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Homogenization of immiscible compressible two–phase flow in random porous media

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Abstract

The paper deals with stochastic homogenization of a system modeling immiscible compressible two-phase, such as water and gas, flow in random porous media. The problem is written in terms of the phase formulation, i.e. the saturation of one phase and the pressure of the second phase are primary unknowns. This formulation leads to a coupled system consisting of a nonlinear degenerate parabolic equation for the gas pressure and a nonlinear degenerate parabolic diffusion-convection equation for the liquid saturation, subject to appropriate boundary and initial conditions. We consider the behavior of compressible two-phase flow in heterogeneous reservoirs with permeability and porosity being realizations of given statistically homogeneous random fields. We derive the effective (macroscopic) problem and prove the convergence of solutions. Our approach relies on stochastic two-scale convergence techniques, the realization-wise notion of stochastic two-scale convergence being used. Also, we exploit various a priori estimates as well as monotonicity and compactness arguments. To our best knowledge, this is the first stochastic homogenization result in the case of compressible two-phase flow in random porous media.

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1. Introduction

The problem of the description of two-phase flows through highly heterogeneous media is faced in many branches of engineering and applied sciences such as groundwater hydrology, petroleum engineering, environmental sciences. More recently, multiphase flow attracted an essential interest of engineers and researchers dealing with gas migration in a nuclear waste repository [37] and sequestration of CO_2 [35].

These applications involve a detailed description of the underground reservoir and running the simulation on the created detailed model. This can be challenging and time consuming for complex systems of flows in highly heterogeneous porous media because of lack of information and description of details that require high computational time.

Since in the subsurface permeability heterogeneity occurs on many different length scales, numerical models of flow cannot, in general, resolve all the plurality of scales. Therefore, approaches based on upscaling or homogenization are required to represent the effect of subgrid scale variations on larger scale flow.

The problem of homogenization of multiphase flow through heterogeneous porous media has quite a long history, a number of methods and approaches have been elaborated. There is a vast literature on this topic. Here we will merely mention some references to mathematical homogenization results for flow and transport in porous media. There were a number of works devoted to the qualitative theory of systems of equations describing incompressible immiscible two-phase flow in porous media. Among them are [1,9,10,17–20,23,33], where the questions of existence, uniqueness and regularity of a weak solution were investigated. Important qualitative results such as existence and regularity of weak solutions for compressible immiscible two-phase flow in heterogeneous porous media were obtained in [5,24–26,32], and in the case of discontinuous capillary pressure in [6]. For the results on homogenization of incompressible and compressible two-phase flow in porous media, we refer for instance to [2–4,7,8,29,30] and the references therein.

The mentioned articles considered homogenization problems in porous media with a periodic microstructure. In particular, the rigorous homogenization results obtained in these papers are valid under the assumption that the corresponding porous medium is periodic or locally periodic. Although the results obtained for systems with periodic coefficients provide an important information on the effective behavior of the two-phase flow of interest, the description of the effective behavior of the flow based on the periodicity assumption usually cannot be accurate except for some special cases.

The properties of the medium, such as the porosity and the permeability, are not known in any precise way due to the lack and accuracy of available measurements. Due to this fact, subsurface flow and transport predictions are plagued by uncertainty. The need for assessing and quantifying uncertainty in subsurface flow has driven research in the stochastic aspects of hydrology and multiphase flow physics, see for instance [22,27,34]. The complex nature of subsurface systems, together with inherent incomplete information about their properties, have resulted in the surge of probabilistic modeling of these uncertainties. Quantifying the uncertainty of the model parameters and modeling them as random variables means that the coefficients of the governing partial differential equation (PDEs) are stochastic rather than deterministic. In order to assess the resulting output uncertainty, a stochastic method is required. One possible option is homogenization which was applied successfully for single phase flow in [12,14,16,28,38] and incompressible immiscible two-phase flow uncertainty quantification in [13,36]. In connection with these problems a new technique called stochastic two-scale convergence in the mean was developed, see [15].

In the case of natural reservoirs the assumption that the porous media are random statistically homogeneous is much more realistic and allows to provide more accurate description of the effective characteristics. This paper focuses on modeling and effective descriptions of immiscible compressible two-phase flows through heterogeneous reservoirs with random statistically homogeneous geometric structure. We will be concerned with a nonlinear degenerate system of convection-diffusion equations in a domain modeling the flow and transport of immiscible compressible fluids through heterogeneous random porous media, taking into account capillary and gravity effects.

More precisely, we consider the flow of an incompressible wetting phase (water) and a compressible nonwetting phase (gas) in a porous reservoir. Here we consider a single rock-type model. The original microscopic model is defined in a domain with random statistically homogeneous ergodic microstructure. In our context it means that both the porosity and the absolute permeability are rapidly oscillating statistically homogeneous random functions of the microscopic variable $y = x/\varepsilon$, where x is the macroscopic variable, and $\varepsilon > 0$ is a small parameter that represents the characteristic length scale of the medium. The problem is formulated in terms of the wetting saturation phase (water) and the nonwetting pressure phase (gas). The corresponding system of equations is derived from the mass conservation laws of both fluids on the one side and from the relations between the velocities and the pressure gradients as well as the gravitational forces. These relations are provided by the Darcy-Muskat law. The resulting system consists of a nonlinear equation for the gas pressure coupled with a degenerate parabolic convection-diffusion equation for the water saturation, both equations are subject to appropriate boundary and initial conditions.

In this system the diffusion operator degenerates due to the capillary effects, the degeneracy of this type can be observed both in compressible and incompressible flows. Another type of degeneracy occurs in the region where the gas saturation vanishes. In this region the gas density cannot be determined by its evolution since the gas phase is not presented.

The degeneracy and strong coupling of the equations in the system of interest make the proof of homogenization result rather involved especially in the case of random coefficients. In particular, we are not able to obtain uniform estimates for the gradients of the phase pressures. To overpass this difficulty we reformulate the studied problem in terms of the global pressure and the saturation. This leads to a weaker coupling between the equations of the system. However, we still do not have uniform estimates for the saturation gradient. In addition, due to degeneracy, solutions do not possess much regularity. As a result, passing to the limit in the studied system is not direct and requires rather delicate arguments.

Homogenization problem for incompressible two-phase flow in a random medium was treated successfully in [13]. However, to our best knowledge, rigorous homogenization results for an immiscible compressible multiphase flow in a random medium are missed in the existing literature.

The paper is organized as follows. Section 2 deals with problem setup. We describe the physical model, introduce the corresponding system of equations and provide the assumptions on the data. In Section 3 we recall the results on the existence of a solution and obtain a number of a priori estimates. Finally, in Section 4 we formulate and prove the homogenization theorem.

2. Problem setup

In this Section we formulate the problem of interest. First, in Subsection 2.1 we introduce the system of equations that describes isothermal immiscible compressible two-phase flow in a random porous medium. Subsection 2.2 provides the main assumption on the data. In Subsection 2.3, we provide two examples of the random model. Finally, the notion of the global pressure is briefly recalled and useful relations are also recalled in Subsection 2.4.

2.1. Governing equations

We consider an immiscible compressible two-phase flow model in a heterogeneous porous medium with a random statistically homogeneous microstructure. We suppose that the porous medium occupies a porous reservoir $\mathcal{Q} \subset \mathbb{R}^d$, $d = 1, 2, 3$, being a bounded connected Lipschitz domain. We focus on the phases water and gas, but the consideration below is also valid for a general wetting phase and a non-wetting phase, each consisting of one component. For presentation simplicity we assume that there are no source/sink terms.

To introduce a random microstructure we assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is a standard probability space equipped with an ergodic dynamical system $\mathcal{T}_x$, $x \in \mathbb{R}^d$, that is

- $\mathcal{T}_{x+y} = \mathcal{T}_x \circ \mathcal{T}_y,\ x, y \in \mathbb{R}^d,\ \mathcal{T}_0 = \mathrm{Id};$
- $\mathbf{P}(\mathcal{T}_x(A)) = \mathbf{P}(A)$ for all $x \in \mathbb{R}^d,\ A \in \mathcal{F};$
- $\mathcal{T}_\cdot : \mathbb{R}^d \times \Omega \mapsto \Omega$ is a measurable mapping from $\mathbb{R}^d \times \Omega$ to Ω, with $\mathbb{R}^d$ being equipped with the Borel σ-algebra.

The ergodicity of $\mathcal{T}_\cdot$ means that the probability of any set $A \in \mathcal{F}$ which is invariant with respect to all $\mathcal{T}_x$, $x \in \mathbb{R}^d$, is equal to zero or one.

Also we assume that there are a positive random variable $\mathbf{\Phi} = \mathbf{\Phi}(\omega)$ and a positive definite random matrix $\boldsymbol{K} = \boldsymbol{K}(\omega)$ and denote

$$\Phi(\cdot, \omega) = \mathbf{\Phi}(\mathcal{T}_x \omega), \qquad K(\cdot, \omega) = \boldsymbol{K}(\mathcal{T}_x \omega),$$

that is $\Phi(\cdot, \omega)$ and $K(\cdot, \omega)$ are realizations of the random porosity function $\mathbf{\Phi}$ and random global permeability tensor $\boldsymbol{K}$.

We then scale this random structure with a parameter ε which represents the ratio of the typical size of local inhomogeneities to the size of the whole region $\mathcal{Q}$, and define the porosity and the absolute permeability tensor by

$$\Phi^\varepsilon(x, \omega) = \Phi\Big(\frac{x}{\varepsilon}, \omega\Big) = \mathbf{\Phi}(\mathcal{T}_{\frac{x}{\varepsilon}} \omega), \qquad K^\varepsilon(x, \omega) = K\Big(\frac{x}{\varepsilon}, \omega\Big) = \boldsymbol{K}(\mathcal{T}_{\frac{x}{\varepsilon}} \omega), \tag{2.1}$$

we assume that $0 < \varepsilon \ll 1$ is a small parameter that tends to zero. Prior to writing down the equations of the model, we introduce the following notation. Let $S^\varepsilon = S_l^\varepsilon(x, t)$ and $S_g^\varepsilon = S_g^\varepsilon(x, t) = 1 - S^\varepsilon$ be the saturations of the wetting and the nonwetting phases, respectively; $k_{r,l} = k_{r,l}(S_l^\varepsilon)$ and $k_{r,g} = k_{r,g}(S_g^\varepsilon)$ stand for the relative permeabilities of the corresponding phases; $p_l^\varepsilon = p_l^\varepsilon(x, t)$, $p_g^\varepsilon = p_g^\varepsilon(x, t)$ are their pressures; and ϱ_l, ϱ_g are the corresponding densities. In what follows we assume that the density of the wetting phase is a constant, without loss of generality this constant is equal to 1 so that $\varrho_l(p_l^\varepsilon) = 1$. We then fix an arbitrary time interval $[0, T]$, $T > 0$, and denote $\mathcal{Q}_T = \mathcal{Q} \times]0, T[$. Standard flow models consist of equations for mass conservation of each phase, the multiphase extension of Darcy's law for fluid flow in porous media and by the capillary pressure law (see, e.g., [9,18,21]):

$$\begin{cases} 0 \leqslant S^\varepsilon \leqslant 1 \quad \text{a.e. in } Q_T, \\ \Phi^\varepsilon(x,\omega)\dfrac{\partial S^\varepsilon}{\partial t} - \operatorname{div}\Big\{K^\varepsilon(x,\omega)\lambda_l(S^\varepsilon)\big(\nabla p_l^\varepsilon - \vec{g}\big)\Big\} = 0 \quad \text{in } Q_T; \\ \Phi^\varepsilon(x,\omega)\dfrac{\partial \Theta^\varepsilon}{\partial t} - \operatorname{div}\Big\{K^\varepsilon(x,\omega)\lambda_g(S^\varepsilon)\varrho_g(p_g^\varepsilon)\big(\nabla p_g^\varepsilon - \varrho_g(p_g^\varepsilon)\vec{g}\big)\Big\} = 0 \quad \text{in } Q_T; \\ P_c(S^\varepsilon) = p_g^\varepsilon - p_l^\varepsilon \quad \text{in } Q_T, \end{cases} \tag{2.2}$$

where Θ^ε is the function given by

$$\Theta^\varepsilon \stackrel{\text{def}}{=} \varrho_g(p_g^\varepsilon)(1 - S^\varepsilon), \tag{2.3}$$

and the velocities of the wetting and the nonwetting phases $\vec{q}_l^{\,\varepsilon}$ and $\vec{q}_g^{\,\varepsilon}$ satisfy the Darcy-Muskat law:

$$\vec{q}_l^{\,\varepsilon} = -K^\varepsilon(x,\omega)\lambda_l(S^\varepsilon)\bigg(\nabla p_l^\varepsilon - \vec{g}\bigg) \qquad \text{with } \lambda_l(s) = \frac{k_{r,l}}{\mu_l}(s); \tag{2.4}$$

$$\vec{q}_g^{\,\varepsilon} = -K^\varepsilon(x,\omega)\lambda_g(S_g^\varepsilon)\bigg(\nabla p_g^\varepsilon - \varrho_g(p_g^\varepsilon)\vec{g}\bigg) \qquad \text{with } \lambda_g(s) = \frac{k_{r,g}}{\mu_g}(s). \tag{2.5}$$

Here the subscripts l and g correspond to liquid or wetting phase and to gas or nonwetting phase; $\vec{g}$, μ_l, μ_g and λ_l, λ_g stand for the gravity vector, the viscosities and the mobilities of the wetting and the nonwetting phases, respectively.

System (2.2)–(2.3) is equipped with the following boundary and initial conditions.

`Boundary conditions`. We suppose that ∂Q consists of two $(d-1)$-dimensional sets $\Gamma_{\rm inj}$ and $\Gamma_{\rm imp}$ with a Lipschitz boundary such that $\Gamma_{\rm inj} \cap \Gamma_{\rm imp} = \emptyset$ and $\partial Q = \overline{\Gamma}_{\rm inj} \cup \overline{\Gamma}_{\rm imp}$. The subscripts come from "injection" and "impervious" parts of the boundary. The boundary conditions are given by:

$$\begin{cases} p_g^\varepsilon(x,t) = p_l^\varepsilon(x,t) = 0 \quad \text{on } \Gamma_{\rm inj} \times (0,T); \\ \vec{q}_l^{\,\varepsilon} \cdot \vec{\nu} = \vec{q}_g^{\,\varepsilon} \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_{\rm imp} \times (0,T), \end{cases} \tag{2.6}$$

where the velocities $\vec{q}_l^{\,\varepsilon}, \vec{q}_g^{\,\varepsilon}$ are defined in (2.4)–(2.5).

`Initial conditions`. The initial conditions read:

$$p_l^\varepsilon(x,0) = p_l^{\mathbf{0}}(x) \quad \text{and} \quad p_g^\varepsilon(x,0) = p_g^{\mathbf{0}}(x) \quad \text{in } Q. \tag{2.7}$$

Notice that from (2.6) and the fact that $P_c(1) = 0$, it follows that $S^\varepsilon = 1$ on $\Gamma_{\rm inj} \times (0,T)$. The initial condition for S^ε is uniquely defined by the equation

$$P_c(S^{\mathbf{0}}(x)) = p_g^{\mathbf{0}}(x) - p_l^{\mathbf{0}}(x). \tag{2.8}$$

Then according to (2.3) the initial condition for Θ^ε reads

$$\Theta^{\mathbf{0}} = \varrho_g(p_g^{\mathbf{0}})(1 - S^{\mathbf{0}}). \tag{2.9}$$

In the next Section we specify the conditions on the data of system (2.2)–(2.9) which ensure the existence of a solution to this system. Our goal is to study the asymptotic behavior of this solution as $\varepsilon \to 0$, and to construct the limit problem.

Since the coefficients $\Phi(\cdot)$ and $K(\cdot)$ are random fields, the solutions of problem (2.2)-(2.7) are also random functions that depend on the realization $\omega \in \Omega$. In the rest of this paper, for the sake of brevity, we follow the convention commonly uses in the literature on stochastic analysis and do not indicate explicitly the dependence on ω. In particular we write $\Phi(x)$ and $K(x)$ instead of $\Phi(x, \omega)$ and $K(x, \omega)$.

2.2. *Main assumptions*

Here we provide our assumptions on the data of system (2.2)-(2.9). In order to formulate these assumptions we need two auxiliary functions. Namely, we denote

$$\alpha(s) = \frac{\lambda_g(s)\,\lambda_w(s)}{\lambda(s)}\left|P_c'(s)\right| \tag{2.10}$$

and

$$\beta(s) = \int_0^s \alpha(r)\,dr. \tag{2.11}$$

We assume that the following conditions are fulfilled:

(A.1) The random variable $\mathbf{\Phi}$ belongs to $L^\infty(\Omega)$; moreover, there are constants ϕ_-, ϕ^+ such that $0 < \phi_- < \phi^+$ and

$$0 < \phi_- \leqslant \mathbf{\Phi} \leqslant \phi^+ < 1 \quad \text{a. s. in } \Omega. \tag{2.12}$$

(A.2) The random field $\boldsymbol{K}$ belongs to $(L^\infty(\Omega))^{d\times d}$, and there exist constants K_-, K^+ such that $0 < K_- < K^+$ and

$$K_-|\xi|^2 \leqslant (\boldsymbol{K}(\omega)\xi, \xi) \leqslant K^+|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{\mathrm{d}}, \text{ a.s. in } \Omega. \tag{2.13}$$

(A.3) The density of the nonwetting phase $\varrho_g = \varrho_g(p)$ is a continuously differentiable increasing function such that $\varrho_g(p) = \varrho_{\min}$ for $p \leqslant p_{\min}$; $\varrho_g(p) = \varrho_{\max}$ for $p \geqslant p_{\max}$; $\varrho_{\min} < \varrho_g(p) < \varrho_{\max}$ for $p_{\min} < p < p_{\max}$; with $0 < \varrho_{\min} < \varrho_{\max} < +\infty$ and $0 < p_{\min} < p_{\max} < +\infty$.

(A.4) The capillary pressure $P_c(s)$ is a $C^1([0, 1]; \mathbb{R}^+)$ function such that $P_c'(s) < 0$ in $[0, 1]$ and $P_c(1) = 0$.

(A.5) The functions λ_l, λ_g are continuous on the interval [0, 1] and possess the following properties:

$$0 \leqslant \lambda_l(s), \lambda_g(s) \leqslant 1 \quad \text{for all } s \in [0,1]; \qquad \lambda_l(0) = \lambda_g(1) = 0; \tag{2.14}$$

there is a constant $L_0 > 0$ such that

$$\lambda(s) = \lambda_l(s) + \lambda_g(s) \geqslant L_0 \quad \text{for all } s \in [0,1]. \tag{2.15}$$

(A.6) The function α defined in (2.10) belongs to $C^1([0,1];\mathbb{R}^+)$. Moreover, $\alpha(s) > 0$ for $s \in (0,1)$. Notice that due to (2.14) we have $\alpha(0) = \alpha(1) = 0$.

(A.7) The function β^{-1}, inverse of β is Hölder continuous on the interval $[0, \beta(1)]$ that is there exist constants $\theta \in (0,1)$ and $C_\beta > 0$ such that for all $s_1, s_2 \in [0, \beta(1)]$ the following inequality holds:

$$\left|\beta^{-1}(s_1) - \beta^{-1}(s_2)\right| \leqslant C_\beta \, |s_1 - s_2|^\theta.$$

(A.8) The initial conditions $p_g^{\mathbf{0}}$ and $p_l^{\mathbf{0}}$ are $L^2(Q)$ functions.

(A.9) The function $S^{\mathbf{0}}$ satisfies the inequality $0 \leqslant S^{\mathbf{0}} \leqslant 1$ a.e. in Q.

In the existing literature conditions similar to those in **(A.1)-(A.9)** are commonly used in the theory of multiphase flow in porous media.

2.3. *Example of random media*

In this Subsection we provide two examples of random porosity functions and absolute permeability tensors. The first example is based on the Bernoulli checkerboard structure and the second one on a Poisson point process.

1. Let ζ_j, $j \in \mathbb{Z}^d$, be a family of i.i.d. (independent identically distributed) random variables such that

$$\zeta_j = \begin{cases} \kappa_0 & \text{with probability } q \\ \kappa_1 & \text{with probability } 1-q, \end{cases}$$

for some $\kappa_0, \kappa_1 \in \mathbb{R}$, $0 < \kappa_0 < \kappa_1$, and $q \in (0,1)$. We set

$$\hat{\Phi}(x) = \zeta_j, \quad \text{if } j_k - \frac{1}{2} < x_k \leq j_k + \frac{1}{2}, \;\; k = 1, \dots, d.$$

The law of $\hat{\Phi}(\cdot)$ is invariant with respect to any integer shift of its argument. In order to make it statistically homogeneous in $\mathbb{R}^d$ we consider a random variable η which is independent on ζ_j, $j \in \mathbb{Z}^d$, and uniformly distributed on the unite cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Letting $\Phi(x) = \hat{\Phi}(x - \eta)$ we obtain a statistically homogeneous porosity function. The ergodicity immediately follows from the fact that $\Phi(\cdot)$ has a finite range of dependence.

Similarly, for positive definite symmetric matrices $\mathcal{K}_0$ and $\mathcal{K}_1$ and for $q_1 \in (0,1)$ we consider a family of i.i.d. random variables M_j, $j \in \mathbb{Z}^d$, with

$$M_j = \begin{cases} \mathcal{K}_0 & \text{with probability } q_1 \\ \mathcal{K}_1 & \text{with probability } 1-q_1. \end{cases}$$

Then we define

$$\hat{K}(x) = M_j \quad \text{if } j_k - \frac{1}{2} < x_k \le j_k + \frac{1}{2}, \ \ k = 1, \dots, d,$$

and $K(x) = \hat{K}(x-\eta)$.

2. Let Ξ be a Poisson point process in $\mathbb{R}^d$ with intensity one. By definition Ξ is a locally finite random set in $\mathbb{R}^d$ such that

i. For any bounded Borel set $Q \subset \mathbb{R}^d$ the random variable $\#(\Xi \cap Q)$ has Poisson distribution with parameter $|Q|$:

$$\mathbf{P}\big\{\#(\Xi \cap Q) = n\big\} = e^{-|Q|}\frac{|Q|^n}{n!}.$$

ii. For any disjoint Borel sets $Q_1, \dots, Q_N$ the random variables $\#(\Xi \cap Q_1), \dots, \#(\Xi \cap Q_N)$ are independent.

Denote by $x_1, x_2, \dots$, the points of this Poisson point process, and by $\mathcal{V}_j$ the cells of the corresponding Voronoi tessellation. We recall that

$$\mathcal{V}_j = \{x \in \mathbb{R}^d : \operatorname{dist}(x, x_j) \le \operatorname{dist}(x, \Xi \setminus \{x_j\})\}.$$

Then we consider i.i.d. random variables ζ_j, $j \in \mathbb{Z}^+$, that take on values on the interval $[\kappa_0, \kappa_1]$ with $0 < \kappa_0 < \kappa_1$, and define the porosity, for $x \in \mathcal{V}_j$, by $\Phi(x) = \zeta_j$. The absolute permeability tensor $K(\cdot)$ is defined in a similar way.

The statistical homogeneity of $\Phi(x)$ and $K(x)$ follows from the shift invariance of the law of Ξ. The ergodicity is a straightforward consequence of the properties of a Poisson point process.

2.4. Global pressure and useful relations

In this section, we rearrange the system of equations in (2.2) using the notion of the so called global pressure [9,18]. The main idea is to replace the studied two-phase flow with a flow of a fictive fluid for which the Darcy law holds with a non-degenerate coefficient. This rearrangement helps us to obtain several important a priori estimates and finally the compactness results.

We are looking for a pressure P^ε and the coefficient $\gamma(s)$ such that $\gamma(s) > 0$ holds true for all $s \in [0, 1]$, and

$$\lambda_l(S^\varepsilon)\nabla p_l^\varepsilon + \lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \gamma(S^\varepsilon)\nabla \mathsf{P}^\varepsilon. \tag{2.16}$$

Now let us define the desired pressure P^ε (the global pressure) and the coefficient γ. The global pressure P^ε is defined by

$$p_l^\varepsilon = \mathsf{P}^\varepsilon + \mathsf{G}_l(S^\varepsilon) \quad \text{and} \quad p_g^\varepsilon = \mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon) \tag{2.17}$$

with

$$\mathsf{G}_g(s) = \mathsf{G}_g(0) + \int_0^s \frac{\lambda_l(r)}{\lambda(r)} P_c'(r)\,dr, \qquad \mathsf{G}_l(s) = \mathsf{G}_g(s) - P_c(s). \tag{2.18}$$

Here the function λ is defined in (2.15).

Due to **(A.4)** the function P^ε is well defined. Since

$$\nabla \mathsf{G}_l(S^\varepsilon) = -\frac{\lambda_g(S^\varepsilon)}{\lambda(S^\varepsilon)} P_c'(S^\varepsilon)\,\nabla S^\varepsilon, \tag{2.19}$$

it is straightforward to check that

$$\lambda_g(S^\varepsilon)\nabla \mathsf{G}_g(S^\varepsilon) + \lambda_l(S^\varepsilon)\nabla \mathsf{G}_l(S^\varepsilon) = 0 \tag{2.20}$$

and, hence

$$\lambda_l(S^\varepsilon)\nabla p_l^\varepsilon + \lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \lambda(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \Big\{\lambda_g(S^\varepsilon)\nabla \mathsf{G}_g(S^\varepsilon) + \lambda_l(S^\varepsilon)\nabla \mathsf{G}_l(S^\varepsilon)\Big\} = \lambda(S^\varepsilon)\nabla \mathsf{P}^\varepsilon.$$

It remains to set $\gamma(s) = \lambda(s)$, and (2.16) follows.

Notice that from (2.18) we get:

$$\lambda_l(S^\varepsilon)\nabla \mathsf{G}_l(S^\varepsilon) = \alpha(S^\varepsilon)\nabla S^\varepsilon \quad \text{and} \quad \lambda_g(S^\varepsilon)\nabla \mathsf{G}_g(S^\varepsilon) = -\alpha(S^\varepsilon)\nabla S^\varepsilon, \tag{2.21}$$

where the function α is given by (2.10).

It is also convenient to introduce the following quantities:

$$\mathfrak{a}(s) = \sqrt{\frac{\lambda_g(s)\,\lambda_l(s)}{\lambda(s)}}\,\big|P_c'(s)\big| \quad \text{and} \quad \mathfrak{b}(s) = \int_0^s \mathfrak{a}(r)\,dr. \tag{2.22}$$

After straightforward computations, considering the definition of the global pressure, (2.11) and (2.22), we obtain

$$\lambda_g(S^\varepsilon)|\nabla p_g^\varepsilon|^2 + \lambda_l(S^\varepsilon)|\nabla p_l^\varepsilon|^2 = \lambda(S^\varepsilon)|\nabla \mathsf{P}^\varepsilon|^2 + \big|\nabla \mathfrak{b}(S^\varepsilon)\big|^2 \tag{2.23}$$

and

$$\lambda_l(S^\varepsilon)\nabla p_l^\varepsilon = \lambda_l(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \nabla\beta(S^\varepsilon), \quad \text{and} \quad \lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \lambda_g(S^\varepsilon)\nabla \mathsf{P}^\varepsilon - \nabla\beta(S^\varepsilon). \tag{2.24}$$

Also, since by condition **(A.5)** the functions λ_l and λ_g are bounded, we have

$$\big|\nabla\beta(S^\varepsilon)\big|^2 = \frac{\lambda_g(S^\varepsilon)\,\lambda_l(S^\varepsilon)}{\lambda(S^\varepsilon)}\,\big|\nabla \mathfrak{b}(S^\varepsilon)\big|^2 \leqslant C\,\big|\nabla \mathfrak{b}(S^\varepsilon)\big|^2. \tag{2.25}$$

It remains to determine the initial and boundary conditions for P^ε. The initial condition can be easily derived from (2.7), (2.8) and (2.17). We leave out the details.

Let us calculate the value of the global pressure function P on Γ_{inj}. Since by condition **(A.4)** we have $P_c(1) = 0$, then $S^\varepsilon = 1$ on Γ_{inj}. Therefore, thanks to (2.17), the function P^ε is equal to a constant on Γ_{inj}. We denote it by P^1.

3. Existence result and estimates of a solution

The question of the existence of a solution to problem (2.2)–(2.9) has been studied in the previous works [2,26]. In the same works a number of important a priori estimates have been obtained. For the reader convenience we formulate here the corresponding existence result and several estimates for the solution.

Denote $H^1_{\Gamma_{\text{inj}}}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_{\text{inj}}\}$. We equip it with the norm $\|u\|_{H^1_{\Gamma_{\text{inj}}}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

From (2.12) and (2.13) if follows that almost surely

$$\phi_- \leqslant \Phi^\varepsilon(x) \leqslant \phi^+ \quad \text{and} \quad K_-|\xi|^2 \leqslant (K^\varepsilon(x)\xi, \xi) \leqslant K^+|\xi|^2 \ \text{ for all } x \in \mathcal{Q}, \ \ \xi \in \mathbb{R}^d, \tag{3.26}$$

with $\Phi^\varepsilon(x) = \boldsymbol{\Phi}(\mathcal{T}_{\frac{x}{\varepsilon}}\omega)$ and $K^\varepsilon(x) = \boldsymbol{K}(\mathcal{T}_{\frac{x}{\varepsilon}}\omega)$, see (2.1). We call $\omega \in \Omega$ for which (3.26) holds *typical*. From now on without mentioning it again we assume that ω is typical.

Definition 3.1. A triple of function $p_g^\varepsilon = p_g^\varepsilon(x,t)$, $p_l^\varepsilon = p_l^\varepsilon(x,t)$ and $S^\varepsilon = S^\varepsilon(x,t)$ is called a solution of problem (2.2)-(2.9) if all of the following holds

$$p_l^\varepsilon, p_g^\varepsilon \in L^2(\mathcal{Q}_T) \quad \text{and} \quad \sqrt{\lambda_l(S^\varepsilon)}\,\nabla p_l^\varepsilon, \ \sqrt{\lambda_g(S^\varepsilon)}\,\nabla p_g^\varepsilon \in L^2(\mathcal{Q}_T); \tag{3.27}$$

$$\beta(S^\varepsilon) \in L^2(0,T; H^1(\mathcal{Q})) \quad \text{and} \quad \mathsf{P}^\varepsilon - \mathsf{P}^1 \in L^2(0,T; H^1_{\Gamma_{\text{inj}}}(\mathcal{Q})); \tag{3.28}$$

$$\Phi^\varepsilon \frac{\partial S^\varepsilon}{\partial t} \in L^2(0,T; H^{-1}(\mathcal{Q})) \quad \text{and} \quad \Phi^\varepsilon \frac{\partial \Theta^\varepsilon}{\partial t} \in L^2(0,T; H^{-1}(\mathcal{Q})); \tag{3.29}$$

$$0 \leqslant S^\varepsilon \leqslant 1 \quad \text{a.e. in } \mathcal{Q}_T, \qquad S^\varepsilon = 1 \quad \text{on } \Gamma_{\text{inj}}; \tag{3.30}$$

for any $\varphi_l, \varphi_g \in C^1([0,T]; H^1(\mathcal{Q}))$ such that $\varphi_l = \varphi_g = 0$ on $\Gamma_{\text{inj}} \times (0,T)$ and $\varphi_l(x,T) = \varphi_g(x,T) = 0$, we have:

$$-\int\limits_{\mathcal{Q}_T} \Phi^\varepsilon(x) S^\varepsilon \frac{\partial \varphi_l}{\partial t}\, dx dt - \int\limits_{\mathcal{Q}} \Phi^\varepsilon(x) S^{\mathbf{0}}(x)\varphi_l(x,0)\, dx + \int\limits_{\mathcal{Q}_T} K^\varepsilon(x)\lambda_l(S^\varepsilon)\nabla p_l^\varepsilon \cdot \nabla\varphi_l\, dxdt - \int\limits_{\mathcal{Q}_T} K^\varepsilon(x)\lambda_l(S^\varepsilon)\,\vec{g}\cdot\nabla\varphi_l\, dxdt = 0 \tag{3.31}$$

and

$$-\int\limits_{\mathcal{Q}_T} \Phi^\varepsilon(x)\Theta^\varepsilon \frac{\partial \varphi_g}{\partial t}\, dxdt - \int\limits_{\mathcal{Q}} \Phi^\varepsilon(x)\Theta^{\mathbf{0}}(x)\varphi_g(x,0)\, dx + \int\limits_{\mathcal{Q}_T} K^\varepsilon(x)\lambda_g(S^\varepsilon)\varrho_g(p_g^\varepsilon)\nabla p_g^\varepsilon \cdot \nabla\varphi_g\, dxdt - \int\limits_{\mathcal{Q}_T} K^\varepsilon(x)\lambda_g(S^\varepsilon)\left[\varrho_g(p_g^\varepsilon)\right]^2 \vec{g}\cdot\nabla\varphi_g\, dxdt = 0 \tag{3.32}$$

with Θ^ε defined in (2.3); the following relation holds

$$P_c(S^\varepsilon) = p_g^\varepsilon - p_l^\varepsilon.$$

The initial conditions are satisfied in the following sense: for any $\psi \in H^1_{\Gamma_{\rm inj}}(\mathcal{Q})$

$$\lim_{t\to 0} \int_{\mathcal{Q}} \Phi^\varepsilon(x) S^\varepsilon(x,t)\,\psi(x)\,dx = \int_{\mathcal{Q}} \Phi^\varepsilon(x) S^{\mathbf{0}}(x)\psi(x)\,dx \tag{3.33}$$

and

$$\lim_{t\to 0} \int_{\mathcal{Q}} \Phi^\varepsilon(x)\Theta^\varepsilon(x,t)\psi(x)\,dx = \int_{\mathcal{Q}} \Phi^\varepsilon(x)\Theta^{\mathbf{0}}(x)\psi(x)\,dx \tag{3.34}$$

with $S^{\mathbf{0}}$ and $\Theta^{\mathbf{0}}$ defined in (2.8) and (2.9), respectively.

Remark 1. As was shown in [2] for any function $\psi \in H^1_{\Gamma_{\rm inj}}(\mathcal{Q})$ the integrals $\int_{\mathcal{Q}} \Phi^\varepsilon(x) S^\varepsilon(x, t)\psi(x)dx$ and $\int_{\mathcal{Q}} \Phi^\varepsilon(x)\Theta^\varepsilon(x,t)\psi(x)dx$ are continuous functions of t on $[0,T]$. Thus, the limits in (3.33) and (3.34) are well defined.

The following result has been proved in [2,26].

Theorem 3.2. *Under assumptions* **(A.1)-(A.9)** *for any* $\varepsilon > 0$ *problem* (2.2)–(2.9) *has a solution* $p_l^\varepsilon = p_l^\varepsilon(x,t)$, $p_g^\varepsilon = p_g^\varepsilon(x,t)$ *and* $S^\varepsilon = S^\varepsilon(x,t)$ *that satisfies Definition 3.1.*

Below we also formulate several estimates for a solution of (2.2)–(2.9) that have been obtained in [2,8].

Theorem 3.3. *Let* p_l^ε, p_g^ε, S^ε *be a solution of problem* (2.2), *and assume that the global pressure* P^ε *is given by (2.17). Then*

$$\int_{\mathcal{Q}_T} \left\{ \lambda_l(S^\varepsilon)|\nabla p_l^\varepsilon|^2 + \lambda_g(S^\varepsilon)|\nabla p_g^\varepsilon|^2 \right\} dxdt \leqslant C, \tag{3.35}$$

$$\int_{\mathcal{Q}_T} \left\{ |\nabla \mathsf{P}^\varepsilon|^2 + |\nabla \beta(S^\varepsilon)|^2 + \left|\nabla \mathfrak{b}(S^\varepsilon)\right|^2 \right\} dxdt \leqslant C, \tag{3.36}$$

$$\|\partial_t(\Phi^\varepsilon\Theta^\varepsilon)\|_{L^2(0,T;H^{-1}(\mathcal{Q}))} + \|\partial_t(\Phi^\varepsilon S^\varepsilon)\|_{L^2(0,T;H^{-1}(\mathcal{Q}))} \leqslant C; \tag{3.37}$$

here the constant C *is deterministic and does not depend on* ε.

4. Homogenization result

In this section we first remind the notion of stochastic two-scale convergence. We use here the realization-wise version of this convergence that was introduced in [39]. Then we provide several compactness results for a solution of problem (2.2)–(2.9). After that we calculate the homogenized coefficients and formulate the homogenization theorem. The proof of this theorem is given in Section 4.3.

4.1. Stochastic two-scale convergence. Compactness results

Changing if necessary the probability space we may assume that Ω is a compact metric space, $\mathcal{F}$ its Borel σ-algebra and the dynamical system T_x is continuous. We give a definition of stochastic two-scale convergence that is adapted to our framework.

Definition 4.1. We say that a family of $L^2(\mathcal{Q}_T)$ functions $u^\varepsilon = u^\varepsilon_{\tilde\omega}(x,t)$ stochastically two-scale converges to a function $u^0(x,t,\omega)$ if the following two conditions are fulfilled:

- There exists $\varepsilon_0 > 0$ such that

$$\|u^\varepsilon\|_{L^2(\mathcal{Q}_T)} \le C_{\tilde\omega} \quad \text{for all } \varepsilon < \varepsilon_0;$$

- Almost surely (for almost all $\tilde\omega \in \Omega$) for any $\varphi \in C^\infty(\mathcal{Q}_T)$ and any $\psi \in C(\Omega)$ we have

$$\int\limits_{\mathcal{Q}_T} u^\epsilon_{\tilde\omega}(x,t)\varphi(x,t)\psi(\mathcal{T}_{\frac{x}{\varepsilon}}\tilde\omega)\,dxdt \longrightarrow \int\limits_{\mathcal{Q}_T} u^0(x,t,\omega)\varphi(x,t)\psi(\omega)\,dxdtd\mathbf{P}(\omega). \tag{4.38}$$

As was shown in the proof of Lemma 5.1 in [39] (see the first paragraph on page 41 for the detailed formulation) for any function $\psi \in L^2(\Omega)$ there is its modification (that is a function that differs from ψ on the set of zero measure $\mathbf{P}$) such that relation (4.38) holds true. In what follows we consider this particular modification of functions from $L^2(\Omega)$.

In order to formulate the main properties of stochastic two-scale convergence we introduce the subspaces $L^2_{\rm pot}(\Omega)$ and $L^2_{\rm sol}(\Omega)$ in the standard way, see [31, Section 7.2]. Let U_x, $x \in \mathbb{R}^d$, be a strongly continuous group of unitary operators in $L^2(\Omega)$ defined by $U_x f(\omega) = f(\mathcal{T}_x\omega)$. The generator of this group along the jth coordinate direction is denoted by ∂_j and its domain by $\mathcal{D}_j$. The set $\mathcal{D} = \bigcap_{j=1}^d \mathcal{D}_j$ is dense in $L^2(\Omega)$. Letting $\nabla_\omega u(\omega) = \big(\partial_1 u(\omega), \dots, \partial_d u(\omega)\big)$ for $u \in \mathcal{D}$ we denote by $L^2_{\rm pot}(\Omega)$ the closure of the set $\{\nabla_\omega u : u \in \mathcal{D}\}$ in $(L^2(\Omega))^d$. The subspace $L^2_{\rm sol}(\Omega)$ is defined as the closure in $(L^2(\Omega))^d$ of the set of vector function $(v_1(\omega), \dots, v_d(\omega))$ such that $v_j \in \mathcal{D}_j$, $j = 1, \dots, d$, and $\sum_1^d \partial_j v_j = 0$.

The subspaces $L^2_{\rm pot}(\Omega)$ and $L^2_{\rm sol}(\Omega)$ are orthogonal in $(L^2(\Omega))^d$, and $(L^2(\Omega))^d = L^2_{\rm pot}(\Omega) \oplus L^2_{\rm sol}(\Omega)$. See, for instance, [31] for further details.

Some properties of the stochastic two-scale convergence are collected in the following statement.

Theorem 4.2. *For any family $u^\varepsilon = u^\varepsilon_{\tilde\omega}(x,t)$ such that $\|u^\varepsilon\|_{L^2(\mathcal{Q}_T)} \le C(\tilde\omega)$ there exists a sequence $\varepsilon_k \to 0$ and a function $u^0 \in L^2(\mathcal{Q}_T \times \Omega)$ such that u^{ε_k} stochastically two-scale converges to u^0, as $k \to \infty$.*

If

$$\|u^\varepsilon\|_{L^2(Q_T)} + \|\nabla_x u^\varepsilon\|_{L^2(Q_T)} \le C(\tilde\omega),$$

then u^0 *does not depend on* ω, $u^0 \in L^2(0,T;H^1(Q))$, *and there exists a function* $u^1 \in L^2(Q_T; L^2_{\rm pot}(\Omega))$ *such that*

$$\nabla_x u^\varepsilon \xrightarrow{s2s} \nabla_x u^0 + u^1;$$

here and later on symbol $\xrightarrow{s2s}$ *denotes stochastic two-scale convergence.*

If

$$\|u^\varepsilon\|_{L^2(Q_T)} + \varepsilon\|\nabla_x u^\varepsilon\|_{L^2(Q_T)} \le C(\tilde\omega),$$

then

$$\varepsilon\nabla_x u^\varepsilon \xrightarrow{s2s} \nabla_\omega u^0(x,t,\omega).$$

The proof of these statements can be found in [39].

We turn to the properties of solutions of problem (2.2)–(2.9).

Theorem 4.3. *Let* S^ε, p_l^ε *and* p_g^ε *be a solution of problem* (2.2)–(2.9), *and assume that conditions* **(A.1)–(A.9)** *are fulfilled. Then there exist a function* $\widehat S = \widehat S(x,t)$, $0 \leqslant \widehat S \leqslant 1$, *a function* $\widehat{\mathsf P} \in L^2(0,T;H^1(Q))$ *and a function* $\widehat\Theta \in L^\infty(Q_T)$ *such that, for a subsequence, as* $\epsilon \to 0$,

$$S^\varepsilon(x,t) \to \widehat S(x,t) \quad \text{in } L^q(Q_T) \quad \text{for all } q \in [1,+\infty); \tag{4.39}$$

$$\mathsf P^\varepsilon(x,t) \rightharpoonup \widehat{\mathsf P}(x,t) \text{ weakly in } L^2(0,T;H^1(Q)); \tag{4.40}$$

$$\Theta^\varepsilon \to \widehat\Theta \quad \text{in } L^2(Q_T). \tag{4.41}$$

Moreover, $\widehat\Theta = (1-\widehat S)\,\varrho_g(P_g)$ *with* $P_g = \widehat{\mathsf P} + G_g(\widehat S)$.

Remark 2. The statement of the latter theorem holds for any typical realization $\tilde\omega$. However, the choice of a convergent subsequence as well as the limit functions $\widehat S$, $\widehat{\mathsf P}$ and $\widehat\Theta$ might depend on $\tilde\omega$.

As an immediate consequence of (4.39) we have

$$\beta(S^\varepsilon) \to \beta(\widehat S) \quad \text{in } L^q(\Omega_T) \quad \text{for all } q \in [1,+\infty). \tag{4.42}$$

Proof of Theorem 4.3. By the Birkhoff ergodic theorem almost surely the functions Φ^ε converge weakly in $L^2(Q_T)$ to a constant equal to $\mathbf{E}\boldsymbol\Phi = \int_\Omega \boldsymbol\Phi(\omega)\,d\mathbf P(\omega)$. Then according to Lemma 4.2 and Remark 1 in [2], Section 4 the families $\{S^\varepsilon\}_{\varepsilon>0}$ and $\{\Theta^\varepsilon\}_{\varepsilon>0}$ are compact in $L^2(Q_T)$. This implies the desired convergence in (4.39) and (4.41). The convergence in (4.40) is an immediate consequence of the estimate (3.36).

The relation $\widehat\Theta = (1-\widehat S)\,\varrho_g(P_g)$ has been justified in Lemma 4.8 in [2], Section 4. □

4.2. Effective system and homogenization theorem

We begin this section by considering an auxiliary problem that reads:
given a vector $\eta \in \mathbb{R}^d$ find $\boldsymbol{\xi}_\eta \in L^2_{\rm pot}(\Omega)$ such that

$$\boldsymbol{K}(\boldsymbol{\xi}_\eta + \eta) \in L^2_{\rm sol}(\Omega).$$

This problem has a unique solution, see [31, Chapter 7.2]. If η is equal to the j-th coordinate vector e_j in $\mathbb{R}^d$, we denote the corresponding solution by $\boldsymbol{\xi}_j$.

Let $\boldsymbol{\xi} = \boldsymbol{\xi}(\omega)$ be a matrix valued function whose j-th column coincides with $\boldsymbol{\xi}_j$, $j = 1, \ldots, d$. We define the effective characteristics

$$K^{\rm hom} = \int\limits_\Omega \boldsymbol{K}(\omega)\big(\boldsymbol{\xi}(\omega) + \mathbf{I}\big)\, d\mathbf{P}(\omega), \quad \Phi^{\rm hom} = \int\limits_\Omega \boldsymbol{\Phi}(\omega)\, d\mathbf{P}(\omega); \tag{4.43}$$

here the symbol $\mathbf{I}$ stands for the unit matrix.

The homogenized system takes the form

$$\begin{cases} 0 \leqslant \widehat{S} \leqslant 1 \quad \text{in } Q_T; \\ \Phi^{\rm hom}\, \dfrac{\partial \widehat{S}}{\partial t} - {\rm div}_x \Big\{ K^{\rm hom}\, \lambda_l(\widehat{S}) \big[\nabla P_l - \vec{g}\big] \Big\} = 0 \quad \text{in } Q_T; \\ \Phi^{\rm hom}\, \dfrac{\partial \widehat{\Theta}}{\partial t} - {\rm div}_x \Big\{ K^{\rm hom}\, \varrho_g(P_g)\, \lambda_g(\widehat{S}) \big[\nabla P_g - \varrho_g(P_g)\vec{g}\big] \Big\} = 0 \quad \text{in } Q_T; \\ P_c(\widehat{S}) = P_g - P_l \quad \text{in } Q_T, \\ \widehat{\Theta} = (1 - \widehat{S})\, \varrho_g(P_g) \quad \text{in } Q_T. \end{cases} \tag{4.44}$$

`Boundary conditions.`

$$\begin{cases} P_g(x,t) = P_l(x,t) = 0 \quad \text{on } \Gamma_{\rm inj} \times (0,T), \\ \widehat{\vec{q}}_l \cdot \vec{\nu} = \widehat{\vec{q}}_g \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_{\rm imp} \times (0,T); \end{cases} \tag{4.45}$$

where the velocities $\widehat{\vec{q}}_l$, $\widehat{\vec{q}}_g$ are defined by

$$\widehat{\vec{q}}_l = -K^{\rm hom}\lambda_l(\widehat{S})\Big(\nabla P_l - \vec{g}\Big) \quad \text{and} \quad \widehat{\vec{q}}_g = -K^{\rm hom}\lambda_g(\widehat{S})\Big(\nabla P_g - \varrho_g(P_g)\vec{g}\Big). \tag{4.46}$$

`Initial conditions`. The initial conditions are the same as for the original system in (2.7). Namely,

$$P_l(x,0) = p_l^{\mathbf{0}}(x) \quad \text{and} \quad P_g(x,0) = p_g^{\mathbf{0}}(x) \quad \text{in } Q. \tag{4.47}$$

Observe that the limit problem is deterministic. The functions $\widehat{S}$, P_l and P_g represent the homogenized wetting phase saturation, the wetting phase pressure and the nonwetting phase pressure, respectively. Concerning the numerical computation of the effective parameters $\Phi^{\rm hom}$ and $K^{\rm hom}$, we refer for instance to [11] where some example computations, for a two-phase flow through a quarter five spot reservoir, comparing the heterogeneous simulations to the global homogenized one are presented.

Theorem 4.4. *Assume that conditions* **(A.1)**-**(A.9)** *hold. Then, almost surely, a solution* $\big(S^\varepsilon,\, p_l^\varepsilon,\, p_g^\varepsilon\big)$ *of problem* (2.2)-(2.9) *converges for a subsequence, as* $\varepsilon\to 0$, *to a solution* $\big(\widehat{S},\, P_l,\, P_g\big)$ *of the homogenized problem in* (4.44)-(4.47) *in the following topology:*

$$\begin{aligned} &S^\varepsilon \to \widehat{S} \quad \text{in } L^q(Q_T) \quad \text{for any } q\in[1,+\infty);\\ &p_l^\varepsilon \rightharpoonup P_l, \quad \text{and} \quad p_g^\varepsilon \rightharpoonup P_g \ \text{ weakly in } L^2(Q_T). \end{aligned}$$

The proof of this theorem is given in the next section.

4.3. Proof of the homogenization theorem

Proof of Theorem 4.4. The rigorous derivation of the limit problem relies on the above a priori estimates and compactness results as well as on stochastic two-scale convergence technique developed in [39].

By the estimates in Theorem 3.3 and Theorem 4.2 we obtain that almost surely for a subsequence

$$\lambda_l(S^\varepsilon)\nabla p_l^\varepsilon = \lambda_l(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \nabla\beta(S^\varepsilon) \overset{s2s}{\rightharpoonup} \lambda_l(\widehat{S})\nabla\widehat{\mathsf{P}} + \nabla\beta(\widehat{S}) + \theta_l \tag{4.48}$$

with $\theta_l=\theta_l(x,t,\omega)$, $\theta_l\in L^2(Q_T;L^2_{\rm pot}(\Omega))$, and

$$\lambda_g(S^\varepsilon)\nabla p_g^\varepsilon = \lambda_l(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \nabla\beta(S^\varepsilon) \overset{s2s}{\rightharpoonup} \lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}} + \nabla\beta(\widehat{S}) + \theta_g \tag{4.49}$$

with $\theta_g=\theta_g(x,t,\omega)$, $\theta_g\in L^2(Q_T;L^2_{\rm pot}(\Omega))$.

Lemma 4.1. *Under our standing assumptions, for a subsequence*

$$K^\varepsilon\big(\lambda_l(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \nabla\beta(S^\varepsilon)\big) \overset{s2s}{\rightharpoonup} \boldsymbol{K}\big(\lambda_l(\widehat{S})\nabla\widehat{\mathsf{P}} + \nabla\beta(\widehat{S}) + \theta_l\big), \tag{4.50}$$

$$K^\varepsilon\varrho_g(\mathsf{P}^\varepsilon + \mathsf{G}_g(S^\varepsilon))\big(\lambda_g(S^\varepsilon)\nabla \mathsf{P}^\varepsilon + \nabla\beta(S^\varepsilon)\big) \overset{s2s}{\rightharpoonup} \boldsymbol{K}\varrho_g(\widehat{\mathsf{P}} + \mathsf{G}_g(\widehat{S}))\big(\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}} + \nabla\beta(\widehat{S}) + \theta_g\big). \tag{4.51}$$

Proof. Since the function K^ε is statistically homogeneous and bounded, the limit relation in (4.50) follows from (4.48). Indeed, it is sufficient to choose for any $\delta>0$ a continuous bounded function $\boldsymbol{K}_\delta\in C(\Omega)$ such that $\|\boldsymbol{K}_\delta-\boldsymbol{K}\|_{L^2(\Omega)}<\delta$. Then by the Birkhoff ergodic theorem for almost all $\widetilde{\omega}\in\Omega$ and for all sufficiently small $\varepsilon>0$ we have $\|K_\delta^\varepsilon-K^\varepsilon\|_{L^2(Q_T)}\le C\delta$. Relation (4.50) holds true if we replace K^ε and $\boldsymbol{K}$ with K_δ^ε and $\boldsymbol{K}_\delta$, respectively. The validity of this relation for K^ε and $\boldsymbol{K}$ can now be obtained by the standard approximation arguments.

Justification of the convergence in (4.51) is more tricky. Denote $\{\widehat{S}=1\}$ the set $\{(x,t)\in\Omega_T:\widehat{S}(x,t)=1\}$, and let $\mathbf{1}_{\{\widehat{S}=1\}}$ be the corresponding characteristic function. From (4.41) it is easy to deduce that $\big(1-\mathbf{1}_{\{\widehat{S}=1\}}\big)\varrho_g(\mathsf{P}^\varepsilon+\mathsf{G}_g(S^\varepsilon))$ converges to $\big(1-\mathbf{1}_{\{\widehat{S}=1\}}\big)\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))$ a.e., as $\varepsilon\to 0$. Considering the boundedness of ϱ_g and the properties of K^ε we conclude that

$$\begin{aligned}\big(1-\mathbf{1}_{\{\widehat{S}=1\}}\big)K^\varepsilon\varrho_g(\mathsf{P}^\varepsilon+\mathsf{G}_g(S^\varepsilon))\big(\lambda_g(S^\varepsilon)\nabla\mathsf{P}^\varepsilon+\nabla\beta(S^\varepsilon)\big)&\overset{s2s}{\rightharpoonup}\\ \overset{s2s}{\rightharpoonup}\big(1-\mathbf{1}_{\{\widehat{S}=1\}}\big)\boldsymbol{K}\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\big(\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}+\nabla\beta(\widehat{S})+\theta_g\big)&.\end{aligned}\tag{4.52}$$

It remains to show that

$$\begin{aligned}\mathbf{1}_{\{\widehat{S}=1\}}K^\varepsilon\varrho_g(\mathsf{P}^\varepsilon+\mathsf{G}_g(S^\varepsilon))\big(\lambda_g(S^\varepsilon)\nabla\mathsf{P}^\varepsilon+\nabla\beta(S^\varepsilon)\big)&\overset{s2s}{\rightharpoonup}\\ \overset{s2s}{\rightharpoonup}\mathbf{1}_{\{\widehat{S}=1\}}\boldsymbol{K}\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\big(\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}+\nabla\beta(\widehat{S})+\theta_g\big)&.\end{aligned}\tag{4.53}$$

Since $\lambda_g(1)=0$ and $\nabla\widehat{\mathsf{P}}^\varepsilon$ is bounded in $L^2(Q_T)$,

$$\mathbf{1}_{\{\widehat{S}=1\}}\lambda_g(S^\varepsilon)\nabla\widehat{\mathsf{P}}^\varepsilon\longrightarrow 0=\mathbf{1}_{\{\widehat{S}=1\}}\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}\quad\text{strongly in } L^2(Q_T).$$

By (3.36) and the first relation in (2.25) we obtain

$$\mathbf{1}_{\{\widehat{S}=1\}}\nabla\beta(S^\varepsilon)\longrightarrow 0=\mathbf{1}_{\{\widehat{S}=1\}}\nabla\beta(\widehat{S})\quad\text{strongly in } L^2(Q_T).$$

Therefore, $\mathbf{1}_{\{\widehat{S}=1\}}\theta_g=0$. Combining the last three relations yields (4.53) and completes the proof of Lemma. □

Next we choose in the integral identity (3.32) a test function of the form $\varphi_g(x,t)=\varepsilon\varphi(x,t)\psi(T_{\frac{x}{\varepsilon}}\omega)$ with $\varphi\in C^\infty(\mathbb{R}^d\times[0,T])$ that has a compact support in $\mathbb{R}^d\times[0,T)$, and $\psi\in\mathcal{D}(\Omega)$. Then the first two integrals on the left-hand side of (3.32) tend to zero as $\varepsilon\to 0$. Passing to the two-scale limit in the last two integrals and considering (4.51) we obtain

$$\begin{aligned}&\int\limits_{Q_T}\int\limits_{\Omega}\varphi\boldsymbol{K}(\omega)\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\big(\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}+\nabla\beta(\widehat{S})+\theta_g\big)\cdot\nabla_\omega\psi(\omega)\,dxdtd\mathbf{P}(\omega)\\&\qquad-\int\limits_{Q_T}\int\limits_{\Omega}\varphi\boldsymbol{K}(\omega)\lambda_g(\widehat{S})\big[\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\big]^2\vec{g}\cdot\nabla_\omega\psi(\omega)\,dxdt\mathbf{P}(\omega)=0\end{aligned}\tag{4.54}$$

Since φ is an arbitrary smooth function with a compact support, then for almost all $(x,t)\in\mathbb{R}^d\times[0,T]$ we have

$$\int\limits_{\Omega}\boldsymbol{K}(\omega)\big[\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}+\nabla\beta(\widehat{S})-\lambda_g(\widehat{S})\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\vec{g}+\theta_g\big]\cdot\Psi(\omega)\,\mathbf{P}(\omega)=0$$

for each $\Psi\in L^2_{\rm pot}(\Omega)$. Taking into account the definition of $\boldsymbol{\xi}(\cdot)$ we arrive at the following formula

$$\theta_g=\boldsymbol{\xi}(\omega)\big[\lambda_g(\widehat{S})\nabla\widehat{\mathsf{P}}+\nabla\beta(\widehat{S})-\lambda_g(\widehat{S})\varrho_g(\widehat{\mathsf{P}}+\mathsf{G}_g(\widehat{S}))\big].\tag{4.55}$$

It remains to choose a smooth test function φ of the form $\varphi = \varphi(x, t)$ with a compact support in $\mathbb{R}^d \times [0, T)$. Substituting the expression on the right-hand side of (4.55) for θ_g in (4.51), taking into account the definition of K^{hom} in (4.43), and recalling the relations between the global pressure and the phase pressures, we pass to the two-scale limit in (3.32) as $\varepsilon \to 0$. This yields the weak formulation of the second equation in (4.44). The first equation can be derived in a similar way with a number of simplifications. The proof of the fact that the boundary and the initial conditions in (4.45)–(4.47) are fulfilled is straightforward. This completes the proof of Theorem 4.4. □

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