HOMOGENIZATION OF RANDOM PARABOLIC OPERATORS

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Abstract: The homogenization problem for a random parabolic operator of the following type

$$A^\varepsilon = \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \frac{\partial}{\partial x_j}$$

is studied; here ε is a small parameter, $\alpha > 0$ and ξ_s is a diffusion process in R^d possessing an invariant measure with density $p(y)$. The matrix $a_{ij}(z,y)$ is supposed to be periodic in z and uniformly elliptic. It is shown that under some additional assumptions on ξ_s the operators A^ε G-converge as $\varepsilon \to 0$ to specific parabolic operator $\bar{A}$ with constant coefficients. It should be noted that the averaging procedure depends crucially on whether $\alpha > 2$, $\alpha = 2$ or $\alpha < 2$. In particular, for $\alpha = 2$ the homogenized matrix $\{\bar{a}_{ij}\}$ can be found in terms of joint distribution of the process ξ_s and the process ruled by the operator $\frac{\partial}{\partial z_i} a_{ij}(z,y)\frac{\partial}{\partial z_j}$.

1. Introduction.

The paper is devoted to homogenization of parabolic operators with rapidly oscillating coefficients which are random in the time variable. In contrast to the standard approach where the coefficients of random operators are the realizations of a transformation group preserving some probability measure, we consider parabolic operators whose coefficients depend on time through some (certain) rapidly oscillating stochastic process. Such equations arise, for example, when studying the effect of random forces on microinhomogeneous medium. Here we study the simplest case when all the coefficients are periodic in spatial variables and the stochastic process is of diffusion type. The corresponding operator takes the form

$$A^\varepsilon = \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \frac{\partial}{\partial x_j} \qquad (1.1)$$

where $a_{ij}(z,y)$ are periodic in $z \in R^n$, ξ_s is a diffusion process with values in R^d (or in a compact Riemannian manifold), $\alpha > 0$ and ε is a small positive parameter.

Our main goal is to prove homogenizaton results for the operators (1.1) and to find the coefficients of the limiting operators (so called effective diffusion matrix). It is shown

that under natural conditions on the coefficients $a_{ij}(x,y)$ and the process ξ_s the operators A^ε converge to a non random operator with constant coefficients. The structure of the averaged operator depends crucially on whether $\alpha = 2$, $\alpha < 2$ or $\alpha > 2$.

In order to prove the convergence results we construct the family of suitable approximate solutions of the equations studied in such a way that their difference with the exact solutions (so called correctors) vanishes as $\varepsilon \to 0$. This idea has already been used in classical homogenization theory [1,2] and in stochastic differential equations [3,4], but there different correctors are used.

In the first section we start with the description of the process ξ_s and then prove a number of auxiliary results. The following sections are devoted to the cases $\alpha = 2$, $\alpha < 2$ and $\alpha > 2$ respectively.

2. Setting of a problem and auxiliary results.

We study the behaviour of solution of the following initial-boundary problem

$$
\frac{\partial}{\partial t} u^\varepsilon - A^\varepsilon u^\varepsilon = \frac{\partial}{\partial t} u^\varepsilon - \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \frac{\partial}{\partial x_j} u^\varepsilon = 0, \quad x \in G, \quad 0 \le t \le T, \tag{2.1}
$$
$$
u^\varepsilon(x,t)|_{\partial G} = 0, \qquad u^\varepsilon|_{t=0} = f(x),
$$

as small positive parameter ε goes to zero; here $G \subset R^n$ is a smooth bounded domain, ξ_t is a diffusion process with values in R^d, defined on a probability space (Ω, F, P), whose infinitesimal generator has a form

$$
L = q_{ij}(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} + b_i(y) \frac{\partial}{\partial y_i};
$$

the coefficients $a_{ij}(z,y)$ are periodic in the first variable z. Below the following conditions on the coefficients of A^ε and L are supposed to hold:

C1. All the coefficients $a_{ij}(z,y)$, $q_{ij}(y)$ and $b_i(y)$ are uniformly bounded as well as their first order derivatives in all variables:

$$
|a_{ij}(z,y)| + |\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le c_1,
$$

$$
|q_{ij}(y)| + |\nabla_y q_{ij}(y)| \le c_1,
$$

here ∇_z and ∇_y mean the gradients in z and y respectively.

C2. Both A^ε and L are uniformly elliptic:

$$
c_2|\zeta|^2 \le a_{ij}(z,y)\zeta_i\zeta_j \le c_3|\zeta|^2, \quad \zeta \in R^n,\ c_2 > 0,
$$

$$
c_2|\zeta|^2 \le q_{ij}(y)\zeta_i\zeta_j \le c_3|\zeta|^2, \quad \zeta \in R^d.
$$

C3. There exist $R > 0$ and $c_4 > 0$ such that

$$
b(y) \cdot y/|y| \le -c_4
$$

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Under above conditions **C1-C3** the process ξ_t has unique invariant probability measure (see [5]). This measure possess a smooth positive density $p(y)$ that forms the kernel of formal adjoint operator L^* of L:

$$L^*p = \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j}(q_{ij}(y)p(y)) - \frac{\partial}{\partial y_i}(b_i(y)p(y)) = 0.$$

Later on we assume that the distribution of ξ_0 coincides with the invariant measure.

The next condition concerns the behaviour of $p(y)$.

C4. There exist $R > 0$ and $c > 0$ such that

$$\frac{1}{p(y)} \frac{y_i}{|y|} \frac{\partial}{\partial y_j}(q_{ij}(y)p(y)) \leq -c < 0$$

for all y, $|y| > R$.

It is also convenient to introduce the diffusion process (η_t, ξ_t) with values in $T^n \times R^d$, whose infinitesimal generator is equal to

$$A + L = \frac{\partial}{\partial z_i} a_{ij}(z,y) \frac{\partial}{\partial z_j} + q_{ij}(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} + b_i(y) \frac{\partial}{\partial y_i};$$

here T^n is the torus of periodicity of $a_{ij}(z,y)$. The invariant density $p(z,y)$ of this process does not depend on z and the equality $p(z,y) = p(y)$ holds so we use the same notation $p(y)$ for both densities.

Denote by $L_p^2(T^N \times R^d)$ a weighted space with a norm

$$\|f\|_p^2 = \int\limits_{T^n} \int\limits_{R^d} f(z,y)p(y)dzdy,$$

by $\bar{L}_p^2$ its subspace $\left\{ f \in L_p^2 \mid \int_{T^n \times R^d} f(z,y)p(y)dzdy = 0 \right\}$ and by $\bar{H}_p^1(T^N \times R^d)$ the space $\{f \in \bar{L}_p^2 \mid |\nabla_z f| + |\nabla_y f| \in L_p^2\}$. The following statement is systematically used below.

Lemma 2.1 *Let* $f(z,y) \in \bar{L}_p^2(T^n \times R^d)$ *and let*

$$|f(z,y)| \leq c(1 + |y|)^k, \qquad k \geq 0.$$

Then the equation $(A + L)u(z,y) = f(z,y)$ *does have unique solution* $u(z,y) \in \bar{H}_p^1$ *and an estimate*

$$|u(z,y)| \leq c_1(1 + |y|)^{k+1}, \qquad k \geq 0$$

holds; moreover, the constant c_1 depends only on c and k.
If, in addition,

$$|\partial_z^\beta \partial_y^\gamma f(z,y)| \leq c(1 + |y|)^k, \qquad k \geq 0, \qquad |\beta| + |\gamma| \leq N$$

for some $N > 0$ then

$$|\partial_z^\beta \partial_y^\gamma u(z,y)| \leq c_1(1+|y|)^{k+1}, \qquad k \geq 0.$$

Proof: The existence an the uniqueness of $u \in \bar{H}_p^1(T^n \times R^d)$ as well as the uniform estimate

$$\|u\|_{H_p^1} \leq c|f|_p \tag{2.2}$$

can be obtained in exactly the same way as in [3]. Denote $B_R = \{y \in R^d : |y| \leq R\}$ where R is taken from **C3**. Considering (2.2) and positiveness of $p(y)$ and applying standard elliptic estimates [6] we find

$$\max_{T^n \times B_R} |u(z,y)| \leq c|f|_p \tag{2.3}$$

It is then clear that in the domain $T^n \times (R^d \setminus B_R)$ the function $u(z,y)$ coincides with the solution of the following boundary problem

$$(A+L)v(z,y) = f(z,y), \quad (z,y) \in T^n \times (R^d \setminus B_R),$$

$$v\big|_{|y|=R} = u\big|_{|y|=R}$$

Under hypothesis **C1-C3** its solution $v(z,y)$ has the following probabilistic representation

$$v(z,y) = -\mathbf{E}\int_0^\tau f(\eta_s, \xi_s)ds + \mathbf{E}u(\eta_\tau, \xi_\tau),$$

here (η_t, ξ_t) starts from the point (z,y) and τ is the exit time from the domain $T^n \times (R^d \setminus B_R)$. To estimate $v(z,y)$ we construct a barrier function. To this end let us consider an auxiliary problem

$$C'\delta_y v'(y) - \left(C'\frac{d}{|y|^2} + \frac{C''}{|y|}\right)y \cdot \nabla v'(y) = c_0\left(C'(k+1)k(1+|y|)^{k-1} - C''(k+1)(1+|y|)^k\right) - c_1,$$

$$v'\big|_{|y|=R} = \sup_{T^n \times \partial B_R} |v(z,y)|$$

and take the constants C', C'', c_0, c_1 to make the following relations hold:
1. $\{q_{ij}(y)\} \geq C'\{\delta_{ij}\}$ for each y;
2. $q_{11}(y)/R + b(y) \cdot y/|y| \leq -C''$ for each y (we assume here that the constant R is sufficiently large);
3. $|f(z,y)| \leq -(c_0\left(C'(k+1)k(1+|y|)^{k-1} - C''(k+1)(1+|y|)^k\right) - c_1)$.
The solution $v'(y)$ can be found explicitly:

$$v'(y) = c_0(1+|y|)^{k+1} + c_1(1+|y|)/C'' + c_2.$$

This function evidently satisfies the estimate

$$0 \leq v'(y) \leq c_3(1+|y|)^{k+1}. \tag{2.4}$$

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It then follows from **1.-3.** that

$$(A + L)(v'(y) - v(z,y)) \leq 0, \qquad v'(y)|_{|y|=R} \geq v(z,y)|_{|y|=R},$$

and by probabilistic representation

$$u(z,y) = v(z,y) \leq v'(y).$$

Similarly, $u(z,y) \geq -v'(y)$. Together with (2.4) this implies the first statement of the lemma. The second one follows from the Schauder estimates. $\qquad\square$

The following statement allows us to pass to the limit when integrating rapidly oscillating functions. With the help of the following statement we will pass to the limit in the integrals of rapidly oscillating functions.

Lemma 2.2 *Let $h(z,y,x)$ be a smooth, periodic in z and compactly supported in x function such that*

$$|h(z,y,x)| + |\nabla_z h(z,y,x)| + |\nabla_y h(z,y,x)| + |\nabla_x h(z,y,x)| \leq c(1 + |y|)^k \qquad (2.5)$$

for some $k > 0$. Then for any $T > 0$

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t \int_{R^d} (h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}(x)) u^\varepsilon(x,s) dx ds \right)^2 = 0,$$

where

$$\bar{h}(x) = \int_{T^n} \int_{R^d} h(z,y,x) p(y) dy dz.$$

Proof: First of all let us show that it suffices to prove the following limiting relation

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t (h(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, \eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)) ds \right)^2 = 0, \qquad (2.6)$$

where $\eta_t^\varepsilon = \varepsilon \eta_{\frac{t}{\varepsilon^2}}$ is a process corresponding to the operator A^ε. Indeed, according to [7, th. 8.1 and 8.7] the function $u^\varepsilon(x,t)$ is a density of conditional distribution of η_t^ε with respect to σ-field $\mathcal{F}_t^\varepsilon = \sigma\{\xi_s, 0 \leq s \leq t/\varepsilon^\alpha\}$. Therefore,

$$\int_0^t \int_{R^n} \left(h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}(x) \right) u^\varepsilon(x,s) dx ds = \int_0^t \mathbf{E} \left([h(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, \eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)] \Big| \mathcal{F}_s^\varepsilon \right) ds. \quad (2.7)$$

Then, by [7, th. 8.1]

$$\int_0^t \mathbf{E} \left([h(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, \eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)] \Big| \mathcal{F}_s^\varepsilon \right) ds = \mathbf{E} \left(\int_0^t \left(h(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, \eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon) \right) ds \Big| \mathcal{F}_t^\varepsilon \right).$$

Now, let us note that the process

$$\mathcal{N}_t^\varepsilon = \mathbf{E}\left(\sup_{0\leq\nu\leq T}\Big|\int_0^\nu \big(h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)\big)ds\Big|\Big|\mathcal{F}_t^\varepsilon\right)$$

is a continuous martingale so by the Doob and Jensen inequalities

$$\mathbf{E}\sup_{0\leq t\leq T}\left(\mathbf{E}\Big\{\int_0^t \big(h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)\big)ds\Big|\mathcal{F}_t^\varepsilon\Big\}\right)^2 \leq$$

$$\mathbf{E}\sup_{0\leq t\leq T}\left(\mathbf{E}\Big\{\sup_{0\leq\nu\leq T}\Big|\int_0^\nu \big(h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)\big)ds\Big|\Big|\mathcal{F}_t^\varepsilon\Big\}\right)^2 = \mathbf{E}\sup_{0\leq t\leq T}\left(\mathcal{N}_t^\varepsilon\right)^2 \leq$$

$$c\mathbf{E}(\mathcal{N}_T^\varepsilon)^2 \leq c\mathbf{E}\left(\mathbf{E}\Big\{\sup_{0\leq t\leq T}\Big(\int_0^\nu \big(h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)\big)ds\Big)^2\Big|\mathcal{F}_T^\varepsilon\Big\}\right) =$$

$$c\mathbf{E}\sup_{0\leq t\leq T}\left(\int_0^t \big(h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon)\big)ds\right)^2$$

Together with (2.6) and (2.7) this implies the statement of the lemma.

To prove (2.6) we consider the cases $\alpha = 2$, $\alpha > 2$ and $\alpha < 2$ independently. For $\alpha = 2$ define a function $g(z,y,x)$ as a solution of the following problem:

$$(A+L)g(z,y,x) = h(z,y,x) - \bar{h}(x), \quad g \in \bar{L}_p^2(T^n \times R^d) \qquad (2.8)$$

By Lemma 2.1 $g(z,y,x)$ does exist and satisfies an estimate

$$|g(z,y,x)| + |\nabla_z g(z,y,x)| + |\nabla_y g(z,y,x)| + |\nabla_x g(z,y,x)| \leq c(1+|y|)^{k+1}$$

Applying Ito's formula to $g(\frac{\eta_t^\varepsilon}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}},\eta_t^\varepsilon)$ we obtain

$$g(\frac{\eta_t^\varepsilon}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}},\eta_t^\varepsilon) - g(\frac{\eta_0^\varepsilon}{\varepsilon},\xi_0,\eta_0^\varepsilon) = \frac{1}{\varepsilon}\int_0^t \frac{\partial}{\partial z_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})dw_1^j(s) +$$

$$\int_0^t \frac{\partial}{\partial x_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})dw_1^j(s) + \frac{1}{\varepsilon}\int_0^t \theta_{ij}(\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial y_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)dw_2^j(s) +$$

$$\int_0^t \frac{\partial^2}{\partial x_i\partial x_j}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds + \frac{2}{\varepsilon}\int_0^t \frac{\partial^2}{\partial z_i\partial x_j}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds +$$

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$$\frac{1}{\varepsilon}\int_0^t \frac{\partial}{\partial x_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\frac{\partial}{\partial z_j}a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds \;+\; \frac{1}{\varepsilon^2}\int_0^t (A+L)g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)ds;$$

here $\sigma_{ij}(z,y) = \sqrt{\{2a_{ij}(z,y)\}}$, $\theta_{ij}(y) = \sqrt{\{2q_{ij}(y)\}}$; $w_1(s)$ and $w_2(s)$ are independent Wiener processes with values on T^n and R^d respectively. Multiplying the last equality by ε^2 and taking into account (2.8) we have

$$\int_0^t (h(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}},\eta_s^\varepsilon) - \bar{h}(\eta_s^\varepsilon))ds \;=\; \varepsilon^2 g(\frac{\eta_t^\varepsilon}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}},\eta_t^\varepsilon) - \varepsilon^2 g(\frac{\eta_0^\varepsilon}{\varepsilon},\xi_0,\eta_0^\varepsilon) \;-$$

$$\varepsilon\int_0^t \frac{\partial}{\partial z_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})dw_1^j(s) \;-\; \varepsilon^2\int_0^t \frac{\partial}{\partial x_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})dw_1^j(s) \;-$$

$$\text{(2.9)}$$

$$\varepsilon\int_0^t \theta_{ij}(\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial y_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)dw_2^j(s) \;+\; \varepsilon^2\int_0^t \frac{\partial^2}{\partial x_i\partial x_j}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds \;+$$

$$2\varepsilon\int_0^t \frac{\partial^2}{\partial z_i\partial x_j}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds \;+\; \varepsilon\int_0^t \frac{\partial}{\partial x_i}g(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}},\eta_s^\varepsilon)\frac{\partial}{\partial z_j}a_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})ds$$

Let us estimate all the terms on the right hand side. These estimates are based on the following

Proposition 2.3 *For any fixed $T > 0$, $k > 0$ and $\beta > 0$*

$$\lim_{\varepsilon\to 0}\mathbf{E}\sup_{0\le t\le T}\varepsilon^\beta|\xi_{\frac{t}{\varepsilon^2}}|^k = 0$$

Proof: Consider an auxiliary operator $\tilde{L} = (c_3\Delta - c_4\frac{y}{|y|}\cdot\nabla)$ with the constants c_3 and c_4 taken from the hypothesis **C2-C3**. Denote by $\tilde{\xi}_t^y$ and ξ_t^y the diffusion processes starting from the point y, whose infinitesimal generators are equal to $\tilde{L}$ and L respectively, and by $\tilde{\tau}^y(r,R)$ and $\tau^y(r,R)$ their exit times from the spherical layer $B_r^R = \{y : R \le |y| \le r\}$. It follows from the maximum principle and the definition of $\tilde{L}$ that for any $r > R$ and $y \in B_r^R$

$$\mathbf{P}\{|\xi_{\tau^y(r,R)}^y| = r\} \le \mathbf{P}\{|\tilde{\xi}_{\tilde{\tau}^y(r,R)}^y| = r\} \le c(R)\exp(-c(r-|y|)), \quad c > 0; \qquad \text{(2.10)}$$

the explicit formula for $\mathbf{P}\{|\tilde{\xi}_{\tilde{\tau}^y(r,R)}^y| = r\}$ was also used here. Using the technique developed in [8], it is easy to derive from (2.10) that uniformly in $r > 0$ and $y \in B_{\frac{r}{\varepsilon^\gamma}}$

$$\mathbf{P}\{\tau^y(\frac{r+1}{\varepsilon^\gamma}) \le \frac{T}{\varepsilon^2}\} \le c\exp(-\frac{c_0}{\varepsilon^\gamma}), \quad c_0 > 0;$$

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here $\tau^y(r)$ is the exit time of ξ_t^y from B_r and γ is a positive number that will be fixed later. Due to the strong Markov property of ξ_t this implies

$$\mathbf{P}\{\tau^y(\frac{r+m}{\varepsilon^\gamma}) \leq \frac{T}{\varepsilon^2}\} \leq c^m \exp(-\frac{mc_0}{\varepsilon^\gamma}), \quad m = 1, 2, 3..., \tag{2.11}$$

for any $y \in B_{\frac{r}{\varepsilon^\gamma}}$. Finally, (2.11) yields

$$\mathbf{E} \sup_{0 \leq t \leq \frac{T}{\varepsilon^2}} \varepsilon^\beta |\xi_t|^k \leq$$

$$\varepsilon^\beta \sum_{m=0}^\infty \sum_{l=0}^\infty \left(\frac{m+l+1}{\varepsilon^\gamma}\right)^k \mathbf{P}\left(\{\frac{m}{\varepsilon^\gamma} \leq |\xi_0| \leq \frac{m+1}{\varepsilon^\gamma}\} \cap \{\frac{m+l}{\varepsilon^\gamma} \leq \sup_{0 \leq t \leq \frac{T}{\varepsilon^2}} |\xi_t| \leq \frac{m+l+1}{\varepsilon^\gamma}\}\right) \leq$$

$$\varepsilon^\beta \sum_{m=0}^\infty \mathbf{P}\{\frac{m}{\varepsilon^\gamma} \leq |\xi_0| \leq \frac{m+1}{\varepsilon^\gamma}\} \sum_{l=0}^\infty \left(\frac{m+l+1}{\varepsilon^\gamma}\right)^k c^l \exp(-lc_0/\varepsilon^\gamma) \leq \varepsilon^\beta \frac{\bar{c}}{k\varepsilon^\gamma};$$

here we also use the strong Markov property of ξ_t and the fact that the density $p(y)$ of ξ_0 decays exponentially as $|y| \to \infty$. To complete the proof it suffices to put $\gamma = \frac{\beta}{2k}$. $\square$

By Proposition 2.3

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} \varepsilon^4 \left(g(\frac{\eta_t^\varepsilon}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}, \eta_t^\varepsilon)\right)^2 \leq \lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} \varepsilon^4 c(1 + |\xi_{\frac{t}{\varepsilon^2}}|)^{2k+2} = 0$$

Then, by the Doob inequality

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} \left(\varepsilon \int_0^t \frac{\partial}{\partial z_i} g(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, \eta_s^\varepsilon) \sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}) dw_1^j(s)\right)^2 \leq$$

$$4 \lim_{\varepsilon \to 0} \mathbf{E}\varepsilon^2 \int_0^T \left(\frac{\partial}{\partial z_i} g(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, \eta_s^\varepsilon) \sigma_{ij}(\frac{\eta_s^\varepsilon}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}})\right)^2 ds \leq$$

$$\lim_{\varepsilon \to 0} 4c\varepsilon^2 \int_0^T \mathbf{E}(1 + |\xi_{\frac{s}{\varepsilon^2}}|)^{2k+2} ds = 0$$

Other terms on the right hand side of (2.9) can be estimated in the same way. Thus, (2.6) holds and for $\alpha = 2$ the statement of the lemma is proved.

In the case $\alpha > 2$ define a function $g_1(z, y, x)$ as follows

$$Lg_1 = h(z, y, x) - \bar{h}_1(z, x), \qquad \bar{h}_1(z, x) = \int_{R^d} h(z, y, x) p(y) dy.$$

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The solvability conditions are obviously satisfied so by Lemma 2.1 the function $g_1(z, y, x)$ does exist and an estimate

$$|g_1(z,y,x)| + |\nabla_z g_1(z,y,x)| + |\nabla_y g_1(z,y,x)| + |\nabla_x g_1(z,y,x)| \le c(1 + |y|)^{k+1}$$

holds. We have

$$\int\limits_0^t \int\limits_{R^n} (h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}(x)) u^\varepsilon(x,s) dx = \int\limits_0^t \int\limits_{R^n} (h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}_1(\frac{x}{\varepsilon}, x)) u^\varepsilon(x,s) dx +$$

$$\int\limits_0^t \int\limits_{R^n} (\bar{h}_1(\frac{x}{\varepsilon}, x) - \bar{h}(x)) u^\varepsilon(x,s) dx.$$

As was already proved (see the case $\alpha = 2$) the second term on the right hand side goes to zero as $\varepsilon \to 0$. The proof of the fact that the first one goes to zero is quite similar to those given for $\alpha = 2$.

If $\alpha < 2$ we define a function $g_2(z, y, x)$ as a solution of the following problem

$$A g_2(z,y,x) = h(z,y,x) - \bar{h}_2(y,x), \qquad \bar{h}_2(y,x) = \int\limits_{T^n} h(z,y,x) dz.$$

The function $g_2(z, y, x)$ does obviously exist and satisfies an estimate

$$|g_2(z,y,x)| + |\nabla_z g_2(z,y,x)| + |\nabla_y g_2(z,y,x)| + |\nabla_x g_2(z,y,x)| \le c(1 + |y|)^{k+1}$$

We have

$$\int\limits_0^t \int\limits_{R^n} (h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}(x)) u^\varepsilon(x,s) dx = \int\limits_0^t \int\limits_{R^n} (h(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}_2(\xi_{\frac{s}{\varepsilon^\alpha}}, x)) u^\varepsilon(x,s) dx +$$

$$\int\limits_0^t \int\limits_{R^n} (\bar{h}_1(\xi_{\frac{s}{\varepsilon^\alpha}}, x) - \bar{h}(x)) u^\varepsilon(x,s) dx.$$

As was proved above the second term on the right hand side goes to zero. Indeed, it suffices to introduce new small parameter $\varepsilon' = \varepsilon^{\alpha/2}$. As to the first term it goes to zero too by the same arguments as for $\alpha = 2$. The lemma is completely proved. $\qquad \square$

Also, we are interested in the limiting behaviour of the following martingale

$$\mathcal{M}_t^\varepsilon = \int\limits_0^t \int\limits_{R^n} h_i \left(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, x \right) u^\varepsilon(x,s) dx dw_2^i(s).$$

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Lemma 2.4 *Let $h(z, y, x)$ be a smooth, periodic in z and compactly supported in x vector-function such that*

$$|h(z, y, x)| + |\nabla_x h(z, y, x)| \leq (1 + |y|)^k$$

and

$$\int_{T^n} h(z, y, x) dz = 0 \tag{2.12}$$

for each y and x. Then,

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \leq t \leq T} |\mathcal{M}_t^\varepsilon| = 0.$$

Proof: According to the Doob inequality it suffices to show that

$$\lim_{\varepsilon \to 0} \mathbf{E} < \mathcal{M}^\varepsilon >_T = 0,$$

where $< \mathcal{M}^\varepsilon >_t$ is a quadratic characteristic of $\mathcal{M}_t^\varepsilon$. Due to assumption (2.12) there exists periodic in z matrix-function $H(z, y, x)$ such that

$$\frac{\partial}{\partial z_i} H_{ij}(z, y, x) = h_j(z, y, x);$$

moreover, $H(z, y, x)$ can be chosen to satisfy an estimate

$$|H(z, y, x)| + |\nabla_x H(z, y, x)| \leq c(1 + |y|)^k. \tag{2.13}$$

Then, in view of the obvious equality

$$\frac{\partial}{\partial x_i} H_{ij}(\frac{x}{\varepsilon}, y, x) = (\frac{1}{\varepsilon} \frac{\partial}{\partial z_i} H_{ij}(z, y, x) + \frac{\partial}{\partial x_i} H_{ij}(z, y, x))|_{z = \frac{x}{\varepsilon}}$$

we can rewrite $\mathcal{M}_t^\varepsilon$ in the following form

$$\mathcal{M}_t^\varepsilon = \int_0^t \int_{R^n} \frac{\partial}{\partial z_i} H_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, x\right) u^\varepsilon(x, s) dx dw_2^j(s) =$$

$$-\varepsilon \int_0^t \int_{R^n} H_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, x\right) \frac{\partial}{\partial x_i} u^\varepsilon(x, s) dx dw_2^j(s) -$$

$$-\varepsilon \int_0^t \int_{R^n} \left(\frac{\partial}{\partial x_i} H_{ij}\left(z, \xi_{\frac{s}{\varepsilon^2}}, x\right)\Big|_{z = \frac{x}{\varepsilon}}\right) u^\varepsilon(x, s) dx dw_2^j(s) = \mathcal{M}_t^{1\varepsilon} + \mathcal{M}_t^{2\varepsilon}$$

By (2.13)

$$\mathbf{E} < \mathcal{M}^{1\varepsilon} >_T = \mathbf{E} \left(\varepsilon \int_0^T \int_{R^n} H_{ij}\left(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, x\right) \frac{\partial}{\partial x_i} u^\varepsilon(x, s) dx dw_2^j(s)\right)^2 =$$

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$$\mathbf{E}\varepsilon^2 \int\limits_0^T \int\limits_{R^n} \left(H\left(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}, x\right) \nabla_x u^\varepsilon(x,s) \right)^2 dx ds \leq$$

$$\mathbf{E}\left(\sup_{0 \leq t \leq T} \varepsilon^2 (1 + |\xi_{\frac{t}{\varepsilon^2}}|)^{2k} \int\limits_0^T \int\limits_{R^n} |\nabla_x u^\varepsilon(x,s)|^2 dx ds \right) \leq C\mathbf{E} \sup_{0 \leq t \leq T} \varepsilon^2 (1 + |\xi_{\frac{t}{\varepsilon^2}}|)^{2k};$$

here the uniform estimate

$$\|u^\varepsilon\|_{L^2(0,T;H^1(G))} \leq c$$

was also used. Finally, by Proposition 2.3

$$\lim_{\varepsilon \to 0} \mathbf{E} < \mathcal{M}^{1\varepsilon} >_T = 0.$$

The martingale $\mathcal{M}_t^{2\varepsilon}$ can be estimated in the same way. The lemma is proved. $\qquad\square$

3. Homogenization of self-similar operators.

In this section we investigate the problem (2.1) in the automodelling case $\alpha = 2$. Denote by V the space $L_w^2(0,T;H_0^1(G)) \cap C(0,T;L_w^2(G))$, where symbol w means that the corresponding space is endowed with its weak topology; the space $C(0,T;L_w^2(G))$ is endowed with the topology of uniform convergence. Let B be Borel σ-field on V. As was shown in [9] V is a Lusin and completely regular space and the Prokhorov criterion of weak compactness for a family of probability measures on V is valid. The main result of the section is the following

Theorem 3.1 *Let $\alpha = 2$. Then, under assumptions* **C1**-**C4** *the solution $u^\varepsilon(x,t)$ of (2.1) converges in probability in the space V to the solution of a problem*

$$\frac{\partial}{\partial t} u_0(x,t) - \bar{a}_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u_0(x,t) = 0$$

$$u|_{\partial G} = 0, \qquad u_0|_{t=0} = f(x) \tag{3.1}$$

with the coefficients $\bar{a}_{ij}$ given by the following formula

$$\bar{a}_{ij} = \int\limits_{T^n} \int\limits_{R^d} \left(a_{ij}(z,y) + a_{ik}(z,y) \frac{\partial}{\partial z_k} \chi_j(z,y) \right) p(y) dz dy;$$

here $\chi(z,y) \in \bar{H}_p^1(T^n \times R^d)$ is the solution of the equation $(A+L)\chi_i(z,y) = -\frac{\partial}{\partial z_j} a_{ij}(z,y)$.

Remark 3.2 *The convergence of $u^\varepsilon(x,t)$ in V implies the strong convergence in $L^2(0,T;H^1(G))$, therefore, $u^\varepsilon(x,t)$ converges in probability to $u^0(x,t)$ in the norm of $L^2(0,T;H^1(G))$.*

Proof: Let us consider a family of Radon probability measures $\{Q^\varepsilon\}$ on (V,B) where Q^ε is the law of $u^\varepsilon(x,t)$, $0 \leq t \leq T$. Using standard estimates for parabolic equations [6] it is easy to show that the family $\{Q^\varepsilon\}$ is relatively compact (see, for example, [3], [9] where

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a number of similar results are proved). We will demonstrate that any limiting point Q of this family is concentrated on the set of weak solutions to the problem (3.1). This will imply the statement of the theorem due to uniqueness of weak solution.

So we fix arbitrary $\varphi(x,t) \in C^{\infty}(0,T;C_0^{\infty}(Q))$ and study the limiting behaviour of the following expression

$$(u^{\varepsilon},\varphi) + \varepsilon\big(\frac{\partial}{\partial x_i}[u^{\varepsilon}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}})],\varphi\big) = (u^{\varepsilon},\varphi) - \varepsilon\big(u^{\varepsilon}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}),\frac{\partial}{\partial x_i}\varphi\big);$$

here and in what follows $(\cdot,\cdot)$ means inner product in $L^2(G)$. By the Ito's formula

$$(u^{\varepsilon}(x,t),\varphi(x,t)) - \varepsilon\big(u^{\varepsilon}(x,t)\chi_i(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}),\frac{\partial}{\partial x_i}\varphi(x,t)\big) - (f,\varphi(x,0)) +$$

$$\varepsilon\big(f\chi_i(\frac{x}{\varepsilon},\xi_0),\frac{\partial}{\partial x_i}\varphi(x,0)\big) = \int_0^t (A^{\varepsilon}u^{\varepsilon}(x,s),\varphi(x,s))ds + \int_0^t (u^{\varepsilon}(x,s),\frac{\partial}{\partial s}\varphi(x,s))ds -$$

$$\varepsilon\int_0^t (A^{\varepsilon}u^{\varepsilon}(x,s),\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_i}\varphi(x,s))ds - \varepsilon\int_0^t (u^{\varepsilon}(x,s)\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_i}\frac{\partial}{\partial s}\varphi(x,s))ds -$$

$$\varepsilon\int_0^t (u^{\varepsilon}(x,s)\frac{\partial}{\partial x_i}\varphi(x,s),\frac{1}{\varepsilon^2}[b_j(\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial y_j}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}}) + q_{jk}(\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})])ds -$$

$$\varepsilon\int_0^t (u^{\varepsilon}(x,s)\frac{\partial}{\partial x_i}\varphi(x,s),\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}}))dw_2^j(s) =$$

$$\int_0^t (u^{\varepsilon},\frac{\partial}{\partial s}\varphi)ds + \int_0^t (u^{\varepsilon},a_{ij}(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi + \frac{1}{\varepsilon}\frac{\partial}{\partial z_i}a_{ij}(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_j}\varphi)ds -$$

$$\int_0^t \Big(u^{\varepsilon},\frac{1}{\varepsilon}A\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_j}\varphi - \frac{\partial}{\partial z_i}[a_{ij}(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\chi_k(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})]\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi +$$

$$a_{ij}(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial z_i}\chi_k(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi - \varepsilon a_{ij}(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\chi_k(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi\Big)ds -$$

$$\frac{1}{\varepsilon}\int_0^t (u^{\varepsilon},L\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial x_i}\varphi)ds - \varepsilon\int_0^t (u^{\varepsilon},\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}})\frac{\partial}{\partial s}\frac{\partial}{\partial x_i}\varphi)ds +$$

$$\int_0^t (u^{\varepsilon}\frac{\partial}{\partial x_i}\varphi,\frac{\partial}{\partial y_j}\chi_i(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^2}}))dw_2^j(s);$$

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here the equation $\frac{\partial}{\partial t}u^\varepsilon = A^\varepsilon u^\varepsilon$ was also used. Collecting the terms of the same order of ε and taking into account the choice of $\chi(z, y)$ we obtain after simple transformation

$$(u^\varepsilon(x,t), \varphi(x,t)) - (f, \varphi(x,0) - \int_0^t (u^\varepsilon, \frac{\partial}{\partial s}\varphi)ds - \int_0^t (u^\varepsilon, \bar{a}_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi)ds \; =$$

$$-\varepsilon(u^\varepsilon(x,t), \chi_i(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\frac{\partial}{\partial x_i}\varphi(x,t)) + \varepsilon(f, \chi_i(\frac{x}{\varepsilon}, \xi_0)\frac{\partial}{\partial x_i}\varphi(x,0))+$$

$$\int_0^t (u^\varepsilon, (a_{ij}(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}) + \frac{\partial}{\partial z_k}[a_{ik}(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\chi_i(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})] + a_{ik}(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\frac{\partial}{\partial z_k}\chi_i(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}) - \bar{a}_{ij})\frac{\partial^2\varphi}{\partial x_i \partial x_j})ds$$

$$+\varepsilon\int_0^t (u^\varepsilon\chi_i(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}), (\frac{\partial}{\partial x_i}(\frac{\partial}{\partial s}\varphi + a_{jk}(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi)ds+$$

$$\int_0^t (u^\varepsilon\frac{\partial}{\partial x_i}\varphi, \frac{\partial}{\partial y_j}\chi_i(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon^2}}))dw_2^j(s).$$

Considering the fact that the operator L commutes with the operator of taking the mean value in z it is easy to show that the function $\nabla_y\chi(z, y, x)$ satisfies condition (2.12). By Lemma 2.2 and 2.4 and Proposition 2.3 all the terms on the right hand side go to zero (here the function $\varphi(x, t)$ depends on t but it is quite easy to show that all the mentioned statements hold true in this case). Thus,

$$\lim_{\varepsilon \to 0} \mathbf{E} \sup_{0 \le t \le T} \left| (u^\varepsilon(x,t), \varphi(x,t)) - (f, \varphi(x,0)) - \int_0^t (u^\varepsilon, \frac{\partial}{\partial t}\varphi)ds - \int_0^t (u^\varepsilon, \bar{a}_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi)ds \right| = 0.$$

$$(3.2)$$

Now, define on V the following functionals

$$\Phi_\varphi(u) = \sup_{0 \le t \le T} \left| (u(x,t), \varphi(x,t)) - (f, \varphi(x,0)) - \int_0^t (u, \frac{\partial}{\partial s}\varphi)ds - \int_0^t (u, \bar{a}_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi)ds \right|$$

and

$$\bar{\Phi}_\varphi(u) = \begin{cases} \Phi_\varphi(u), & \text{if } \Phi_\varphi(u) \le 1 \\ 1, & \text{otherwise} \end{cases}.$$

By (3.2)

$$\lim_{\varepsilon \to 0} \mathbf{E}\bar{\Phi}_\varphi(u^\varepsilon) = 0$$

or, in terms of Q^ε,

$$\lim_{\varepsilon \to 0} \mathbf{E}^{Q^\varepsilon}\bar{\Phi}_\varphi(u) = 0 \qquad (3.3)$$

The functional $\bar{\Phi}_\varphi(u)$ is bounded and continuous. Therefore, passing to the limit in (3.3) we deduce

$$\mathbf{E}^Q \bar{\Phi}_\varphi(u) = 0$$

for any limiting point Q of Q^ε and for any $\varphi \in C^\infty(0, T; C_0^\infty(G))$. In view of non negativity of $\bar{\Phi}_\varphi(u)$ the last equality implies

$$Q\{u \in V | \bar{\Phi}_\varphi(u) = 0\} = 1$$

for any φ. This, in turn, leads to the relation

$$Q\{u \in V | \Phi_\varphi(u) = 0 \text{ for any } \varphi \in C^\infty(0, T; C_0^\infty(G))\} = 1$$

Thus, Q is concentrated on the weak solution of (3.1). The theorem is proved. $\qquad\square$

Corollary 3.3 *The sequence of operators* $\left(\frac{\partial}{\partial t} - A^\varepsilon\right)$ *G-converges in probability to the operator* $\frac{\partial}{\partial t} - \bar{A}$.

Remark 3.4 The statements of the theorem and of the corollary remain unchanged if instead of assumption $a_{ij}(z, y) \in C_b^1(T^n \times R^d)$ in **C1** we suppose that $a_{ij}(z, y)$ are measurable and uniformly bounded. Indeed, it suffices to approximate the matrix $a_{ij}(z, y)$ by the sequence of smooth matrices.

4. Homogenization of non self-similar operators.

In the section we formulate the homogenization results for non automodel operators i.e. for $\alpha \neq 2$. We start with the case $\alpha < 2$. In this case the oscillation in spatial variables is in some sense faster then the oscillation in time. Denote by $\hat{a}_{ij}(y)$ the coefficients of a homogenized (with respect to $\frac{x}{\varepsilon}$) operator of the family $\frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, y) \frac{\partial}{\partial x_j}$, here y is a parameter.

Theorem 4.1 *Let* $0 < \alpha < 2$. *Then under assumptions* **C1**-**C4** *the solution* u^ε *of (2.1) converges in probability in the space* V *to the solution of the problem (3.1) with the coefficients* $\bar{a}_{ij}$ *equal to the mean value of* $\hat{a}_{ij}(y)$:

$$\bar{a}_{ij} = \int\limits_{R^d} \hat{a}_{ij}(y) p(y) dy$$

In the case $\alpha > 2$ denote by $\tilde{a}_{ij}(z)$ the mean value of $a_{ij}(z, y)$ in y:

$$\tilde{a}_{ij}(z) = \int\limits_{R^d} a_{ij}(z, y) p(y) dy$$

Here the following assertion takes place

Theorem 4.2 *Let* $\alpha > 2$. *Then under assumptions* **C1**-**C4** *the solution* u^ε *of (2.1) converges in probability in the space* V *to the solution of the problem (3.1) where the*

coefficients $\bar{a}_{ij}$ coincide with the coefficients of the homogenized operator of the family $\frac{\partial}{\partial x_i} \tilde{a}_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}$.

The proofs of Theorems 4.1 and 4.2 are based on the same ideas as the proof of Theorem 3.1.

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