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HOMOGENIZATION OF A NONLINEAR CONVECTION-DIFFUSION EQUATION WITH RAPIDLY OSCILLATING COEFFICIENTS AND STRONG CONVECTION

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ABSTRACT

A Cauchy problem for a nonlinear convection-diffusion equation with periodic rapidly oscillating coefficients is studied. Under the assumption that the convection term is large, it is proved that the limit (homogenized) equation is a nonlinear diffusion equation which shows dispersion effects. The convergence of the homogenization procedure is justified by using a new version of a two-scale convergence technique adapted to rapidly moving coordinates.

1. *Introduction*

This paper is devoted to homogenization of a model semilinear parabolic equation of convection-diffusion type with periodic, rapidly oscillating coefficients. The material is stratified, that is oscillations are allowed in all but one direction. We assume that the convection term is large, which is related to the self-similar diffusive scaling in the equation. For this strong convection term we do not suppose that the convection velocity is divergence free, nor that the effective drift is zero. As described in [**8**, Chapter 2] in such a situation the convection might dominate the diffusion, and we cannot expect nontrivial convergence of the family of solutions $u^\varepsilon(t,x)$ for a fixed spatial frame x but only in moving coordinates $x + B^\varepsilon(t)$. Due to the choice of scaling, in appropriate moving coordinates the homogenized problem shows the diffusive dynamics. As a consequence of the presence of strong nonlinear convection, the dispersion effects appear, that is the diffusion coefficients of the limit quasilinear problem depend on the convection velocity (see, for example [**2**] or [**11**] for the formal asymptotic explanation of the dispersion). To prove the result we adapt the two-scale convergence method introduced by Nguetseng [**12**] and Allaire [**1**], to the case of rapidly moving coordinates, and we combine it with the appropriate choice of test functions depending on the solution of the adjoint auxiliary problem (5).

Also, since classical theorems on compactness of embedding of Sobolev spaces in bounded domains do not apply in the whole space, in our case the compactness result for the family of solutions is not a straightforward consequence of a priori estimates. We show that the uniform localization of solutions holds in moving coordinates, and, in this way, we gain the compactness in the moving coordinates.

The problem under consideration appears, for instance, when studying the long-term behaviour of the nonlinear convection-diffusion model in stratified periodic media. In this case, letting $\varepsilon = 1/\sqrt{T}$ and making self-similar rescaling, we arrive at our homogenization problems. The desired long-term behaviour can now be described in terms of the effective characteristics of this problem.

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Previously, homogenization problems for linear convection-diffusion models with a zero mean drift were considered in [**4**] and then in many other works. The case of divergence-free convection term has been widely studied in the existing literature; for instance, [**3**, **7**] dealt with equations with small diffusion coefficients. The homogenization result for general linear periodic convection-diffusion operators with nontrivial effective drift was obtained in [**13**].

2. *Setting of the problem*

We study the asymptotic behaviour of solutions of the Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(\mathbf{A}^\varepsilon(t,x)\nabla u^\varepsilon) + \varepsilon^{-1} b^\varepsilon(t,x,u^\varepsilon)\cdot\nabla u^\varepsilon = 0 \qquad \text{in }]0,T[\times\mathbf{R}^n, \tag{1}$$

$$u^\varepsilon(0,x) = \varphi(x), \qquad x\in\mathbf{R}^n, \tag{2}$$

with $\mathbf{A}^\varepsilon(t,x) = \mathbf{A}(t,x'/\varepsilon)$ and

$$b^\varepsilon(t,x,v) = \Big(a\Big(t,\frac{x'}{\varepsilon}\Big), h\Big(t,\frac{x'}{\varepsilon}\Big)f(v)\Big), \tag{3}$$

where $x = (x',x_n)\in\mathbf{R}^n$, $x'\in\mathbf{R}^{n-1}$, $x_n\in\mathbf{R}$. The matrix function $\mathbf{A}(t,y) = [\mathbf{A}_{ij}(t,y)]$, vector function $a(t,y) = (a_1(t,y),\dots,a_{n-1}(t,y))$ and scalar function $h(t,y)$ are assumed to be 1-periodic in $y=(y_1,\dots,y_{n-1})$, and thus can be identified with the corresponding functions on the $(n-1)$-dimensional torus denoted by Y. We suppose throughout this paper that the following hold.

(1) Coefficients $\mathbf{A}, a$ and h are of class $C^2_{\mathrm{per}}(Y)$.
(2) The nonlinearity $f\in C^2(\mathbf{R})$.
(3) The initial condition $\varphi\in C_0^\infty(\mathbf{R}^n)$.
(4) The diffusion tensor $\mathbf{A}$ is uniformly positive definite, that is there exists a constant $c_0>0$ such that for any $(t,y)\in]0,T[\ \times\ Y$ and $\xi\in\mathbf{R}^n$

$$\xi\cdot\mathbf{A}\,\xi = \sum_{i,j=1}^n \mathbf{A}_{ij}(t,y)\,\xi_i\,\xi_j \geqslant c_0\,|\xi|^2. \tag{4}$$

Since the convection term here is not divergence free, the following auxiliary problem plays an important role in further analysis:

$$\operatorname{div}_y(\mathbf{A}^t\,\nabla_y z + a\,z) = 0 \qquad \text{in } Y, \tag{5}$$

where we use the following notation for partial differential operators with respect to y.

$$\operatorname{div}_y v = \sum_{\alpha=1}^{n-1}\frac{\partial v_\alpha}{\partial y_\alpha}, \qquad \nabla_y\,\chi = \Big(\frac{\partial\chi}{\partial y_1},\dots,\frac{\partial\chi}{\partial y_{n-1}}\Big),\ (\mathbf{A}^t\,\nabla_y z)_\alpha(t,y) = \sum_{\beta=1}^{n-1} A_{\alpha\beta}(t,y)\frac{\partial z}{\partial y_\beta},$$

and we always assume that any function depending on y is Y periodic, that is defined on the torus $Y = T^{n-1}$. Equation (5) is linear and has a nontrivial solution in the space of Y-periodic functions. Furthermore, under the normalization

$$\int_Y z(t,y)\,dy = 1 \tag{6}$$

the solution is unique and, due to the zero maximum principle (see [**8**, Chapter 1]), strictly positive.

In addition to the above assumptions, for the nonlinear convection term we assume that the function h satisfies the hypothesis

$$\int_Y z(t,y)\, h(t,y)\, dy = 0, \qquad \forall\, t \in [0,T]. \tag{7}$$

In case of divergence-free linear convection velocity $\operatorname{div}_{x'} a = 0$ the auxiliary function z is a constant and the above hypothesis reduces to the assumption that the mean value of h is zero.

REMARK 1. The above assumption should be understood in the sense that the average, taken with respect to the corresponding invariant measure z dy, of the last nonlinear component of convection velocity, is zero.

The particular form of the nonlinear convection term (3) is a technical assumption which enables the separation of scales in the homogenized problem and it does not seem to be essential. On the contrary, hypothesis (7) seems to be crucial. Under this hypothesis the (large) principal term of the effective convection does not depend on the profile of the solution which makes it possible to introduce moving coordinates. It turns out that, without condition (7), the formal homogenization procedure leads to an ill-posed problem.

3. *Solvability of the ε-problem*

In this section we present the existence and uniqueness result for the Cauchy problem (1), (2). Since a priori estimates are easy to derive (see Section 5), it is a simple exercise to prove the existence of the weak solution using the compactness argument. We can also combine [**9**, Theorem 8.1] (about classical solutions) and [**9**, Theorem 5.2] (about weak solutions) to get the following result.

THEOREM 1. *Suppose that $f \in C^2(\mathbf{R})$, $h \in C^{1,\beta}([0,T]; C^{1,\beta}_{\mathrm{per}}(Y))$, $a \in C^{1,\beta}([0,T]; C^{1,\beta}_{\mathrm{per}}(Y)^{n-1})$, $\varphi \in C^{2,\beta}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ and $\mathbf{A} \in C^{1,\beta}([0,T]; C^{1,\beta}_{\mathrm{per}}(Y)^{n\times n})$ for some $\beta \in\,]0,1[$, and let the uniform ellipticity condition* (4) *be fulfilled. Then problem* (1), (2) *has a unique classical solution*:

$$\begin{aligned} u^\varepsilon \in H^{2+\beta,1+\beta/2}(\mathbf{R}^n_T) = \{ & v \in C([0,T]\times \mathbf{R}^n); \\ & \left(\frac{\partial}{\partial t}\right)^r D^s_x\, v \in C([0,T]\times \mathbf{R}^n), 2r+s < 2+\beta\}. \end{aligned}$$

Furthermore, if $\varphi \in L^2(\mathbf{R}^n)$ then there is a solution $u^\varepsilon \in L^2(0,T; H^1(\mathbf{R}^n)) \cap L^\infty(\,]0,T[\, \times \mathbf{R}^n)$, and it is unique in this class.

4. *Main result*

We are interested in the macroscopic behaviour of the solution u^ε. The main result of the paper is summarized in the following theorem.

THEOREM 2. *Let u^ε be the solution of the problem* (1), (2), *and let* $w^\varepsilon(t,x) = u^\varepsilon(t, x+\varepsilon^{-1}B(t))$, *where B is defined by*

$$B(t) = \int_0^t \int_Y (a(s,y) - \operatorname{div}\mathbf{A}(s,y))\, z(s,y)\, dy\, ds.$$

Then

$$w^\varepsilon \to w^0 \ \text{ weakly in } L^2(0,T;H^1(\mathbf{R}^n)) \text{ and strongly in } L^2(0,T;L^2(\mathbf{R}^n)).$$

The limit w^0 is the unique solution of homogenized problem (40) *below. This homogenized problem is well-posed.*

The proof of this theorem is given in Section 6.

5. *A priori estimates*

As was explained in the introduction, there is no hope of obtaining a nontrivial convergence result for u^ε itself. In order to improve the situation, we introduce moving coordinates $(t,x) \to (t, x+\varepsilon^{-1}B(t))$ and study the function

$$w^\varepsilon(t,x) = u^\varepsilon(t, x+\varepsilon^{-1}B(t)),$$

where

$$B(t) = \int_0^t \int_Y (a(s,y) - \operatorname{div}\mathbf{A}(s,y))\, z(s,y)\, dy\, ds. \tag{8}$$

We also denote by

$$\bar{b}(t) = \frac{d}{dt}B(t) = \int_Y (a(t,y) - \operatorname{div}\mathbf{A}(t,y))\, z(t,y)\, dy \tag{9}$$

the effective drift vector (see [**8**]).

To avoid the dimension inconsistency, we also introduce the n-dimensional vectors $(B,0) = (B_1,\dots,B_{n-1},0)$, $(\bar{b},0) = (\bar{b}_1,\dots,\bar{b}_{n-1},0)$, $(a^\varepsilon,0) = (a_1^\varepsilon,\dots,a_{n-1}^\varepsilon,0)$, etc. Abusing slightly the notation we will use, for these extended vectors, the same symbols B, $\bar{b}$, a^ε, etc. if it does not lead to ambiguity.

We are now ready to prove the a priori estimates.

PROPOSITION 1. *There exists a constant $C>0$, independent of ε, such that*

$$|w^\varepsilon|_{L^2(0,T;H^1(\mathbf{R}^n))} \leqslant C \tag{10}$$

$$|w^\varepsilon|_{L^\infty(]0,T[\,\times\mathbf{R}^n)} \leqslant C. \tag{11}$$

Proof. We define

$$z^\varepsilon(t,x) = z\left(t, \frac{x'}{\varepsilon}\right).$$

Using $z^\varepsilon u^\varepsilon$ as a test function in the variational formulation of problem (1), (2), we arrive at the relation

$$\int_0^T \int_{\mathbf{R}^n} z^\varepsilon\, \mathbf{A}^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon\, dx\, dt = \frac{1}{2}\int_{\mathbf{R}^n} [\varphi(0,x)^2 z^\varepsilon(0,x) - u^\varepsilon(T,x)^2 z^\varepsilon(T,x)]\, dx - \int_0^T \int_{\mathbf{R}^n} (u^\varepsilon)^2 \frac{\partial z^\varepsilon}{\partial t}\, dx\, dt,$$

which implies, by the Gronwall lemma, the uniform in ε bound (10). Estimate (11) follows from the maximum principle (see [**9**] or [**10**]). $\square$

The above estimates only imply weak compactness of w^ε. Since our equation is nonlinear, we need some strong compactness result in order to pass to the limit. The problem is posed in the whole space $\mathbf{R}^n$ which is inconvenient for proving strong compactness. Thus, our first goal is to restrict our study to a compact subset. We introduce the notation

$$Q_R =]-R, R[^{\,n}.$$

LEMMA 1. *For any $\delta > 0$, there exists $R(\delta) > 0$ such that*

$$|w^\varepsilon|_{L^2(]0,T[\,\times\,(\mathbf{R}^n\setminus Q_{R(\delta)}))} \leqslant \delta. \tag{12}$$

Proof. Let $\varphi \in C^\infty(\mathbf{R})$ be a cut-off function, such that $0 \leqslant \varphi \leqslant 1$, $\varphi(s) = 0$ for $s \leqslant 1$ and $\varphi = 1$ for $s \geqslant 2$. We denote by $\varphi_r(x) = \varphi(|x|/r)$ and by $\varphi_r^\varepsilon(t,x) = \varphi_r(x - \varepsilon^{-1}B(t))$, where $r > 0$ will be chosen later on. Multiplying (1) by $u^\varepsilon\, z^\varepsilon\, \varphi_r^\varepsilon$ and integrating by parts, we get on the left-hand side

$$\begin{aligned}
\int_0^T \int_{\mathbf{R}^n} \frac{\partial u^\varepsilon}{\partial t}\, u^\varepsilon\, z^\varepsilon \varphi_r^\varepsilon\, dx\, dt &= -\frac{1}{2}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2 \varphi_r^\varepsilon \frac{\partial z^\varepsilon}{\partial t}\, dx\, dt \\
&+ \frac{1}{2\varepsilon}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2\, \overline{b}(t)\cdot\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)(\varphi_r^\varepsilon)'\, z^\varepsilon\, dx\, dt \\
&+ \frac{1}{2}\int_{\mathbf{R}^n} |u^\varepsilon(T,\,\cdot)|^2\, \varphi_r^\varepsilon(T,\,\cdot)\, z^\varepsilon(T,\,\cdot)\, dx - \frac{1}{2}\int_{\mathbf{R}^n} |u_0|^2\, \varphi_r\, z^\varepsilon(0,\,\cdot)\, dx,
\end{aligned} \tag{13}$$

where $(\varphi_r^\varepsilon)'(t,x) = (d/ds)\varphi(s)\Big|_{s=(r^{-1}\,|x-\varepsilon^{-1}B(t)|)}$. By choosing sufficiently large r we obtain

$$\int_{\mathbf{R}^n} |u_0|^2\, \varphi_r\, z^\varepsilon(0,\,\cdot)\, dx = 0.$$

On the right-hand side we have

$$\begin{aligned}
&\int_0^T\!\!\int_{\mathbf{R}^n} u^\varepsilon\, \varphi_r^\varepsilon z^\varepsilon\, [\operatorname{div} \mathbf{A}^\varepsilon \nabla u^\varepsilon - \varepsilon^{-1} b^\varepsilon \cdot \nabla u^\varepsilon]\, dx\, dt \\
&\quad = -\frac{1}{2\varepsilon}\int_0^T \int_{\mathbf{R}^n} h^\varepsilon\, f(u^\varepsilon)\, u^\varepsilon\, \frac{\partial u^\varepsilon}{\partial x_n}\, \varphi_r^\varepsilon\, z^\varepsilon\, dx\, dt \\
&\quad - \int_0^T\!\!\int_{\mathbf{R}^n} z^\varepsilon \varphi_r^\varepsilon \mathbf{A}^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon - \int_0^T \int_{\mathbf{R}^n} u^\varepsilon z^\varepsilon\, (\varphi_r^\varepsilon)'\, \mathbf{A}^\varepsilon\!\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\cdot \nabla u^\varepsilon\, dx\, dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2 (\varphi_r^\varepsilon)'\, \mathbf{A}^\varepsilon \left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\cdot \nabla z^\varepsilon\, dx\, dt \\
&\quad + \frac{1}{2\varepsilon}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2 z^\varepsilon a^\varepsilon \cdot \left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)(\varphi_r^\varepsilon)'\, dx\, dt \\
&\quad + \frac{1}{2\varepsilon}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2\, \varphi_r^\varepsilon \operatorname{div}(z^\varepsilon\, a^\varepsilon)\, dx\, dt + \frac{1}{2}\int_0^T \int_{\mathbf{R}^n} |u^\varepsilon|^2\, \varphi_r^\varepsilon \operatorname{div}(\mathbf{A}^t \nabla z^\varepsilon)\, dx\, dt.
\end{aligned}$$

We recall that a^ε in this formula stands for $(a_1^\varepsilon, \ldots, a_{n-1}^\varepsilon, 0)$. The last two integrals mutually cancel due to the definition of z. With the remaining terms, (13) can be written in the form

$$\frac{1}{2}\int_{\mathbf{R}^n} |u^\varepsilon(T,\cdot)|^2\,\varphi_r^\varepsilon(T,\cdot)\,z^\varepsilon(T,\cdot)\,dx + \int_0^T\int_{\mathbf{R}^n} \mathbf{A}^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon\; z^\varepsilon\;\varphi_r^\varepsilon\,dx\,dt$$

$$= -\frac{1}{2\varepsilon}\int_0^T\int_{\mathbf{R}^n} \varphi_r^\varepsilon z^\varepsilon h^\varepsilon f(u^\varepsilon)\,u^\varepsilon\,\frac{\partial u^\varepsilon}{\partial x_n}\,dx\,dt \tag{14}$$

$$-\frac{1}{2}\int_0^T\int_{\mathbf{R}^n} z^\varepsilon\,(\varphi_r^\varepsilon)'\mathbf{A}^\varepsilon\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\cdot\nabla|u^\varepsilon|^2\,dx\,dt \tag{15}$$

$$+\frac{1}{2\varepsilon}\int_0^T\int_{\mathbf{R}^n} |u^\varepsilon|^2(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\cdot\left(-z^\varepsilon\;\overline{b}+(\mathbf{A}^t\nabla_y z)^\varepsilon+z^\varepsilon\;a^\varepsilon\right)dx\,dt.$$

The last integral can be controlled using the fact that there exists a periodic in y matrix-function $\eta(t,y)$, such that

$$\mathrm{div}_y\,\eta = (a-\overline{b})\,z + \mathbf{A}^t\nabla_y z. \tag{16}$$

Indeed, due to the definition of $\overline{b}$, the right-hand side has a zero mean for any $t\in[0,T]$

$$\int_Y (a-\overline{b}+\mathbf{A}^t\nabla_y z) = \int_Y (a-\overline{b}-\mathrm{div}\,\mathbf{A})\,z\;dy = 0,$$

which is a necessary and sufficient condition for the existence of such a function η. Then we have for $\eta^\varepsilon(t,x)=\eta(t,x'/\varepsilon)$

$$J_1^\varepsilon = \frac{1}{2\varepsilon}\int_0^T\int_{\mathbf{R}^n} |u^\varepsilon|^2(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\cdot(-\overline{b}\,z^\varepsilon+(\mathbf{A}^t\nabla_y z)^\varepsilon+a^\varepsilon z^\varepsilon)\,dx\;dt$$

$$= -\frac{1}{2}\int_0^T\int_{\mathbf{R}^n} |u^\varepsilon|^2(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\;\cdot\;\mathrm{div}\,\eta^\varepsilon\,dx\,dt$$

$$= \frac{1}{2}\int_0^T\int_{\mathbf{R}^n}\left\{(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\;\cdot\;\eta^\varepsilon\nabla\,|u^\varepsilon|^2\right.$$

$$\left. +|u^\varepsilon|^2\;(\eta^\varepsilon\nabla)\cdot\left[(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\right]\right\}dx\,dt.$$

Since

$$\nabla\left[(\varphi_r^\varepsilon)'\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)\right] = \varphi''(r^{-1}\,|x-\varepsilon^{-1}B|)\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)$$

$$\otimes\left(\frac{x-\varepsilon^{-1}B}{r\,|x-\varepsilon^{-1}B|}\right)+\varphi(r^{-1}\,|x-\varepsilon^{-1}B|)\;|x-\varepsilon^{-1}B|^{-1}\,r^{-2}$$

$$\times\left(\mathbf{I}+\frac{(x-\varepsilon^{-1}B)\otimes(x-\varepsilon^{-1}B)}{|x-\varepsilon^{-1}B|^2}\right)$$

we have

$$\left|J_1^\varepsilon\right| \leqslant C\,(r^{-1}+r^{-2}) \tag{17}$$

with $C>0$ independent of ε. Next we treat the integral

$$J_2^\varepsilon = \frac{1}{2\varepsilon}\int_0^T\int_{\mathbf{R}^n} \varphi_r^\varepsilon z^\varepsilon h^\varepsilon f(u^\varepsilon)u^\varepsilon\frac{\partial u^\varepsilon}{\partial x_n}\;dx\,dt.$$

We first notice that, due to assumption (7), there exists a function $\psi(t,y)$ such that

$$\mathrm{div}_y\psi = h\,z \qquad \text{in }\;]0,T[\,\times Y. \tag{18}$$

We let $\psi^\varepsilon(t,x) = \psi(t,x'/\varepsilon)$ and, furthermore, define the function

$$G(\tau) = \int_0^\tau s\, f(s)\, ds.$$

Then

$$2\, J_2^\varepsilon = \int_0^T \int_{\mathbf{R}^n} \varphi_r^\varepsilon \operatorname{div} \psi^\varepsilon \frac{\partial}{\partial x_n} G(u^\varepsilon)\, dx\, dt = -\int_0^T \int_{\mathbf{R}^n} \frac{\partial \varphi_r^\varepsilon}{\partial x_n} \operatorname{div} \psi^\varepsilon\, G(u^\varepsilon)\, dx\, dt$$
$$= \int_0^T \int_{\mathbf{R}^n} \left(\frac{\partial \varphi_r^\varepsilon}{\partial x_n}\, \psi^\varepsilon\, f(u^\varepsilon)\, u^\varepsilon\, \nabla_{x'} u^\varepsilon + G(u^\varepsilon)\, \psi^\varepsilon\, \nabla_{x'} \frac{\partial \varphi_r^\varepsilon}{\partial x_n} \right) \leqslant C\,(r^{-1} + r^{-2}),$$

again with $C > 0$ independent of ε. Since the remaining integral on the right-hand side of (15) does not have a big factor, it is straightforward to see that this integral is of order r^{-1}, so that

$$\tfrac{1}{2} \int_{\mathbf{R}^n} |u^\varepsilon(T,\,\cdot)|^2\, \varphi_r^\varepsilon(T,\,\cdot)\, z^\varepsilon(T,\,\cdot) + \int_0^T \int_{\mathbf{R}^n} \mathbf{A}^\varepsilon \nabla u^\varepsilon\, \nabla u^\varepsilon\, z^\varepsilon\, \varphi_r^\varepsilon \leqslant C\,(r^{-1} + r^{-2}),$$

proving the claim. □

Our next step is to introduce the orthonormal basis $\{e_j\}_{j\in\mathbf{Z}^n}$ in $L^2(]0,1[^n)$, such that $e_j \in C_0^\infty([0,1]^n)$. Then the functions $\{e_{jk}\}_{j,k\in\mathbf{Z}^n}$, where $e_{jk}(x) = e_j(x-k)$, form an orthonormal basis in $L^2(\mathbf{R}^n)$. Now we introduce the following, time-dependent, Fourier coefficients:

$$\mu_{kj}^\varepsilon(t) = (w^\varepsilon(t,\,\cdot)\, \tilde z^\varepsilon(t,\,\cdot),\, e_{kj})_{L^2(\mathbf{R}^n)}$$
$$\nu_{kj}^\varepsilon(t) = (w^\varepsilon(t,\,\cdot)\,,\, e_{kj}\,)_{L^2(\mathbf{R}^n)}$$

with $\tilde z^\varepsilon(t,x) = z((x + \varepsilon^{-1}B(t))/\varepsilon)$. We now explore the properties of these Fourier coefficients.

LEMMA 2. *There are constants $C_{jk} > 0$, independent of ε such that*

$$|\mu_{kj}(t) - \mu_{kj}(s)| = \ |(w^\varepsilon(t,\,\cdot), \tilde z^\varepsilon(t,\,\cdot)\, e_{kj})_{L^2(\mathbf{R}^n)} - (w^\varepsilon(s,\,\cdot), \tilde z^\varepsilon(s,\,\cdot) e_{kj})_{L^2(\mathbf{R}^n)}\,| \leqslant C_{kj}\, \sqrt{t-s}. \tag{19}$$

.

Proof. Denote $\tilde e_{kj}(t,x) = e_{kj}(t, x - B(t)/\varepsilon)$. Using $z^\varepsilon\ \tilde e_{kj}$ as a test function in the variational formulation of (1), we arrive at

$$(w^\varepsilon(t,\,\cdot)\tilde z^\varepsilon(t,\,\cdot) - w^\varepsilon(s,\,\cdot)\tilde z^\varepsilon(s,\,\cdot),\ e_{kj})_{L^2(\mathbf{R}^n)}$$
$$= \int_s^t \int_{\mathbf{R}^n} \left[\frac{\partial}{\partial t}(u^\varepsilon\, z^\varepsilon)(\tau,x) + \varepsilon^{-1}\bar b(\tau)\cdot \nabla(u^\varepsilon\, z^\varepsilon)(\tau,x) \right]\ \tilde e_{kj}(x)\, dx\, d\tau$$
$$= \int_s^t \int_{\mathbf{R}^n} u^\varepsilon(\tau,x)\, \frac{\partial z^\varepsilon}{\partial t}(\tau,x)\, \tilde e_{kj}(x)\, dx\, d\tau$$
$$- \int_s^t \int_{\mathbf{R}^n} (z^\varepsilon\, \mathbf{A}^\varepsilon \nabla u^\varepsilon \cdot \nabla \tilde e_{kj} + \tilde e_{kj}\, (\mathbf{A}^\varepsilon)^t \nabla z^\varepsilon \cdot \nabla u^\varepsilon)\, dx\, d\tau$$
$$+ \varepsilon^{-1} \int_s^t \int_{\mathbf{R}^n} \left(u^\varepsilon\, \tilde e_{kj} \operatorname{div}(a^\varepsilon\, z^\varepsilon) + z^\varepsilon\, u^\varepsilon (a^\varepsilon \cdot \nabla)\, \tilde e_{kj} + \tilde e_{kj} \bar b \cdot \nabla(u^\varepsilon z^\varepsilon) \right)\, dx\, d\tau$$
$$- \varepsilon^{-1} \int_s^t \int_{\mathbf{R}^n} z^\varepsilon\, h^\varepsilon\, f(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_n}\, \tilde e_{kj}\, dx\, d\tau.$$

We proceed as in the proof of Lemma 1. Define η^ε by (16) and ψ^ε by (18), and let $F(\tau) = \int_0^\tau f(s)ds$. Then

$$\begin{aligned}(w^\varepsilon(t,\cdot)\tilde z^\varepsilon(t,\cdot) - w^\varepsilon(s,\cdot)\tilde z^\varepsilon(s,\cdot),\ e_{kj})_{L^2(\mathbf{R}^n)} &= \int_s^t \int_{\mathbf{R}^n} \Big\{ \eta^\varepsilon : \nabla (u^\varepsilon \nabla \tilde e_{kj}) \\ &\quad - z^\varepsilon \mathbf{A}^\varepsilon \nabla u^\varepsilon \nabla \tilde e_{kj} - \psi^\varepsilon \cdot \nabla_{x'} \\ &\quad \times \left(F(u^\varepsilon) \frac{\partial \tilde e_{kj}}{\partial x_n} \right) + u^\varepsilon \tilde e_{kj} \frac{\partial z^\varepsilon}{\partial t} \Big\} \\ &\leqslant C \int_s^t |u^\varepsilon(\tau)|_{H^1(\mathbf{R}^n)}\, d\tau \\ &\leqslant C\sqrt{s-t}\,. \end{aligned}$$

□

Notice that the above result gives the equicontinuity in time for $w^\varepsilon\,\tilde z^\varepsilon$. Our next aim is estimating the difference between μ^ε_{kj} and ν^ε_{kj}.

Lemma 3. *For each $k, j \in \mathbf{Z}^n$, there exists a constant $C_{kj} > 0$, independent of ε, such that*

$$|(w^\varepsilon,\ e_{kj}\ (1 - \tilde z^\varepsilon)_{L^2(\mathbf{R}^n)}|_{L^2(0,T)} \leqslant C_{kj}\,\varepsilon.$$

Proof. Making use of relation (6), one can solve the problem

$$\mathrm{div}_y \Psi = 1 - z \qquad \text{in } Y \times]0,T[.$$

Thus we have $\varepsilon \mathrm{div}_{x'} \Psi(t, x'/\varepsilon) = 1 - z^\varepsilon(t,x)$ implying the claim. □

According to the Arzela–Ascoli theorem, Lemma 2 implies that, for any $k, j \in \mathbf{Z}^n$, and each sequence $\varepsilon_i \to 0$, there is a subsequence $\varepsilon_i' \to 0$ and some $\overline{\mu}_{kj} \in C([0,T])$ such that

$$\mu^{\varepsilon_i'}_{kj} \to \overline{\mu}_{kj} \qquad \text{in } C([0,T]). \tag{20}$$

Furthermore, we can use the diagonal procedure to get the subsequence

$$\varepsilon_i'' \to 0 \tag{21}$$

such that (20) holds for all $j, k \in \mathbf{Z}^n$. Due to the uniform boundedness of $\tilde z^\varepsilon\, w^\varepsilon$ in $L^2(]0,T[\times \mathbf{R}^n)$, the function

$$\overline{w}(t,x) = \sum_{j,k} \overline{\mu}_{jk}(t)\, e_{jk}(x)$$

is an element of $L^2(]0,T[\times\mathbf{R}^n)$. Moreover, we have the following.

Lemma 4. *For any $N > 0$ and any $\delta > 0$ there exists $K(\delta) > 0$ such that*

$$\left| w^\varepsilon \chi_{Q_N} - \sum_{|k|\leqslant K(\delta)} \sum_{|j|\leqslant N} \nu^\varepsilon_{kj}\, e_{kj} \right|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \delta,$$

where χ_{Q_N} denotes the characteristic function of a set Q_N.

Proof. This is a straightforward consequence of the embedding $H^1(Q_N) \hookrightarrow L^2(Q_N)$, bound (10) and Lemma 1. □

THEOREM 3. *For any $\varepsilon_i \to 0$ the sequence $\{w^{\varepsilon_i}\}_{i\in\mathbf{N}}$ has a subsequence strongly convergent in $L^2(]0,T[\times\mathbf{R}^n)$.*

Proof. Let $\{\varepsilon_i''\}$ be the subsequence constructed in (21). We intend to show that the subsequence $\{w^{\varepsilon_i''}\}_{\varepsilon_i''\to 0}$ tends to $\overline{w}$ strongly in $L^2(]0,T[\times\mathbf{R}^n)$. Let $\delta > 0$ be an arbitrary number. We first note that, due to Lemma 1, there exists $N > 0$ such that

$$|w^\varepsilon - w^\varepsilon\,\chi_{Q_N}|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \frac{\delta}{8}. \tag{22}$$

Now, by Lemma 4, there exists $K(\delta) > 0$ such that

$$\left| w^\varepsilon\,\chi_{Q_N} - \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \nu_{kj}^\varepsilon\, e_{kj}\right|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \frac{\delta}{8}. \tag{23}$$

By Lemma 3, for ε small enough

$$\left| \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \nu_{kj}^\varepsilon\, e_{kj} - \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \mu_{kj}^\varepsilon\, e_{kj}\right|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \frac{\delta}{8}. \tag{24}$$

Finally, the convergence (20) implies the existence of a $\varepsilon_0(\delta) > 0$, such that, for all $\varepsilon_i'' \leqslant \varepsilon_0$ (or equivalently for all $i \geqslant i_0(\delta)$) one has

$$\left| \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \mu_{kj}^{\varepsilon_i''}\, e_{kj} - \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \overline{\mu}_{kj}\, e_{kj}\right|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \frac{\delta}{8}. \tag{25}$$

It remains to chose $K(\delta)$ large enough, so that

$$\left| \overline{w} - \sum_{|k|\leqslant K(\delta)}\sum_{|j|\leqslant N} \overline{\mu}_{kj}\, e_{kj}\right|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant \frac{\delta}{8}. \tag{26}$$

Summing up (22)–(26) we achieve the convergence of $\{w^{\varepsilon_i''}\}$ towards $\overline{w}$ in the norm of $L^2(]0,T[\times\mathbf{R}^n)$. $\square$

6. *Convergence of the homogenization procedure*

Two-scale convergence, introduced by Nguetseng in [**12**], has to be slightly modified to make an appropriate tool for our problem. In fact we modify its evolutional version from [**6**]. We define the two-scale convergence for oscillating functions and fast traveling frame. We give its definition: we say that the sequence $\varphi^\varepsilon \in L^2(]0,T[\times\mathbf{R}^n)$, $|\varphi^\varepsilon|_{L^2(]0,T[\times\mathbf{R}^n)} \leqslant C$, two-scale converges to $\varphi^0 \in L^2(]0,T[\times\mathbf{R}^n\times Y)$ in moving coordinates $(x+\varepsilon^{-1}B(t))$ if

$$\int_0^T\int_{\mathbf{R}^n} \varphi^\varepsilon(t,x)\psi(t,x-\varepsilon^{-1}B(t),\varepsilon^{-1}x')\,dx\,dt \to \int_0^T\int_{\mathbf{R}^n}\int_Y \varphi^0(t,x,y)\psi(t,x,y)\,dy\,dx\,dt, \tag{27}$$

for any test function $\psi \in L^2(\mathbf{R}^n\times]0,T[;C_{\mathrm{per}}(Y))$.

REMARK 2. Our modification consists of allowing the test function to be taken in the form

$$\psi^\varepsilon(t,x) = \psi(t, x - \varepsilon^{-1}B(t), \varepsilon^{-1}x').$$

It is clear that

$$\lim_{\varepsilon\to 0} |\psi^\varepsilon|_{L^2(\mathbf{R}^n \times]0,T[)} = |\psi|_{L^2(\mathbf{R}^n \times]0,T[\times Y)}.$$

Hence the shifted test functions admit uniform bound, and the compactness arguments used in the theory of two-scale convergence, also work in our case. However, we repeat the proof of the compactness theorem for the reader's convenience.

We also modify slightly the compactness result from [**1**] and [**6**].

PROPOSITION 2. *Let functions $\varphi^\varepsilon \in L^2(0,T; H^1(\mathbf{R}^n))$ satisfy the bound*

$$|\varphi^\varepsilon|_{L^2(0,T;H^1(\mathbf{R}^n))} \leqslant C.$$

Then there exist a subsequence, denoted by the same symbol $\{\varphi^\varepsilon\}$, and functions $\varphi^0 \in L^2(0,T; H^1(\mathbf{R}^n))$ and $\varphi^1 \in L^2(]0,T[\times \mathbf{R}^n\,;\, H^1(Y))$ such that

$$\varphi^\varepsilon \to \varphi^0 \qquad \text{two-scale} \tag{28}$$

$$\nabla\varphi^\varepsilon \to \nabla_x\varphi^0 + \nabla_y\varphi^1 \quad \text{two-scale.} \tag{29}$$

Proof. We proceed as in [**1**] to find the bound for the integral

$$\begin{aligned}\left|\int_0^T \int_{\mathbf{R}^n} \varphi^\varepsilon(t,x)\, \psi(t, x-\varepsilon^{-1}B(t), \varepsilon^{-1}x')\, dx\, dt\right| \\ \leqslant C\, |\psi(t, x-\varepsilon^{-1}B(t), \varepsilon^{-1}x')|_{L^2(]0,T[\times \mathbf{R}^n)} \\ \leqslant C\, |\psi(t,x,y)|_{L^2(]0,T[\times \mathbf{R}^n; C_{\mathrm{per}}(Y))}.\end{aligned}$$

Therefore the sequence of measures defined by

$$\langle \ell_\varepsilon, \psi\rangle \equiv \int_0^T \int_{\mathbf{R}^n} \varphi^\varepsilon(t,x)\, \psi(t, x-\varepsilon^{-1}B(t), \varepsilon^{-1}x')\, dx\, dt$$

admits a subsequence that converges weakly* in $L^2(]0,T[\times \mathbf{R}^n; \mathcal{M}_{\mathrm{per}}(Y))$ to some

$$\ell_0 \in L^2(]0,T[\times \mathbf{R}^n; \mathcal{M}_{\mathrm{per}}(Y)),$$

where $\mathcal{M}_{\mathrm{per}}(Y) = (C_{\mathrm{per}}(Y))'$. Using the property stated in Remark 2, that is,

$$\lim_{\varepsilon\to 0} |\psi^\varepsilon|_{L^2(]0,T[\times \mathbf{R}^n)} = |\psi|_{L^2(]0,T[\times \mathbf{R}^n \times Y)},$$

we conclude that ℓ_0 is continuous functional on $L^2(\,]0,T[\,\times \mathbf{R}^n \times Y)$. Thus it can be represented as

$$\langle \ell_0, \psi\rangle = \int_0^T \int_{\mathbf{R}^n} \int_Y \varphi^0(t,x,y)\, \psi(t,x,y)\, dx\, dt\, dy$$

for some $\varphi^0 \in L^2(]0,T[\times \mathbf{R}^n \times Y)$ implying (28). By the same reasons we can associate with the sequence $\nabla\varphi^\varepsilon$ a function $\zeta^0 \in L^2(]0,T[\times \mathbf{R}^n \times Y)^n$ such that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbf{R}^n} \nabla\varphi^\varepsilon(t,x)\,\psi(t,x-\varepsilon^{-1}B(t),\varepsilon^{-1}x')\,dx\,dt$$
$$=\int_0^T\int_{\mathbf{R}^n}\int_Y \zeta^0(t,x,y)\,\psi(t,x,y)\,dx\,dt\,dy.$$

Integration by parts leads to

$$\varepsilon\int_0^T\int_{\mathbf{R}^n} \nabla\varphi^\varepsilon(t,x)\,\psi(t,x-\varepsilon^{-1}B(t),\varepsilon^{-1}x')\,dx\,dt$$
$$=-\int_0^T\int_{\mathbf{R}^n} \varphi^\varepsilon(t,x)[\operatorname{div}_y\psi(t,x-\varepsilon^{-1}B(t),\varepsilon^{-1}x')$$
$$+\varepsilon\operatorname{div}_x\psi(t,x-\varepsilon^{-1}B(t),\varepsilon^{-1}x')]\,dx\,dt.$$

Thus

$$\int_0^T\int_{\mathbf{R}^n}\int_Y \varphi^0(t,x,y)\operatorname{div}_y\psi(t,x,y)\,dx\,dt\,dy=0,$$

implying that φ^0 does not depend on y. Taking ψ such that $\operatorname{div}_y\psi=0$ and passing to the limit gives

$$\int_0^T\int_{\mathbf{R}^n}\int_Y(\zeta^0\,\psi+\varphi^0\operatorname{div}_x\psi)=0,$$

that is, $\zeta^0-\nabla\varphi^0 \perp \psi$ for all $\psi \in C_0^\infty(]0,T[\times\mathbf{R}^n;C_{\mathrm{per}}^\infty(Y))$ such that $\operatorname{div}_y\psi(t,x,y)=0$. Consequently $\zeta^0(t,x,y)=\nabla_x\varphi^0(t,x)+\nabla_y\varphi^1(t,x,y)$ for some $\varphi^1\in L^2(]0,T[\times\mathbf{R}^n\,;H^1_{\mathrm{per}}(Y))$. □

Remark 3. If we rewrite the functions φ^ε in moving coordinates: $\tilde\varphi^\varepsilon(x,t)=\varphi^\varepsilon(x+\varepsilon^{-1}B(t),t)$, then relation (27) takes the form

$$\int_0^T\int_{\mathbf{R}^n}\tilde\varphi^\varepsilon(t,x)\psi(t,x,\varepsilon^{-1}(x'+\varepsilon^{-1}B(t)))\,dx\,dt$$
$$\to\int_0^T\int_{\mathbf{R}^n}\int_Y\varphi^0(t,x,y)\,\psi(t,x,y)\,dy\,dx\,dt.$$

Abusing slightly the notation we will write in this case $\tilde\varphi^\varepsilon\to\varphi^0$ two-scale.

Estimates (10), (11) and Theorem 3 derived in the previous section imply the following convergence.

Lemma 5. *There exist $w^0\in L^2(0,T;H^1(\mathbf{R}^n))$ and $w^1\in L^2(]0,T[\times\mathbf{R}^n\times;H^1_{\mathrm{per}}(Y))$ such that*

$$\begin{aligned} w^\varepsilon &\to w^0 && \textit{weakly in } L^2(0,T;H^1(\mathbf{R}^n))\\ & && \textit{strongly in } L^2(0,T;L^2(\mathbf{R}^n))\\ & && \textit{two-scale,} \qquad (30)\\ \nabla w^\varepsilon &\to \nabla_x w^0+\nabla_y w^1 && \textit{two-scale.} \qquad (31)\end{aligned}$$

Further analysis will rely on the following statement.

LEMMA 6. *Let $p \in L^2(]0,T[\times Y)$ have zero average in y, that is,*

$$\int_Y p(t,y)\,dy = 0$$

for (almost everywhere) $t \in (0,T)$. Then for any $\phi \in C_0^\infty(]0,T[\times \mathbf{R}^n)$ the following hold.

(i)

$$\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n} \tilde{p}^\varepsilon\, w^\varepsilon\, \phi\, dx\, dt \longrightarrow \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y p\, w^1\, dy\Big)\phi\, dx\, dt.$$

(ii)

$$\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n} \tilde{p}^\varepsilon\, f(w^\varepsilon)\,\frac{\partial w^\varepsilon}{\partial x_n}\,\phi\, dx\, dt \longrightarrow \int_0^T\int_{\mathbf{R}^n}\phi\Big(\int_Y p\frac{\partial}{\partial x_n}(f(w^0)\, w^1)\, dy\Big)dx\, dt,$$

where $\tilde{p}^\varepsilon(t,x) = p(t,\varepsilon^{-1}x' + \varepsilon^{-2}B(t))$.

Proof. (i) Since $p \in L^2(0,T;L_0^2(Y))$ there exists $q \in L^2(0,T;H^1_{\mathrm{per}}(Y))$ such that

$$\mathrm{div}_y\, q = \sum_{\alpha=1}^{n-1}\frac{\partial q_\alpha}{\partial y_\alpha} = p$$

and $\int_Y q = 0$. Then for the function

$$\tilde{q}^\varepsilon(t,x) = (q_1(t,\varepsilon^{-1}x' + \varepsilon^{-2}B(t)),\ldots,q_{n-1}(t,\varepsilon^{-1}x' + \varepsilon^{-2}B(t)))$$

we have

$$\mathrm{div}_{x'}\,\tilde{q}^\varepsilon = \sum_{\alpha=1}^{n-1}\frac{\partial \tilde{q}^\varepsilon_\alpha}{\partial x_\alpha} = \varepsilon^{-1}\,\tilde{p}^\varepsilon.$$

Thus

$$\begin{aligned}
\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n}\tilde{p}^\varepsilon\, w^\varepsilon\,\phi\, dx\, dt &= \int_0^T\int_{\mathbf{R}^n}\mathrm{div}_{x'}\,\tilde{q}^\varepsilon\, w^\varepsilon\,\phi\, dx\, dt\\
&= -\int_0^T\int_{\mathbf{R}^n}\Big(q^\varepsilon\cdot\nabla_{x'}w^\varepsilon\,\phi + w^\varepsilon\,\tilde{q}^\varepsilon\cdot\nabla_{x'}\phi\Big)dx\, dt\\
&\to -\int_0^T\int_{\mathbf{R}^n}\Big(\int_Y\{q\cdot\nabla_{x'}w^0 + q\cdot\nabla_y w^1\}\, dy\Big)\phi\, dx\, dt\\
&\quad -\int_0^T\int_{\mathbf{R}^n}\Big(\int_Y q\, dy\Big)w^0\,\nabla_{x'}\phi\, dx\, dt\\
&= -\int_0^T\int_{\mathbf{R}^n}\phi\Big(\int_Y q\cdot\nabla_y w^1\, dy\Big)dx\, dt = \int_0^T\int_{\mathbf{R}^n}\phi\Big(\int_Y \mathrm{div}_y q\, w^1\, dy\Big)dx\, dt\\
&= \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y p\, w^1\, dy\Big)\phi\, dx\, dt.
\end{aligned}$$

(ii) For the second assertion we combine the same idea with the strong convergence (30). Defining q as in (i), we have

$$\begin{aligned}
\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n}\tilde{p}^\varepsilon\, f(w^\varepsilon)\,\frac{\partial w^\varepsilon}{\partial x_n}\;\phi^0\,dx\,dt &= \varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n}\tilde{p}^\varepsilon\,\frac{\partial F(w^\varepsilon)}{\partial x_n}\;\phi^0\,dx\,dt\\
&= -\int_0^T\int_{\mathbf{R}^n}F(w^\varepsilon)\,\mathrm{div}_{x'}\tilde{q}^\varepsilon\;\frac{\partial\phi^0}{\partial x_n}\,dx\,dt\\
&= \int_0^T\int_{\mathbf{R}^n}\tilde{q}^\varepsilon\cdot\left[\nabla_{x'}\left(\frac{\partial\phi^0}{\partial x_n}\right)F(w^\varepsilon)+\nabla_{x'}w^\varepsilon\,f(w^\varepsilon)\frac{\partial\phi^0}{\partial x_n}\right]dx\,dt\\
&\to \int_0^T\int_{\mathbf{R}^n}\frac{\partial\phi^0}{\partial x_n}\,f(w^0)\bigg(\int_Y q\cdot\nabla_y w^1\,dy\bigg)dx\,dt\\
&= -\int_0^T\int_{\mathbf{R}^n}\frac{\partial\phi^0}{\partial x_n}\,f(w^0)\left(\int_Y p\;w^1\right). \qquad\square
\end{aligned}$$

Our next step is to pass to the limit in equation (1).

We take the test function of the form

$$\psi^\varepsilon(t,x)=z^\varepsilon(t,x)\phi^0(t,x-\varepsilon^{-1}B(t))+\varepsilon\phi^1\left(t,x-\varepsilon^{-1}B(t),\frac{x'}{\varepsilon}\right),$$

where $\phi^0\in H^1(0,T;H^1(\mathbf{R}^n))$ is such that $\phi^0(T,x)=0$ and $\phi^1\in H^1_0(]0,T[\times\mathbf{R}^n;H^1_{\mathrm{per}}(Y))$.

Thus we obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbf{R}^n}\Bigg[-u^\varepsilon\left(\frac{\partial z^\varepsilon}{\partial t}(\phi^0)^\varepsilon+z^\varepsilon\left(\frac{\partial\phi^0}{\partial t}\right)^\varepsilon\right)+\mathbf{A}^\varepsilon\nabla u^\varepsilon\;\{\nabla z^\varepsilon\,(\phi^0)^\varepsilon\\
&\quad+(\nabla\phi^0)^\varepsilon\,z^\varepsilon+(\nabla_y\,\phi^1)^\varepsilon\}\Bigg]\,dx\,dt\\
&\quad-\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n}[(a^\varepsilon-\overline{b})\cdot\,(\nabla_x\phi^0)^\varepsilon\,z^\varepsilon+\mathrm{div}(z^\varepsilon a^\varepsilon)\,(\phi^0)^\varepsilon]\,u^\varepsilon\,dx\,dt\\
&\quad+\int_0^T\int_{\mathbf{R}^n}(a^\varepsilon\cdot\nabla u^\varepsilon(\phi^1)^\varepsilon+\overline{b}\cdot(\nabla_x\phi^1)^\varepsilon\;u^\varepsilon\;)\,dx\,dt \qquad (32)\\
&\quad+\varepsilon^{-1}\int_0^T\int_{\mathbf{R}^n}h^\varepsilon\,f(u^\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_n}\,z^\varepsilon\,(\phi^0)^\varepsilon\,dx\,dt-\int_0^T\int_{\mathbf{R}^n}h^\varepsilon\,f(u^\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_n}\,(\phi^1)^\varepsilon\,dx\,dt \qquad (33)\\
&=\int_{\mathbf{R}^n}\varphi(x)\,z\left(0,\frac{x}{\varepsilon}\right)\phi^0(0,x)\,dx+O(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
(\phi^1)^\varepsilon(t,x)&=\phi^1(t,x-\varepsilon^{-1}B,\varepsilon^{-1}x'),\\
(\nabla_y\,\phi^1)^\varepsilon(t,x)&=(\nabla_y\,\phi^1)(t,x-\varepsilon^{-1}B,\varepsilon^{-1}x')\\
(\nabla_x\,\phi^1)^\varepsilon(t,x)&=(\nabla_x\,\phi^1)(t,x-\varepsilon^{-1}B,\varepsilon^{-1}x')\\
(\phi^0)^\varepsilon(t,x)&=\phi^1(t,x-\varepsilon^{-1}B),\quad(\nabla_x\,\phi^0)^\varepsilon(t,x)=(\nabla_x\,\phi^0)(t,x-\varepsilon^{-1}B).
\end{aligned}$$

In some of the above integrals we change the variables $x \to x - \varepsilon^{-1} B(t)$; then we denote

$$\tilde{a}^\varepsilon(t,x) = a\Big(t, \frac{x'}{\varepsilon} + \frac{B(t)}{\varepsilon^2}\Big), \quad \tilde{\mathbf{A}}^\varepsilon(t,x) = \mathbf{A}\Big(t, \frac{x'}{\varepsilon} + \frac{B(t)}{\varepsilon^2}\Big).$$

Now, by Lemma 5, we get for the first two terms on the left-hand side of (32)

$$\int_0^T \int_{\mathbf{R}^n} u^\varepsilon\, z^\varepsilon\, \Big(\frac{\partial \phi^0}{\partial t}\Big)^\varepsilon = \int_0^T \int_{\mathbf{R}^n} w^\varepsilon\, \tilde{z}^\varepsilon\, \frac{\partial \phi^0}{\partial t} \to \int_0^T \int_{\mathbf{R}^n} w^0\, \frac{\partial \phi^0}{\partial t},$$
$$\int_0^T \int_{\mathbf{R}^n} u^\varepsilon\, \frac{\partial z^\varepsilon}{\partial t}\, (\phi^0)^\varepsilon \to \int_0^T \int_{\mathbf{R}^n} w^0\, \phi^0\, \frac{\partial}{\partial t} \Big(\int_Y z\Big) = 0.$$

Next, we group together the terms

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^n} \mathbf{A}^\varepsilon \nabla u^\varepsilon \nabla z^\varepsilon\, (\phi^0)^\varepsilon\, dx\, dt - \varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} (a^\varepsilon - \overline{b}) \cdot (\nabla_x \phi^0)^\varepsilon u^\varepsilon\, z^\varepsilon\, dx\, dt \\
&\quad - \varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \operatorname{div}(a^\varepsilon z^\varepsilon)\, (\phi^0)^\varepsilon\, u^\varepsilon\, dx\, dt = -\varepsilon^{-2} \int_0^T \int_{\mathbf{R}^n} u^\varepsilon (\phi^0)^\varepsilon \{(\operatorname{div}_y(\mathbf{A}^t \nabla_y z))^\varepsilon \\
&\quad + (\operatorname{div}_y(z\, a))^\varepsilon\}\, dx\, dt - \varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \{(\mathbf{A}^t \nabla_y z)^\varepsilon + (a^\varepsilon - \overline{b}) z^\varepsilon\} \cdot (\nabla_x \phi^0)^\varepsilon u^\varepsilon\, dx\, dt \\
&= -\varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \{(\mathbf{A}^t \nabla_y z)^\varepsilon + (a^\varepsilon - \overline{b}) z^\varepsilon\} \cdot (\nabla_x \phi^0)^\varepsilon u^\varepsilon\, dx\, dt;
\end{aligned}$$

equation (5) has also been used here. By the definition of $\overline{b}$, we have

$$-\int_Y \{A^t \nabla_y z + (a - \overline{b}) z\}\, dy = \int_Y \big(\operatorname{div}_y \mathbf{A} - a + \overline{b}\big)\, (t,y)\; z(t,y)\, dy\; = 0.$$

Therefore, item (i) of Lemma 6 applies and we obtain

$$\begin{aligned}
&-\varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \{(\mathbf{A}^t \nabla_y z)^\varepsilon + (a^\varepsilon - \overline{b}) z^\varepsilon\} \cdot (\nabla_x \phi^0)^\varepsilon u^\varepsilon\, dx\, dt \\
&\quad = -\varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \Big[\tilde{\mathbf{A}}^\varepsilon (\widetilde{\nabla_y z})^\varepsilon + (\tilde{a}^\varepsilon - \overline{b})\, \tilde{z}^\varepsilon\Big]\, (t,x) \cdot \nabla \phi^0(t,x)\; w^\varepsilon(t,x)\, dx\, dt \\
&\quad \to - \int_0^T \int_{\mathbf{R}^n} \int_Y w^1\, (\mathbf{A}^t \nabla_y z + (a - \overline{b}) z) \cdot \nabla \phi^0\; dy\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbf{R}^n} \int_Y z\; \big\{w^1\, (\operatorname{div}_y \mathbf{A} - a + \overline{b}) + \mathbf{A}^t \nabla_y w^1\big\} \cdot \nabla \phi^0\; dy\, dx\, dt.
\end{aligned}$$

We proceed with other terms in (32). Directly from (31) we get

$$\int_0^T \int_{\mathbf{R}^n} \tilde{\mathbf{A}}^\varepsilon \nabla w^\varepsilon \nabla \phi^0 \tilde{z}^\varepsilon\, dx\, dt \to \int_0^T \int_{\mathbf{R}^n} \Big(\int_Y z \mathbf{A} (\nabla_x w^0 + \nabla_y w^1)\Big) \nabla \phi^0\; dy\, dx\, dt.$$

The next integral is easy to handle:

$$\begin{aligned}
\int_0^T \int_{\mathbf{R}^n} \mathbf{A}^\varepsilon\, \nabla u^\varepsilon\, [\nabla_y \phi^1]^\varepsilon\, dx\, dt &\to \int_0^T \int_{\mathbf{R}^n} \Big(\int_Y \mathbf{A}^t\, \nabla_y \phi^1\Big) \cdot \nabla w^0\; dy\, dx\, dt \\
&\quad + \int_0^T \int_{\mathbf{R}^n} \Big(\int_Y \mathbf{A}\, \nabla_y w^1\, \nabla_y \phi^1\Big) dy\, dx\, dt.
\end{aligned}$$

The linear convection terms related to ϕ^1 give

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^n} (a^\varepsilon \cdot \nabla u^\varepsilon \, (\phi^1)^\varepsilon + \overline{b} \cdot (\nabla_x \phi^1)^\varepsilon \, u^\varepsilon) \, dx \, dt \\
&\quad \to \int_0^T \int_{\mathbf{R}^n} \int_Y a \cdot \Big(\nabla_x w^0 + \nabla_y w^1\Big) \phi^1 \, dx \, dt \, dy + \int_0^T \int_{\mathbf{R}^n} \int_Y \overline{b} \cdot \nabla_x \phi^1 \, w^0 \, dx \, dt \, dy \\
&\quad = \int_0^T \int_{\mathbf{R}^n} \left[\left(\left(\int_Y \phi^1 \, (a - \overline{b}) \right) \cdot \nabla w^0 \, dx \, dt \, dy + \left(\int_Y \phi^1 \, (a \cdot \nabla_y w^1) \right) \right) \right] dx \, dt \, dy.
\end{aligned}$$

We now have to deal with the nonlinear terms. Thanks to hypothesis (7), strong convergence (30), and item (ii) of Lemma 6, we obtain

$$\varepsilon^{-1} \int_0^T \int_{\mathbf{R}^n} \tilde{h}^\varepsilon \, \tilde{z^\varepsilon} \, f(w^\varepsilon) \, \frac{\partial w^\varepsilon}{\partial x_n} \, \phi^0 \, dx \, dt \to - \int_0^T \int_{\mathbf{R}^n} \left(\int_Y h \, z \, w^1 \, dy \right) f(w^0) \, \frac{\partial \phi^0}{\partial x_n} \, dx \, dt.$$

Finally, for the last integral on the left-hand side of (32), considering (11), we get

$$\int_0^T \int_{\mathbf{R}^n} h^\varepsilon \, f(u^\varepsilon) \, \frac{\partial u^\varepsilon}{\partial x_n} \, (\phi^1)^\varepsilon \, dx \, dt \to \int_0^T \int_{\mathbf{R}^n} f(w^0) \, \frac{\partial w^0}{\partial x_n} \left(\int_Y h \, \phi^1 \, dy \right) dx \, dt.$$

Since z is periodic in variable y, we have

$$\int_{\mathbf{R}^n} \varphi(x) \, \phi^0(0,x) \, z\left(0, \frac{x'}{\varepsilon}\right) dx \to \int_{\mathbf{R}^n} \varphi(x) \, \phi^0(0,x) \, dx.$$

Combining all the above convergences we prove the following statement.

PROPOSITION 3. *The limit functions w^0 and w^1 satisfy the following coupled problem.*

$$\begin{aligned}
&\int_{\mathbf{R}^n} \varphi(x) \phi^0(0,x) \, dx = - \int_0^T \int_{\mathbf{R}^n} w^0 \, \frac{\partial \phi^0}{\partial t} dx \, dt \\
&\quad + \int_0^T \int_{\mathbf{R}^n} \left\{ \left(\int_Y w^1 \, z \, (\mathrm{div}_y \, \mathbf{A} - a + \overline{b}) \, dy \right) \cdot \nabla \phi^0 + \left(\int_Y z \, \mathbf{A} \, dy \right) \nabla w^0 \nabla \phi^0 \right\} dx \, dt \\
&\quad + \int_0^T \int_{\mathbf{R}^n} \left\{ \left(\int_Y z \, (\mathbf{A} + \mathbf{A}^t) \nabla_y w^1 \, dy \right) \nabla \phi^0 + \left(\int_Y \phi^1 \, a \cdot (\nabla w^0 + \nabla_y w^1) \, dy \right) \right. \\
&\quad \left. - \left(\int_Y \phi^1 \, dy \right) \overline{b} \cdot \nabla_x w^0 \right\} dx \, dt + \int_0^T \int_{\mathbf{R}^n} \left(\int_Y \mathbf{A}(\nabla w^0 + \nabla_y w^1) \cdot \nabla_y \phi^1 \right) dx \, dt \\
&\quad - \int_0^T \int_{\mathbf{R}^n} \left(\int_Y h \, z \, w^1 \, dy \right) f(w^0) \, \frac{\partial \phi^0}{\partial x_n} \, dx \, dt + \int_0^T \int_{\mathbf{R}^n} f(w^0) \, \frac{\partial w^0}{\partial x_n} \left(\int_Y h \, \phi^1 \, dy \right) dx \, dt,
\end{aligned} \tag{34}$$

for any $\phi \in L^2(0,T; H^1(\mathbf{R}^n))$ and $\phi^1 \in L^2(]0,T[\times \mathbf{R}^n \, ; H^1_{\mathrm{per}}(Y))$.

REMARK 4. It could be more convenient to rewrite (34) in a symmetric form by choosing $\phi^1(t,x,y) = z(t,y)\,\psi(t,x,y)$. Thus (34) becomes

$$\begin{aligned}
\int_{\mathbf{R}^n} \varphi(x)\phi^0(0,x)\,dx = &-\int_0^T\int_{\mathbf{R}^n} w^0\,\frac{\partial\phi^0}{\partial t}\,dx\,dt + \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y z\mathbf{A}^t\,\nabla_y w^1\,dy\Big)\,\nabla_x\phi^0\,dx\,dt \\
&+ \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y z\mathbf{A}(\nabla_x w^0 + \nabla_y w^1)\,(\nabla_x\phi^0 + \nabla_y\psi)\,dy\Big)\,dx\,dt \\
&- \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y z\mathbf{A}^t\nabla_y\psi\,dy\Big)\,\nabla_x w^0\,dx\,dt + \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y [w^1\,z\,(\operatorname{div}_y\mathbf{A} - a + \overline{b})\cdot\nabla\phi^0 \\
&+ z\,\psi(a - \overline{b} - \operatorname{div}_y\mathbf{A})\cdot\nabla w^0]\,dy\Big)dx\,dt \\
&+ \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y [z\,a + \mathbf{A}^t\,\nabla_y z\,]\cdot\nabla_y w^1\,\psi\,dy\Big)\,dx\,dt \\
&- \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y h\,z\,w^1\,dy\Big)\,f(w^0)\,\frac{\partial\phi^0}{\partial x_n}\,dx\,dt + \int_0^T\int_{\mathbf{R}^n}\Big(\int_Y z\,h\,\psi\Big)\,f(w^0)\,\frac{\partial w^0}{\partial x_n}\,dx\,dt.
\end{aligned} \tag{35}$$

To justify the convergence of the homogenization procedure it suffices to prove the uniqueness of the homogenized limit. That way all the subsequences w^ε have the same limit w^0. We prove the uniqueness by decoupling. The idea is as follows: first we notice that the coupled problem (35) is linear with respect to w^1. Therefore, for fixed w^0, there exists a unique w^1 such that (w^0, w^1) solves the coupled problem. Furthermore, such w^1 admits a separation of 'fast' and 'slow' variables, see (36) below. Substituting w^1 of this form in (35) reduces the above two-scale problem to the quasilinear parabolic equation (40) (the homogenized problem). It then remains to prove that (40) has only one solution.

7. *Decoupling the homogenized problem*

We write down the solution w^1 in the form

$$w^1(t,x,y) = \sum_{\alpha=1}^{n-1}\chi_\alpha(t,y)\,\frac{\partial w^0}{\partial x_\alpha}(t,x) + \theta(t,y)\,f(w^0(t,x))\,\frac{\partial w^0}{\partial x_n}(t,x). \tag{36}$$

By substituting the above ansatz in (35) and choosing $\phi^0 = 0$, we find that χ_α and θ must satisfy the equations

$$a\cdot\nabla(\chi_\alpha + y_\alpha) - \operatorname{div}[\mathbf{A}\nabla(\chi_\alpha + y_\alpha)] = \overline{b}_\alpha \tag{37}$$

$$a\cdot\nabla\theta - \operatorname{div}(\mathbf{A}\nabla\theta) + h = 0. \tag{38}$$

Those two equations are stationary, linear convection-diffusion equations on the torus Y. Due to the choice of $\overline{b}$ and condition (7), each of them has a unique (up to a constant) classical solutions on Y or, equivalently, in the space of 1-periodic functions. This statement relies on Fredholm's alternative and reads as follows.

PROPOSITION 4. *Problems* (37) *and* (38) *have unique* (*up to a constant*) *solutions if and only if*

$$\int_Y (a - \overline{b} - \operatorname{div}\mathbf{A})\,z\,dy = 0$$

and

$$\int_Y z\,h\,dy = 0.$$

Furthermore, $\chi_\alpha, \theta \in C^2([0,T] \times Y)$.

Clearly, $w^1(t,x,y)$ is defined up to an additive function of the arguments t and x. In order to make the choice of this additive function fixed, later on we always impose the normalization condition $\int_Y w^1(t,x,y)dy = 0$.

Lemma 7. *Given the first component* $w^0 \in L^2(0,T;H^1(\mathbf{R}^n)) \cap L^\infty(]0,T[\times \mathbf{R}^n)$ *of a solution to problem* (35), *the second component*

$$w^1 \in L^2\big(]0,T[\times \mathbf{R}^n\,; H^1_{\text{per}}(Y)\big)$$

is uniquely defined. Furthermore, such w^1 *can be written in the form* (36), *with* χ_α *and* θ *defined by* (37) *and* (38), *respectively.*

Proof. Let $w^0 \in L^2(0,T;H^1(\mathbf{R}^n)) \cap L^\infty(]0,T[\times \mathbf{R}^n)$ and $w^1 \in L^2(]0,T[\times \mathbf{R}^n;$ $H^1_{\text{per}}(Y))$ be a solution of the coupled problem (the existence is granted by Theorem 3). For fixed w^0 the problem for w^1 can be written in the form

$$\begin{aligned} -\text{div}_y(z\,\mathbf{A}\,\nabla_y w^1) &+ (z\,a + \mathbf{A}^t\,\nabla_y z)\,\cdot\,\nabla_y w^1 \\ &= z\,(\overline{b} - a + \text{div}_y\,\mathbf{A})\,\cdot\,\nabla_x\,w^0 - z\,h\,f(w^0)\,\frac{\partial w^0}{\partial x_n}. \end{aligned} \tag{39}$$

This is a linear elliptic equation with respect to y (where (t,x) are only parameters). It has a unique solution $w^1 \in L^2(]0,T[\times \mathbf{R}^n\,; H^1_{\text{per}}(Y))$. The representation (36) is a straightforward consequence of solvability of the equations (37) and (38). $\square$

Taking $\phi^1 = 0$ in (35), substituting w^1 in the form (36), we get the macroscopic (homogenized) problem for w^0 in the form of a nonlinear diffusion equation. In fact, we have the following.

Lemma 8. *Let* (w^0, w^1) *be a solution of coupled problem* (35). *Then* w^0 *satisfies the quasilinear equation*

$$\frac{\partial w^0}{\partial t} - \text{div}\,(\mathcal{A}(w^0)\,\nabla w^0) = 0 \quad \text{in }]0,T[\times \mathbf{R}^n, \quad w^0(0,\,\cdot\,) = \varphi \ \text{ in } \mathbf{R}^n, \tag{40}$$

where the homogenized diffusion tensor $\mathcal{A}(w^0)$ *has the form*

$$\mathcal{A}(w^0) = \mathcal{A}^0 + f(w^0)\,\mathcal{A}^1 + f(w^0)^2\,\mathcal{A}^2, \tag{41}$$

with

$$\mathcal{A}^0_{\alpha\beta} = \int_Y \left\{ z \left[\mathbf{A}_{\alpha\beta} + \sum_{\gamma=1}^{n-1} (\mathbf{A}_{\alpha\gamma} + \mathbf{A}_{\gamma\alpha}) \frac{\partial \chi_\beta}{\partial y_\gamma} + (\text{div}\,\mathbf{A} - a + \overline{b})_\beta\ \chi_\alpha \right] \right\} \tag{42}$$

$$\mathcal{A}^1_{\alpha\beta} = \int_Y \left\{ z \left[(\text{div}\,\mathbf{A} - a + \overline{b})_\alpha\ \theta \delta_{\beta n} + \sum_{\gamma=1}^{n-1} (\mathbf{A}_{\alpha\gamma} + \mathbf{A}_{\gamma\alpha}) \frac{\partial \theta}{\partial y_\gamma}\,\delta_{\beta n} + h\,\chi_\beta\,\delta_{\alpha n} \right] \right\} \tag{43}$$

$$\mathcal{A}^2_{\alpha\beta} = -\left(\int_Y h\,z\,\theta \right) \delta_{\alpha n}\,\delta_{\beta n}\,. \tag{44}$$

We know that (40) has a solution. To prove the uniqueness we shall use, again, [**9**, Theorem 8.1]. We should verify its conditions.

LEMMA 9. *There are $c(w^0) > 0$ and $C > 0$ such that*

$$\xi \cdot \mathcal{A}(w^0)\, \xi \geqslant c(w^0)\, |\xi|^2 \geqslant C|\xi|^2,$$

for any $\xi \in \mathbf{R}^n$.

Proof. Multiplying (37) by $z \;\; \chi_\beta\, \xi_\alpha\, \xi_\beta$, integrating over Y and taking the summation over α and β, we obtain

$$\sum_{\alpha,\beta=1}^{n} \mathcal{A}^0_{\alpha\beta}\, \xi_\alpha\, \xi_\beta = \int_Y z\, \mathbf{A}\nabla(\hat{\chi}+y)\cdot\xi\; \nabla(\hat{\chi}+y)\cdot\xi > 0, \tag{45}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ and $\hat{\chi} = (\chi_1, \ldots, \chi_{n-1}, 0)$. Analogously, multiplying (38) by $z\,\theta$ and integrating over Y we obtain

$$-\int_Y hz\theta = \int_Y z\, \mathbf{A}\nabla\theta\, \nabla\theta \geqslant 0. \tag{46}$$

Finally, combining (37) and (38), we get

$$\sum_{\alpha,\beta=1}^{n} \mathcal{A}^1_{\alpha\beta}\, \xi_\alpha\, \xi_\beta = \xi_n \int_Y z\, (\mathbf{A}+\mathbf{A}^t)\, \nabla\theta\, \nabla[(\hat{\chi}+y)\cdot\xi]. \tag{47}$$

Summing up the above relations we obtain

$$\sum_{\alpha,\beta=1}^{n} \mathcal{A}(w^0)_{\alpha\beta}\, \xi_\alpha\, \xi_\beta = \int_Y z\, \mathbf{A}\, \nabla_y\, \omega \cdot \nabla_y\, \omega, \tag{48}$$

where

$$\omega = (\hat{\chi}+y)\cdot\xi + f(w^0)\, \theta\, \xi_n. \qquad \square$$

THEOREM 4. *The homogenized problem* (40) *has a unique classical solution* $w^0 \in H^{2+\beta,1+\beta/2}(\mathbf{R}^n_T) \cap L^2(0,T;H^1(\mathbf{R}^n))$.

Proof. This follows from [**9**, Theorems 8.1 and 5.2]. $\square$

This completes the proof of the main theorem, Theorem 2.

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