlimsup_{\varepsilon\to 0} \left| \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx - \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u_h^\varepsilon, \nabla u_h^\varepsilon\big)\,dx \right| \leqslant Ch. \tag{3.18}$$

Indeed by (3.8) and (3.9), the following inequality holds

$$\begin{aligned}
&\left| \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx - \int\limits_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u_h^\varepsilon, \nabla u_h^\varepsilon\big)\,dx \right| \\
&\quad\leqslant \int\limits_{\Pi_h \cap \Omega_{h,\Gamma}} \big|\mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\big|\,dx + \int\limits_{\Pi_h \cap \Omega_{h,\Gamma}} \big|\mathsf{F}_{\pi_\varepsilon}\big(x, u_h^\varepsilon, \nabla u_h^\varepsilon\big)\big|\,dx \\
&\quad\leqslant Ch + C \int\limits_{\Pi_h \cap \Omega_{h,\Gamma}} \left\{1 + \big|\zeta^\varepsilon\big|^{\pi_\varepsilon(x)} + \big|\nabla\zeta^\varepsilon\big|^{\pi_\varepsilon(x)} + \frac{1}{h^{2\pi_\varepsilon(x)}}\big|\zeta^\varepsilon\big|^{\pi_\varepsilon(x)} + \big|\zeta^\varepsilon\big|\right\}dx.
\end{aligned}$$

By the definition of ζ^ε,

$$\lim_{\varepsilon\to 0}\int\limits_{\Pi_h\cap\Omega_{h,\Gamma}}\left\{\left|\zeta^\varepsilon\right|^{\pi_\varepsilon(x)}+\frac{1}{h^{2\pi_\varepsilon(x)}}\left|\zeta^\varepsilon\right|^{\pi_\varepsilon(x)}+\left|\zeta^\varepsilon\right|\right\}dx=0.$$

Then considering (3.9) we get

$$\int\limits_{\Pi_h\cap\Omega_{h,\Gamma}}\left\{1+\left|\nabla\zeta^\varepsilon\right|^{\pi_\varepsilon(x)}\right\}dx\leqslant Ch.$$

This yields (3.18).

Inequalities (3.17) and (3.18) imply that

$$\begin{aligned}\varliminf_{\varepsilon\to 0};\int\limits_{\Omega_{h,\Gamma}}\mathsf{F}_{\pi_\varepsilon}\left(x,u^\varepsilon,\nabla u^\varepsilon\right)dx&\geqslant\varliminf_{\varepsilon\to 0}\int\limits_{\Omega_{h,\Gamma}}\mathsf{F}_{\pi_\varepsilon}\left(x,u_h^\varepsilon,\nabla u_h^\varepsilon\right)dx-Ch\\&=\varliminf_{\varepsilon\to 0}\sum_\alpha\int\limits_{K_h^\alpha\subset\Omega_{h,\Gamma}}\mathsf{F}_{\pi_\varepsilon}\left(x,u_h^\varepsilon,\nabla u_h^\varepsilon\right)dx-Ch.\end{aligned}\tag{3.19}$$

Since u_h^ε converges strongly to the function u in the space $L^{\pi_\varepsilon(\cdot)}(\Omega)$ and $\nabla u=A_j$, then, for any $K_h^\alpha\subset\Omega_{h,\Gamma}$, we have:

$$\begin{aligned}\varliminf_{\varepsilon\to 0}\int\limits_{K_h^\alpha}\mathsf{F}_{\pi_\varepsilon}\left(x,u_h^\varepsilon,\nabla u_h^\varepsilon\right)dx&\geqslant\varliminf_{\varepsilon\to 0}\int\limits_{K_h^\alpha}a_{p_\varepsilon}^\varepsilon(x)\left|A_j+\nabla\left(\varphi^h\zeta^\varepsilon\right)\right|^{\pi_\varepsilon(x)}dx\\&\quad+\int\limits_{K_h^\alpha}\left\{\beta_p(x)|u|^{p_0(x)}-f(x)u\right\}dx.\end{aligned}\tag{3.20}$$

Denote

$$\pi_h^\alpha=\min_{y\in\bar K_h^\alpha}\pi_\varepsilon(y),\qquad p_h^\alpha=\max_{y\in\bar K_h^\alpha}p_\varepsilon(y).\tag{3.21}$$

Then the first term on the right-hand side of (3.20) satisfies the estimate

$$\varliminf_{\varepsilon\to 0}\int\limits_{K_h^\alpha}a_{p_\varepsilon}^\varepsilon(x)\left|A_j+\nabla\left(\varphi^h\zeta^\varepsilon\right)\right|^{\pi_\varepsilon(x)}dx\geqslant\varliminf_{\varepsilon\to 0}\int\limits_{K_h^\alpha}\frac{1}{p_h^\alpha}a\left(\frac{x}{\varepsilon}\right)\left|A_j+\nabla\left(\varphi^h\zeta^\varepsilon\right)\right|^{\pi_h^\alpha}dx.\tag{3.22}$$

Since $\varphi^h\zeta^\varepsilon=0$ on ∂K_h^α, the integral on the right-hand side of the last inequality can be estimated as follows

$$\begin{aligned}\int\limits_{K_h^\alpha}\frac{1}{p_h^\alpha}a\left(\frac{x}{\varepsilon}\right)\left|A_j+\nabla\left(\varphi^h\zeta^\varepsilon\right)\right|^{\pi_h^\alpha}dx&\geqslant\min_{v\in W_\#^{1,\pi_h^\alpha}(K_h^\alpha)}\int\limits_{K_h^\alpha}\frac{1}{p_h^\alpha}a\left(\frac{x}{\varepsilon}\right)|A_j+\nabla v|^{\pi_h^\alpha}dx\\&=\frac{h^n}{p_h^\alpha}\min_{v\in W_\#^{1,\pi_h^\alpha}(Y)}\int\limits_Y a(y)\left|A_j+\nabla v(y)\right|^{\pi_h^\alpha}dy,\end{aligned}$$

the last equality here follows from the convexity of the integrand. Denote

$$L(p,\vec\xi)=\min_{v\in W_\#^{1,p}(Y)}\int\limits_Y a(y)\left|\vec\xi+\nabla v(y)\right|^p dy.$$

Let us prove that this function is continuous in p.

Lemma 3.1. *For all $\vec{\xi} \in \mathbb{R}^n$ the function $L(p, \vec{\xi})$ is a continuous function of the variable p on the interval $(1, +\infty)$.*

Proof. The statement of the lemma is a direct consequence of the Meyers estimate (see, e.g., [12]). □

Combining now (3.19), (3.20), (3.22), the statement of Lemma 3.1 and the fact that π_h^α and p_h^α converge to $p_0(x)$ as $\varepsilon \to 0$ and then $h \to 0$, we conclude that

$$\varliminf_{\varepsilon\to 0} \int_\Omega \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant \varliminf_{\varepsilon\to 0} \int_{\Omega_{h,\Gamma}} \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx$$
$$\geqslant \int_{\Omega_{h,\Gamma}} \frac{1}{p_0(x)} L\big(p_0(x), \nabla u\big)\,dx + \int_{\Omega_{h,\Gamma}} \big\{\beta_p(x)|u|^{p_0(x)} - f(x)u\big\}\,dx - \kappa(h),$$

where $\kappa(h)$ tends to zero as $h \to 0$. Considering now the definition of $\mathsf{T}(x, \vec{\xi})$, estimates (3.12) and (3.14) and the properties of the limit Lagrangian, we obtain

$$\varliminf_{\varepsilon\to 0} \int_\Omega \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant J_{\mathrm{hom}}[u] - \kappa_1(h),$$

where $\kappa_1(h)$ tends to zero, as $h \to 0$. Since the left-hand side and the first term on the right-hand side of the last inequality do not depend on h, this yields the desired estimate (3.5).

By the definition of $\pi_\varepsilon(x)$ we have $\pi_\varepsilon(x) \leqslant p_0(x)$. Therefore, the family $\widetilde{J}^\varepsilon[u]$ is uniformly, in ε, continuous in $W^{1,p_0(\cdot)}(\Omega)$ topology, and the fact that (3.5) holds for all $u \in \mathcal{A}(\Omega)$ implies that (3.5) holds for all $u \in W^{1,p_0(\cdot)}(\Omega)$. This completes the proof of the "lim inf"-inequality for $\widetilde{J}^\varepsilon[u]$.

Finally, the "lim inf"-inequality for the functionals $J^\varepsilon[u]$ is a consequence of the following inequality: for any sequence u^ε that converges in $L^{p_0(\cdot)}(\Omega)$ it holds

$$\varliminf_{\varepsilon\to 0} \int_\Omega \mathsf{F}_{p_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx \geqslant \varliminf_{\varepsilon\to 0} \int_\Omega \mathsf{F}_{\pi_\varepsilon}\big(x, u^\varepsilon, \nabla u^\varepsilon\big)\,dx.$$

3.2. Proof of the "lim sup"-inequality

Since the limit functional $J_{\mathrm{hom}}[u]$ is continuous in $W^{1,p_0(x)}$ norm and piecewise affine functions are dense in $W^{1,p_0(x)}(\Omega)$, it suffices to prove the "lim sup"-inequality only for piecewise affine functions u. Let K_h^α be a periodic cubic grid in $\mathbb{R}^n$ with period $(h - h^2)$, and denote by $\{x^\alpha\}$ the periodic set of points being the centers of the cubes K_h^α. We associate with this covering a partition of unity $\{\varphi_\alpha\}$: $0 \leqslant \varphi_\alpha(x) \leqslant 1$; $\varphi_\alpha(x) = 0$ for $x \notin K_h^\alpha$; $\varphi_\alpha(x) = 1$ for $x \in K_h^\alpha \setminus \bigcup_{\beta\neq\alpha} K_h^\beta$; $\sum_\alpha \varphi_\alpha(x) = 1$ for $x \in \Omega$; $|\nabla\varphi_\alpha(x)| \leqslant Ch^{-2}$.

Denote by $U_\alpha(y)$ a minimizer in (2.12) with $x = x^\alpha$, $\vec{\xi} = \nabla u(x^\alpha)$, $U_\alpha(y)$ is a Y-periodic function. In the domain Ω we define

$$w_h^\varepsilon(x) = u(x) + \varepsilon \sum_\alpha U_\alpha\left(\frac{x}{\varepsilon}\right)\varphi_\alpha(x). \tag{3.23}$$

From the Meyers estimate (see [1]) it follows that there exist $\delta > 0$ and $h_0 > 0$ such that

$$\big\|w_h^\varepsilon\big\|_{W^{1,p_\varepsilon(\cdot)+\delta}(\Omega)} \leqslant C \tag{3.24}$$

for all $h < h_0$ and $\varepsilon < h^2$. Indeed, by [1] the functions $U_\alpha(y)$ admit the bound

$$\|U_\alpha\|_{W^{1,p_0(x^\alpha)+2\delta}(Y)} \leqslant C$$

for some $\delta > 0$. Since $p_0(x)$ is uniformly continuous in $\overline{\Omega}$, there is $h_0 > 0$ such that

$$\|U_\alpha\|_{W^{1,p^\alpha+\delta}(Y)} \leqslant C$$

for all $h < h_0$; here and later on $p^\alpha = \max_{K_h^\alpha} p_0(x)$. This implies the inequality

$$\left\|u(x) + \varepsilon \sum_\alpha U_\alpha\left(\frac{x}{\varepsilon}\right)\varphi_\alpha(x)\right\|_{W^{1,p_\varepsilon(x)+\delta}(\Omega)} \leqslant C\big(1 + \varepsilon h^{-2}\big).$$

Under the assumption that $\varepsilon < h^2$ the last estimate yields the bound (3.24).

Denote by Ω_h^- the set $\Omega_{h,\Gamma} \setminus \Pi_h$, where the sets $\Omega_{h,\Gamma}$ and Π_h have been defined in (3.16) and (3.7), respectively. Letting $h = \varepsilon^{1/4}$ and considering the structure of w_h^ε, we conclude that

$$\|w^\varepsilon_{\varepsilon^{1/4}}\|_{W^{1,p_\varepsilon(x)}(\Omega)} \leqslant C, \qquad \|w^\varepsilon_{\varepsilon^{1/4}}\|_{W^{1,p_0(x)+\delta}(\Omega)} \leqslant C, \tag{3.25}$$

$$\int\limits_{\Pi_{\varepsilon^{1/4}}} \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_\varepsilon(x)}\,dx \leqslant C\varepsilon^{1/4}, \tag{3.26}$$

$$\left|\int\limits_{\Omega^-_{\varepsilon^{1/4}}} \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_\varepsilon(x)}\,dx - \sum_\alpha \int\limits_{\Omega^-_{\varepsilon^{1/4}}\cap K^\alpha_{\varepsilon^{1/4}}} \frac{1}{p_0(x^\alpha)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_0(x^\alpha)}\,dx\right| \leqslant \kappa_2(\varepsilon), \tag{3.27}$$

where $\kappa_2(\varepsilon)$ converges to zero as $\varepsilon \to 0$. Since $\mathsf{T}(x,\vec{\xi})$ is a continuous function of x, we have

$$\left|\int\limits_{\Omega^-_{\varepsilon^{1/4}}} \mathsf{T}\big(x, \nabla u(x)\big)\,dx - \sum_\alpha \int\limits_{\Omega^-_{\varepsilon^{1/4}}\cap K^\alpha_{\varepsilon^{1/4}}} \mathsf{T}\big(x^\alpha, \nabla u(x)\big)\,dx\right| \leqslant \kappa_3(\varepsilon), \tag{3.28}$$

where $\kappa_3(\varepsilon)$ also converges to zero as $\varepsilon \to 0$. Taking into account the relation

$$\int\limits_{\Omega^-_{\varepsilon^{1/4}}\cap K^\alpha_{\varepsilon^{1/4}}} \frac{1}{p_0(x^\alpha)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_0(x^\alpha)}\,dx = \int\limits_{\Omega^-_{\varepsilon^{1/4}}\cap K^\alpha_{\varepsilon^{1/4}}} \mathsf{T}\big(x^\alpha, \nabla u(x)\big)\,dx$$

and combining (3.25)–(3.28), we deduce the estimate

$$\left|\int\limits_{\Omega^-_{\varepsilon^{1/4}}} \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_\varepsilon(x)}\,dx - \int\limits_{\Omega^-_{\varepsilon^{1/4}}} \mathsf{T}\big(x, \nabla u(x)\big)\,dx\right| \leqslant \kappa_4(\varepsilon)$$

with $\kappa_4(\varepsilon)$ vanishing as $\varepsilon \to 0$. It remains to notice that

$$\left|\int\limits_{\Omega\setminus\Omega^-_{\varepsilon^{1/4}}} \frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right)\left|\nabla w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_\varepsilon(x)}\,dx\right| + \left|\int\limits_{\Omega\setminus\Omega^-_{\varepsilon^{1/4}}} \mathsf{T}\big(x, \nabla u(x)\big)\,dx\right| \leqslant C\varepsilon^{1/4},$$

and that

$$\int\limits_\Omega \left(\frac{1}{p_\varepsilon(x)} b\left(\frac{x}{\varepsilon}\right)\left|w^\varepsilon_{\varepsilon^{1/4}}\right|^{p_\varepsilon(x)} - f(x) w^\varepsilon_{\varepsilon^{1/4}}\right)dx \underset{\varepsilon\to 0}{\longrightarrow} \int\limits_\Omega \big(\beta_p(x)|u|^{p_0(x)} - f(x)u\big)\,dx,$$

and the desired "lim sup"-inequality follows. This completes the proof of the statement (C.1) of Theorem 2.2.

3.3. Convergence of minimizers

Consider the variational problem corresponding to the functional J^ε:

$$J^\varepsilon\big[u^\varepsilon\big] \to \inf, \quad u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega). \tag{3.29}$$

It is known from [1–3,6] that, for each $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ of problem (3.12). From (3.3), (2.10), and the assumption (A5) on the function f, we have that

$$\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C \quad \text{and} \quad A_{p_\varepsilon(\cdot)}\big(u^\varepsilon\big) \leqslant C. \tag{3.30}$$

Now it follows from assumption (A3), (3.30), and Meyers' estimates in Sobolev spaces with variable exponents (see, e.g., [1]) that

$$\|u^\varepsilon\|_{W^{1,p_0(\cdot)}(\Omega)} \leqslant C. \tag{3.31}$$

This means that $\{u^\varepsilon\}$ is a weakly compact set in $W^{1,p_0(\cdot)}(\Omega)$. Hence, one can extract a subsequence $\{u^\varepsilon,\ \varepsilon = \varepsilon_k \to 0\}$ that converges to a function $u \in W^{1,p_0(\cdot)}(\Omega)$ weakly in $W^{1,p_0(\cdot)}(\Omega)$ and strongly in the space $L^{p_0(\cdot)}(\Omega)$. Let us show that $u = u(x)$ is a solution of the variational problem

$$J_{\mathrm{hom}}[u] \to \inf, \quad u \in W^{1,p_0(\cdot)}(\Omega). \tag{3.32}$$

First, since u^ε is the solution of the variational problem (3.29), then

$$J^\varepsilon\big[u^\varepsilon\big] \leqslant J^\varepsilon\big[w^\varepsilon\big]$$

for any sequence w^ε. For any $w \in W^{1,p_0(\cdot)}(\Omega)$, consider a sequence w^ε which converges to w, as $\varepsilon \to 0$, in $L^{p_0}(\Omega)$ and satisfies the Γ-limsup inequality. Taking into account also the "lim inf"-inequality, we get

$$J_{\mathrm{hom}}[u] \leqslant \varliminf_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[u^\varepsilon\big] \leqslant \varlimsup_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[u^\varepsilon\big] \leqslant \varlimsup_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[w^\varepsilon\big] \leqslant J_{\mathrm{hom}}[w]. \tag{3.33}$$

The last inequality implies that u is a solution to problem (3.32). Letting $w = u$ in the last relation yields

$$J_{\mathrm{hom}}[u] = \varliminf_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[u^\varepsilon\big] = \varlimsup_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[u^\varepsilon\big]. \tag{3.34}$$

Since the limit problem (3.32) has a unique solution, then the whole sequence $\{u^\varepsilon\}$ converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ and strongly in $L^{p_0(\cdot)}(\Omega)$ to the function u, as $\varepsilon \to 0$. This completes the proof of Theorem 2.2.

Remark 1. The above mentioned results may be proved in the same way for more general functionals

$$J^\varepsilon[u] = \int_\Omega \left(\frac{1}{p_\varepsilon(x)} a\left(\frac{x}{\varepsilon}\right) \big|\nabla u(x)\big|^{p_\varepsilon(x)} + \frac{1}{\sigma_\varepsilon(x)} b\left(\frac{x}{\varepsilon}\right) \big|u(x)\big|^{\sigma_\varepsilon(x)} - f(x)u(x) \right) dx,$$

with

$$1 < \sigma^{(-)} \leqslant \sigma_\varepsilon(x) \leqslant \sigma^{(+)} < \frac{np_\varepsilon(x)}{n - p_\varepsilon(x)}$$

satisfying the conditions (2.2)–(2.4).

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