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Homogenization of random convolution energies

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Abstract

We prove a homogenization theorem for a class of quadratic convolution energies with random coefficients. Under suitably stated hypotheses of ergodicity and stationarity, we prove that the Γ-limit of such energy is almost surely a deterministic quadratic Dirichlet-type integral functional, whose integrand can be characterized through an asymptotic formula. The proof of this characterization relies on results on the asymptotic behaviour of subadditive processes. The proof of the limit theorem uses a blow-up technique common for local energies, which can be extended to this 'asymptotically local' case. As a particular application, we derive a homogenization theorem on random perforated domains.

1. *Introduction*

In this paper, we consider random energies of convolution type. Such energies and the corresponding stationary and evolution non-local equations may be interpreted for example in the context of mathematical models in population dynamics where macroscopic properties can be reduced to studying the evolution of the first-correlation functions describing the population density u in the system [21, 29]. Also, non-local problems of this type are used in biology to model a swarm [33] and in image processing for image regularization [24] and for image denoising and deblurring [28]. Among other applications are mathematical finance models based on optimal control theory [4], particle systems [6], coagulation models.

Our model energies are defined on L^2-functions in a reference domain D and are of the form

$$\frac{1}{\varepsilon^{d+2}}\int_{D\times D} B^\omega\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\, a\Big(\frac{y-x}{\varepsilon}\Big)(u(y)-u(x))^2 dy\, dx, \tag{1.1}$$

or

$$\frac{1}{\varepsilon^{d+2}}\int_{(D\cap\varepsilon E^\omega)\times(D\cap\varepsilon E^\omega)} a\Big(\frac{y-x}{\varepsilon}\Big)(u(y)-u(x))^2 dy\, dx. \tag{1.2}$$

Here $a:\mathbb{R}^d\to\mathbb{R}$ is a *convolution kernel* which describes the strength of the interaction at a given distance and ε is a scaling parameter. In order that the limit of energies above be well defined on $H^1(\Omega)$, we require that

$$\int_{\mathbb{R}^d} a(\xi)(1+|\xi|^2)\, dx < +\infty. \tag{1.3}$$

In (1.1), the strictly positive coefficient B^ω represents the features of the environment, while in (1.2) E^ω is a random perforated domain giving the regions where interaction actually occurs, both depending on the realization of a random variable. Note that functionals (1.2) can be also written as (1.1) with the degenerate coefficient $B^\omega(x,y)=\chi_{E^\omega}(x)\chi_{E^\omega}(y)$, where χ_E denotes

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the characteristic function of E. Note that more in general we may consider oscillations on a different scale than ε; for example, taking coefficients $B^\omega(x/\delta, y/\delta)$ with $\delta = \delta_\varepsilon$, but the case when these two scales differ can be treated more easily by a separation-of-scale argument.

The effect of the scaling parameter ε as $\varepsilon \to 0$ is twofold, on one hand producing a local limit model as the convolution kernel concentrates, and on the other hand ensuring a homogenization effect through the oscillations provided by B^ω. To illustrate the first issue, we may consider the underlying energies (those with the perturbation B^ω set to 1)

$$\frac{1}{\varepsilon^{d+2}} \int_{D\times D} a\Big(\frac{y-x}{\varepsilon}\Big)(u(y)-u(x))^2 dy\, dx. \tag{1.4}$$

We note that if $u \in C^1(D)$, then $u(y) - u(x) \approx \langle \nabla u(x), y-x\rangle$. Here and in what follows $\langle\cdot,\cdot\rangle$ stands for the standard scalar product in $\mathbb{R}^d$. Then, using the change of variables $y = x + \varepsilon\xi$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{d+2}} \int_{D\times D} a\Big(\frac{y-x}{\varepsilon}\Big)(\langle \nabla u(x), y-x\rangle)^2 dy\, dx = \int_D \int_{\mathbb{R}^d} a(\xi)(\langle \nabla u(x), \xi\rangle)^2 d\xi\, dx, \tag{1.5}$$

so that the quadratic functional

$$\int_D \langle A\nabla u, \nabla u\rangle\, dx, \qquad \text{with the matrix } A \text{ defined by } \quad \langle Az, z\rangle = \int_{\mathbb{R}^d} a(\xi)(\langle z,\xi\rangle)^2 d\xi, \tag{1.6}$$

gives an approximation of (1.4). Conversely, we may think of (1.4) as giving a more general form of quadratic energies allowing for interactions between points at scale ε. In terms of Γ-convergence, this computation can be extended to a Γ-limit result and obtain the corresponding convergence of minimum problems. To that end, we will suppose that $a : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$0 \leqslant a(\xi) \leqslant C \frac{1}{(1+|\xi|)^{d+2+\kappa}}, \tag{1.7}$$

for some $C, \kappa > 0$ (which is a quantified version of (1.3)), and

$$a(\xi) \geqslant c > 0 \quad \text{if } |\xi| \leqslant r_0 \tag{1.8}$$

for some $r_0 > 0$ and $c > 0$.

In a Γ-convergence context, energies (1.4) have been considered as an approximation of a Dirichlet-type integral in phase-transition problems (see, for example, [**1**]) and more recently in connection with minimal-cut problems in Data Science [**23**]. Limits of energies similar to (1.4), of the form

$$\frac{1}{\varepsilon^{d}} \int_{D\times D} a\Big(\frac{y-x}{\varepsilon}\Big)\Big|\frac{u(y)-u(x)}{y-x}\Big|^2 dy\, dx, \tag{1.9}$$

have also been studied by Bourgain *et al.* as an alternative definition of the L^p-norm of the gradient of a Sobolev function [**7, 20**], within a general interest towards non-local functionals [**19**]. The relation between the regularity of functions u and the convergence of the functionals

$$\int_D \int_D W\Big(\frac{u(x)-u(y)}{x-y}\Big)\rho^\varepsilon(x-y)dxdy$$

with continuous W and ρ^ε converging to the δ-function was considered in [**36**]. In the context of Free-Discontinuity Problems, functionals of the form

$$\frac{1}{\varepsilon^{d}} \int_{D\times D} a\Big(\frac{y-x}{\varepsilon}\Big) \min\Big\{\Big|\frac{u(y)-u(x)}{y-x}\Big|^2, \frac{1}{\varepsilon}\Big\} dy\, dx, \tag{1.10}$$

have been proved to provide an approximation of the Mumford–Shah functionals by Gobbino [25] after a conjecture by De Giorgi. Furthermore, discrete counterparts of functionals (1.4), that is, energies of the form

$$\frac{1}{\varepsilon^{d+2}} \sum_{i,j\in\varepsilon\mathcal{L}} a_{ij}(u_i - u_j)^2, \tag{1.11}$$

where $\mathcal{L}$ is a d-dimensional lattice, have been widely investigated (see, for example, [2, 11, 16, 34]) as a discrete approximation of quadratic integral functionals. Such type of functionals or the corresponding operators have been analysed in different ways under various inhomogeneity and randomness assumptions (see, for example, [3, 5, 13, 15, 23, 30, 32, 34]).

The homogenization results for linear elliptic random difference operators in $\mathbb{Z}^d$ were first obtained in [32] and [30] and further developed in [34] and other works. A comparison with results in deterministic discrete G-closure problems can be obtained from [15].

The paper [37] deals with an approximation of Mumford–Shah functional in the context of random stationary lattices. Under some uniform geometric assumptions on the lattice, the nearest Voronoi neighbour interaction model is considered. In contrast with $\mathbb{Z}^d$ models, in the case of stochastically isotropic lattices the Γ-limit is isotropic.

In the work [3], discrete-to-continuous Γ-limits are investigated for energies defined on random stationary lattices (for random thin-film energies, see, for example, [13]). The energies admit both nearest neighbours and long distance interactions, it is assumed that the nearest-neighbour terms satisfy p-growth conditions and for the long distance terms proper moment conditions are fulfilled. Under the assumptions that the lattices satisfy some uniform geometric estimates, the authors prove the almost-sure existence of the Γ-limit and study its properties. That work represents somehow a discrete counterpart of our study in the case of uniformly elliptic media. In general, the discrete-to-continuum analysis on lattices developed in the last twenty years provides a number of useful techniques and results in parallel with the case of convolution-type energies, with notable differences due to having interactions at all scales in the convolution case.

In the recent paper [35], linear convolution type operators are studied in random stationary uniformly elliptic media. The approach used in that paper relies on a corrector technique. However, constructing correctors in a random perforated environment encounters serious technical problems. It is an interesting open question.

In the mentioned works, the case of random perforated domains was not addressed. To our best knowledge, this case was not studied in the existing literature (for some discrete deterministic analog on lattices see [12], where, however, 'perforated sets' may be treated in a quite simple way due to their discreteness). It should be noted that the presence of perforation weakens the coerciveness properties of the studied functionals and makes the proof of compactness results more delicate.

In our case, we will prove a general homogenization result, which, under proper stationarity and ergodicity assumptions, will comprise both random coefficients and random perforated domains as in (1.1), assuming that B^ω satisfies $0 < \lambda_1 \leqslant B^\omega(x,y) \leqslant \lambda_2 < +\infty$, and (1.2), where E^ω is a *random perforated domain* consisting of a unique connected component, see Definition 3 for further details. The limit behaviour of these energies is described by their Γ-limit in the $L^2(D)$ topology as a standard elliptic integral, of the form

$$F_{\mathrm{hom}}(u) = \int_D \langle A_{\mathrm{hom}}\nabla u, \nabla u\rangle\, dx. \tag{1.12}$$

The matrix A_{hom} is characterized by an asymptotic formula obtained using a limit theorem for subadditive processes, and can be compared to those in [27, 39, 40]. The choice of the $L^2(D)$ topology is justified by the coerciveness of the convolution energies, which ensures the convergence of minimum problems.

The plan of the paper is as follows. In Section 2, we define the general form of the random functionals that we are going to consider. Section 3 is devoted to the statement and proof of a compactness theorem. The proof of this result follows closely that of the compactness result for non-linear convolution energies used to approximate Free-Discontinuity Problems obtained by Gobbino [**8, 25**]; due to the quadratic growth conditions on the energies, we can improve that result from L^1 to L^2 compactness. In Section 4, we prove Poincaré and Poincaré–Wirtinger inequalities, which, together with the compactness result, justify the application of the direct method of the Calculus of Variations to minimum problems, and hence the asymptotic study of convolution energies in terms of Γ-convergence.

More precisely, given a family of minimization problems for functionals whose principal terms are defined by (1.1) or (1.2), we can claim that the corresponding minimizers converge to the minimizer of the functional obtained as the Γ-limit of this family, if the set of minimizers is compact in $L^2(D)$. The desired compactness follows from the results of Section 3 under the condition that the set of minimizers is bounded in $L^2(D)$, and the proof of boundedness relies on the Poincaré inequalities justified in Section 4. At the end of Section 7, we provide an example of minimization problem that illustrates this approach.

In Section 5, we use the stationarity and ergodicity properties of the energies to prove the existence of an asymptotic homogenization formula giving a deterministic homogeneous integrand using results on the asymptotic behaviour of almost-subadditive processes in [**31**]. The formula is used in Section 6 to prove the homogenization theorem using an adaptation to (non-local) homogenization problems of the blow-up technique of Fonseca and Müller [**17, 22**]. Finally, in Section 7 we remark that the result can be applied to the homogenization of random perforated domains.

2. *Setting of the problem*

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability set, and assume that τ_x, $x \in \mathbb{R}^d$ is a *measure-preserving dynamical system* on this probability space; that is, $\{\tau_x\}_{x\in\mathbb{R}^d}$ is a group of measurable mappings $\tau_x : \Omega \mapsto \Omega$ such that:

- $\tau_x \circ \tau_y = \tau_{x+y}, \quad \tau_0 = \mathrm{Id}$;
- $\mathbf{P}(\tau_x A) = \mathbf{P}(A)$ for all $x \in \mathbb{R}^d$ and $A \in \mathcal{F}$;
- $\tau_\cdot : \mathbb{R}^d \times \Omega \mapsto \Omega$ is a measurable map. We assume here that $\mathbb{R}^d \times \Omega$ is equipped with a product σ-algebra $\mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ is a Borel σ-algebra in $\mathbb{R}^d$.

We also assume that $\{\tau_x\}$ is *ergodic*; that is, the measure of any set $A \in \mathcal{F}$ which is invariant with respect to τ_x for all $x \in \mathbb{R}^d$ is equal to 0 or 1.

Given an open subset D of $\mathbb{R}^d$, for all $\varepsilon > 0$ and $u \in L^2(D)$ we will consider convolution-type energies of the form

$$F_\varepsilon^\omega(u) = \frac{1}{\varepsilon^{d+2}} \int_D \int_D b^\omega\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)(u(y) - u(x))^2 \, dy \, dx, \tag{2.1}$$

where b^ω are stationary ergodic integrands satisfying

$$0 \leqslant b^\omega(x, y) \leqslant C \frac{1}{(1 + |x - y|)^{d+2+\kappa}}. \tag{2.2}$$

More precisely, we assume that

$$b^\omega(x, y) = \mathtt{b}(\tau_x\omega, \tau_y\omega, x - y), \tag{2.3}$$

where $\mathtt{b}(\omega_1, \omega_2, \xi)$ is a function defined on $\Omega \times \Omega \times \mathbb{R}^d$ such that

$$0 \leqslant \mathtt{b}(\omega_1, \omega_2, \xi) \leqslant C \frac{1}{(1 + |\xi|)^{d+2+\kappa}}. \tag{2.4}$$

In order to make the definition of a function b in (2.3) well defined, we need additional assumptions on $\mathtt{b}$. One option is to assume that $\mathtt{b}(\omega_1, \omega_2, \xi) = \mathtt{b}_1(\omega_1)\mathtt{b}_2(\omega_2)a(\xi)$, where $\mathtt{b}_1$ and $\mathtt{b}_2$ are non-negative bounded random variables, and $a(\xi)$ is a measurable function in $\mathbb{R}^d$ that satisfies estimate (1.7). Another option is to assume that Ω is a topological space, the group $\tau_x\omega$ is continuous in x, and the function $\mathtt{b} = \mathtt{b}(\omega_1, \omega_2, \xi)$ is continuous in ω_1 and ω_2 and measurable in ξ and $\mathtt{b}(\omega_1, \omega_2, \xi) \leqslant a(\xi)$ with a function $a(\cdot)$ satisfying estimate (1.7). In both cases, the definition of b^ω in (2.3) makes sense.

In order to obtain coerciveness properties which allow to include in our results both types of models (1.1) and (1.2), that is, with integrands

- $b^\omega(x, y) = B^\omega(x, y)a(x - y)$ with $0 < \lambda_1 \leqslant B^\omega(x, y) \leqslant \lambda_2 < +\infty$; or
- $b^\omega(x, y) = \chi_{\cdot E^\omega}(x)\chi_{\cdot E^\omega}(y)a(x - y)$ with E^ω being a random perforated domain (see Section 7 for the precise definition of random perforated domain);

we will make the following abstract assumption.

DEFINITION 1. We say that b^ω is a *coercive energy function* if there exist constants C and Ξ_0 such that for all U open subsets of $\mathbb{R}^d$, $z \in \mathbb{R}^d$, $\Xi \geqslant \Xi_0$ and $u \in L^2(U)$ satisfying the boundary condition

$$u(x) = \langle z, x\rangle \quad \text{if} \;\; \operatorname{dist}(x, \partial U) < \Xi,$$

there exists a function $v \in L^2(U)$ satisfying the boundary condition

$$v(x) = \langle z, x\rangle \quad \text{if} \;\; \operatorname{dist}(x, \partial U) < \Xi/2,$$

such that

$$\int_{U\times U} b^\omega(x, y)(v(y) - v(x))^2\, dy\, dx \leqslant \int_{U\times U} b^\omega(x, y)(u(y) - u(x))^2\, dy\, dx, \tag{2.5}$$

and

$$\int_{\{x,y\in U: |x-y|<1\}} (v(y) - v(x))^2\, dy\, dx \leqslant C \int_{U\times U} b^\omega(x, y)(v(y) - v(x))^2\, dy\, dx. \tag{2.6}$$

REMARK 1. Note that if $b^\omega(x, y) \geqslant C > 0$ when $|x - y| < 1$, or if $b^\omega(x, y) = \chi_{E^\omega}(x)\chi_{E^\omega}(y)a(x - y)$ with E^ω a realization of a random perforated domain, then b^ω is coercive. Indeed, in the former case, we take $u = v$ in the definition above, while in the latter case, v is a suitable extension of u in the perforation that was constructed in [**18**].

REMARK 2 (coerciveness). The terminology in Definition 1 is justified by the Compactness Theorem in Section 3, which ensures that if b^ω is a coercive energy function, then sequences bounded in $L^2(D)$ and for which the energy on the left-hand side of (2.6) is equibounded admit $L^2_{\mathrm{loc}}(D)$ converging subsequences and their limit is in $H^1(D)$.

2.1. *Notation*

Unless otherwise stated C denotes a generic strictly positive constant independent of the parameters of the problem taken into account.

$Q_T = [-T/2, T/2]^d$ denotes the d-dimensional coordinate cube centred in 0 and with side-length T. If $T = 1$, then we write $Q = Q_1$.

If $x, y \in \mathbb{R}^d$, then $|y - x|_1 = \sum_{j=1}^d |y_j - x_j|$.

$\lfloor t\rfloor$ denotes the integer part of $t \in \mathbb{R}$.

χ_A denotes the characteristic function of the set A.

For all $t > 0$ and D open subset of $\mathbb{R}^d$, we denote $D(t) = \{x \in D : \operatorname{dist}(x, \partial D) > t\}$.

As a shorthand, the notation $\{P(\xi)\}$ will stand for $\{\xi \in \mathbb{R}^d : P(\xi) \text{ holds}\}$ if no confusion may arise.

3. *A compactness theorem*

Let D be an open set with Lipschitz boundary. We show that families of functions that have bounded energies of the type (1.4) are compact in $L^2_{\rm loc}(D)$. To this end, for $0 < r \leqslant \sigma$, we define the functional

$$F_\varepsilon^{\sigma,r}(w) = \int_{D(\sigma)} \int_{\{|\xi|\leqslant r\}} \left(\frac{w(x+\varepsilon\xi) - w(x)}{\varepsilon}\right)^2 d\xi\, dx, \quad w \in L^2(D).$$

In the case when $D = \mathbb{R}^d$, the $L^1_{\rm loc}$-compactness can be directly obtained by comparison with finite-difference energies approximating the Mumford–Shah functional studied by Gobbino [**25**]. Here we follow his proof, to deduce the $L^2_{\rm loc}$-compactness.

THEOREM 3.1 (compactness theorem). *Let D be an open set with Lipschitz boundary, and assume that for a family $\{w_\varepsilon\}_{\varepsilon>0}$, $w_\varepsilon \in L^2(D)$, the estimate*

$$F_\varepsilon^{k\varepsilon,r}(w_\varepsilon) := \int_{D(k\varepsilon)} \int_{\{|\xi|\leqslant r\}} \left(\frac{w_\varepsilon(x+\varepsilon\xi) - w_\varepsilon(x)}{\varepsilon}\right)^2 d\xi\, dx \leqslant C \tag{3.1}$$

is satisfied with some $k > 0$ and $r > 0$. Assume moreover that the family $\{w_\varepsilon\}$ is bounded in $L^2(D)$. Then for any sequence ε_j such that $\varepsilon_j > 0$ and $\varepsilon_j \to 0$, as $j \to \infty$, and for any open subset $D' \Subset D$ the set $\{w_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is relatively compact in $L^2(D')$ and every limit point of the sequence $\{w_{\varepsilon_j}\}$ is in $H^1(D)$.

Before proving the theorem, we prove some auxiliary results. We first introduce the local average of a function $u \in L^2(D)$ by

$$\mathring{u}_\delta = \int_{\{|\xi|\leqslant 1\}} u(x+\delta\xi)\phi(\xi)\, d\xi,$$

where ϕ is a symmetric non-negative C_0^∞ function in $\mathbb{R}^d$ supported in the unit ball centred at the origin, $\int \phi(\xi)\, d\xi = 1$. In our framework, the function $\mathring{u}_\delta$ is well defined in $D(\delta)$. The properties of the local average operator are described in the following statement.

PROPOSITION 3.2. *Let δ and σ be positive numbers with $\delta < \sigma$. Then we have*

$$\|\mathring{u}_\delta - u\|^2_{L^2(D(\sigma))} \leqslant C_\phi \delta^2 F_\delta^{\sigma,1}(u). \tag{3.2}$$

For any $\delta > 0$ such that $D' \subset D(\delta)$, the function $\mathring{u}_\delta$ is smooth in D' and satisfies the inequalities

$$\|\mathring{u}_\delta\|_{L^\infty(D')} \leqslant C_\phi \delta^{-\frac{d}{2}} \|u\|_{L^2(D)}, \qquad \|\nabla \mathring{u}_\delta\|_{L^\infty(D')} \leqslant C_\phi \delta^{-\frac{d}{2}-1} \|u\|_{L^2(D)}. \tag{3.3}$$

Proof. For any $u \in L^2(D)$ by the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\|\mathring{u}_\delta - u\|^2_{L^2(D(\sigma))} &= \int_{D(\sigma)} \int_{\{|\xi|\leqslant 1\}} \int_{\{|\eta|\leqslant 1\}} (u(x+\delta\xi) - u(x))\,(u(x+\delta\eta) - u(x))\,\phi(\xi)\,\phi(\eta)\, d\eta\, d\xi\, dx \\
&\leqslant \delta^2 \left(\int_{D(\sigma)} \int_{\{|\xi|\leqslant 1\}} \int_{\{|\eta|\leqslant 1\}} \left(\frac{u(x+\delta\xi) - u(x)}{\delta}\right)^2 (\phi(\xi))^2 dx d\xi d\eta \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{D(\sigma)} \int_{\{|\xi|\leqslant 1\}} \int_{\{|\eta|\leqslant 1\}} \left(\frac{u(x+\delta\eta) - u(x)}{\delta}\right)^2 (\phi(\eta))^2 dx d\xi d\eta \right)^{\frac{1}{2}} \\
&\leqslant C_\phi \delta^2 F_\delta^{\sigma,1}(u).
\end{aligned}$$

The estimates in (3.3) are standard. ∎

PROPOSITION 3.3. *For any $j \in \mathbb{N}$ such that $j\varepsilon \leqslant \operatorname{dist}(D', \partial D) - k\varepsilon$, the following inequality holds:*

$$F_{j\varepsilon}^{(j+k)\varepsilon,1}(u) \leqslant F_{\varepsilon}^{k\varepsilon,1}(u) \tag{3.4}$$

for all $u \in L^2(D)$.

Proof. Representing $u(x + j\varepsilon\xi) - u(x)$ as $(u(x + j\varepsilon\xi) - u(x + (j-1)\varepsilon\xi)) + (u(x + (j-1)\varepsilon\xi) - u(x + (j-2)\varepsilon\xi)) + \cdots + (u(x + \varepsilon\xi) - u(x))$, we obtain

$$F_{j\varepsilon}^{(j+k)\varepsilon,1}(u) \leqslant j \int_{D((j+k)\varepsilon)} \int_{\{|\xi|\leqslant 1\}} \sum_{m=1}^{j} \frac{\left(u(x + m\varepsilon\xi) - u(x + (m-1)\varepsilon\xi)\right)^2}{(j\varepsilon)^2} dx d\xi$$

$$\leqslant j^2 \int_{D(k\varepsilon)} \int_{\{|\xi|\leqslant 1\}} \frac{\left(u(x + \varepsilon\xi) - u(x)\right)^2}{(j\varepsilon)^2} dx d\xi = F_{\varepsilon}^{k\varepsilon,1}(u)$$

as desired. □

Proof of Theorem 3.1. One may assume without loss of generality that $r = 1$. In order to prove the compactness result, it suffices to show that, fixed D', for each $\delta > 0$ there exists a relatively compact set $\mathcal{K}_\delta$ in $L^2(D')$ such that for any $j \in \mathbb{N}$, we have

$$\|w_{\varepsilon_j} - h_j\|_{L^2(D')} \leqslant \delta \tag{3.5}$$

for some $h_j \in \mathcal{K}_\delta$.

We define $\mathcal{K}_\delta$ as follows. If $\varepsilon_j \geqslant \delta$, we set $h_j = w_{\varepsilon_j}$; otherwise,

$$h_j = \mathring{w}_{\varepsilon_j,\delta_j} = \int_{\{|\xi|\leqslant 1\}} w_{\varepsilon_j}(x + \delta_j\xi)\phi(\xi)\, d\xi,$$

where $\delta_j = \lfloor \frac{\delta}{\varepsilon_j} \rfloor \varepsilon_j$. Note that $\frac{1}{2}\delta < \delta_j \leqslant \delta$ for any j such that $\varepsilon_j < \delta$. We finally set $\mathcal{K}_\delta = \bigcup_{j=1}^{\infty} \{h_j\}$.

It is convenient to represent $\mathcal{K}_\delta$ as a union $\mathcal{K}_\delta = \mathcal{K}_{\delta,1} \cup \mathcal{K}_{\delta,2}$ with

$$\mathcal{K}_{\delta,1} = \bigcup_{\{j\,:\,\varepsilon_j \geqslant \delta\}} h_j, \qquad \mathcal{K}_{\delta,2} = \bigcup_{\{j\,:\,\varepsilon_j < \delta\}} h_j$$

Since ε_j tends to zero as $j \to \infty$, the first set consists of a finite number of elements and thus is compact. By (3.3) for any $h_j \in \mathcal{K}_{\delta,2}$, we obtain

$$|h_j(x)| \leqslant C(\delta), \quad |\nabla h_j(x)| \leqslant C(\delta) \quad \text{for all } x \in D'.$$

Therefore, by the Arzelà–Ascoli theorem, the set $\mathcal{K}_{\delta,2}$ is relatively compact in $C(D')$. Consequently, this set is also relatively compact in $L^2(D')$. This yields the desired relative compactness of $\mathcal{K}_\delta$.

If $\varepsilon_j \geqslant \delta$, then $h_j = w_{\varepsilon_j}$, and (3.5) holds. If $\varepsilon_j < \delta$, then by (3.2) we get

$$\|w_{\varepsilon_j} - h_j\| \leqslant C_\phi \delta_j F_{\delta_j,1}^{(\delta_j + k\varepsilon_j)}(w_{\varepsilon_j}).$$

Combining this inequality with (3.4) and recalling that $\delta_j = \lfloor \frac{\delta}{\varepsilon_j} \rfloor \varepsilon_j$, we obtain

$$\|w_{\varepsilon_j} - h_j\| \leqslant C_\phi \delta_j F_{\varepsilon_j,1}^{k\varepsilon_j}(w_{\varepsilon_j}) \leqslant C\delta_j \leqslant C\delta;$$

here we have also used (3.1). The last inequality implies (3.5).

It remains to show that each limit point w is in $H^1(D)$. To that end we may use the 'slicing technique' (see, for example, [8] Section 4.1, [9] Chapter 15 or [10] Section 3.4). This general method allows to reduce the analysis to that of one-dimensional sections, and recover a lower bound by integrating over all sections. It has already been applied in [25] to sequences of non-linear functionals of the form

$$\frac{1}{\varepsilon^{d+1}}\int_D\int_D a\Big(\frac{y-x}{\varepsilon}\Big) f\Big(\frac{(u(y)-u(x))^2}{\varepsilon}\Big)dy\,dx \tag{3.6}$$

in order to obtain compactness in spaces of functions with bounded variation. In our case, we are in a simplified situation with f equal the identity and we can improve the result to compactness in $H^1(D)$.

In the one-dimensional case, it is not restrictive to study functionals of the form

$$G_\varepsilon(u)=\int_{(0,1)}\int_{(-1,1)}\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)^2 d\xi\,dx, \tag{3.7}$$

and regard all functions as defined on $\mathbb{R}$. With Fatou's lemma in mind, in order to have a lower bound it suffices to examine separately the functionals

$$G^\xi_\varepsilon(u)=\int_{(0,1)}\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)^2 dx \tag{3.8}$$

for fixed $\xi\in(-1,1)$.

For simplicity, we treat the case $\xi\in(0,1)$. We may suppose that $u_\varepsilon\to u$ in $L^2(\mathbb{R})$. Note that for almost all $t\in(0,1)$ the piecewise-constant functions $u_{\varepsilon,\xi,t}$ defined by

$$u_{\varepsilon,\xi,t}(x)=u_\varepsilon(\varepsilon\xi t+\varepsilon\xi k)\qquad \text{if } \varepsilon\xi k\leqslant x<\varepsilon\xi(k+1)$$

converge to u in $L^2(\mathbb{R})$, and we have

$$\begin{aligned}G^\xi_\varepsilon(u_\varepsilon)&\geqslant\sum_{k=1}^{\lfloor 1/\varepsilon\xi\rfloor-1}\int_{k\varepsilon\xi}^{(k+1)\varepsilon\xi}\Big(\frac{u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)}{\varepsilon}\Big)^2 dt\\ &=\sum_{k=1}^{\lfloor 1/\varepsilon\xi\rfloor-1}\int_0^1\varepsilon\xi\Big(\frac{u_\varepsilon((k+1)\varepsilon\xi+t\varepsilon\xi)-u_\varepsilon(k\varepsilon\xi+t\varepsilon\xi)}{\varepsilon}\Big)^2 dt\\ &=\xi^2\int_{(0,1)}\sum_{k=1}^{\lfloor 1/\varepsilon\xi\rfloor-1}\varepsilon\xi\Big(\frac{u_{\varepsilon,\xi,t}((k+1)\varepsilon\xi)-u_{\varepsilon,\xi,t}(k\varepsilon\xi)}{\varepsilon\xi}\Big)^2 dt\\ &\geqslant\xi^2\int_{(0,1)}\int_{(\delta,1-\delta)}(u'_{\varepsilon,\xi,t}(x))^2\,dx\,dt,\end{aligned} \tag{3.9}$$

eventually for all $\delta>0$ fixed, where we have identified the discrete function $k\varepsilon\xi\mapsto u_{\varepsilon,\xi,t}(k\varepsilon\xi)$ defined on $\varepsilon\xi\mathbb{Z}$ with its piecewise-affine interpolation. Note that for almost all t, this functions still converge to u. From (3.9), we deduce that $u\in H^1(\delta,1-\delta)$. By the arbitrariness of δ and the uniformity of the bound on the L^2-norm of u', we deduce that $u\in H^1(0,1)$. For more details on this proof, we refer to [8], where the non-linear case is treated.

The deduction of the d-dimensional lower bound from the one-dimensional one can be obtained by repeating word for word the proof of [8, Theorem 5.19] with G^ξ_ε in the place of F^1_ε in the notation therein. This completes the proof of the compactness. □

4. *Poincaré inequalities*

We first prove a Poincaré–Wirtinger inequality as follows.

THEOREM 4.1 (Poincaré–Wirtinger inequality). *Let D be a Lipschitz bounded domain. For each fixed $r_0 > 0$, there exists a constant $C > 0$ such that for any $v \in L^2(D)$, we have*

$$\int_D (v(x) - v_D)^2\,dx \leqslant C \int_D \int_{\{\xi:|\xi|\leqslant r_0, x+\varepsilon\xi\in D\}} \left(\frac{v(x+\varepsilon\xi) - v(x)}{\varepsilon}\right)^2 d\xi\,dx, \tag{4.1}$$

and v_D is the average of v over D. The constant C does not depend on ε.

Proof of Theorem. 4.1 We set

$$F_\varepsilon^0(r,v) = \int_D \int_{\{\xi:|\xi|\leqslant r, x+\varepsilon\xi\in D\}} \left(\frac{v(x+\varepsilon\xi) - v(x)}{\varepsilon}\right)^2 d\xi\,dx$$

and

$$F^1(G_1,G_2,v) = \int_{G_1}\int_{G_2} (v(x)-v(y))^2 dx\,dy.$$

In what follows, the notation D^ε is used for $\frac{1}{\varepsilon}D$.

We first consider the case when D is a cube, $D = (-\frac{L}{2}, \frac{L}{2})^d$, and r is a sufficiently large number, say $r \geqslant 3\sqrt{d}$. We also assume that $\frac{L}{2\varepsilon}$ is an integer number.

Denote $\mathcal{S}^\varepsilon = \{j \in \mathbb{Z}^d : j + [-\frac12, \frac12]^d\} \cap D^\varepsilon \neq \emptyset$. For any $i \in \mathcal{S}^\varepsilon$ and $j \in \mathcal{S}^\varepsilon$, construct a path $\boldsymbol{\gamma}(i,j) = \{j_k\}_{k=1}^N$ in $\mathbb{Z}^d$ such that $j_1 = i$, $j_N = j$, $|j_k - j_{k+1}| = 1$. The path is constructed in such a way that it starts along the first coordinate direction until the first coordinate of j_k coincides with the first coordinate of j, then it follows the second coordinate direction and so on. We then have:

(i) the length of each path is not greater than $d\frac{L}{\varepsilon}$;
(ii) for each $j \in \mathcal{S}^\varepsilon$ the total number of paths $\{\gamma(i,l) : i, l \in \mathcal{S}^\varepsilon\}$ that pass through j is not greater than $(\frac{L}{\varepsilon})^{d+1}$:

$$\#\{\boldsymbol{\gamma}(i,l) : i, l \in \mathcal{S}^\varepsilon,\ j \in \boldsymbol{\gamma}(i,l)\} \leqslant d\left(\frac{L}{\varepsilon}\right)^{d+1}. \tag{4.2}$$

Property (i) is evident. In order to justify (ii), consider all paths that come to j along mth coordinate direction, that is all paths $j_1, \dots, j_N$ such that for some $k \in \mathbb{Z}^+$ we have $j_k = j$ and $j_k - j_{k-1}$ coincides with the mth coordinate vector. By construction, the number of starting points for such paths does not exceed $(j + \frac{L}{2\varepsilon}) \times (\frac{L}{\varepsilon})^{m-1}$. Similarly, the number of end points for such paths does not exceed $(\frac{L}{\varepsilon} - j + 1) \times (\frac{L}{\varepsilon})^{d-m}$. Since starting and end points define the corresponding path uniquely, the total number of paths that come to j along mth coordinate direction is not greater than $(j + \frac{L}{2\varepsilon}) \times (\frac{L}{\varepsilon} - j + 1) \times (\frac{L}{\varepsilon})^{d-1} < (\frac{L}{\varepsilon})^{d+1}$. Summing up over m, we arrive at (4.2).

For any $j \in \mathcal{S}^\varepsilon$ denote $Q_j = \varepsilon j + \varepsilon[-\frac12, \frac12]^d$. For i and j in $\mathcal{S}^\varepsilon$, the 'interaction energy of the cubes Q_i and Q_j' can be estimated as follows. We consider a path $\boldsymbol{\gamma}(i,j)$, denote the length of this path by N and its elements by $\gamma_1, \gamma_2, \dots, \gamma_N$, and introduce the variables $\eta_2, \dots, \eta_{N-1}$, $\eta_k \in Q_0$. Then we have

$$\begin{aligned}
&\int_{\varepsilon Q_i}\int_{Q_j} \left(\frac{u(x) - u(\varepsilon\xi)}{\varepsilon}\right)^2 d\xi dx \\
&= \varepsilon^{d-2} \int_{Q_0}\int_{Q_0} (u(\varepsilon\gamma_1 + \varepsilon\eta_1) - u(\varepsilon\gamma_N + \varepsilon\eta_N))^2\, d\eta_1 d\eta_N \\
&= \varepsilon^{d-2} \int_{Q_0} \cdots \int_{Q_0} (u(\varepsilon\gamma_1 + \varepsilon\eta_1) - u(\varepsilon\gamma_2 + \varepsilon\eta_2) + u(\varepsilon\gamma_2 + \varepsilon\eta_2) - \dots
\end{aligned}$$

$$
\begin{aligned}
&-u(\varepsilon\gamma_N+\varepsilon\eta_N))^2 d\eta_1 d\eta_2\cdots d\eta_N\\
&\leqslant N\varepsilon^{d-2}\sum_{i=1}^{N-1}\int_{Q_0}\int_{Q_0}(u(\varepsilon\gamma_i+\varepsilon\eta_i)-u(\varepsilon\gamma_{i+1}+\varepsilon\eta_{i+1}))^2 d\eta_i d\eta_{i+1}\\
&\leqslant (Ld)\varepsilon^{d-3}\sum_{i=1}^{N-1}\int_{Q_0}\int_{Q_0}(u(\varepsilon\gamma_i+\varepsilon\eta)-u(\varepsilon\gamma_{i+1}+\varepsilon\xi))^2 d\xi d\eta\\
&\leqslant (Ld)\varepsilon^{-3}\sum_{i=1}^{N-1}\int_{\varepsilon Q_0}\int_{\{\xi:\varepsilon\gamma_i+x+\varepsilon\xi\in D,|\xi|<r\}}(u(\varepsilon\gamma_i+x)-u(\varepsilon\gamma_i+x+\varepsilon\xi))^2 d\xi dx,
\end{aligned}
$$

the last inequality here follows from the fact that for any $x\in\varepsilon Q_0$ the set $\{\varepsilon\gamma_{i+1}+\varepsilon\xi\,:\,\xi\in Q_0\}$ is a subset of $\{\xi\,:\,\varepsilon\gamma_i+x+\varepsilon\xi\in D,|\xi|<r\}$, if $r>2\sqrt{d}$.

Considering (4.2), we deduce from the last inequality that

$$
\begin{aligned}
&\int_D\int_D (u(x)-u(y))^2 dx\,dy\\
&=\sum_{i,\,l\in\mathcal{S}^\varepsilon}\varepsilon^{d+2}\int_{\varepsilon Q_i}\int_{Q_l}\left(\frac{u(x)-u(\varepsilon\xi)}{\varepsilon}\right)^2 d\xi dx\\
&\leqslant (Ld)\varepsilon^{d-1}\left(\frac{L}{\varepsilon}\right)^{d+1}\sum_{j\in\mathcal{S}^\varepsilon}\int_{x\in\varepsilon Q_0}\int_{\{\xi:x+\varepsilon\xi\in D,|\xi|<r\}}(u(\varepsilon j+x)-u(\varepsilon j+x+\varepsilon\xi))^2 dx d\xi\\
&\leqslant L^{d+2}d\int_{x\in D}\int_{\{\xi:x+\varepsilon\xi\in D,|\xi|<r\}}\left(\frac{u(x)-u(x+\varepsilon\xi)}{\varepsilon}\right)^2 dx d\xi.
\end{aligned}
$$

Since

$$
\int_D\int_D (u(x)-u(y))^2 dx\,dy = 2\int_D (u(x)-u_D)^2 dx,
$$

this yields the desires inequality in the case of a cubic domain.

The case of an arbitrary $r>0$ and $L>0$ can be reduced to the one just studied by standard scaling arguments.

If D is a strongly star-shaped domain, then there exists a cube $\mathbf{B}$ and a Lipschitz isomorphism $J:D\mapsto\mathbf{B}$ such that $|J(x)-J(y)|\leqslant\ell|x-y|$, $|\frac{\partial J}{\partial x}|\leqslant\ell$, $|(\frac{\partial J}{\partial x})^{-1}|\leqslant\ell$ for some $\ell>0$. This statement follows from [**38**, Theorem 2]. For an arbitrary $u\in L^2(D)$ denote $u_J(x)=u(J^{-1}(x))$ and $u_{\mathbf{B},J}=\int_{\mathbf{B}}u_J(x)dx$. Also, we set $r_1=r/\ell$. Since the desired inequality has been proved for cubic domains, we have

$$
\begin{aligned}
&\int_D\int_D (u(x)-u(y))^2\,dx\,dy\\
&=\int_{\mathbf{B}}\int_{\mathbf{B}}(u_J(x)-u_J(y))^2\left|\frac{\partial J^{-1}}{\partial x}(x)\right|\left|\frac{\partial J^{-1}}{\partial x}(y)\right|dx\,dy\\
&\leqslant \ell^2\int_{\mathbf{B}}\int_{\mathbf{B}}(u_J(x)-u_J(y))^2\,dx\,dy\\
&\leqslant C\varepsilon^{-d}\ell^2\int_{\mathbf{B}}\int_{\{y\in\mathbf{B}:|y-x|<\varepsilon r_1\}}\left(\frac{u_J(x)-u_J(y)}{\varepsilon}\right)^2 dy\,dx
\end{aligned}
$$

$$\leqslant C\varepsilon^{-d}\ell^2 \int_D \int_{\{\xi: x+\varepsilon\xi\in D, |\xi|<r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 \left|\frac{\partial J}{\partial x}(x)\right| \left|\frac{\partial J}{\partial x}(y)\right| dy\, dx$$

$$\leqslant C\varepsilon^{-d}\ell^4 \int_D \int_{\{\xi: x+\varepsilon\xi\in D, |\xi|<r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\, dx,$$

where the constant C depends only on the size of $\mathbf{B}$, r_1 and d.

It remains to consider an arbitrary bounded Lipschitz set D. Such a set can be represented as a union of a finite number of strongly star shaped domains, we denote these domains $D_1, \dots, D_N$.

We first consider the case $N = 2$, we denote by $\widetilde{\mathbf{B}}$ a cube such that $\widetilde{\mathbf{B}} \subset D$, $|\widetilde{\mathbf{B}} \cup D_1| \geqslant \frac{1}{2}|\widetilde{\mathbf{B}}|$, $|\widetilde{\mathbf{B}} \cup D_2| \geqslant \frac{1}{2}|\widetilde{\mathbf{B}}|$. Note that $|\widetilde{\mathbf{B}} \cup D_1| = |\widetilde{\mathbf{B}} \cup D_2| = \frac{1}{2}|\widetilde{\mathbf{B}}|$ if the interiors of D_1 and D_2 do not intersect. In the rest of the proof, the symbols $\widetilde{\mathbf{B}}_1$ and $\widetilde{\mathbf{B}}_2$ stand for $\widetilde{\mathbf{B}} \cup D_1$ and $\widetilde{\mathbf{B}} \cup D_2$, respectively.

If we denote

$$\overline{u}_k = \frac{1}{|D_k|}\int\limits_{D_k} u(x)\,dx, \quad k=1,2; \qquad \overline{u}_{0,k} = \frac{1}{|\widetilde{\mathbf{B}}_k|}\int\limits_{\widetilde{\mathbf{B}}_k} u(x)\,dx, \quad k=1,2; \qquad \overline{u}_0 = \frac{1}{|\widetilde{\mathbf{B}}|}\int\limits_{\widetilde{\mathbf{B}}} u(x)\,dx$$

then

$$(\overline{u}_1 - \overline{u}_{0,1})^2 = \left(\frac{1}{|\widetilde{\mathbf{B}}_1|\,|D_1|}\int_{\widetilde{\mathbf{B}}_1}\int_{D_1} u(x)\,dx\,dy - \frac{1}{|\widetilde{\mathbf{B}}_1|\,|D_1|}\int_{\widetilde{\mathbf{B}}_1}\int_{D_1} u(y)\,dx\,dy\right)^2$$

$$\leqslant \frac{1}{|\widetilde{\mathbf{B}}_1|\,|D_1|}\int_{\widetilde{\mathbf{B}}_1}\int_{D_1} (u(x)-u(y))^2\,dx\,dy$$

$$\leqslant \frac{1}{|\widetilde{\mathbf{B}}_1|\,|D_1|}\int_{D_1}\int_{D_1} (u(x)-u(y))^2\,dx\,dy$$

$$\leqslant C\varepsilon^{-d}\int_{D_1}\int_{\{y\in D_1: |y-x|<\varepsilon r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\,dx$$

$$\leqslant C\varepsilon^{-d}\int_{D}\int_{\{y\in D: |y-x|<\varepsilon r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\,dx;$$

here we have used inequality (4.1) in D_1 that holds because D_1 is a strongly star shaped domain. In the same way, we prove that

$$(\overline{u}_{0,1} - \overline{u}_{0,2})^2 \leqslant C\varepsilon^{-d}\int_{D}\int_{\{y\in D: |y-x|<\varepsilon r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\,dx,$$

and

$$(\overline{u}_{0,2} - \overline{u}_2)^2 \leqslant C\varepsilon^{-d}\int_{D}\int_{\{y\in D: |y-x|<\varepsilon r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\,dx.$$

Therefore,

$$(\overline{u}_1 - \overline{u}_2)^2 \leqslant C\varepsilon^{-d}\int_{D}\int_{\{y\in D: |y-x|<\varepsilon r\}} \left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dy\,dx.$$

Since $u_D \in (\overline{u}_1, \overline{u}_2)$, the last inequality yields

$$\begin{aligned}\int_D (u(x)-u_D)^2\,dx &\leqslant \sum_{k=1}^{2}\Big(2\int_{D_k}(u(x)-\overline{u}_k)^2\,dx + 2|D_k|(\overline{u}_k-u_D)^2\Big)\\ &\leqslant 2\sum_{k=1}^{2}\int_{D_k}(u(x)-\overline{u}_k)^2\,dx + 2|D|(\overline{u}_1-\overline{u}_2)^2\\ &\leqslant C\varepsilon^{-d}\int_D\int_{\{y\in D:|y-x|<\varepsilon r\}}\Big(\frac{u(x)-u(y)}{\varepsilon}\Big)^2\,dy\,dx.\end{aligned}$$

The case $N > 2$ can be achieved by induction. □

We next consider functions with given boundary data.

Lemma 4.2 (Poincaré inequality). *Let D be a bounded set and let $u \in L^2(D)$ be such that $u = 0$ on a 2ε-neighbourhood of ∂D (and extended to 0 outside D). Then there exists a constant C depending only on the diameter of D such that*

$$\int_D |u(x)|^2\,dx \leqslant C\frac{1}{\varepsilon^{d+2}}\int_D\int_{\{|\xi|\leqslant\varepsilon\}}(u(x+\xi)-u(x))^2 d\xi dx\,. \tag{4.3}$$

Proof. It suffices to treat the case $d = 1$ and $D = (0, 1)$, the general case being recovered from this one by considering one-dimensional stripes. For notational convenience, we replace ε by 2ε, so that our claim becomes that

$$\int_0^1 |u(x)|^2\,dx \leqslant C\frac{1}{\varepsilon^3}\int_{-\infty}^{+\infty}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx\,, \tag{4.4}$$

keeping in mind that the first integral in the right-hand side is indeed restricted to (0,1).

For all $k \in \mathbb{N}$, we note that, since

$$(x-2\varepsilon, x+2\varepsilon) \supset (k\varepsilon-\varepsilon, k\varepsilon+\varepsilon) \text{ if } x \in (k\varepsilon-\varepsilon, k\varepsilon+\varepsilon),$$

we have

$$\begin{aligned}&\int_{k\varepsilon-\varepsilon}^{k\varepsilon+\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx\\ &\geqslant \int_{k\varepsilon-\varepsilon}^{k\varepsilon+\varepsilon}\int_{k\varepsilon-\varepsilon}^{k\varepsilon+\varepsilon}(u(y)-u(x))^2 dy\,dx\\ &\geqslant \int_{k\varepsilon-\varepsilon}^{k\varepsilon}\int_{k\varepsilon}^{k\varepsilon+\varepsilon}(u(y)-u(x))^2 dy\,dx\\ &= \varepsilon\int_{k\varepsilon-\varepsilon}^{k\varepsilon}|u(x)|^2\,dx - 2\int_{k\varepsilon-\varepsilon}^{k\varepsilon}u(x)\,dx\int_{k\varepsilon}^{k\varepsilon+\varepsilon}u(y)dy + \varepsilon\int_{k\varepsilon}^{k\varepsilon+\varepsilon}|u(y)|^2 dy\\ &\geqslant \varepsilon\Bigg(\int_{k\varepsilon-\varepsilon}^{k\varepsilon}|u(x)|^2\,dx - 2\sqrt{\int_{k\varepsilon-\varepsilon}^{k\varepsilon}|u(x)|^2\,dx}\sqrt{\int_{k\varepsilon}^{k\varepsilon+\varepsilon}|u(y)|^2 dy} + \int_{k\varepsilon}^{k\varepsilon+\varepsilon}|u(y)|^2 dy\Bigg)\\ &= \varepsilon\Bigg(\sqrt{\int_{k\varepsilon-\varepsilon}^{k\varepsilon}|u(x)|^2\,dx} - \sqrt{\int_{k\varepsilon}^{k\varepsilon+\varepsilon}|u(y)|^2 dy}\Bigg)^2.\end{aligned} \tag{4.5}$$

Note that for $k = 0$ this gives

$$\sqrt{\int_0^{\varepsilon} |u(y)|^2\,dy} \leqslant \sqrt{\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon} (u(y)-u(x))^2 dy\,dy},$$

and for $k > 0$

$$\sqrt{\int_{k\varepsilon}^{k\varepsilon+\varepsilon} |u(y)|^2\,dy} - \sqrt{\int_{k\varepsilon-\varepsilon}^{k\varepsilon} |u(y)|^2\,dy} \leqslant \sqrt{\frac{1}{\varepsilon}\int_{k\varepsilon-\varepsilon}^{k\varepsilon+\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon} (u(y)-u(x))^2 dy\,dy},$$

By a recursive argument from $k = 0$, we deduce that

$$\begin{aligned}\int_{k\varepsilon}^{k\varepsilon+\varepsilon} |u(y)|^2\,dy &\leqslant \frac{1}{\varepsilon}\left(\sum_{j=0}^{k}\sqrt{\int_{j\varepsilon-\varepsilon}^{j\varepsilon+\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx}\right)^2 \\ &\leqslant \frac{1}{\varepsilon^2}\sum_{j=0}^{k}\int_{j\varepsilon-\varepsilon}^{j\varepsilon+\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx \\ &\leqslant \frac{2}{\varepsilon^2}\int_{-\infty}^{+\infty}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx,\end{aligned}$$

where the factor 2 takes into account that the intervals $(j\varepsilon-\varepsilon, j\varepsilon+\varepsilon)$ overlap for consecutive values of j. Noting that indeed the term with $k = 0$ is 0 by our assumptions on the values of u close to the boundary, it suffices now to sum up the contribution over all $k \in \{1, \dots, \lfloor 1/\varepsilon\rfloor\}$ to obtain

$$\int_0^1 |u(y)|^2\,dy \leqslant 2\frac{\lfloor 1/\varepsilon\rfloor}{\varepsilon^2}\int_{-\infty}^{+\infty}\int_{x-2\varepsilon}^{x+2\varepsilon}(u(y)-u(x))^2 dy\,dx,$$

which gives (4.4) with $C = 2$. Note that if the interval (0,1) is substituted by any interval, then we can take C as twice the length of the interval. □

5. *Definition of the homogenized energy density*

Let b be as in Section 2. For all $K \in \mathbb{N}$, we set

$$b_K^{\omega}(x,y) = \begin{cases} b^{\omega}(x,y) & \text{if } |x-y| < K \\ 0 & \text{otherwise,}\end{cases} \tag{5.1}$$

and, for $z \in \mathbb{R}^d$, U open subset of $\mathbb{R}^d$, and $K \in \mathbb{N}$, we define

$$\mathcal{M}_K^{\omega}(z,U) = \inf\left\{\int_U\int_{\mathbb{R}^d} b_K^{\omega}(x,y)(v(x)-v(y))^2 dx\,dy : v(x) = \langle z,x\rangle \text{ if } \operatorname{dist}(x,\partial U) < K\right\}. \tag{5.2}$$

Note that, using $v(x) = \langle z,x\rangle$ as a test function, we get

$$\mathcal{M}_K^{\omega}(z, x+Q_R) \leqslant CR^d|z|^2 \tag{5.3}$$

for all x and R.

We now recall that the function b^{ω} defined in (2.3) and thus b_K^{ω} are statistically homogeneous and satisfy estimates (2.2). This allows us to prove the following statement:

LEMMA 5.1. *For all K and z, the limit*

$$\gamma_K(z) = \lim_{R\to+\infty} \frac{\mathcal{M}_K^\omega(z, Q_R)}{R^d} \tag{5.4}$$

exists almost surely, it is independent of ω, and $K \mapsto \gamma_K(z)$ is an increasing function. Moreover, there exists an increasing function f_K with

$$\lim_{R\to+\infty} f_K(R) = +\infty$$

such that

$$\gamma_K(z) = \lim_{R\to+\infty} \frac{\mathcal{M}_K^\omega(z, x_R + Q_R)}{R^d} \tag{5.5}$$

for all $\{x_R\}$ such that $|x_R| \leqslant R f_K(R)$.

Proof. Our arguments rely on a uniform version of the sub-additive ergodic theorem, see [**31**, Theorem 1]. For any $j \in \mathbb{Z}^{d,+} = \{0, 1, 2, \ldots\}^d$, we define $Q^j = j + \bar{\frac{1}{2}} + Q$, where $\bar{\frac{1}{2}}$ is the vector $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. For any finite subset $\mathcal{A}$ of $\mathbb{Z}^{d,+}$ denote $Q^{\mathcal{A}} = \bigcup_{j\in\mathcal{A}} Q^j$, and $\Phi_K^\omega(z, \mathcal{A}) = \mathcal{M}_K^\omega(z, Q^{\mathcal{A}})$.

From definition (5.2) for any non-intersecting finite sets $\mathcal{A}$ and $\mathcal{B}$, we have

$$\Phi_K^\omega(z, \mathcal{A}\cup\mathcal{B}) \leqslant \Phi_K^\omega(z, \mathcal{A}) + \Phi_K^\omega(z, \mathcal{B}).$$

REMARK 3. Observe that the last inequality holds for the truncated kernels b_K^ω, however, it need not hold for the generic kernel b^ω if its support does not belong to the set $\{(x, y) : |x - y| \leqslant K\}$.

Since $b_K^\omega(x, y)$ is statistically homogeneous, the family $\{\Phi_K(z, \mathcal{A})\}$ is stationary; that is, for any $j \in \mathbb{Z}^{d,+}$ and any finite collection $\mathcal{A}_1, \ldots, \mathcal{A}_N$ the joint law of $\{\Phi_K(z, \mathcal{A}_1 + j), \ldots, \Phi_K(z, \mathcal{A}_N + j)\}$ is the same as the joint law of $\{\Phi_K(z, \mathcal{A}_1), \ldots, \Phi_K(z, \mathcal{A}_N)\}$.

In order to justify the stationarity, consider first an arbitrary $\mathcal{A}_1 \subset \mathbb{Z}^{d,+}$ and a vector $j \in \mathbb{Z}^{d,+}$. By the definition of $\Phi_K^\omega(z, \mathcal{A}_1)$, we have

$$\begin{aligned}
&\Phi_K^\omega(z, \mathcal{A}_1 + j) = \mathcal{M}_K^\omega(z, Q^{\mathcal{A}_1+j}) = \mathcal{M}_K^\omega(z, Q^{\mathcal{A}_1} + j) \\
&\quad = \inf\left\{\int_{Q^{\mathcal{A}_1}+j}\int_{\mathbb{R}^d} b_K^\omega(x, y)(v(x) - v(y))^2 dx\, dy : v(x) = \langle z, x\rangle \text{ if } \operatorname{dist}(x, \partial(Q^{\mathcal{A}_1} + j)) < K\right\} \\
&\quad = \inf\left\{\int_{Q^{\mathcal{A}_1}}\int_{\mathbb{R}^d} b_K^\omega(x - j, y - j)(v(x) - v(y))^2 dx\, dy : v(x) = \langle z, x\rangle - \langle z, j\rangle \text{ if } \operatorname{dist}(x, \partial(Q^{\mathcal{A}_1}))<K\right\} \\
&\quad = \inf\left\{\int_{Q^{\mathcal{A}_1}}\int_{\mathbb{R}^d} b_K^{\tau_{-j}\omega}(x, y)(v(x) - v(y))^2 dx\, dy : v(x) = \langle z, x\rangle \text{ if } \operatorname{dist}(x, \partial(Q^{\mathcal{A}_1})) < K\right\} \\
&\quad = \mathcal{M}_K^{\tau_{-j}\omega}(z, Q^{\mathcal{A}_1} = \Phi_K^{\tau_{-j}\omega}(z, \mathcal{A}_1).
\end{aligned}$$

The case of a finite collection of sets can be considered in the same way.

Then according to [**31**, Theorem 1], there exists $\gamma_K(z)$ such that for any $N > 0$, we have

$$\lim_{R\to\infty} \sup\left\{\left|\frac{\mathcal{M}_K^\omega(z, R(x + Q))}{R^d} - \gamma_K(z)\right| : |x| \leqslant N\right\} = 0 \tag{5.6}$$

almost surely. This implies (5.4); moreover, since $b^\omega > 0$, $K \mapsto \gamma_K(z)$ is an increasing function.

Note that we can choose a (slowly growing) sequence $N = N^\omega(R)$ such that (5.6) still holds, which yields (5.5). □

DEFINITION 2 (homogenized energy function). We define

$$\gamma(z) = \lim_{K\to+\infty} \gamma_K(z) = \sup_{K>0} \gamma_K(z).$$

For $z \in \mathbb{R}^d$, U open subset of $\mathbb{R}^d$, and $K \in \mathbb{N}$, we set

$$\widetilde{\mathcal{M}}_K^\omega(z,U) = \inf\left\{\int_U\int_U b^\omega(x,y)(v(x)-v(y))^2 dx\,dy : v(x) = \langle z,x\rangle \text{ if } \operatorname{dist}(x,\partial U) < K\right\}. \tag{5.7}$$

Note that $\widetilde{\mathcal{M}}_K^\omega(z,U)$ cannot be directly compared with $\mathcal{M}_K^\omega(z,U)$ as defined in (5.2) since on one side $b_K^\omega \leqslant b^\omega$ while the second integral is performed on U and not $\mathbb{R}^d$. However, still using $v(x) = \langle z,x\rangle$ as a test function, we get

$$\widetilde{\mathcal{M}}_K^\omega(z,x+Q_R) \leqslant CR^d|z|^2 \tag{5.8}$$

for all x and R.

LEMMA 5.2. *Let b^ω be coercive. For all K and z, we have*

$$\gamma(z) = \lim_{K\to+\infty}\limsup_{R\to+\infty}\frac{\widetilde{\mathcal{M}}_K^\omega(z,Q_R)}{R^d} = \lim_{K\to+\infty}\liminf_{R\to+\infty}\frac{\widetilde{\mathcal{M}}_K^\omega(z,Q_R)}{R^d} \tag{5.9}$$

almost surely.

The proof of this lemma is based on the following proposition.

PROPOSITION 5.3. *If U is a cube in $\mathbb{R}^d$ and $v \in L^2(U)$, then we have*

$$\int_{\{x,y\in U:|x-y|>K\}} b^\omega(x,y)(v(x)-v(y))^2 dx\,dy \leqslant CK^{-\kappa}\int_{\{x,y\in U:|x-y|<1\}}(v(x)-v(y))^2 dx\,dy, \tag{5.10}$$

with the same κ as in (2.4) and the constant C depending only on the bounds on b^ω and the dimension d.

Proof of Proposition 5.3. Without loss of generality, we may assume that the cube U is centred at the origin; that is, $U = Q_T$ for some $T > 0$. Furthermore, we may suppose that T is integer, and cover Q_T with the set of unit cubes $Q(j) = Q + j$, $j \in \mathbb{Z}^d \cap U$. If $K > T$, the statement trivially holds. Otherwise, for any j' and j'' such that $|j'-j''|_1 = n$ with $n \geqslant K$, we consider a path (that is, an array of points in $\mathbb{Z}^d$), $j' = j_0, j_1, \dots, j_n = j''$, with $|j_i - j_{i+1}|_1 = 1$, that has the following properties: in the starting segment of this path $j_0, j_1, \dots, j_{n_1}$ only the first coordinate is changed until it is equal to the first coordinate of j'' (that is, $n_1 = j''_1 - j'_1$, and $j_{i+1} = j_i + (1,0,\dots,0)$). Then we proceed with the second coordinate, and so on.

In order to estimate the contribution to the energy of the interaction between the cubes $Q(j')$ and $Q(j'')$, with fixed n we first estimate the integral

$$\begin{aligned}
&\int_{\{(y_0,y_n)\in Q\times Q\}}(v(y_0+j_0)-v(y_n+j_n))^2 dy_0 dy_n\\
&= \int_Q\cdots\int_Q\left(\sum_{i=0}^{n-1}(v(y_i+j_i)-v(y_{i+1}+j_{i+1}))\right)^2 dy_0 dy_1\dots dy_n\\
&\leqslant n\int_Q\int_Q\sum_{i=0}^{n-1}(v(x+j_i)-v(y+j_{i+1}))^2 dx\,dy\,.
\end{aligned}$$

Note that each pair of neighbouring points in $U\cap\mathbb{Z}^d$ belongs to not more than n^d paths as described above for some pair j', j'' in U such that $|j'-j''|_1=n$. Taking this into account and summing up over all j', j'' in $U\cap\mathbb{Z}^d$ with $|j'-j''|_1=n$, we obtain

$$\sum_{\substack{j',\,j''\in U\cap\mathbb{Z}^d\\ |j'-j''|_1=n}}\int_{Q\times Q}(v(x+j')-v(y+j''))^2dx\,dy\leqslant n^{d+1}\int_{(U\times U)\cap\{|x-y|_1\leqslant 2\}}(v(x)-v(y))^2dx\,dy.$$

Taking (2.4) into account, we have

$$\begin{aligned}&\int_{\{(x,y)\in U\times U:|x-y|>K\}}b^\omega(x,y)(v(x)-v(y))^2dx\,dy\\ &\quad\leqslant C\sum_{n=K}^{T}\frac{n^{d+1}}{(1+n)^{d+2+\kappa}}\int_{\{(x,y)\in U\times U:|x-y|_1\leqslant 2\}}(v(x)-v(y))^2dx\,dy\\ &\quad\leqslant CK^{-\kappa}\int_{\{(x,y)\in U\times U:|x-y|_1\leqslant 2\}}(v(x)-v(y))^2dx\,dy.\end{aligned}$$

The desired statement follows from the last inequality by a scaling argument. □

Proof of Lemma 5.2. Denote

$$\overline{\mathcal{M}}^\omega_K(z,U)=\inf\left\{\int_U\int_U b^\omega_K(x,y)(v(x)-v(y))^2dx\,dy: v(x)=\langle z,x\rangle\text{ if }\operatorname{dist}(x,\partial U)<K\right\}. \tag{5.11}$$

Then

$$\begin{aligned}0\leqslant\mathcal{M}^\omega_K(z,U)-\overline{\mathcal{M}}^\omega_K(z,U)&=\int_U\int_{\mathbb{R}^d\setminus U}b^\omega_K(x,y)\langle z,(x-y)\rangle^2dx\,dy\\ &\leqslant C|z|^2K^{1-\kappa}\mathcal{H}^{d-1}(\partial U).\end{aligned}\tag{5.12}$$

Let u be a minimizer for $\mathcal{M}^\omega_{2K}(z,U)$ (which we may assume exists). Let v be given by Definition 1 with $\Xi=2K$. We then have

$$\begin{aligned}\widetilde{\mathcal{M}}^\omega_K(z,U)&\leqslant\int_U\int_U b^\omega(x,y)(v(x)-v(y))^2dx\,dy\\ &=\int_U\int_U b^\omega_{2K}(x,y)(v(x)-v(y))^2dx\,dy\\ &\quad+\int_{\{x,y\in U:|x-y|>2K\}}b^\omega(x,y)(v(x)-v(y))^2dx\,dy\\ &\leqslant\overline{\mathcal{M}}^\omega_{2K}(z,U)+CK^{-\kappa}\int_{\{x,y\in U:|x-y|<1\}}(v(x)-v(y))^2dx\,dy\\ &\leqslant\overline{\mathcal{M}}^\omega_{2K}(z,U)+CK^{-\kappa}\int_{U\times U}b(x,y)(v(x)-v(y))^2dx\,dy\\ &\leqslant\overline{\mathcal{M}}^\omega_{2K}(z,U)+CK^{-\kappa}|z|^2|U|\\ &\leqslant\mathcal{M}^\omega_{2K}(z,U)+CK^{-\kappa}|z|^2|U|+C|z|^2K^{1-\kappa}\mathcal{H}^{d-1}(\partial U).\end{aligned}\tag{5.13}$$

Conversely, since $\overline{\mathcal{M}}^{\omega}_K(z,U) \leqslant \widetilde{\mathcal{M}}^{\omega}_K(z,U)$, we have

$$\mathcal{M}^{\omega}_K(z,U) \leqslant \widetilde{\mathcal{M}}^{\omega}_K(z,U) + C|z|^2 K^{1-\kappa}\mathcal{H}^{d-1}(\partial U). \tag{5.14}$$

Dividing by R^d, taking the upper limit in (5.13) and the lower limit in (5.14) with $U = Q_R$, we obtain

$$\begin{aligned}\gamma_K(z) = \liminf_{R\to+\infty} \frac{\mathcal{M}^{\omega}_K(z,Q_R)}{R^d} &\leqslant \liminf_{R\to+\infty} \frac{\widetilde{\mathcal{M}}^{\omega}_K(z,Q_R)}{R^d}\\ &\leqslant \limsup_{R\to+\infty} \frac{\widetilde{\mathcal{M}}^{\omega}_K(z,Q_R)}{R^d}\\ &\leqslant \limsup_{R\to+\infty} \frac{\mathcal{M}^{\omega}_{2K}(z,Q_R)}{R^d} + CK^{-\kappa}|z|^2\\ &= \gamma_{2K}(z) + CK^{-\kappa}|z|^2.\end{aligned}$$

Taking the limit as $K \to +\infty$, we obtain the claim. □

6. *Homogenization*

We now state and prove a homogenization result with respect to the strong L^2-convergence.

THEOREM 6.1. *Let D be an open set with Lipschitz boundary, and let F^{ω}_{ε} be given by (2.1) on $L^2(\Omega)$. Then F^{ω}_{ε} almost surely Γ-converge with respect to the L^2-convergence to the functional*

$$F_{\rm hom}(u) = \int_D \langle A_{\rm hom}\nabla u, \nabla u\rangle\, dx \tag{6.1}$$

on $H^1(D)$, where $A_{\rm hom}$ is a symmetric matrix which satisfies

$$\langle A_{\rm hom} z, z\rangle = \gamma(z). \tag{6.2}$$

The proof of this theorem will make use of a 'convolution version' of a classical lemma by De Giorgi that allow to match the boundary values of a target function (see [**18**, Proposition 2.2])

PROPOSITION 6.2 (Treatment of boundary values). *Let A be a bounded open set with Lipschitz boundary and let $v_\eta \to v$ in $L^2(A)$ with $v \in H^1(A)$. For every $\delta > 0$, there exist v^δ_η converging to v in $L^2(A)$ such that*

$$v^\delta_\eta = v \text{ in } A \setminus A(\delta), \qquad v^\delta_\eta = v_\eta \text{ in } A(2\delta)$$

and

$$\limsup_{\eta\to 0}(F^{\omega}_{\eta}(v^\delta_\eta) - F^{\omega}_{\eta}(v_\eta)) \leqslant o(1)$$

as $\delta \to 0$.

Proof of Theorem. 6.1 By Remark 2, it suffices to describe the Γ-limit in $H^1(D)$.

We note that F^{ω}_{ε} are quadratic functionals, so that also their Γ-limit is a quadratic functional (see [**9**]). Then, if we prove that the Γ-limit exists and admits the representation

$$F_{\rm hom}(u) = \int_D \gamma(\nabla u)\, dx, \tag{6.3}$$

then also γ must be a quadratic form on $\mathbb{R}^d$, from which the existence of a matrix $A_{\rm hom}$ satisfying (6.2) follows.

We now prove (6.3), first showing a lower bound. We fix ω, $u \in H^1(D)$ and a sequence $u_\varepsilon \to u$ with bounded $F_\varepsilon(u_\varepsilon)$. As in [**18**], we use a variation of the Fonseca–Müller blow-up technique [**22**]. We first define the measures on D given by

$$\mu_\varepsilon(A) = \frac{1}{\varepsilon^{d+2}} \int_A \int_D b^\omega\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)(u_\varepsilon(y) - u_\varepsilon(x))^2 d\xi\, dx.$$

Since $\mu_\varepsilon(D) = F_\varepsilon(u_\varepsilon)$, these measures are equibounded, and we may suppose that they converge weakly* to some measure μ. We now fix an arbitrary Lebesgue point x_0 for u and ∇u, and set $z = \nabla u(x_0)$. The lower bound inequality is proved if we show that

$$\frac{d\mu}{dx}(x_0) \geqslant \gamma(z). \tag{6.4}$$

Upon a translation argument, it is not restrictive to suppose that x_0 be a Lebesgue point of all u_ε (upon passing to a subsequence), and that $u_\varepsilon(x_0) = u(x_0) = 0$. We note that for almost all $\rho > 0$, we have $\mu_\varepsilon(x_0 + Q_\rho) \to \mu(x_0 + Q_\rho)$. Since

$$\frac{d\mu}{dx}(0) = \lim_{\rho\to 0^+} \frac{\mu(x_0 + Q_\rho)}{\rho^d},$$

and for almost all $\rho > 0$

$$\mu(Q_\rho) = \lim_{\varepsilon\to 0} \mu_\varepsilon(x_0 + Q_\rho)$$

we may choose (upon passing to a subsequence) $\rho = \rho_\varepsilon$ with $1 >> \rho >> \varepsilon$ such that

$$\frac{d\mu}{dx}(0) = \lim_{\varepsilon\to 0^+} \frac{\mu_\varepsilon(x_0 + Q_\rho)}{\rho^d}.$$

Note that we may choose ρ_ε tending to zero ‘arbitrarily slow’; that is, for all f with $\lim\limits_{\varepsilon\to 0} f(\varepsilon) = 0$, we may choose ρ_ε with

$$\rho_\varepsilon \geqslant f(\varepsilon). \tag{6.5}$$

Note moreover that

$$\begin{aligned}\mu_\varepsilon(x_0 + Q_\rho) &= \frac{1}{\varepsilon^d} \int_{x_0+Q_\rho} \int_D b^\omega\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\Big(\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}\Big)^2 dx\, dy \\ &\geqslant \frac{1}{\varepsilon^d} \int_{x_0+Q_\rho} \int_{x_0+Q_\rho} b^\omega\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\Big(\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}\Big)^2 dx\, dy.\end{aligned}$$

We now change variables and set

$$v_\varepsilon(y) = \frac{u_\varepsilon(x_0 + \rho y)}{\rho} \text{ for } y \in Q_1\,.$$

Note that, since x_0 is a Lebesgue point of both u and ∇u and we have assumed that $u(x_0) = 0$, then $\frac{u(x_0+\rho y)}{\rho}$ converges (for a subsequence) to $\langle z, y\rangle$ as $\rho \to 0$ in $L^2(Q_1)$. Since we also have assumed that $u_\varepsilon(x_0) = 0$, we may choose $\rho = \rho_\varepsilon$ above so that

$$v_\varepsilon \to \langle z, y\rangle \text{ in } L^2(Q_1).$$

By Proposition 6.2 above, applied with $v = \langle z, x\rangle$, $A = Q_1$ and $\eta = \varepsilon/\rho$, for all $\delta > 0$ there exists a sequence v^δ_ε such that $v^\delta_\varepsilon(y) = \langle z, y\rangle$ on $Q_1 \setminus Q_{1-\delta}$ and

$$\frac{1}{\varepsilon^d\rho^d}\int_{Q_\rho}\int_{Q_\rho} b^\omega\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{\varepsilon}\Big)^2\,dx\,dy$$
$$\geqslant \frac{\rho^d}{\varepsilon^d}\int_{Q_1}\int_{Q_1} b^\omega\Big(\frac{x_0}{\varepsilon}+\frac{x}{\varepsilon/\rho},\frac{x_0}{\varepsilon}+\frac{y}{\varepsilon/\rho}\Big)\Big(\frac{v^\delta_\varepsilon(x)-v^\delta_\varepsilon(y)}{\varepsilon/\rho}\Big)^2\,dx\,dy + o(1)$$

as $\delta \to 0$ uniformly in ε.

If we set $R = R_\varepsilon = \rho/\varepsilon$ and change variables, we get

$$\frac{1}{\rho^d}\mu_\varepsilon(x_0+Q_\rho) \geqslant \frac{1}{R^d}\int_{\frac{x_0}{\varepsilon}+Q_{\frac{\rho}{\varepsilon}}}\int_{\frac{x_0}{\varepsilon}+Q_{\frac{\rho}{\varepsilon}}} b^\omega(x,y)(v_R(x)-v_R(y))^2\,dx\,dy + o(1)$$

as $\delta \to 0$, where

$$v_R(x) = v^\delta_\varepsilon\Big(\frac{x}{R}-\frac{x_0}{\rho}\Big).$$

For every fixed $K > 0$, we have that

$$v_R(x) = \langle z, x\rangle \text{ if } \operatorname{dist}\Big(x, \partial\Big(\frac{x_0}{\varepsilon}+Q_{\frac{\rho}{\varepsilon}}\Big)\Big) < K$$

for ε small enough (and hence R large enough). Hence, we may use v_R as a test function in the definition on $\widetilde{\mathcal{M}}^\omega_K(z, Q_R)$. We also note that suitably choosing f in (6.5) we have that $x_R = x_0/\rho$ satisfies $|x_R| \leqslant R f_K(R)$ in Lemma 5.1, so that we finally obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\rho^d}\mu_\varepsilon(x_0+Q_\rho) \geqslant \lim_{R\to+\infty}\frac{\mathcal{M}^\omega_K(z, x_R+Q_R)}{R^d} + o(1) = \gamma_K(z) + o(1)$$

as $\delta \to 0$. Hence we have

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \geqslant \int_U \gamma_K(\nabla u)\,dx + o(1).$$

By taking the supremum in K, using the Monotone Convergence Theorem, and by the arbitrariness of δ, we get the desired lower bound.

The proof of the upper bound is obtained by a standard density argument by piecewise-affine functions (see also [**18**]) once it is shown for D a d-dimensional simplex S and $u(x) = \langle z, x\rangle$ a linear function. We consider L large enough so that $Q_L \supset D$ for some $L > 0$. We fix $m \in \mathbb{N}$ and subdivide Q_L into m^d cubes $Q^m_i = x^m_i + Q_{L/m}$ of side-length L/m and disjoint interiors. With fixed $K \in \mathbb{N}$ we choose $u^i_\varepsilon \in L^2(\frac{1}{\varepsilon}Q^m_i)$ such that $v(x) = \langle z, x\rangle$ if $\operatorname{dist}(x, \frac{1}{\varepsilon}\partial Q^m_i) < K$ and

$$\int_{\frac{1}{\varepsilon}Q^m_i\times\frac{1}{\varepsilon}Q^m_i} b^\omega(x,y)(u^i_\varepsilon(x)-u^i_\varepsilon(y))^2 dx\,dy \leqslant \mathcal{M}^\omega_K\Big(z, \frac{1}{\varepsilon}x^m_i + Q_{\frac{L}{m\varepsilon}}\Big) + 1$$
$$\leqslant \frac{L^d}{m^d\varepsilon^d}(\gamma_K(z)+o(1)) + 1 \tag{6.6}$$

as $\varepsilon \to 0$ and $K \to +\infty$.

We then define $u^m_\varepsilon \in L^2(Q)$ by setting

$$u^m_\varepsilon(x) = \varepsilon\, u^i_\varepsilon\Big(\frac{x}{\varepsilon}\Big) \text{ if } x \in Q^m_i\,.$$

We set

$$I^m = \{I : Q^m_i \cap D \neq \emptyset\},$$

and compute

$$F^\omega_\varepsilon(u^m_\varepsilon) \leqslant \sum_{i\in I^m} \frac{1}{\varepsilon^{d+2}} \int_{Q^m_i\times Q^m_i} b^\omega\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)(u^m_\varepsilon(x)-u^m_\varepsilon(y))^2 dx\,dy$$

$$+\frac{1}{\varepsilon^{d+2}}\sum_{i\neq j}\int_{\{x\in Q^m_i:\mathrm{dist}(x,\partial Q^m_i)<\varepsilon K\}}\int_{\{y\in Q^m_i:\mathrm{dist}(y,\partial Q^m_i)<\varepsilon K\}} b^\omega\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)|z|^2|x-y|^2\,dx\,dy$$

$$+\frac{1}{\varepsilon^{d+2}}\int_{\{x,y\in Q_L:|x-y|>\varepsilon K\}} b^\omega\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)(u^m_\varepsilon(x)-u^m_\varepsilon(y))^2 dx\,dy$$

$$\leqslant \sum_{i\in I^m}\varepsilon^d\int_{\frac{1}{\varepsilon}Q^m_i\times\frac{1}{\varepsilon}Q^m_i} b^\omega(x,y)(u^i_\varepsilon(x)-u^i_\varepsilon(y))^2dx\,dy + CKm\varepsilon|z|^2+CK^{-\eta}$$

$$\leqslant \Big(|U|+O\Big(\frac{1}{m}\Big)\Big)\gamma_K(z)+o(1)+CKm\varepsilon|z|^2+CK^{-\eta}.$$

Note that we have used assumption (2.2) to estimate the second term in the sum, and Proposition 5.3 with $U=\frac{L}{\varepsilon}Q$ and the coerciveness of b^ω to estimate the third term in the sum.

We may now choose $m=m_\varepsilon\to+\infty$ such that

$$\limsup_{\varepsilon\to0} F^\omega_\varepsilon(u^m_\varepsilon)\leqslant L^d\gamma_K(z)+o(1)$$

as $K\to+\infty$. Note that, since $u^m_\varepsilon(x)=\langle z,x\rangle$ if $\mathrm{dist}(x,\bigcup_i\partial(Q^m_i))<\varepsilon K$, then $u^m_\varepsilon\to\langle z,x\rangle$ in $L^2(D)$ and we obtain an upper bound with $\gamma_K(z)+o(1)$. Letting $K\to+\infty$, we finally have the desired estimate. □

7. *Random perforated domains*

In this section, we note that Theorem 6.1 can be applied to the homogenization on randomly perforated domains.

First we define random sets in $\mathbb{R}^d$.

DEFINITION 3 (Random sets and random perforations). We say that E^ω is a *random set* in $\mathbb{R}^d$ if there exists $\Omega_1\in\mathcal{F}$ with $\mathbf{P}(\Omega_1)\mathbf{P}(\Omega\setminus\Omega_1)>0$ such that $E^\omega=\{x\in\mathbb{R}^d : \chi_{\Omega_1}(\tau_x\omega)=1\}$ for each $\omega\in\Omega$.

A random set E^ω is called a *random perforated domain* if it possesses the following properties.

(i) Almost surely $\mathbb{R}^d\setminus E^\omega$ is a union of bounded open sets in $\mathbb{R}^d$.
(ii) The diameters of these sets are uniformly bounded.
(iii) The distance between any two distinct sets is bounded from below by a positive constant.
(iv) The boundary of these sets are uniformly Lipschitz continuous; that is, there exist constants $L>0$ and ρ_1, $\rho_2>0$ such that for any point $x\in\partial E^\omega$ there exists a set C which, up to translation by x and rotation, is of the form $(-\rho_1,\rho_1)^{d-1}\times(-\rho_2,\rho_2)$ such that $C\cap E^\omega$ is the sub-graph of a L-Lipschitz function defined on $(-\rho_1,\rho_1)^{d-1}$.

There is a great variety of random perforated domains. We consider here two examples to clarify a possible structure of such domains.

(i) Consider a random point set in $\mathbb{R}^d$ that was used in [**3, 37**]. Recall that this set is stationary, and there exist constants $r>0$ and $R>r$ such that the distance between any two points of this set is almost surely greater than r, and any ball of radius R contains at least one point of this set. Then, for each point of the set, we place a ball of radius $r/2$ centred at this point and take the union of such balls. The complement to this union gives us an example of random perforated domain.

(ii) For a Poisson point process $X(\omega) = \{x_j^\omega\}$ in $\mathbb{R}^d$ with intensity one, we consider the corresponding Voronoi tessellation $V(\omega) = v_j^\omega$ with $v_j = \{x \in \mathbb{R}^d : \mathrm{dist}(x, x_j) < \mathrm{dist}(x, X \setminus x_j)\}$. Then for positive r and R, $0 < r < R$, we select all the indices j such that the ball of radius r centred at x_j belongs to v_j and $\mathrm{diam}(v_j) < R$. We introduce a random perforation as the union over these j of the sets $\tilde{v}_j = \{x \in \mathbb{R}^d : \mathrm{dist}(x, x_j) < \frac{1}{2}\mathrm{dist}(x, X \setminus x_j)\}$ and define a random perforated set as its complement.

We now assume that E^ω is a random perforated domain, and we set

$$b^\omega(x, y) = \chi_{E^\omega}(x)\chi_{E^\omega}(y)a(x - y). \tag{7.1}$$

The key observation is that such b^ω is coercive. This is implied by the following theorem in [**18**, Theorem 3.2].

Theorem 7.1 (extension theorem). *Let E^ω be a random perforated domain that satisfies condition* (1)–(4) *above. Let b^ω be defined by* (7.1)*. Then there exists $k > 0$ and $r > 0$ such that almost surely for all $u \in L^2(D \cap \varepsilon E^\omega)$ there exists $v \in L^2(D)$ such that*

$$v = u \quad \text{on } D \cap \varepsilon E^\omega, \tag{7.2}$$

$$\int_{D(k\varepsilon)} \int_{\{|\xi| \leqslant r\}} \left(\frac{v(x + \varepsilon\xi) - v(x)}{\varepsilon} \right)^2 d\xi\, dx \leqslant C F_\varepsilon^\omega(u) \tag{7.3}$$

and

$$\int_{D(k\varepsilon)} |v|^2\, dx \leqslant C \int_{D \cap \varepsilon E} |u|^2\, dx. \tag{7.4}$$

Theorem 6.1 can be rephrased as follows.

Theorem 7.2. *Let D be an open set with Lipschitz boundary, let E^ω be a random perforated domain as above, and let F_ε^ω be given by*

$$F_\varepsilon^\omega(u) = \frac{1}{\varepsilon^{d+2}} \int_{(D \cap \varepsilon E^\omega) \times (D \cap \varepsilon E^\omega)} a\left(\frac{x - y}{\varepsilon}\right) (u(y) - u(x))^2 dy\, dx. \tag{7.5}$$

Then F_ε^ω almost surely Γ-converge with respect to the L^2-convergence to the functional (6.1) *on $H^1(D)$, where A_{hom} is a symmetric matrix which satisfies*

$$\langle A_{\mathrm{hom}} z, z \rangle = \lim_{K \to +\infty} \lim_{R \to +\infty} \frac{1}{R^d} \inf \left\{ \int_{Q_R \cap E^\omega} \int_{E^\omega} a(x - y)(v(x) - v(y))^2 dx\, dy : v(x) = \langle z, x \rangle \text{ if } \mathrm{dist}(x, \partial Q_R) < K \right\}. \tag{7.6}$$

In order to deal with minimization problems in perforated domains, we need a version of Poincaré inequality adapted to the perforated domains geometry. The following statement is a corollary of Theorem 4.1.

Corollary 7.3. *Let $r_0 > 0$ be defined in* (1.8)*. Let $k > 0$ and $r > 0$ be the same as in Theorem* 7.1*. Then for any $u \in L^2(D)$, the following inequality holds:*

$$\int_{D(k\varepsilon) \cap \varepsilon E} \left(u(x) - u_{\{D(k\varepsilon) \cap \varepsilon E\}}\right)^2 dx \leqslant C F_\varepsilon(u); \tag{7.7}$$

here

$$u_{\{D(k\varepsilon)\cap\varepsilon E\}} = \frac{1}{|D(k\varepsilon)\cap\varepsilon E|}\int_{\{D(k\varepsilon)\cap\varepsilon E\}} u(x)\,dx.$$

Proof. Consider an extension of function u that belongs to $L^2(D)$ and satisfies conditions (7.2)–(7.4). We denote this extension by v, its existence is granted by Theorem 7.1. Denoting

$$v_{\{D(k\varepsilon)\}} = \frac{1}{|D(k\varepsilon)|}\int_{D(k\varepsilon)} v(x)dx,$$

we have

$$\begin{aligned}
&\int_{D(k\varepsilon)\cap\varepsilon E}\big(u(x)-u_{\{D(k\varepsilon)\cap\varepsilon E\}}\big)^2\,dx \\
&\leqslant \int_{D(k\varepsilon)\cap\varepsilon E}\big(u(x)-v_{\{D(k\varepsilon)\}}\big)^2\,dx \leqslant \int_{D(k\varepsilon)}\big(v(x)-v_{\{D(k\varepsilon)\}}\big)^2\,dx \\
&\leqslant C\int_{D(k\varepsilon)}\int_{\{|\xi|\leqslant r_0\}}\left(\frac{v(x+\varepsilon\xi)-v(x)}{\varepsilon}\right)^2 d\xi\,dx \leqslant C_1F_\varepsilon(u);
\end{aligned}$$

here we have used Theorem 4.1 and inequality (7.3). □

We provide now an example of homogenization of a minimization problem. Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, and assume that E^ω is a random perforated domain. From the definition of E^ω, it follows that $D\setminus\varepsilon E^\omega$ is a union of a finite number of bounded open sets whose diameter does not exceed $c\varepsilon$ with a deterministic constant $c>0$. We denote these sets $S_1^\varepsilon,\dots S_N^\varepsilon$ with $N=N(\varepsilon,\omega)$, and define

$$\tilde D_\varepsilon = D\setminus\bigcup_{\{j\,:\,\mathrm{dist}(S_j^\varepsilon,\partial D)>\sqrt{\varepsilon}\}} S_j.$$

Denote $\mathring{L}^2(\tilde D_\varepsilon)=\{u\in L^2(\tilde D_\varepsilon):\int_{L^2(\tilde D_\varepsilon)}u dx=0\}$ and $\mathring{L}^2(D)=\{u\in L^2(D):\int_{L^2(D)}u dx=0\}$. Given $f\in\mathring{L}^2(D)$, consider the following minimization problem

$$\frac{1}{\varepsilon^{d+2}}\int_{\tilde D_\varepsilon\times\tilde D_\varepsilon} a\Big(\frac{x-y}{\varepsilon}\Big)(u(y)-u(x))^2dxdy-\int_{\tilde D_\varepsilon}f(x)u(x)dx \;\longrightarrow\; \min, \tag{7.8}$$

where the minimum is taken over $\mathring{L}^2(\tilde D_\varepsilon)$.

PROPOSITION 7.4. *Under assumptions (1.7) and (1.8), problem (7.8) has a unique minimizer u^ε. Moreover, as $\varepsilon\to0$, u^ε converges in $\mathring{L}^2(\tilde D_\varepsilon)$ towards the unique minimizer u^0 of the problem*

$$F_{\rm hom}(u)-\mathbf{P}(\Omega_1)\int_D f(x)u(x)dx \;\longrightarrow\; \min,$$

where the minimum is taken over $\mathring{L}^2(D)\cap H^1(D)$, and $F_{\rm hom}$ is defined in (7.6).

Proof. By [**18**, Theorem 3.2] for any $u\in\mathring{L}^2(\tilde D_\varepsilon)$, there exist $k>0$, $r>0$ and an extension v^ε of u to D such that

$$\int_{(D(k\varepsilon)\times D(k\varepsilon))\cap\{|x-y|\leqslant r\varepsilon\}}(v^\varepsilon(x)-v^\varepsilon(y))^2dxdy\leqslant C\int_{\tilde D_\varepsilon\times\tilde D_\varepsilon}a\Big(\frac{x-y}{\varepsilon}\Big)(u(y)-u(x))^2dxdy$$

with $D(k\varepsilon) = \{x \in D : \operatorname{dist}(x, \partial D) > k\varepsilon\}$. Since for all sufficiently small ε in $D \setminus D(k\varepsilon)$, we have $v^\varepsilon(x) = u(x)$ and the last inequality yields

$$\int_{(D\times D)\cap\{|x-y|\leqslant r\varepsilon\}} (v^\varepsilon(x) - v^\varepsilon(y))^2 dxdy \leqslant C \int_{\tilde{D}_\varepsilon \times \tilde{D}_\varepsilon} a\left(\frac{x-y}{\varepsilon}\right)(u(y) - u(x))^2 dxdy. \qquad (7.9)$$

By the Poincaré inequality stated in Theorem 4.1, we obtain

$$\frac{1}{\varepsilon^{d+2}} \int_{(D\times D)\cap\{|x-y|\leqslant r\varepsilon\}} (v^\varepsilon(x) - v^\varepsilon(y))^2 dxdy \geqslant c \int_D (v^\varepsilon(x) - v^\varepsilon_D)^2 \, dx$$

$$\geqslant c \int_{\tilde{D}_\varepsilon} (u(x) - v^\varepsilon_D)^2 \, dx \geqslant c \int_{\tilde{D}_\varepsilon} (u(x))^2 \, dx.$$

Therefore,

$$\frac{1}{\varepsilon^{d+2}} \int_{\tilde{D}_\varepsilon \times \tilde{D}_\varepsilon} a\left(\frac{x-y}{\varepsilon}\right)(u(y) - u(x))^2 dxdy \geqslant \int_{\tilde{D}_\varepsilon} (u(x))^2 \, dx.$$

This implies that for each $\varepsilon > 0$ the functional in (7.8) has a unique minimum point, and, denoting this minimum point u^ε, we have $\|u^\varepsilon\|_{L^2(\tilde{D}_\varepsilon)} \leqslant C$. Taking one more time the extension of u^ε to D and applying Theorem 3.1, we conclude that the said extensions are relatively compact in $L^2(D')$ for any open set D' such that $\overline{D'} \subset D$. The desired statement now follows from Theorem 7.2 and we should also take into account the relation

$$\int_{\tilde{D}_\varepsilon} f u^\varepsilon dx \to \mathbf{P}(\Omega_1) \int_D f u^0 dx, \quad \text{as } \varepsilon \to 0,$$

which is a consequence of the Birkhoff theorem. □

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