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Homogenization of surface and length energies for spin systems

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Abstract

We study the homogenization of lattice energies related to Ising systems of the form

$$E_\varepsilon(u) = -\sum_{ij} c^\varepsilon_{ij} u_i u_j,$$

with u_i a spin variable indexed on the portion of a cubic lattice $\Omega \cap \varepsilon\mathbb{Z}^d$, by computing their Γ-limit in the framework of surface energies in a BV setting. We introduce a notion of homogenizability of the system $\{c^\varepsilon_{ij}\}$ that allows to treat periodic, almost-periodic and random statistically homogeneous models (the latter in dimension two), when the coefficients are positive (ferromagnetic energies), in which case the limit energy is finite on $BV(\Omega; \{\pm 1\})$ and takes the form

$$F(u) = \int_{\Omega\cap\partial^*\{u=1\}} \varphi(\nu)\, d\mathcal{H}^{d-1}$$

(ν is the normal to $\partial^*\{u = 1\}$), where φ is characterized by an asymptotic formula. In the random case φ can be expressed in terms of first-passage percolation characteristics. The result is extended to coefficients with varying sign, under the assumption that the areas where the energies are antiferromagnetic are well-separated. Finally, we prove a dual result for discrete curves.

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1. Introduction

In this paper we study surface energies defined on lattice systems through bond interactions. These energies are related to Ising energies, commonly written in the form

$$E(u) = -\sum_{ij} c_{ij} u_i u_j,$$

where u_i is a spin variable taking values $+1$ or -1 and i, j are indices varying in (a suitable subset of) a square lattice $\mathbb{Z}^d$ (see e.g. [19]). If the coefficients c_{ij} are supposed to be positive then ground states are constant. Even with this assumption, if boundary conditions or additional constraint are added, minimizers are not trivial and it is interesting to determine their behaviour when the minimization process involves an increasingly large number of indices. This problem can be set in a variational framework involving energies on lattice subsets, after identifying a function $\{u_i\}$ with the set $A = \{i \in \mathbb{Z}^d\colon u_i = 1\}$. To that end, note that the energy above can be written equivalently as

$$E(u) = \sum_{ij} c_{ij} (u_j - u_j)^2,$$

upon addition of a constant. Under the simplifying assumption that the relevant interactions are those between nearest neighbours (i.e., that we may assume $c_{ij} = 0$ if $|i - j| \neq 1$) then this energy can be seen as a discrete surface energy, concentrated on the boundary of A; i.e., on nearest-neighbour pairs (i, j) such that $u_i \neq u_j$. If moreover c_{ij} is constant for nearest-neighbour interactions, then we have the prototypical *ferromagnetic energy* of a subset A of the lattice $\mathbb{Z}^d$, defined as

$$E(A) = \#\bigl\{(i, j)\colon i \in A,\ j \notin A,\ |i - j| = 1\bigr\}.$$

A continuous approximation of such energies can be obtained in the framework of surface energies defined on sets of finite perimeter. In fact, if we identify each lattice subset A with the union of (coordinate) cubes $\bigcup_{i\in A}(i + Q)$, where $Q = (-1/2, 1/2)^d$ is the unit coordinate cube centered in 0, and we still denote this set by A with a slight abuse of notation, then

$$E(A) = \mathcal{H}^{d-1}(\partial A)$$

coincides with the $(d-1)$-dimensional measure of the boundary of A (i.e., the *perimeter* of A). The overall behaviour of such energy for large sets (compared with the lattice spacing) can be described by scaling it as

$$E_\varepsilon(A) = \varepsilon^{d-1}\#\bigl\{(i, j)\colon i \in A,\ j \notin A,\ |i - j| = \varepsilon\bigr\} \quad \text{for } A \subset \varepsilon\mathbb{Z}^d,$$

and using the methods of Γ-convergence (see [8,9]). These energies are approximated as $\varepsilon \to 0$ by the surface energy

$$\int_{\partial^* A} \|\nu\|_1 \, d\mathcal{H}^{d-1},$$

with the family of sets of finite perimeter (see [1]) as domain. The anisotropic energy density $\|\nu\|_1 = \sum_{n=1}^d |\nu_1|$ depends on the normal ν to the essential boundary of A, and inherits the symmetries of the underlying cubic lattice.

In this paper we consider an inhomogeneous discrete environment modeled on the energies above, where the main assumption is their dependence only on nearest-neighbour interactions (this assumption can be weakened – see below). This is done by considering positive weights c_{ij} (without loss of generality we can assume $c_{ij} = c_{ji}$) for every pair of nearest neighbours (i, j) in $\mathbb{Z}^d$ (i.e., such that $|i - j| = 1$), or equivalently coefficients c_k for every index k in the dual lattice

$$\mathcal{Z} = \left\{k\colon \text{there exist } i, j \in \mathbb{Z}^d,\ |i - j| = 1,\ k = \frac{i + j}{2}\right\}$$

corresponding to the bond between the nodes labeled by i and j. We can also localize our energies by considering a smooth bounded open set $\Omega \subset \mathbb{R}^d$ and the (scaled) energies

$$E_\varepsilon(A) = \varepsilon^{d-1} \sum \left\{c_{ij}\colon \varepsilon i \in A,\ \varepsilon j \in \Omega_\varepsilon \setminus A,\ |i - j| = 1\right\} \tag{1}$$

defined on sets $A \subset \Omega_\varepsilon$, where

$$\Omega_\varepsilon = \varepsilon\mathbb{Z}^d \cap \Omega.$$

The main goal of the paper is then to characterize the limit of these energies as $\varepsilon \to 0$. We note that if we identify each $A \subset \Omega_\varepsilon$ with $\bigcup_{\varepsilon i \in A} \varepsilon(i + Q)$ (and we still denote this set by A with the same slight abuse of notation as above) then E_ε is defined on the family of such unions of cubes, and $E_\varepsilon(A)$ is related to the continuous perimeter functional

$$E_\varepsilon(A) = \int_{\partial A \cap \Omega} a\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1},$$

where

$$a(x) = c_{ij} \quad \text{if } x \in \partial(i + Q) \cap \partial(j + Q)$$

(some care has to be taken to deal with the part of ∂A close to the boundary of Ω). The nearest-neighbour assumption can be relaxed to assuming the non-negativeness of all c_{ij}, the strict positiveness of nearest-neighbour coefficients, and a decay estimate on c_{ij} as $|i - j| \to +\infty$, but the general case of c_{ij} changing sign arbitrarily is not addressed in this paper, involving microscopical homogenization and surface relaxation (see [1,10]).

We first deal with the elliptic case when $0 < \alpha \leqslant c_{ij} \leqslant \beta < +\infty$, in which the energies are equi-coercive with respect to the L^1-convergence of sets (or equivalently the weak*-BV convergence of characteristic functions). We introduce a general notion of a 'homogenizable system' of coefficients $\{c_{ij}\}$, which is satisfied in particular by periodic and almost-periodic systems, but also covers statistically homogeneous random systems. With this definition we can develop a discrete analog of the arguments already introduced for the continuous case (see e.g. Ambrosio and Braides [4]) to prove the existence of a limit anisotropic energy on sets of finite perimeter in Ω, of the form

$$F(A) = \int_{\partial A \cap \Omega} \varphi(\nu)\, d\mathcal{H}^{d-1}.$$

In the case of ergodic statistically homogeneous random systems in dimension two the coefficients $\{c_{ij}^{\omega}\}$ depend on the realization of a random variable; in that case we prove that the limit is deterministic and is characterized by a *first-passage percolation formula*. The main technical tool is a use of the Fonseca and Müller [15] blow-up technique extended to the homogenization setting by Braides, Maslennikov and Sigalotti [14], adapted to cover the new homogenizability condition. The result can be compared with a variational percolation theorem for defects by Braides and Piatnitski [12] (see also [13]), where the surface tension is given by the *chemical distance* on the weak cluster, noting that different growth assumptions on the variational problems involve correspondingly different types of percolation issues.

In the elliptic case above, the limit energy density φ can be expressed in terms of the limit of minimum problems involving 'discrete hypersurfaces', so that we have a direct analogy with the continuous theory. This analogy is lost in the case of interactions c_{ij} with changing signs. Note that in the continuous case energy densities must be always positive, otherwise the corresponding functionals are not bounded from below. Also in the case of discrete interactions we may not have a uniform lower bound; nevertheless, it is possible again to define a Γ-limit after the addition of suitable positive constants if the regions where the coefficients are negative are sufficiently small and well-separated. In fact, even though we do not have any bound in those regions we may find an infinite connected set $\mathcal{W}$ with points with positive interaction and define a convergence $u_\varepsilon \to u$ as the weak convergence of $(u_\varepsilon - u)\chi_{\varepsilon W}$. The estimates obtained on $\mathcal{W}$ are then sufficient to prove that such a limit u is in $BV(\Omega; \{\pm 1\})$, and to obtain a limit that again can be described as a surface integral of the same form as for the elliptic case. The type of arguments is similar to the ones used for the homogenization of surface energies in perforated domains, where the surface energy is zero on well-separated 'holes' (see [21]); in the discrete case though it is possible also to have negative surface energies. It is interesting to note that the surface energy density φ is described by a limit of problems that cannot be interpreted as minimal surface problems and involve both surface and bulk terms.

A final part of the paper concerns the homogenization of lattice energies defined on discrete paths $\gamma = \{\varepsilon k_j\}$; i.e., arrays of points in $\varepsilon\mathcal{Z}$ ($\mathcal{Z}$ the dual lattice to $\mathbb{Z}^d$) such that $\|k_j - k_{j-1}\|_1 = 1$, of the form

$$E_\varepsilon(\gamma) = \varepsilon \sum_j c_{k_j}.$$

Note that paths can be identified with piecewise-affine curves parameterized by arc-length. If the coefficients are elliptic and satisfy another suitable notion of 'homogenizability' (which includes periodic and almost-periodic systems) then we prove the existence of a Γ-limit with domain consisting of the $W^{1,\infty}$-curves γ which satisfy $\|\gamma'\|_1 \leqslant 1$ almost everywhere. This Γ-limit has the form

$$F(\gamma) = \int_0^{L(\gamma)} \psi\big(\gamma'(t)\big)\,dt.$$

Again the function ψ can be characterized by an asymptotic formula, and in the case of random interactions (without any restriction on the dimension d) we prove that is a deterministic integrand. The formula for ψ turns out to be different from the one obtained for surface integrals, and underlines the fact that the Γ-limit is obtained with respect to a stronger topology (for a comparison with a continuous problem we refer to Braides and Defranceschi [11, Chapter 16.1]).

2. Homogenization of spin systems

We consider the energies E_ε defined in (1) under the growth hypothesis

$$0 < \alpha \leqslant c_{ij} \leqslant \beta < +\infty \tag{2}$$

(for weaker hypotheses see Remark 2.5 below).

Remark 2.1 *(Compactness).* We identify each subset $A \subset \varepsilon\mathbb{Z}^d$ with the set

$$\bigcup_{\varepsilon i \in A} \big(\varepsilon(i + Q)\big),$$

which we still denote by A. If A_ε is such that $E_\varepsilon(A_\varepsilon) \leqslant C$ then for all ε small enough we obtain the estimate

$$\mathcal{H}^{d-1}\big(\partial A_\varepsilon \cap \Omega'\big) \leqslant \frac{C}{\alpha}$$

for all $\Omega' \Subset \Omega$. This implies that, up to subsequences, the sets A_ε converge to a set A of finite perimeter in Ω; i.e., $|A \triangle A_\varepsilon| \to 0$ (see, e.g., [7]).

We introduce the following notion of "homogenizable system". The notation $Q_T^\nu(x)$ stands for a cube with centre x, side length T, and one face orthogonal to ν.

Definition 2.2 *(Homogenizable system).* We say that the set of coefficients $\{c_{ij}\}$ is a *homogenizable system* if for every fixed vector $\nu \in S^{d-1}$ and for every choice of a family (x_T) of points on $\mathbb{R}^d$ with

$$\sup_{T>0} \frac{|x_T|}{T} < +\infty \tag{3}$$

there exists the function

$$\varphi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{d-1}} \inf_A \Bigg\{ \sum_{ij} c_{ij}\colon i \in A,\ j \in \mathbb{R}^d \setminus A,\ i \in Q_T^\nu(x_T)$$

$$\text{or } j \in Q_T^\nu(x_T),\ |i-j| = 1 \Bigg\}, \tag{4}$$

the minimum being taken over all sets $A \subset \mathbb{Z}^d$ such that

$$A \setminus Q_T^\nu(x_T) = \big(\Pi^\nu(x_T) \cap \mathbb{Z}^d\big) \setminus Q_T^\nu(x_T), \tag{5}$$

and where

$$\Pi^\nu(x) = \big\{ y \in \mathbb{R}^d\colon \langle y - x, \nu\rangle \geqslant 0 \big\} \tag{6}$$

denotes the *half space* through x orthogonal to ν, and this limit is independent of such (x_T).

Note that in the minimum problem (4) we take into account both interactions internal to $Q_T^\nu(x_T)$ (when both i and j belong to $Q_T^\nu(x_T)$) and interactions crossing its boundary (when only one of the two indices belongs to $Q_T^\nu(x_T)$).

Remark 2.3. Note that, by (3) and a compactness argument, in the definition of φ we may limit our choice to families (x_T) with $x_T = Tx + o(T)$ for some $x \in \mathbb{R}^d$.

The function φ defined in (4), if it exists, enjoys some properties that are of easy verification from its definition:

(i) the positively homogeneous extension of degree one of φ is convex. In particular it is Lipschitz continuous;
(ii) $\alpha\|\nu\|_1 \leqslant \varphi(\nu) \leqslant \beta\|\nu\|_1$.

As a consequence, if we define the energy

$$F(A) = \int_{\Omega\cap\partial^* A} \varphi(\nu)\, d\mathcal{H}^{d-1}, \tag{7}$$

this is a lower-semicontinuous functional on sets of finite perimeter (we denote by $\partial^* A$ the *reduced boundary* of a set of finite perimeter (see [7])).

We have the following convergence result.

Theorem 2.4 *(Homogenization of discrete perimeters). Let c_{ij} be a homogenizable system as in Definition 2.2 satisfying (2). Then there exists the Γ-limit*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(A) = F(A), \tag{8}$$

where F is defined in (7).

Proof. We begin with the liminf inequality, and fix a family A_ε such that $A_\varepsilon \to A$ and $\liminf_{\varepsilon\to 0} E_\varepsilon(A_\varepsilon) < +\infty$. Up to subsequences, we may suppose that such liminf is actually a limit.

For all ε we consider the set in the dual lattice $\varepsilon\mathcal{Z}$ of $\varepsilon\mathbb{Z}^d$ defined by

$$\widetilde{A}_\varepsilon = \left\{\varepsilon k\colon\ k = \frac{i+j}{2},\ |i-j| = 1,\ \varepsilon i \in A_\varepsilon,\ \varepsilon j \in \Omega_\varepsilon \setminus A_\varepsilon\right\}$$

and the measure

$$\mu_\varepsilon = \sum_{\varepsilon k \in \widetilde{A}_\varepsilon} \varepsilon c_k \delta_{\varepsilon k}.$$

Note that the family of measures μ_ε is equibounded with

$$\alpha\mathcal{H}^{d-1}(\Omega \cap \partial A_\varepsilon) \leqslant \mu_\varepsilon(\Omega) \leqslant \beta\mathcal{H}^{d-1}(\Omega \cap \partial A_\varepsilon).$$

Hence, up to further subsequences we can assume that μ_ε converges weakly* to a finite measure μ. We will estimate from below the part of μ that is concentrated on ∂A.

With fixed $h \in \mathbb{N}$ we can consider the collection $\mathcal{Q}_h$ of cubes $Q^\nu_\rho(x)$ such that the following conditions are satisfied:

(i) $x \in \partial^* A$ and $\nu = \nu(x)$;
(ii) $|(Q^\nu_\rho(x) \cap A) \triangle \Pi^\nu(x)| \leqslant \frac{1}{h}\rho^d$;
(iii) $|\frac{\mu(Q^\nu_\rho(x))}{\rho^{d-1}} - \frac{d\mu}{d\mathcal{H}^{d-1} \llcorner \partial^* A}(x)| \leqslant \frac{1}{h}$;
(iv) $|\frac{1}{\rho^{d-1}}\int_{Q^\nu_\rho(x)\cap\partial^* A} \varphi(\nu(y))\, d\mathcal{H}^{d-1}(y) - \varphi(\nu(x))| \leqslant \frac{1}{h}$;
(v) $\mu(Q^\nu_\rho(x)) = \mu(\overline{Q^\nu_\rho(x)})$.

Note that for fixed $x \in \partial^* A$ and for ρ small enough (ii) is satisfied by the definition of reduced boundary, (iii) follows from the Besicovitch Derivation Theorem provided that

$$\frac{d\mu}{d\mathcal{H}^{d-1} \llcorner \partial^* A}(x) < +\infty;$$

(iv) holds by the same reason, and (v) is satisfied for almost all $\rho > 0$ since μ is a finite measure. We deduce that $\mathcal{Q}_h$ is a fine covering of the set

$$\partial^* A_\mu = \left\{x \in \partial^* A\colon\ \frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x) < +\infty\right\},$$

so that (by Morse lemma, see [18]) there exists a countable family of disjoint closed cubes $\{\overline{Q^{\nu_j}_{\rho_j}(x_j)}\}$ still covering $\partial^* A_\mu$. Note that we have

$$\mathcal{H}^{d-1}\big(\partial^* A \setminus \partial^* A_\mu\big) = 0$$

since $\mu(\partial^* A) < +\infty$.

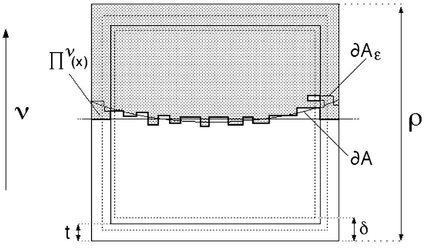

Fig. 1. Construction of a test set.

We now fix one of such cubes $Q^\nu_\rho(x)$. Since $A_\varepsilon \to A$, for ε small enough we have

$$\left|\left(Q^\nu_\rho(x)\cap A_\varepsilon\right)\triangle \Pi^\nu(x)\right| \leqslant \frac{2}{h}\rho^d \tag{9}$$

by (ii) above.

For simplicity of notation we can suppose that $\nu = e_2$ and $x = 0$. With fixed $\delta < 1/2$, from (9) we have in particular

$$\left|\left(\left(Q^\nu_\rho(x)\cap A_\varepsilon\right)\triangle \Pi^\nu(x)\right)\cap\left\{y\colon \rho\frac{\delta}{2}\leqslant \operatorname{dist}\left(y,\partial Q^\nu_\rho(x)\right)\leqslant\rho\delta\right\}\right| \leqslant \frac{2}{h}\rho^d. \tag{10}$$

We deduce that there exists

$$t\in\left[\frac{\rho\delta}{2},\rho\delta\right]$$

such that

$$\mathcal{H}^{d-1}\left(\left(\left(Q^\nu_\rho(x)\cap A_\varepsilon\right)\triangle \Pi^\nu(x)\right)\cap\left\{y\colon \operatorname{dist}\left(y,\partial Q^\nu_\rho(x)\right)=t\right\}\right)\leqslant\frac{4}{h\delta}\rho^{d-1}. \tag{11}$$

We can then define the subset $A^1_\varepsilon \subset Q^\nu_\rho(x)$ by

$$A^1_\varepsilon = \begin{cases} A_\varepsilon & \text{on } Q^\nu_{\rho-t}(x),\\ \Pi^\nu(x) & \text{otherwise.}\end{cases} \tag{12}$$

With the choice of $\nu = e_2$ the set A^1_ε is pictured in Fig. 1. Note that the only change when ν is not a basis vector is that we have a more wiggly shape of the (discretization of the) boundary of $Q^\nu_{\rho-t}(x)$.

Note that

$$\mathcal{H}^1\left(\left(\partial A^1_\varepsilon\setminus\partial A_\varepsilon\right)\cap Q^\nu_\rho(x)\right)\leqslant\frac{4}{h\delta}\rho^{d-1}+\frac{\delta}{2}\rho^{d-1}. \tag{13}$$

We can use the sets $\frac{1}{\varepsilon}A^1_\varepsilon$ in the definition of φ, with $T=\rho/\varepsilon$ and $x_T=\frac{1}{\varepsilon}x=\frac{1}{\rho}Tx$, deducing that

$$\liminf_{\varepsilon\to 0}\mu_\varepsilon\big(Q^\nu_\rho(x)\big)\geqslant \rho^{d-1}\varphi(\nu)-\beta\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)\rho^{d-1}.$$

By (iv) above we then have

$$\liminf_{\varepsilon\to 0}\mu_\varepsilon\big(Q^\nu_\rho(x)\big)\geqslant \int\limits_{Q^\nu_\rho(x)\cap\partial^*E}\varphi\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-\left(\beta\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)+\frac{1}{h}\right)\rho^{d-1},$$

and we finally deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mu_\varepsilon(\Omega)&\geqslant\sum_j\liminf_{\varepsilon\to 0}\mu_\varepsilon\big(Q^{\nu_j}_{\rho_j}(x_j)\big)\\&\geqslant\sum_j\int\limits_{Q^{\nu_j}_{\rho_j}(x_j)\cap\partial^*E}\varphi\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-C\left(\beta\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)+\frac{1}{h}\right)\\&=\int\limits_{\Omega\cap\partial^*E}\varphi\big(\nu(y)\big)\,d\mathcal{H}^{d-1}(y)-C\left(\beta\left(\frac{4}{h\delta}+\frac{\delta}{2}\right)+\frac{1}{h}\right),\end{aligned}$$

which gives the liminf inequality letting first $h\to+\infty$ and then $\delta\to 0$.

The construction of a recovery sequence can be performed just for polyhedral sets, since they are dense in energy in the class of sets of finite perimeter. We only perform the construction when the set is of the form $\Pi^\nu(x)\cap\Omega$ since this construction is easily generalized to each face of a polyhedral boundary.

It is no restriction to suppose that $\Pi^\nu(x)=\Pi^\nu(0)=:\Pi^\nu$, that ν is a rational direction (i.e., there exits S such that $S\nu\in\mathbb{Z}^d$), and that

$$\mathcal{H}^{d-1}\big(\partial\Omega\cap\partial\Pi^\nu\big)=0, \tag{14}$$

since also with these restrictions we obtain a dense class of sets. We denote by $\mathcal{L}:=\partial\Pi^\nu\cap\mathbb{Z}^d$ the lattice of integer points on $\partial\Pi^\nu$; this set can be mapped isometrically to $\delta\mathbb{Z}^{d-1}$, for some $\delta>0$.

With fixed $\eta>0$ we can therefore consider the points

$$x^\varepsilon_k=\delta\left(\left\lfloor\frac{\eta}{\varepsilon\delta}\right\rfloor+1\right)k,\quad k\in\mathbb{Z}^{d-1},$$

and the corresponding points on $\mathcal{L}$, for which we still use the same notation (we use the notation $\lfloor t\rfloor$ for the integer part of t).

Denoted $M=\sup\{|x|:\ x\in\Omega\}$, consider the collection of points

$$\left\{x^\varepsilon_k\in\mathcal{L}:\ \left|x^\varepsilon_k\right|\leqslant\frac{2M}{\varepsilon}\right\}.$$

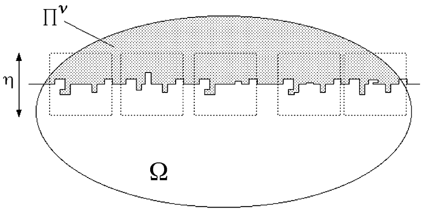

Fig. 2. Construction of the set A^η_ε.

We can take $T = \frac{\eta}{\varepsilon}$ and $x_T = x^\varepsilon_k$ in the definition of homogenizability, and find for all such x^ε_k a cube $Q^\nu_{\frac{\eta}{\varepsilon}}(x^\varepsilon_k)$ centered in x^ε_k and of side length $\frac{\eta}{\varepsilon}$, and a set $A^\varepsilon_k \subset \mathbb{Z}^d$ such that

$$A \setminus Q^\nu_{\frac{\eta}{\varepsilon}}\big(x^\varepsilon_k\big) = \big(\Pi^\nu \cap \mathbb{Z}^d\big) \setminus Q^\nu_{\frac{\eta}{\varepsilon}}\big(x^\varepsilon_k\big),$$
$$\sum_{ij} c_{ij} \leqslant \left(\frac{\eta}{\varepsilon}\right)^{d-1} \big(\varphi(\nu) + o(1)\big),$$

where the sum is performed on nearest neighbours $i \in A^\varepsilon_k$, $j \in \mathbb{R}^d \setminus A^\varepsilon_k$, and $i \in Q^\nu_{\frac{\eta}{\varepsilon}}(x^\varepsilon_k)$ or $j \in Q^\nu_{\frac{\eta}{\varepsilon}}(x^\varepsilon_k)$. We can then define $A^\eta_\varepsilon \subset \varepsilon\mathbb{Z}^d$ as (see Fig. 2)

$$\frac{1}{\varepsilon} A^\eta_\varepsilon \cap Q^\nu_{\frac{\eta}{\varepsilon}}\big(x^\varepsilon_k\big) = A^\varepsilon_k, \qquad \frac{1}{\varepsilon} A^\eta_\varepsilon = \Pi^\nu \cap \mathbb{Z}^d \quad \text{elsewhere.}$$

With fixed η, the family A^η_ε is precompact, so that we may suppose that there exists a limit A^η. We have

$$\begin{aligned} F''\big(A^\eta\big) &:= \Gamma\text{-}\limsup_{\varepsilon\to 0^+} E_\varepsilon\big(A^\eta\big) \\ &\leqslant \limsup_{\varepsilon\to 0^+} E_\varepsilon\big(A^\eta_\varepsilon\big) \\ &\leqslant \varphi(\nu)\mathcal{H}^{d-1}\big(\big\{x \in \partial\Pi^\nu\colon\ \big(x + 2\eta Q^\nu\big) \cap \Omega \neq \emptyset\big\}\big). \end{aligned}$$

Since we have $\Pi^\nu + \eta\nu \subset A^\eta \subset \Pi^\nu - \eta\nu$, by the lower semicontinuity of the functional F'' we deduce then that

$$F''(A) \leqslant \liminf_{\eta\to 0^+} F''\big(A^\eta\big) \leqslant \varphi(\nu)\mathcal{H}^{d-1}\big(\partial\Pi^\nu \cap \overline{\Omega}\big).$$

Eventually, we obtain the desired inequality recalling that $\mathcal{H}^{d-1}(\overline{\Omega} \cap \partial\Pi^\nu) = \mathcal{H}^{d-1}(\Omega \cap \partial\Pi^\nu)$ by (14). □

Remark 2.5 *(Long-range interactions).* The assumption that only nearest-neighbour interactions are taken into account can be substituted by a coerciveness condition on nearest-neighbour interactions (that ensures that the limit energy be defined on sets of finite perimeter), and a decay condition (that ensures that the limit is local, and then of an integral form). Namely we may suppose that

(a) $c_{ij} \geqslant \alpha > 0$ if $|i-j| = 1$;
(b) $0 \leqslant c_{ij} \leqslant \beta_{i-j}$ and $\sum_{k\in\mathbb{Z}^d} \beta_k < +\infty$.

The statement of the homogenizability condition in Definition 2.2 can be then modified simply by requiring that there exists the function

$$\varphi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{d-1}} \inf_A \left\{ \sum_{ij} c_{ij}\colon\ i\in A,\ j\in\mathbb{R}^d\setminus A,\ i\in Q_T^\nu(x_T) \text{ or } j\in Q_T^\nu(x_T) \right\}, \tag{15}$$

with the infimum taken on the same competing sets.

The proof of Theorem 2.4 needs some additional technical modifications as follows: we define the measure

$$\mu_\varepsilon = \sum_{\varepsilon i,\varepsilon j\in\widetilde{A}_\varepsilon} \varepsilon \frac{1}{2} c_{ij} (\delta_{\varepsilon i} + \delta_{\varepsilon j}),$$

and proceed as in the first part of the proof of Theorem 2.4 (the second condition above ensuring that μ_ε is an equibounded family of positive measures). The proof must be then modified in the choice of the point t in (10): for fixed $M\in\mathbb{N}$ we deduce from (9) that there exists $t\in[\frac{\rho\delta}{2}, \rho\delta]$ such that

$$\begin{aligned} &\left|\left(\left(Q_\rho^\nu(x)\cap A_\varepsilon\right)\triangle\Pi^\nu(x)\right)\cap\left\{y\colon\ t-\varepsilon M\leqslant \operatorname{dist}\left(y,\partial Q_\rho^\nu(x)\right)\leqslant t+\varepsilon M\right\}\right| \\ &\quad\leqslant \frac{C}{h\delta}\rho^{d-1}\varepsilon M. \end{aligned} \tag{16}$$

Defining again the subset $A_\varepsilon^1\subset Q_\rho^\nu(x)$ as in (12) we can compute the error due to the change in the boundary values by splitting the contributions into the interactions between pairs inside $\{y\colon\ t-\varepsilon M\leqslant \operatorname{dist}(y,\partial Q_\rho^\nu(x))\leqslant t+\varepsilon M\}$ (for which we use condition (b) above), and those outside this set (using in particular that the distance between points $\varepsilon i, \varepsilon j$ is at least $2\varepsilon M$ whenever $\operatorname{dist}(\varepsilon i,\partial Q_\rho^\nu(x))\geqslant t+\varepsilon M$ and $\operatorname{dist}(\varepsilon j,\partial Q_\rho^\nu(x))\leqslant t-\varepsilon M$), obtaining in the end

$$\liminf_{\varepsilon\to0}\mu_\varepsilon\left(Q_\rho^\nu(x)\right)\geqslant\rho^{d-1}\varphi(\nu)-C\left(\frac{1}{h\delta}+\delta+\sum_{|k|\geqslant M}\beta_k\right)\rho^{d-1}.$$

We leave the details of this computation to the interested reader – and also refer to similar computations in the cut-off argument of Proposition 3.7 in [2] (see also [20]). From (b) and the arbitrariness of M, the liminf inequality follows as in Theorem 2.4. The construction of the recovery sequences for the limsup inequality is essentially unchanged.

2.1. Homogenization of periodic and almost-periodic spin systems

Periodic and almost-periodic coefficients provide nice examples of homogenizable systems, for which moreover the function φ defined in (4) exists also for sequences not satisfying assumption (3).

2.1.1. The periodic case

We check that periodic coefficients give a homogenizable system. To this end, we suppose that there exists $N \in \mathbb{N}$ such that

$$c_{(k+Ne_j)} = c_k \quad \text{for all } k \in \mathbb{Z}^d \text{ and } j = 1, \dots, d. \tag{17}$$

We then choose a sequence (x_T) and remark that by a translation argument by an element of $N\mathbb{Z}^d$ we may suppose that $x_T \in [-\frac{N}{2}, \frac{N}{2}]^d$. With fixed ν, we denote

$$\begin{aligned}\varphi_T(x_T, \nu) = \frac{1}{T^{d-1}} \inf_A \Bigg\{ \sum_{ij} c_{ij} \colon i \in A,\ j \in \mathbb{R}^d \setminus A,\ i \in Q_T^\nu(x_T) \\ \text{or } j \in Q_T^\nu(x_T),\ |i-j| = 1 \Bigg\},\end{aligned} \tag{18}$$

with the minimum taken over the sets A satisfying (5). We then have

$$\left(\frac{T+N}{T}\right)^{d-1} \varphi_{T+N}(0,\nu) - \frac{C}{T} \leqslant \varphi_T(x_T, \nu) \leqslant \left(\frac{T-N}{T}\right)^{d-1} \varphi_{T-N}(0,\nu) + \frac{C}{T},$$

so that it is sufficient to prove the convergence with $x_T = 0$ for all T.

We can also additionally suppose that ν is a rational direction. In fact note that taking sets A' defined by

$$A' = \begin{cases} A & \text{in } Q_T^\nu(0), \\ \Pi^{\nu'}(0) & \text{in } \mathbb{R}^d \setminus Q_T^\nu(0) \end{cases}$$

in the definition of $\varphi_{T'}(0, \nu')$ for $T' > T$ such that $Q_T^\nu(0) \subset Q_{T'}^{\nu'}(0)$ we easily prove that

$$\varphi_{T'}\big(0, \nu'\big) \leqslant \varphi_T(0, \nu) + \omega\big(\big|\nu - \nu'\big|\big) + \omega\left(\left|\frac{T'}{T} - 1\right|\right)$$

for some modulus of continuity ω. From such a continuity estimate we easily deduce that if the limit defining $\varphi(\nu)$ exists for all ν rational direction, then it exists for all $\nu \in S^{d-1}$.

We can now use a classical subadditivity argument to prove the existence of the limit of $\varphi_T(0, \nu)$ (see [4]). To that end it is not restrictive to consider $\nu = e_d$ and any test set A_T for $\varphi_T(0, e_d)$; we then construct the test set A_S for $\varphi_S(0, e_d)$ by

$$A_S = \bigcup_i \big(A_T + [T+1](i,0)\big) \cup \left(\Pi_0^{e_d} \setminus \bigcup_i \big(Q_T^{e_d}(0) + [T+1](i,0)\big)\right),$$

where the set of indices $i \in \mathbb{Z}^{d-1}$ concurring in the two unions above is defined by

$$Q_T^{e_d}(0) + [T+1](i,0) \subset Q_S^{e_d}(0).$$

We then deduce that

$$\varphi_S(0,\nu) \leqslant \varphi_T(0,\nu) + r(S,T),$$

with $\limsup_{T\to+\infty} \limsup_{S\to+\infty} r(S,T) = 0$, and then the existence of the desired limit.

2.1.2. The almost-periodic case

The same method used above can be extended to cover some almost-periodic cases. We then suppose that

$$c_{ij} = f\left(\frac{i+j}{2}\right)$$

with f a uniformly almost-periodic function (Bohr almost periodic) (see [5,16]).

We fix $\eta > 0$. Then there exists a set $T_\eta \subset \mathbb{Z}^d$ of η-*almost periods* of f and $N_\eta \in \mathbb{N}$ such that

$$T_\eta + [0, N_\eta]^d = \mathbb{R}^d$$

and

$$\left| f(x+\varkappa) - f(x) \right| \leqslant \eta \quad \text{for all } x \in \mathbb{R}^d$$

for all $\varkappa \in T_\eta$.

Given x_T let $\varkappa \in T_\eta \cap (x_T + [-N_\eta/2, N_\eta/2]^d)$; then we have

$$\left| \varphi_T(x_T,\nu) - \varphi_T(x_T - \varkappa, \nu) \right| \leqslant \eta \tag{19}$$

and $x_T - \varkappa \in [-N_\eta/2, N_\eta/2]^d$. It suffices then to prove the existence of the limit in (4) for points x_T satisfying $x_T \in [-N_\eta/2, N_\eta/2]^d$, the general case then following by (19) and the arbitrariness of η. As above we can then further restrict to the case $x_T = 0$.

The estimate of $\varphi_S(0,\nu)$ in terms of $\varphi_T(0,\nu)$ for $T < S$ is slightly more complex than in the periodic case. Note first that we cannot directly reduce to ν being a rational direction. We then fix a lattice $\mathcal{L}$ in $\{\langle x,\nu\rangle = 0\}$ isometric to $(T+N_\eta)\mathbb{Z}^{d-1}$, and for each $i \in \mathcal{L}$ with $|i| \in Q^\nu_{(S-(T+N_\eta)/2)}(0)$, we choose $\varkappa_i \in T_\eta \cap (i + [-N_\eta/2, N_\eta/2]^d)$. Then, given a test set A_T for $\varphi_T(0,\nu)$ we construct the set A_S by

$$A_S = \bigcup_i (A_T + \varkappa_i) \cup \left(\Pi^\nu_0 \setminus \bigcup_i \left(Q^\nu_T(0) + \varkappa_i\right)\right),$$

where the set of indices concurring in the two unions are now the $i \in \mathcal{L}$ defined above.

We then deduce that

$$\varphi_S(0,\nu) \leqslant \varphi_T(0,\nu) + \eta C + \frac{1}{S^d}\left(S^d - \left(\left\lfloor \frac{S}{T+N_\eta}\right\rfloor - 2\right)^d T^d\right) + \frac{1}{S^d}\left\lfloor \frac{S}{T+N_\eta}\right\rfloor^d T^d N_\eta$$

the last term due to the misplacement of the centers of the cubes $\varkappa_i + Q_T(0)$ from the plane $\{\langle x,\nu\rangle = 0\}$ (by at most CN_η), and then the existence of the desired limit.

2.2. Homogenization of random spin systems

In this section we will consider only the two-dimensional case for random interactions. To this end we first introduce the probabilistic framework (in any dimension d for future reference). Given a probability space $(\Sigma, \mathcal{F}, \mathbf{P})$ we consider an ergodic stationary discrete random process c_k^ω, $k \in \mathcal{Z}$ (the dual lattice of $\mathbb{Z}^d$). In what follows we will assume that the probability space is equipped with a discrete ergodic dynamical system T_z, $z \in \mathbb{Z}^d$; that is, a group of measurable transformations of Σ such that

- $T_0 = \mathrm{Id}$, $T_{z+y} = T_z \circ T_y$,
- T_z preserves the measure $\mathbf{P}$ for all $z \in \mathbb{Z}^d$.

Hence, there exists a random variable c^ω such that $c_k^\omega = c^{T_k\omega}$.

We are going to compute the Γ-limit of the two-dimensional energies

$$E_\varepsilon^\omega(A) = \varepsilon \sum \bigl\{c_{ij}^\omega \colon \varepsilon i \in A,\ \varepsilon j \in \Omega_\varepsilon \setminus A,\ |i-j| = 1\bigr\} \tag{20}$$

(with the usual identification $c_{ij}^\omega = c_k^\omega$), where $\Omega_\varepsilon = \varepsilon\mathbb{Z}^2 \cap \Omega$.

For any vector $\tau \in \mathbb{R}^2$, $m \in \mathbb{N}$ and $\omega \in \Sigma$ we denote

$$\psi^\omega(x,y) = \min\left\{\sum_{n=1}^{K} c_{i_n i_{n-1}}^\omega \colon i_0 = x,\ i_K = y,\ K \in \mathbb{N}\right\}, \tag{21}$$

where the minimum is taken over all paths joining x and $y \in \mathbb{Z}^2$. The following statement holds.

Lemma 2.6. *(See Boivin [6].) For any $\tau \in \mathbb{R}^2$ the following limit exists almost surely and does not depend on ω*

$$\psi_0(\tau) = \lim_m \frac{1}{m}\psi^\omega\bigl(0, \lfloor m\tau \rfloor\bigr), \tag{22}$$

where $\lfloor m\tau \rfloor_k = \lfloor m\tau_k \rfloor$ is the integer part of the k-th component of $m\tau$.

With this result in mind we can state the convergence theorem.

Theorem 2.7 *(Random homogenization). Let c_{ij}^ω be defined as above and satisfy* (2); *then the Γ-limit $F^\omega = \Gamma\text{-}\lim BV_{\varepsilon\to 0} F_\varepsilon^\omega$ exists almost surely, is deterministic and is given by*

$$F^\omega(A) = \int_{\Omega\cap\partial^* A} \varphi(\nu)\, d\mathcal{H}^1, \tag{23}$$

where

$$\varphi(\nu) = \psi_0\bigl(\nu^\perp\bigr), \tag{24}$$

and $\nu^\perp$ denotes the clockwise rotation of $\pi/2$ of ν.

Before proving the theorem we make some preliminary remarks. We first note that in the case $d = 2$ the notion of homogenizable system given in Definition 2.2 in the previous section can be stated alternatively as follows.

Definition 2.8 *(Homogenizable system).* We say that the set of coefficients c_{ij} is a *homogenizable system* if for every fixed vector $\tau \in \mathbb{R}^d$ and for every choice of sequences (x_m), (y_m) of points on $\mathbb{Z}^d$ with

$$\sup_m \left\{ \frac{|x_m|}{m} + \frac{|y_m|}{m} \right\} < +\infty, \quad \text{and} \quad y_m - x_m = m\tau + o(m) \tag{25}$$

there exists the function

$$\psi_0(\tau) = \lim_m \frac{1}{m} \min\left\{ \sum_{n=1}^{K} c_{i_n i_{n-1}}\colon\ i_0 = x_m,\ i_K = y_m,\ K \in \mathbb{N} \right\}, \tag{26}$$

the minimum being taken over all paths with arbitrary length joining x_m and y_m, and is independent of (x_m), (y_m).

Remark 2.9. The definition above can be seen to be equivalent to the one in the previous section, upon remarking that in the two-dimensional case we can always reduce to boundary of test sets for the definition of φ that are curves joining two opposite sides of the square $Q_T^\nu(x_T)$.

The function ψ_0 defined in (26), if it exists, can be compared with ψ defined later in (48), in which definition we have a restriction on the length of the paths. Hence we have $\psi_0 \leqslant \psi$.

Furthermore, ψ_0 enjoys some properties that are of easy verification from its definition:

(i) ψ_0 is positively homogeneous of degree one and convex. In particular it is Lipschitz continuous;
(ii) $\alpha\|\tau\|_1 \leqslant \psi_0(\tau) \leqslant \beta\|\tau\|_1$.

As a consequence, if we define φ as in (24) the energy

$$F(A) = \int_{\Omega\cap\partial^* A} \varphi(\nu)\, d\mathcal{H}^1, \tag{27}$$

is a lower-semicontinuous functional on sets of finite perimeter.

The translation invariance of the function ψ_0 required in (26) will be ensured in the random case by the following proposition.

Proposition 2.10. *For any $x \in \mathbb{R}^2$ and $\tau \in \mathbb{R}^2$ the limit relation*

$$\lim_m \frac{1}{m} \psi^\omega\big(\lfloor mx \rfloor, \lfloor mx + m\tau \rfloor\big) = \psi_0(\tau) \tag{28}$$

holds almost surely.

Proof. We first prove the statement of the proposition for the vectors $x = (0,1)$ and $\tau = (1,0)$. With this choice of x and τ for any $\epsilon > 0$ we define the events

$$\mathcal{Q}_N = \left\{ \omega \in \Sigma\colon \forall m \geqslant N \text{ it holds } \left| \frac{1}{m}\psi^\omega(0, m\tau) - \psi^0(\tau) \right| < \epsilon \right\}.$$

By Lemma 2.6 for any $\delta > 0$ there is $N_0(\delta)$ such that

$$\mathbf{P}\{\mathcal{Q}_{N_0}\} > 1 - \delta.$$

By the ergodic theorem, almost surely for any $\gamma > 0$ there is $m_0 = m_0(\omega, \gamma)$ such that for any $m \geqslant m_0$ it holds

$$\left| \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{\mathcal{Q}_{N_0}}(T_{ix}\omega) - \mathbf{P}(\mathcal{Q}_{\mathbf{N_0}}) \right| < \gamma. \tag{29}$$

Due to some technical reason we will assume that both $\delta < 1/4$ and $\gamma < 1/4$.

For $m \geqslant m_0(\omega, \gamma)$ denote by L the maximum of integer numbers j such that $j > m$ and for all $i \in (m, j) \cap \mathbb{N}$ it holds $T_{ix}\omega \notin \mathcal{Q}_{N_0}$. In order to estimate L from above, we denote by $\tilde{m}$ the number of unities in the sequence $\{\mathbf{1}_{\mathcal{Q}_{N_0}}(T_{ix}\omega)\}_{i=1}^{m}$. Then

$$\gamma > \left| \frac{\tilde{m}}{m + L} - \mathbf{P}(\mathcal{Q}_{N_0}) \right| = \left| 1 - \mathbf{P}(\mathcal{Q}_{N_0}) - \frac{L + (m - \tilde{m})}{m + L} \right|.$$

Therefore,

$$\frac{L + (m - \tilde{m})}{m + L} < \gamma + \bigl(1 - \mathbf{P}(\mathcal{Q}_{N_0})\bigr) < \gamma + \delta.$$

Since $(m - \tilde{m}) \geqslant 0$ and $\gamma + \delta \leqslant 1/2$, the last inequality yields the upper bound $L < 2(\gamma + \delta)m$.

If we choose $\widetilde{L} = 3(\gamma + \delta)m$ and arbitrary $m > \max(m_0(\omega, \gamma), N_0(\delta))$, then there is $n \in [m, m + \widetilde{L}]$ such that $T_{nx}\omega \in \mathcal{Q}_{N_0}$. Notice that if $T_{ix}\omega \in \mathcal{Q}_{N_0}$, then for any $m > N_0$

$$\left| \frac{1}{m}\psi^\omega(ix, ix + m\tau) - \psi^0(\tau) \right| < \epsilon.$$

Also, since $0 < n - m < 3(\gamma + \delta)m$, by the definition of ψ^ω we have

$$\begin{aligned} \psi^\omega(nx, nx + n\tau) &\leqslant \psi^\omega(nx, mx) + \psi^\omega(mx, mx + m\tau) + \psi^\omega(mx + m\tau, nx + n\tau) \\ &\leqslant \psi^\omega(mx, mx + m\tau) + 9(\gamma + \delta)m\beta, \end{aligned}$$

$$\begin{aligned} \psi^\omega(mx, mx + m\tau) &\leqslant \psi^\omega(mx, nx) + \psi^\omega(nx, nx + n\tau) + \psi^\omega(nx + n\tau, mx + m\tau) \\ &\leqslant \psi^\omega(nx, nx + n\tau) + 9(\gamma + \delta)m\beta. \end{aligned}$$

Thus, $|\psi^\omega(mx, mx + m\tau) - \psi^\omega(nx, nx + n\tau)| \leqslant 9(\gamma + \delta)m\beta$. Taking this into account, we obtain

$$\begin{aligned}
&\left|\frac{1}{m}\psi^\omega(mx, mx + m\tau) - \psi^0(\tau)\right| \\
&\quad\leqslant \left(\frac{1}{m} - \frac{1}{n}\right)\psi^\omega(mx, mx + m\tau) + \frac{1}{n}\left|\psi^\omega(mx, mx + m\tau) - \psi^\omega(nx, nx + n\tau)\right| \\
&\qquad + \left|\frac{1}{n}\psi^\omega(nx, nx + n\tau) - \psi^0(\tau)\right| \\
&\quad\leqslant \frac{3(\gamma + \delta)}{n}m\beta + 9(\gamma + \delta)\beta + \epsilon = 12(\gamma + \delta)\beta + \epsilon.
\end{aligned}$$

Since γ, δ and ϵ are arbitrary positive numbers, this implies the desired limit relation.

One can easily check that the above arguments remain valid in the case of x and τ with integer coordinates. Indeed, we have only used in the proof the ergodicity arguments and the fact that any integer multiplier of x and τ has integer coordinates.

Since ψ_0 is positive one-homogeneous, this implies, in turn, that (28) holds for any vectors x and τ with rational coordinates.

This result can be easily generalized to arbitrary vectors x and τ with the help of continuity arguments. □

Proof of Theorem 2.7. In view of Definition 2.8, Remark 2.9 and Proposition 2.10 the desired statement is a consequence of Theorem 2.4. □

3. Homogenization of spin systems: interactions with changing sign

In this section we consider interactions with changing sign. In this case our energies are a priori not coercive in the space of sets with finite perimeter, even though in the end we will be able to recover a limit form of the same type as in the previous section. It is convenient then to consider again as variables in place of the sets $A \subset \Omega_\varepsilon$ spin functions $u : \Omega_\varepsilon \to \{-1, 1\}$. The scaled energies now read as

$$E_\varepsilon(u) = \frac{1}{8}\sum_{i,j} \varepsilon^{d-1}c_{ij}(u_i - u_j)^2, \tag{30}$$

where now we only suppose

$$|c_{ij}| \leqslant \beta. \tag{31}$$

In this case the identification of u with a set of finite perimeter as for the case of positive interactions does not bring along a compactness property, since we cannot infer any estimates on the perimeter from energy bounds.

Remark 3.1 *(The case of only negative signs).* Note that the case of strictly negative coefficients can be reduced to the case of positive coefficients, upon the change of variables $v_i = (-1)^{i_1+\cdots+i_d}u_i$, since we have

$$(u_i - u_j)^2 = 2 - 2u_i u_j = 2 + 2v_i v_j = 4 - (v_i - v_j)^2$$

when $|i - j| = 1$.

In general the energies E_ε are not bounded from below, with only an estimate of the form $\inf E_\varepsilon \geqslant -\beta\frac{1}{\varepsilon}$ holding. We then need additional assumptions to guarantee both some compactness properties and a suitable normalization of the energy

We say that I is a *connected set* in $\mathbb{Z}^d$ if for each pair of i, j points of I there exists a path from i to j; i.e., there exist $K \in \mathbb{N}$ and $\{i_n\colon n = 0, \dots, K\}$ with $i_0 = i$, $i_K = j$ and $\|i_n - i_{n-1}\|_1 = 1$. The *boundary* of I is

$$\partial I = \big\{i \in I\colon \exists j \in \mathbb{Z}^d \setminus I,\ \|i - j\|_1 = 1\big\}.$$

Definition 3.2 *(Ground states).* We will suppose that the set

$$\mathcal{N} = \big\{i \in \mathbb{Z}^d\colon \exists j \text{ such that } c_{ij} \leqslant 0\big\} \tag{32}$$

can be decomposed as

$$\mathcal{N} = \bigcup_m K_m,$$

where K_m is a connected set in $\mathbb{Z}^d$ such that

(i) $\sup_m \# K_m < +\infty$;
(ii) for all m there exists a connected *safe zone* $N_m \supset K_m$ such that all minimizers $\bar{u}$ of

$$\min\bigg\{\sum_{i,j \in N_m} c_{ij}(u_i - u_j)^2\bigg\} \tag{33}$$

satisfy either $\bar{u}_i = 1$ or $\bar{u}_i = -1$ identically on ∂N_m. Furthermore, we assume that N_m are disjoint and there exists $\kappa > 0$ such that

$$\sum_{i,j \in \cap N_m} c_{ij}\big((u_i - u_j)^2 - (\bar{u}_i - \bar{u}_j)^2\big) \geqslant \kappa \tag{34}$$

if u is not a minimizer. Note that this is not a restriction for example in the periodic setting, or if c_{ij} take a finite number of values.

A (positive) *ground state* for E_ε is defined as a function $\bar{u}$ which coincides with a minimizer of (33) satisfying $\bar{u}_i = 1$ identically on ∂N_m for all m, and equal to 1 elsewhere; a negative ground state is a function of the form $-\bar{u}$, with $\bar{u}$ a positive ground state.

Normalization of the energy. From now on we will fix a positive ground state $\bar{u}$. We then normalize E_ε by setting (with an abuse of notation)

$$E_\varepsilon(u) = \sum_{i,j \in \Omega_\varepsilon} \varepsilon^{d-1} c_{ij}\big((u_i - u_j)^2 - (\bar{u}_i - \bar{u}_j)^2\big). \tag{35}$$

Note that with this normalization we have

$$\begin{aligned}E_\varepsilon(u) &\geqslant \sum\Big\{\sum_{i,j\in\Omega_\varepsilon\cap N_m} c_{ij}\varepsilon^{d-1}\big((u_i-u_j)^2-(\bar u_i-\bar u_j)^2\big):\ N_m\cap\partial\Omega_\varepsilon\neq\emptyset\Big\}\\ &\geqslant -4\beta\varepsilon^{d-1}\sum\{\#N_m:\ N_m\cap\partial\Omega_\varepsilon\neq\emptyset\}\\ &\geqslant -c\mathcal{H}^{d-1}(\partial\Omega).\end{aligned}$$

Connectedness hypothesis on the set $\mathcal{P}:=\mathbb{Z}^d\setminus\mathcal{N}$. We suppose that if $m\neq m'$ then $\operatorname{dist}(N_m, N_{m'})\geqslant 2$. This implies that the set $\mathcal{P}$ contains the connected set

$$\mathcal{C}=\mathbb{Z}^d\setminus\bigcup_m N_m.$$

Note that, after possibly passing to a subsequence there exists the weak* (local) limit μ (in the sense of measures) of

$$\mu_\varepsilon=\sum_{i\in\mathcal{C}}\varepsilon^d\delta_{\varepsilon i}.$$

By our hypotheses then this limit can be written as $\mu=\theta\mathcal{L}^d$ for some strictly positive density $\theta>0$.

Lemma 3.3 *(Compactness). Suppose that* $c_{ij}\geqslant\alpha>0$ *for bonds connecting points in* $\mathcal{C}$. *Let* (u_ε) *be a sequence with equibounded energy. Then, up to subsequences, there exists a function* $u\in BV(\Omega;\{-1,+1\})$ *such that we have*

$$u_\varepsilon\chi_{\varepsilon\mathcal{C}}\rightharpoonup^* u\theta \tag{36}$$

with respect to the weak-convergence in* $L^\infty(\Omega)$.

Proof. By the boundedness in L^∞ of (u_ε) it is enough to prove that (36) holds locally in Ω. We consider the sets of indices

$$M=\{m:\ N_m\subset\Omega\},\qquad M_b=\{m:\ N_m\not\subset\Omega,\ N_m\cap\Omega\neq\emptyset\}$$

and the corresponding collection of “safe zones”. We further subdivide those in three families corresponding to

$$M_\varepsilon^+=\{m:\ u_\varepsilon=+1\ \text{ on }\partial N_m\}, \tag{37}$$

$$M_\varepsilon^-=\{m:\ u_\varepsilon=-1\text{ on }\partial N_m\}, \tag{38}$$

$$M_\varepsilon^0=M\setminus\big(M_\varepsilon^+\cup M_\varepsilon^-\big). \tag{39}$$

Note that for each $m \in M_\varepsilon^0$ the inequality in (34) holds, so that in particular

$$\#\big(M_\varepsilon^0\big) \leqslant \frac{C}{\varepsilon^{d-1}}.$$

We then consider the functions $\hat{u}_\varepsilon$ defined as

$$\hat{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{on } \Omega_\varepsilon \cap \varepsilon\mathcal{C}, \\ 1 & \text{on } N_m \text{ if } m \in M_\varepsilon^+, \\ -1 & \text{on } N_m \text{ if } m \in M \setminus M_\varepsilon^+. \end{cases}$$

We have the estimate

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\geqslant -c\mathcal{H}^{d-1}(\partial\Omega) + \alpha \sum_{i,j\in\varepsilon\mathcal{C}\cap\Omega_\varepsilon} \varepsilon^{d-1}\big((u_\varepsilon)_i - (u_\varepsilon)_j\big)^2 \\ &\geqslant -c\mathcal{H}^{d-1}(\partial\Omega) + \alpha \sum_{i,j\in\Omega_\varepsilon} \varepsilon^{d-1}\big((\hat{u}_\varepsilon)_i - (\hat{u}_\varepsilon)_j\big)^2 - c\varepsilon^{d-1} \sum_{m\in M_\varepsilon^0} \#(N_m) \\ &\geqslant \alpha \sum_{i,j\in\Omega_\varepsilon} \varepsilon^{d-1}\big((\hat{u}_\varepsilon)_i - (\hat{u}_\varepsilon)_j\big)^2 - c \\ &= 4\alpha\mathcal{H}^{d-1}\big(\partial\{\hat{u}_\varepsilon = 1\} \cap \Omega\big) - c, \end{aligned}$$

so that the sequence $(\hat{u}_\varepsilon)$ is equibounded in BV, and hence strongly precompact in L^1. We can therefore suppose that $\hat{u}_\varepsilon \to u$.

We then have

$$u_\varepsilon \chi_{\varepsilon\mathcal{C}} = \hat{u}_\varepsilon \chi_{\varepsilon\mathcal{C}} \rightharpoonup^* u\theta$$

as desired. □

We can state a definition of homogenizable system analogous to the one in the previous sections.

Definition 3.4 *(Homogenizable system).* Let the set of coefficients $\{c_{ij}\}$ satisfy the conditions above; we say that $\{c_{ij}\}$ is a *homogenizable system* if for every fixed vector $\nu \in S^{d-1}$ and for every choice of a family (x_T) of points on $\mathbb{R}^d$ with

$$\sup \frac{|x_T|}{T} < +\infty \tag{40}$$

there exists the function

$$\varphi(\nu) = \lim_{T\to+\infty} \inf \frac{1}{T^{d-1}} \sum_{ij} c_{ij}\big((u_i - u_j)^2 - (\bar{u}_i - \bar{u}_j)^2\big) \tag{41}$$

the minimum being taken over all functions $u : \mathbb{Z}^d \to \{\pm 1\}$ such that

$$u=\bar u \quad \text{in } \big(\Pi_\nu(x_T)\setminus Q^\nu_T(x_T)\big)\cup\bigcup\big\{N_m\colon\ N_m\cap\Pi^\nu(x)\neq\emptyset,\ N_m\not\subset Q^\nu_T(x_T)\big\}, \tag{42}$$

$$u=-\bar u \quad \text{in } \big(\big(\mathbb{Z}^d\setminus\Pi_\nu(x_T)\big)\setminus Q^\nu_T(x_T)\big)\cup\bigcup\big\{N_m\colon\ N_m\cap\Pi^\nu(x)=\emptyset,\ N_m\not\subset Q^\nu_T(x_T)\big\}, \tag{43}$$

and the sum is taken on all pairs (i,j) with $i\in Q^\nu_T(x_T)$ or $j\in Q^\nu_T(x_T)$ and $|i-j|=1$, and this limit is independent of the choice of (x_T).

With this definition we have a homogenization theorem with respect to the convergence (36). With respect to the previous result for positive interactions, we may have a contribution resulting from the optimization of the interactions close to the boundary of Ω. This term depends on the shape of the "safe zones" intersecting the boundary, and requires then a passage to a subsequence (see Remark 3.6 below). We here state an "interior homogenization theorem" where this boundary term does not appear. To this end we define the interior approximation of Ω as

$$\widehat{\Omega}_\varepsilon=\Omega_\varepsilon\setminus\bigcup\{N_m\colon\ N_m\not\subset\Omega_\varepsilon\}$$

and correspondingly the energy

$$\widehat{E}_\varepsilon(u)=\sum_{i,j\in\widehat{\Omega}_\varepsilon}c_{ij}\big((u_j-u_j)^2-(\bar u_i-\bar u_j)^2\big).$$

Theorem 3.5 *(Interior homogenization theorem). Let Ω be a bounded open set with Lipschitz boundary. Let (c_{ij}) be a homogenizable system according to Definition 3.4, and let $\inf c_{ij}>0$ on $\mathcal{C}$. Then the family (E_ε) Γ-converges with respect to the convergence (36) to the energy*

$$F(u)=\int_{S(u)\cap\Omega}\varphi(\nu_u)\,d\mathcal{H}^{d-1},$$

defined on $u\in BV(\Omega;\{\pm1\})$, where φ is given by (41), $S(u)$ is the set of discontinuity points of u (coinciding with $\partial^*\{u=1\}$) and ν_u is the normal to $S(u)$.

Proof. The proof follows almost word for word the proof of Theorem 2.4 substituting sets A_ε with functions u_ε. The only care to be taken is in the construction of test functions for the definition of φ from converging sequences u_ε. Since we control the energy only on the set $\mathcal{C}$, the definition of u^1_ε (corresponding to (12)) must be given as

$$u^1_\varepsilon=\begin{cases}u_\varepsilon & \text{in } Q^\nu_{\rho-t}(x)\setminus\bigcup\{N_m\colon\ N_m\not\subset Q^\nu_{\rho-t}(x)\},\\ u^\nu_x & \text{otherwise,}\end{cases} \tag{44}$$

where u^ν_x is defined as

$$u^\nu_x=\begin{cases}\bar u & \text{in } \Pi^\nu(x)\setminus\bigcup\{N_m\colon\ N_m\cap\Pi^\nu(x)\neq\emptyset\},\\ -\bar u & \text{otherwise}\end{cases}$$

(giving the desired boundary datum for the definition of φ), and t is suitably chosen so as to have estimate (11) hold, up to a multiplicative constant, with the boundary of

$$Q^{\nu}_{\rho-t}(x)\setminus\bigcup\bigl\{N_m\colon\ N_m\not\subset Q^{\nu}_{\rho-t}(x)\bigr\}$$

in place of $\{y\colon\ \mathrm{dist}(y,\partial Q^{\nu}_{\rho}(x))=t\}$. $\square$

Remark 3.6.

(i) If we consider the energies E_ε, the Γ-limit as $\varepsilon\to 0$ in general does not exist. From each (ε_j) we can extract a subsequence such that the Γ-limit F exists along such subsequence. In this case it has the form

$$F(u)=\int\limits_{S(u)\cap\Omega}\varphi(\nu_u)\,d\mathcal{H}^{d-1}+\int\limits_{\partial\Omega}g(x,u)\,d\mathcal{H}^{d-1}$$

(in this last integral u stands for its inner trace) for some g. This g satisfies $|g(x,u)|\leqslant C$, with C depending only on d and $\sup\#N_m$.

(ii) The conclusions of Theorem 3.5 hold unchanged if we consider E_ε as defined on the whole Ω_ε but changing c_{ij} to (arbitrary) positive values on the N_m intersecting $\mathbb{Z}^d\setminus\Omega_\varepsilon$, and setting $\bar u_i=1$ accordingly on such sets.

Remark 3.7 *(The periodic case).* If c_{ij} are periodic of period N the hypotheses of Theorem 3.5 are satisfied if we assume

(a) the minimizers of

$$\min\sum_{i,j\in\{1,\dots,N\}^d}c_{ij}(u_i-u_j)^2$$

satisfy identically $u_i=1$ (or identically $u_i=-1$) on the boundary of $\{1,\dots,N\}^d$;

(b) $c_{ij}>c>0$ if $i\in\{0,\dots,N\}^d\setminus\{1,\dots,N\}^d$.

In the special case when $c_{ij}=\pm 1$ this condition is satisfied if the set of bonds such that $c_{ij}=-1$ inside $\{0,\dots,N\}^d$ is contained in a cube of size k with $k\leqslant\sqrt[d]{(N/2)}$.

4. Homogenization over curves in a discrete environment

We now include a treatment of energies defined on curves in a discrete setting, or rather on parameterization of curves, which is in a sense dual to surface energies. In dimension $d=2$ this can be seen as an alternative way to treat boundaries, when parameterized by a finite family of curves. Note that the discrete energies we consider and their limits are in a sense not defined on geometrical objects, since they depend crucially on the parameterization of the curves. In the simplest situation, when all interaction coefficients are equal, the discrete energies we are going to consider can be seen as the length functional restricted to curves interpolating nearest-neighbour points on the lattice $\varepsilon\mathbb{Z}^d$ and parameterized by arc-length. This is clearly a parameterization-dependent energy. Note that if we also fix the total length L_ε of the curves (i.e., we set the energy equal to $+\infty$ on all other curves) then the energies at ε-scale take only the value L_ε or $+\infty$ and if $L_\varepsilon \to L$ then the Γ-limit is finite and equal to L on functions which are parameterizations of curves with "1-length" not greater than L. Hence, in the limit we have a description depending on the velocity of the curve; the limit energy can be written as

$$\int_0^L \psi\bigl(\gamma'\bigr)\,dt,$$

where simply

$$\psi(v) = \begin{cases} 1 & \text{if } \|v\|_1 \leqslant 1, \\ +\infty & \text{otherwise.} \end{cases}$$

The same type of representation, with non-constant ψ, holds also when the coefficients oscillate at scale ε. While in the case of constant coefficients it is 'energetically convenient' to use parameterizations with $\|v\|_1 = 1$ (i.e., these are the ones with a lower value of the Γ-limit for the same curve), when the coefficients oscillate then the 'optimal velocity' may correspond to $\|v\|_1 < 1$, and is described through a homogenization asymptotic formula. It may be clarifying to think of discrete curves as objects with a mass, proportional to the number of nodes interpolated by the curve, so that a limit continuous curve inherits the limit mass of the discrete approximations (which is inversely proportional to the velocity of the parameterization). More complex functionals depending explicitly on pairs curves–measures close in spirit to our approach and deriving from a discrete model of ternary interactions can be found in [3].

We define a path γ in Ω_ε as an array of points

$$\varepsilon i_0, \varepsilon i_1, \dots, \varepsilon i_{N-1}, \varepsilon i_N \in \Omega_\varepsilon, \quad N \in \mathbb{N},$$

such that

$$|i_n - i_{n-1}|_1 = 1,$$

where $|\cdot|_1$ stands for the L^1 norm in $\mathbb{Z}^d$. Each such path can be identified by the piecewise-affine continuous curve $\gamma : [0, \varepsilon N] \to \mathbb{R}^d$ satisfying $\gamma(\varepsilon n) = \varepsilon i_n$ for $n = 0, 1, \dots, N$, parameterized by arc-length. We say that a path γ joins x to y if $\gamma(0) = \varepsilon i_0 = x$ and $\gamma(\varepsilon N) = \varepsilon i_N = y$.

The energy of a path γ in Ω_ε is

$$F_\varepsilon(\gamma) = \sum_{n=1}^{N} \varepsilon c_{i_n i_{n-1}}. \tag{45}$$

Note that if we suppose that

$$0 < \alpha \leqslant c_{ij} \leqslant \beta < +\infty \tag{46}$$

then if $C_1 \leqslant F_\varepsilon(\gamma) \leqslant C_2$ we have

$$\frac{C_1}{\beta} \leqslant \varepsilon N \leqslant \frac{C_2}{\alpha}.$$

As a consequence, if we consider a family of paths $\gamma_\varepsilon : [0, \varepsilon N_\varepsilon] \to \mathbb{R}^d$ with equibounded energy, then, up to subsequences, we have $\varepsilon N_\varepsilon \to L > 0$, and γ_ε converge weakly* in $W^{1,\infty}$ to a curve $\gamma : [0, L] \to \mathbb{R}^d$ (to this end we may need to extend $\gamma_\varepsilon(s)$ as a constant for $s \geqslant \varepsilon N_\varepsilon$).

In order to define a limit energy on γ as the Γ-limit of F_ε we need some asymptotic properties of c_{ij}.

Definition 4.1 *(Homogenizable system).* We say that the set of coefficients c_{ij} is a *homogenizable system* if for every fixed direction $\tau \in \mathbb{R}^d$ with $\|\tau\|_1 = \sum_{j=1}^d |\tau_j| < 1$, any $M > 1$, and for every choice of sequences (x_m), (y_m) of points in $\mathbb{Z}^d$ with

$$\sup_m \left\{ \frac{|x_m|}{m} + \frac{|y_m|}{m} \right\} \leqslant M < +\infty, \quad \text{and} \quad y_m - x_m = m\tau + o(m) \tag{47}$$

there exists the function

$$\psi(\tau) = \lim_m \frac{1}{m} \inf\left\{ \sum_{n=1}^m c_{i_n i_{n-1}} \colon i_0 = x_m,\ i_m = y_m \right\}, \tag{48}$$

the minimum being taken over all paths with length m that join x_m and y_m. The limit is independent of the choice of (x_m) and (y_m), and exists uniformly in such (x_m) and (y_m).

Remark 4.2. We note that in the definition of ψ above we have not included the ‘boundary case’ when $\|\tau\|_1 = 1$. In fact, in that case the set of admissible paths in (48) may be empty or not, depending on (x_m) and (y_m), so that the requirement of the existence of the limit $\psi(\tau)$ is not satisfied even for c_{ij} constant. Such a drawback can be easily overcome by extending ψ to $\|\tau\|_1 = 1$ by (lower semi)continuity.

If $d = 2$ we can obtain the relation

$$\psi_0(\tau) = \inf\left\{ \lambda\psi\left(\frac{\tau}{\lambda}\right) \colon \lambda > \|\tau\|_1 \right\} \tag{49}$$

between the surface energy density for interfaces ψ_0 and the function ψ above by comparing formulas (48) and (26).

The function ψ defined in (48), if it exists, enjoys some properties on $\{\|\tau\|_1 < 1\}$ that are of easy verification from its definition:

(i) we have $\alpha \leqslant \psi(\tau) \leqslant \beta$, this inequality being proven by comparing with the trivial infimum when c_{ij} are identically α (or β, respectively);
(ii) ψ is a convex and Lipschitz function (with Lipschitz norm not exceeding $d\beta$).

As a consequence, we can extend ψ by continuity for $\|\tau\|_1 = 1$ and by $\psi(\tau) = +\infty$ for $\|\tau\|_1 > 1$; the function thus defined (which we still denote by ψ) is a convex and lower-semicontinuous function on $\mathbb{R}^d$, so that the energy

$$F(\gamma) = \int_0^L \psi(\gamma')\,dt, \tag{50}$$

is a lower-semicontinuous functional on curves with respect to the weak* $W^{1,\infty}$ topology, whose domain are curves such that $\|\gamma'\|_1 \leqslant 1$ a.e.

For $\gamma \in W^{1,\infty}((0,L);\mathbb{R}^d)$ denote $\|\gamma'\|_{\infty,1} = \operatorname{esssup}_{0<t<L} \|\gamma'(t)\|_1$.

We can then state our Γ-convergence result as follows.

Theorem 4.3 *(Homogenization over discrete paths). Let* (46) *be satisfied and let the system* $\{c_{ij}\}$ *be homogenizable as in Definition* 4.1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^d$. *Then the* Γ*-limit of the energies* F_ε *in* (45) *is the energy* F *in* (50). *More precisely,*

(i) *if* $\sup_\varepsilon F_\varepsilon(\gamma_\varepsilon) < +\infty$ *and* $\gamma_\varepsilon(0)$ *is equibounded, then, up to subsequences, there exist* $L \geqslant 0$ *and a curve* $\gamma \in W^{1,\infty}((0,L);\mathbb{R}^d)$ *with* $\|\gamma'\|_{\infty,1} \leqslant 1$ *such that* $\gamma_\varepsilon \rightharpoonup^* \gamma$ *(all curves are extended as constants outside their intervals of definition and this convergence is then understood as weakly* in* $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$*);*

(ii) *if* $\gamma_\varepsilon \rightharpoonup^* \gamma$ *as in* (i) *then*

$$F(\gamma) \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(\gamma_\varepsilon);$$

(iii) *for all* $\gamma \in W^{1,\infty}((0,L);\Omega)$ *with* $\|\gamma'\|_1 \leqslant 1$ *a.e. there exists a sequence* γ_ε *such that* $\gamma_\varepsilon \rightharpoonup^* \gamma$ *as in* (i) *and*

$$F(\gamma) \geqslant \limsup_{\varepsilon\to 0} F_\varepsilon(\gamma_\varepsilon).$$

Proof. Since (i) is proven by standard weak* compactness arguments, we need only to prove (ii) and (iii).

Note preliminarily that, if we fix an arbitrary $M > 1$ and define the functions

$$\psi_m(\tau) = \frac{1}{m}\min\Biggl\{\sum_{n=1}^m c_{i_n i_{n-1}}\colon\ i_0 = x_m,\ i_m = y_m,\ \frac{|x_m|_1}{m} \leqslant M,\ \frac{|y_m|_1}{m} \leqslant M,\ |y_m - x_m - m\tau|_1 \leqslant d\Biggr\}, \tag{51}$$

then (ψ_m) is an equibounded family on $D_1 := \{\tau \in \mathbb{R}^d\colon\ \|\tau\|_1 \leqslant 1\}$. Furthermore, for any $s < 1$ the estimate

$$\bigl|\psi_m(\tau) - \psi_m(\tau')\bigr| \leqslant \beta\bigl\|\tau - \tau'\bigr\|_1 + o(1) \tag{52}$$

holds true for $\tau, \tau' \in D_s := \{\tau \in \mathbb{R}^d\colon\ \|\tau\|_1 < s\}$; here $o(1)$ tends to zero as $m \to \infty$.

The equiboundedness is evident. In order to prove (52) denote $r = \tau' - \tau$. Let γ_m and γ_m' be the paths of length m that provide a minimum in the expressions

$$\min\left\{ \sum_{i=1}^{m} c_{i_n i_{n-1}}\colon\ i_0 = 0,\ \|i_m - m\tau\|_1 \leqslant d \right\}$$

and

$$\min\left\{ \sum_{n=1}^{m} c_{i_n i_{n-1}}\colon\ i_0 = 0,\ \left\|i_m - m\tau'\right\|_1 \leqslant d \right\},$$

respectively. Denote by $\tilde{\gamma}_m$ the shortest path that connects the end points of γ_m and γ_m'. Clearly, $|\tilde{\gamma}_m| \leqslant m\|r\|_1 + 2d$. Also, denote $\hat{m} = m + |\tilde{\gamma}_m|$. Since the system $\{c_{ij}\}$ is homogenizable, we have

$$\begin{aligned}
\psi_m\left(\tau'\right) &= \psi_m(\tau + r) = \psi_{(m+|\tilde{\gamma}_m|)}(\tau + r) + o(1) \\
&= \frac{1}{\hat{m}} \min\left\{ \sum_{n=1}^{\hat{m}} c_{i_n i_{n-1}}\colon\ i_0 = 0,\ \left\|i_{\hat{m}} - \hat{m}\tau'\right\|_1 \leqslant d \right\} + o(1) \\
&\leqslant \frac{1}{\hat{m}} \sum_{\gamma_m \cup \tilde{\gamma}_m} c_{i_n i_{n-1}} + o(1) \leqslant \frac{1}{m} \sum_{\gamma_m} c_{i_n i_{n-1}} + \frac{m\|r\|_1 + 2d}{m}\beta + o(1) \\
&\leqslant \psi(\tau) + \|r\|_1 \beta + o(1).
\end{aligned}$$

Thus,

$$\psi_m\left(\tau'\right) - \psi(\tau) \leqslant \|r\|_1 \beta + o(1).$$

Similarly,

$$\psi_m(\tau) - \psi\left(\tau'\right) \leqslant \|r\|_1 \beta + o(1).$$

This yields (52).

Hence, $\psi_m \to \psi$ uniformly on each D_s with $s < 1$.

We now prove (ii). Let $\gamma_\varepsilon : [0, \varepsilon N_\varepsilon] \to \mathbb{R}^d$ converge to γ and suppose, without loss of generality, that $\varepsilon N_\varepsilon \to L > 0$. Fix $J \in \mathbb{N}$; we may also suppose for the sake of simplicity that $N_\varepsilon / J \in \mathbb{N}$ so that we may consider the value $\gamma_\varepsilon(x_j^\varepsilon)$ at the points $x_j^\varepsilon = \varepsilon j N_\varepsilon / J \in \Omega_\varepsilon$. Note that $x_j^\varepsilon \to x_j = jL/J$, and, by the uniform convergence of γ_ε to γ, that $\gamma_\varepsilon(x_j^\varepsilon) \to \gamma(jL/J)$. We then have (taking $m = N_\varepsilon / J$)

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon(\gamma_\varepsilon) &= \liminf_{\varepsilon\to 0} \sum_{j=1}^{J} \varepsilon \sum_{n=(j-1)N_\varepsilon/J+1}^{jN_\varepsilon/J} c_{i_n i_{n-1}} \\
&\geqslant \liminf_{\varepsilon\to 0} \sum_{j=1}^{J} \frac{\varepsilon N_\varepsilon}{J} \psi_{N_\varepsilon/J}\left(\frac{\gamma(x_j^\varepsilon) - \gamma(x_{j-1}^\varepsilon)}{x_j^\varepsilon - x_{j-1}^\varepsilon} \right)
\end{aligned}$$

$$= \sum_{j=1}^{J} \frac{L}{J} \psi\left(\frac{\gamma(x_j) - \gamma(x_{j-1})}{L/J}\right)$$
$$= \int_0^L \psi(\gamma_J')\, dt,$$

where γ_J is the piecewise-affine interpolation of γ at the points x_j. Since $\gamma_J \rightharpoonup^* \gamma$ in $W^{1,\infty}((0, L); \mathbb{R}^d)$ we obtain (ii) by lower semicontinuity.

The proof of (iii) proceeds by density. It is enough to construct a recovery sequence when $L = 1$, γ is linear and $\gamma(s) = s\tau$ with $\|\tau\|_1 < 1$. With fixed $\eta > 0$ let m be large enough so that

$$\sup_{D_1} |\psi_m - \psi| \leqslant \eta.$$

We can then find points $\tau_0^\varepsilon, \tau_1^\varepsilon, \ldots, \tau_{[1/m\varepsilon]}^\varepsilon \in \varepsilon\mathbb{Z}^d$ such that

$$\left\| \tau_j^\varepsilon - \frac{j}{m\varepsilon}\tau \right\|_1 \leqslant d\varepsilon,$$

and for each of those find a path $\{c_n^{\varepsilon,j}\colon n = 0, \ldots, m\}$ joining $x_{j-1}^\varepsilon = \frac{1}{\varepsilon}\tau_{j-1}^\varepsilon$ to $x_j^\varepsilon = \frac{1}{\varepsilon}\tau_j^\varepsilon$ such that

$$\sum_{n=1}^{m} c_{i_n i_{n-1}}^{\varepsilon,j} \leqslant m\big(\psi_m(\tau) + o(1)\big).$$

We can then construct the corresponding γ_ε by assembling those paths on $[0, \varepsilon m\lfloor\frac{1}{m\varepsilon}\rfloor]$, so that

$$\gamma_\varepsilon(0) = 0, \qquad \gamma_\varepsilon\left(\varepsilon m\left\lfloor\frac{1}{m\varepsilon}\right\rfloor\right) = \tau_{[1/m\varepsilon]}^\varepsilon \to \tau$$

and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\gamma_\varepsilon) = \limsup_{\varepsilon\to 0} \sum_{j=1}^{[1/m\varepsilon]} \sum_{n=1}^{m} \varepsilon c_{i_n i_{n-1}}^{\varepsilon,j} \leqslant \limsup_{\varepsilon\to 0} \sum_{j=1}^{[1/m\varepsilon]} \varepsilon m \psi_m(\tau) \leqslant \psi_m(\tau).$$

By letting $m \to +\infty$ we prove (iii) for γ linear. In the same way we can construct a recovery sequence for γ piecewise affine, and eventually conclude by a density argument. □

4.1. The almost-periodic case

The proof of the homogenizability for (periodic and) almost-periodic systems requires some extra care due to the constraint on the path length in the minimum problem in (48). We will prove that the limit defining the function ψ exists (also without assuming the first condition of (47)). We will suppose that the almost periodicity condition in Section 2.1.2 holds, and we adapt the notation therein for almost periods.

Let $\|\tau\|_1 < 1$ and let (x_m) and (y_m) be sequences satisfying $y_m - x_m = m\tau + o(m)$. With fixed m, let $\{i_n^m\colon 0 \leqslant n \leqslant m\}$ be a path of length m with $i_0^m = x_m$, $i_m^m = y_m$ realizing the minimum in

$$\psi_m(\tau) = \frac{1}{m} \inf\left\{ \sum_{n=1}^{m} c_{i_n i_{n-1}}\colon i_0 = x_m,\ i_m = y_m \right\}. \tag{53}$$

We now construct a test path $\{i_n^M\colon 0 \leqslant n \leqslant M\}$ for the related problem for $\psi_M(\tau)$ with $M > m$ sufficiently large. To this end, let $\eta > 0$ and let $\varkappa_j$ be η-almost periods for f in T_η defined recursively as

$$\varkappa_0 \in x_M + [0, N_\eta]^d, \qquad \varkappa_j \in \varkappa_{j-1} + (y_m - x_m) + [0, N_\eta]^d;$$

we also define

$$y_0^M = x_M, \qquad y_j^M = \varkappa_{j-1} + (y_m - x_m) \quad \text{for } j \geqslant 1.$$

We fix K (to be determined later) and we define the path i_n^M as follows: Let $\{y_n^j\colon 0 \leqslant n \leqslant J_j\}$ be a path joining y_j^M to $\varkappa_j$ for $j = 0, \dots, K-1$ (with length J_j at most dN_η). We set

$$I_0 = 0, \qquad I_j = I_{j-1} + m + J_{j-1} = jm + \sum_{i=0}^{j-1} J_j \quad \text{for } j \geqslant 1,$$

and, for $0 \leqslant n \leqslant J_{K-1} = Km + \sum_{j=0}^{K-1} J_j$,

$$i_n^M = \begin{cases} y_{n-I_j}^j & \text{if } I_j \leqslant n \leqslant I_j + J_j, \\ i_{n-(I_j+J_i)}^m & \text{if } I_j + J_j \leqslant n \leqslant I_{j+1}. \end{cases}$$

Note that the endpoint of such (i_n^M) is

$$i_{I_K}^M = x_K^M = K(y_m - x_m) + \sum_{j=0}^{K-1} \left(\varkappa_j - x_j^M\right) + x_M.$$

In order to being able to complete this path to a path of length M and with final point y_M it is sufficient to check that the inequality

$$M - \left(Km + \sum_{j=0}^{K-1} J_j\right) > \left\|y_M - i_{K-1}^M\right\|_1 \tag{54}$$

is satisfied. Since $\sum_{j=0}^{K-1} J_j \leqslant dKN_\eta$ and also $\|\sum_{j=0}^{K-1} (\varkappa_j - x_j^M)\|_1 \leqslant dKN_\eta$ we obtain the condition

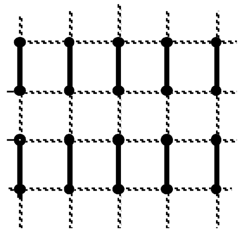

Fig. 3. Layers of vertical connections in the horizontal direction.

$$K \leqslant \frac{M(1-\|\tau\|_1)+o(M)}{m(1-\|\tau\|_1)+2dN_\eta}$$

for some $o(M)$ depending on $y_M - x_M$ and $y_m - x - m$. We then choose

$$K = \left\lfloor \frac{M(1-\|\tau\|_1-\eta)}{m(1-\|\tau\|_1)+2dN_\eta} \right\rfloor,$$

so that this condition is satisfied for M large.

We can then estimate

$$\begin{aligned}
\psi_M(\tau) &\leqslant \frac{Km}{M}\big(\psi_m(\tau)+\eta\big) + \beta\left(\frac{K}{M}dN_\eta + (M - I_{K-1})\right)\\
&\leqslant \psi_m(\tau)+\eta+\beta\frac{1}{M}\left(\frac{M(1-\|\tau\|_1-\eta)}{m(1-\|\tau\|_1)}dN_\eta + \frac{\eta(m+dN_\eta)}{(1-\|\tau\|_1)}\right)\\
&\leqslant \psi_m(\tau)+\eta+\beta\frac{1}{m(1-\|\tau\|_1)}dN_\eta + \beta\frac{1}{M}\frac{\eta(m+dN_\eta)}{(1-\|\tau\|_1)}.
\end{aligned}$$

By letting first $M \to +\infty$ and then $m \to +\infty$ the usual 'subadditive' argument and the arbitrariness of η allow to conclude the proof.

Example 4.4. We give some examples with $c_{ij} \in \{\alpha, \beta\}$ with $\alpha < \beta$ in dimension two. In Figs. 3 and 4 the β-connections are represented by bold lines while the α-connections are represented by dashed lines.

(1) We first consider a layered medium in the horizontal direction as in Fig. 3. In this case optimal paths for $\psi(\tau)$ use the α horizontal connections in proportion $1-\tau_2$, so that we easily obtain

$$\psi(\tau_1, \tau_2) = \alpha + \left(\frac{\beta-\alpha}{2}\right)|\tau_2|.$$

(2) We consider now the more complex geometry in Fig. 4 with period N (in the figure $N = 9$) characterized by the fact that the minimal path of α-connection in the direction $(\pm 1/2, 1/2)$ is of minimal length. One of such paths and a minimal path of α-connection in the direction e_1 are shown in the figure.

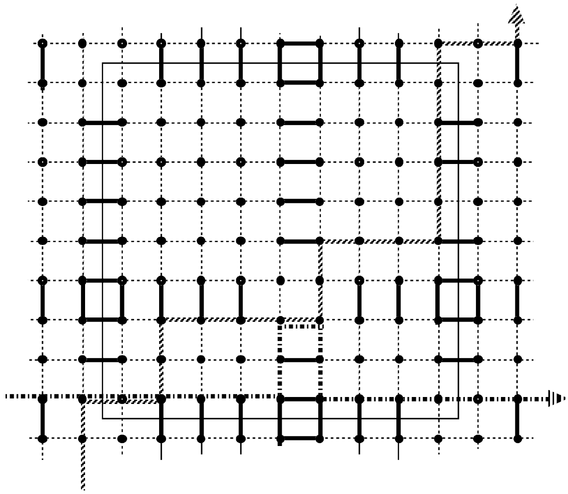

Fig. 4. A geometry giving a non-trivial ψ and minimal paths of α-connections.

As a consequence, we have $\psi(\pm 1/2, 1/2) = \alpha$. Conversely, if we denote by τ_0 the inverse of the length of the minimal path of α-connection in direction e_1, we see that

$$\psi(te_1) = \hat{\psi}(t) := \begin{cases} \alpha & \text{if } |t| \leqslant \tau_0, \\ \alpha + \frac{(|t|-\tau_0)}{1-\tau_0}\frac{(N-1)\alpha+\beta}{N} & \text{if } \tau_0 \leqslant |t| \leqslant 1, \end{cases}$$

where the last coefficient is due to the fact that for $t = 1$ the minimal path is a horizontal straight path with exactly one β-connection in each period.

The computation for a general τ is obtained after decomposing

$$\tau = \min\{|\tau_1|, |\tau_2|\}(\operatorname{sign}\tau_1, \operatorname{sign}\tau_2) + \big(\tau - \min\{|\tau_1|, |\tau_2|\}(\operatorname{sign}\tau_1, \operatorname{sign}\tau_2)\big)$$

and noting that the second vector is a multiple of a coordinate vector with modulus $\|\tau\|_\infty - \min\{|\tau_1|, |\tau_2|\}$, thus obtaining

$$\psi(\tau) = 2\alpha\min\{|\tau_1|, |\tau_2|\} + \hat{\psi}\left(\frac{\|\tau\|_\infty - \min\{|\tau_1|, |\tau_2|\}}{1 - 2\min\{|\tau_1|, |\tau_2|\}}\right)\big(1 - 2\min\{|\tau_1|, |\tau_2|\}\big).$$

4.2. The random case

We adapt the notation of Section 2.2 to the present setting (in this case valid for all dimensions d) by introducing random coefficients c_{ij}^ω and energies F_ε^ω of paths in Ω_ε as

$$F_\varepsilon^\omega(\gamma) = \sum_{n=1}^{N} \varepsilon c_{i_n i_{n-1}}^\omega. \tag{55}$$

In the random case formula (48) reads

$$\psi^\omega(\tau) = \lim_m \frac{1}{m} \inf\left\{ \sum_{n=1}^m c^\omega_{i_n i_{n-1}}\colon\ i_0 = x_m,\ i_m = y_m \right\}, \tag{56}$$

where x_m and y_m satisfy (47). It is convenient to introduce the notation

$$\psi^\omega_m(x, y) = \frac{1}{m} \inf\left\{ \sum_{n=1}^m c^\omega_{i_n i_{n-1}}\colon\ i_0 = x,\ i_m = y \right\}.$$

Our analysis relies on the following result.

Lemma 4.5. *For any $\tau \in \mathbb{R}^d$ with $\|\tau\|_1 < 1$ and for any $x \in \mathbb{R}^d$ there exists the limit in (56) where $x_m = \lfloor mx \rfloor$ and $y_m = \lfloor mx + m\tau \rfloor$. Moreover this limit, denoted by $\psi(\tau)$, is deterministic, and does not depend on x.*

Proof. We first prove the statement of Lemma 4.5 in the special case $x = 0$. We also assume that τ has rational coordinates and that $m = \check{m} j_0$ with $\check{m} = 1, 2, \dots,$ and j_0 being such that the vector $j_0\tau$ has integer coordinates. Then by the Subadditive Ergodic Theorem (see, for instance, [17]) under our standing stationarity and ergodicity assumptions the limit

$$\psi(\tau) = \lim_{m\to\infty} \frac{1}{m} \inf\left\{ \sum_{n=1}^m c_{i_n i_{n-1}}\colon\ i_0 = 0,\ i_m = m\tau \right\} \tag{57}$$

exists a.s. and is deterministic. The fact that (57) also holds for any sequence i_m such that $|i_m - m\tau| = o(m)$ trivially follows from the boundedness of c_{ij}.

The existence of a deterministic limit in (57) for any $\tau \in \mathbb{R}^d$ with $\|\tau\|_1 < 1$ can be easily deduced by the continuity arguments.

It remains to show that for any $x \in \mathbb{R}^d$

$$\lim_{m\to\infty} \frac{1}{m} \inf\left\{ \sum_{n=1}^m c_{i_n i_{n-1}}\colon\ i_0 = \lfloor mx \rfloor,\ \lfloor i_m = mx + m\tau \rfloor \right\} = \psi(\tau).$$

We follow the same scheme as in the proof of Proposition 2.10. First we assume that x and τ have rational coordinates. For arbitrary x and τ the desired statement can be easily obtained by continuity arguments.

For the sake of definiteness, we set $x = (0, \dots, 0, 1)$ and $\tau = (1/2, 0, \dots, 0)$. With this choice of x and τ for any $\epsilon > 0$ we define the events

$$\mathcal{Q}_N = \left\{ \omega \in \Sigma\colon\ \forall m \geqslant N \text{ it holds } \left|\psi^\omega_{2m}(0, 2m\tau) - \psi(\tau)\right| < \epsilon \right\}.$$

The a.s. existence of a deterministic limit in (57) implies that for any $\delta > 0$ there is $N_0(\delta)$ such that

$$\mathbf{P}\{\mathcal{Q}_{N_0}\} > 1 - \delta.$$

By the ergodic theorem, almost surely for any $\gamma > 0$ there is $m_0 = m_0(\omega, \gamma)$ such that for any $m \geqslant m_0$ it holds

$$\left| \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{\mathcal{Q}_{N_0}}(T_{2ix}\omega) - \mathbf{P}(\mathcal{Q}_{\mathbf{N_0}}) \right| < \gamma. \tag{58}$$

For $m \geqslant m_0(\omega, \gamma)$ denote by L the maximum of integer numbers j such that $j > m$ and for all $i \in (m, j) \cap \mathbb{N}$ it holds $T_{2ix}\omega \notin \mathcal{Q}_{N_0}$. In order to estimate L from above, we denote by $\tilde{m}$ the number of unities in the sequence $\{\mathbf{1}_{\mathcal{Q}_{N_0}}(T_{2ix}\omega)\}_{i=1}^{m}$. Then

$$\gamma > \left| \frac{\tilde{m}}{m+L} - \mathbf{P}(\mathcal{Q}_{N_0}) \right| = \left| 1 - \mathbf{P}(\mathcal{Q}_{N_0}) - \frac{L + (m - \tilde{m})}{m+L} \right|.$$

Therefore,

$$\frac{L + (m - \tilde{m})}{m+L} < \gamma + \bigl(1 - \mathbf{P}(\mathcal{Q}_{N_0})\bigr) < \gamma + \delta.$$

Since $(m - \tilde{m}) \geqslant 0$ and $\gamma + \delta \leqslant 1/2$, the last inequality yields the upper bound $L < 2(\gamma + \delta)m$.

If we choose $\widetilde{L} = 3(\gamma + \delta)m$ and arbitrary $m > \max(m_0(\omega, \gamma), N_0(\delta))$, then there is $n \in [m, m + \widetilde{L}]$ such that $T_{2nx}\omega \in \mathcal{Q}_{N_0}$. Notice that if $T_{2ix}\omega \in \mathcal{Q}_{N_0}$, then for any $m > N_0$

$$\bigl|\psi_{2m}^{\omega}(2ix, 2ix + 2m\tau) - \psi^0(\tau)\bigr| < \epsilon.$$

Taking this into account, we obtain

$$\begin{aligned}
&\bigl|\psi^{\omega}(2mx, 2mx + 2m\tau) - \psi^0(\tau)\bigr| \\
&\quad \leqslant \left(1 - \frac{m}{n}\right)\psi^{\omega}(2mx, 2mx + 2m\tau) \\
&\qquad + \frac{1}{2n}\bigl|2m\psi^{\omega}(2mx, 2mx + 2m\tau) - 2n\psi^{\omega}(2nx, 2nx + 2n\tau)\bigr| \\
&\qquad + \bigl|\psi^{\omega}(2nx, 2nx + 2n\tau) - \psi^0(\tau)\bigr| \\
&\quad \leqslant 3(\gamma + \delta)\beta + 12(\gamma + \delta)\beta + \epsilon = 15(\gamma + \delta)\beta + \epsilon;
\end{aligned}$$

here we have also used the inequalities

$$2n\psi^{\omega}(2nx, 2nx + 2n\tau) \leqslant 4\beta(n - m) + 2m\psi^{\omega}(2mx, 2mx + 2m\tau)$$

and

$$2m\psi^{\omega}(2mx, 2mx + 2m\tau) \leqslant 12\beta(n - m) + 2n\psi^{\omega}(2nx, 2nx + 2n\tau),$$

which are valid for all sufficiently small $\delta > 0$ and $\gamma > 0$. Since γ, δ and ϵ are arbitrary sufficiently small positive numbers, this implies the desired limit relation. □

The homogenization theorem for discrete curves in the random case reads as follows.

Theorem 4.6. *Under the ellipticity condition* (46) *and the ergodicity conditions in Section* 2.2 *the Γ-limit of the energies F_ε^ω exists almost surely, is deterministic and is given by* (50)*, with ψ defined from Lemma* 4.5*.*

Proof. To prove the theorem it is enough to notice that Lemma 4.5 ensures the homogenizability of c_{ij}^ω, after remarking that by a compactness argument it suffices to check the existence of the limit in (48) only when $x_m = \lfloor mx \rfloor$ and $y_m = \lfloor mx + m\tau \rfloor$. The fact that the limit is deterministic is then also ensured by the lemma. $\square$

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