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Homogenized model of reaction–diffusion in a porous medium

Leonid Pankratov [a,b], Andrey Piatnitskii [c,d], Volodymyr Rybalko [a]

[a] Département de mathématiques, B.Verkin Institut des Basses Températures (FTINT), 47, av. Lénine, 61103, Kharkov, Ukraine
[b] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, pl. Jussieu 75252 Paris cedex 05, France
[c] Narvik University College, HiN, 8505, Narvik, Norway
[d] Lebedev Physical Institute RAS, 53, Leninski prospect, 117333, Moscow, Russia

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Abstract

We study the initial boundary value problem for the reaction–diffusion equation,

$$\partial_t u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) + g(u^\varepsilon) = h^\varepsilon$$

in a bounded domain Ω with periodic microstructure $\mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$, where $a^\varepsilon(x)$ is of order 1 in $\mathcal{F}^{(\varepsilon)}$ and $\kappa(\varepsilon)$ in $\mathcal{M}^{(\varepsilon)}$ with $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Combining the method of two-scale convergence and the variational homogenization we obtain effective models which depend on the parameter $\theta = \lim_{\varepsilon\to 0} \kappa(\varepsilon)/\varepsilon^2$. In the case of strictly positive finite θ the effective problem is nonlocal in time that corresponds to the memory effect. ***To cite this article: L. Pankratov et al., C. R. Mecanique 331 (2003).***

Résumé

Un modèle homogénéisé de réaction–diffusion dans un milieu poreux. On étudie le problème aux limites pour l'équation de réaction–diffusion

$$\partial_t u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) + g(u^\varepsilon) = h^\varepsilon$$

dans un ouvert borné Ω avec une microstructure périodique $\mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$, où $a^\varepsilon(x)$ vaut 1 dans $\mathcal{F}^{(\varepsilon)}$ et $\kappa(\varepsilon)$ dans $\mathcal{M}^{(\varepsilon)}$ avec $\kappa(\varepsilon) \to 0$ quand $\varepsilon \to 0$. En combinant la méthode de convergence à double échelle et l'homogénéisation variationnelle, on obtient des modèles macroscopiques qui dépendent du paramètre $\theta = \lim_{\varepsilon\to 0} \kappa(\varepsilon)/\varepsilon^2$. Lorsque θ est strictement positif et fini, le problème macroscopique est non local en temps ce qui correspond à l'effet de mémoire. ***Pour citer cet article : L. Pankratov et al., C. R. Mecanique 331 (2003).***

Keywords: Computational solid mechanics; Reaction–diffusion equation; Homogenised model; Memory effect

Mots-clés : Mécanique des solides numérique ; Équation de réaction–diffusion ; Modèle homogénéisé ; Effet de mémoire

E-mail addresses: pankratov@ilt.kharkov.ua, pankrato@ann.jussieu.fr (L. Pankratov), andrey@sci.lebedev.ru (A. Piatnitskii), vrybalko@ilt.kharkov.ua (V. Rybalko).

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On considère une équation parabolique sémi-linéaire avec un coefficient de diffusion périodique fortement contrasté. Cette équation intervient en chimie et en biologie et peut modéliser, par exemple, le processus de réaction–diffusion dans un milieu poreux Ω périodique composé de blocs (matrices) $\mathcal{M}^{(\varepsilon)}$ entourés de fissures $\mathcal{F}^{(\varepsilon)}$ et s'écrit :

$$\begin{cases} \partial_t u^\varepsilon - \nabla\cdot\big(a^\varepsilon(x)\nabla u^\varepsilon\big) + g(u^\varepsilon) = h^\varepsilon & \text{dans } \Omega_T \\ \nabla u^\varepsilon\cdot\vec{\nu} = 0 & \text{sur } (0,T)\times\partial\Omega \\ u^\varepsilon(0,x) = U^\varepsilon(x) & \text{dans } \Omega \end{cases} \tag{1}$$

où $g(u)$ est une fonction regulière dans $\mathbf{R}$ vérifiant les conditions (3) ; $\Omega = \mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$, $\Omega_T = (0,T)\times\Omega$. Le coefficient $a^\varepsilon(x)$, défini par (2), fait apparaître un grand contraste $(1/\kappa(\varepsilon))$ entre les perméabilités des fissures et des blocs. Le résultat principal est le suivant :

Theorem 3.1. *Soient h^ε et U^ε des fonctions vérifiant les conditions du Théorème* 2.1. *Supposons que les suites $\{h^\varepsilon\}$ et $\{U^\varepsilon\}$ convergent à double échelle vers $h = h(x,y)$ et $U = U(x,y)$, et que $\|h^\varepsilon\|_{2,\Omega}$ et $\|U^\varepsilon\|_{2,\Omega}$ convergent vers $\|h\|_{2,\Omega\times Y}$ et $\|U\|_{2,\Omega\times Y}$. Alors $U(x,y) = U_f(x)$ si $y \in \mathcal{F}$ et on a* : (i) *Si $\kappa(\varepsilon)/\varepsilon^2 \to \theta < +\infty$, alors, pour tout $t \in (0,T)$, les solutions de* (1) *convergent à double échelle dans $L^p(\Omega)$ vers $u_f + u_m$, oú $u_m(t,x,y) = 0$ si $y \in \mathcal{F}$, et le couple (u_f, u_m) est la solution unique du problème homogénéisé* (6)–(9). (ii) *Si $\kappa(\varepsilon)/\varepsilon^2 \to +\infty$, alors, pour tout $t \in (0,T)$, les solutions de* (1) *convergent à double échelle dans $L^p(\Omega)$ vers la fonction u vérifiant* (10).

Comme u^ε et $\partial_t u^\varepsilon$ sont bornés dans $L^2(\Omega_T)$, cette convergence ponctuelle à temps implique la convergence (11) à espace-temps (voir Remarque 2).

1. Introduction

We consider a model problem describing nonstationary reaction–diffusion process in media with a high contrast periodic microstructure. For a rather general class of nonlinear reaction terms we deduce the effective (homogenized) problem and establish the convergence results.

Problems of this type often called double porosity problems, appear when studying a chemical reaction in a catalyst pellet, in the theory of flames and in some biological models (see, for example, [1] and the bibliography there).

Previously linear double porosity models were investigated in [2–5] by asymptotic decomposition and two-scale convergence methods. Then a number of interesting homogenization results has been obtained in [5,6] for nonlinear stationary equations that admit a variational formulation; the techniques used in these works rely essentially on variational form of the problem.

In this paper we consider nonstationary double porosity equations with nonlinear reaction term and combining the two scale convergence method and variational techniques, we reduce the original problem to a variational problem with parameter. This allows us to pass to the limit in the nonlinear term and to obtain an effective model which contains an additional (hidden) variable y. The elimination of this variable leads, as in the linear case, to a nonlocal in time model which exhibits a memory phenomenon (see Remark 3). However, in contrast to the linear case, the structure of the nonlocal in time term is more complicated and cannot be reduced to a time convolution operator.

2. Problem statement and a priori estimates

Let Ω be a bounded domain in $\mathbf{R}^n$ $(n \geqslant 2)$ with a smooth boundary $\partial\Omega$. We assume that, in the standard periodicity cell $Y = (0,1)^n$, there is an obstacle $\mathcal{M} \Subset Y$ with a piecewise smooth boundary $\partial\mathcal{M}$. The remainder is denoted by $\mathcal{F} = Y \setminus \mathcal{M}$. We assume that this geometry is repeated periodically all over $\mathbf{R}^n$. The geometric structure within the domain Ω is then obtained by intersecting the ε-multiple of this periodic geometry with Ω, ε being a small positive parameter. We assume that a^ε in (1) is given by $a^\varepsilon = a^\varepsilon(x, \frac{x}{\varepsilon})$ with

$$a^\varepsilon(x, y) = a(x, y)\mathbf{I}_{\mathcal{F}}(y) + \kappa(\varepsilon)a(x, y)\mathbf{I}_{\mathcal{M}}(y) \tag{2}$$

where $a(x, y)$ is a uniformly positive Y-periodic in y function of the class $C(\overline{\Omega}; L^\infty_\#(Y))$, and $\kappa(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$; $\mathbf{I}_{\mathcal{F}}$ and $\mathbf{I}_{\mathcal{M}}$ are the indicators of the sets $\mathcal{F}$ and $\mathcal{M}$ (extended Y-periodically to the whole $\mathbf{R}^n$), respectively.

In the cylinder $\Omega_T = (0, T) \times \Omega$ we consider the initial boundary value problem (1), where $h^\varepsilon, U^\varepsilon : \Omega \to \mathbf{R}$ are given functions and $\vec{\nu}$ is the outer unit normal on $\partial\Omega$.

We suppose that $g = g(u)$ is a $C^1(\mathbf{R})$ function such that

$$\mu_0|u|^{p-1} - A_1 \leqslant \big|g(u)\big| \leqslant \mu_1\big(|u|^{p-1} + 1\big), \quad -\infty < \Lambda \equiv \inf_{u\in\mathbf{R}} g'(u) \tag{3}$$

with $\mu_0 > 0$ and $2 \leqslant p < \frac{2n}{n-2}$ if $n \geqslant 3$ and $2 \leqslant p < +\infty$ if $n = 2$.

Using the standard approach relying on the compactness principle and the Galerkin method (see, e.g., [7]) we prove the following result. (From now on $\|\cdot\|_{p,Q}$ stands for the norm in the space $L^p(Q)$, where Q is a measurable set in $\mathbf{R}^n$ or in $\mathbf{R} \times \mathbf{R}^n$ or in $\mathbf{R}^n \times \mathbf{R}^n$.)

Theorem 2.1. *Let us suppose that the source functions h^ε are uniformly bounded in $L^2(\Omega)$ and the initial data U^ε verify $\|U^\varepsilon\|^2_{p,\Omega} + \|\sqrt{a^\varepsilon(x)}|\nabla U^\varepsilon|\|^2_{2,\Omega} \leqslant A_2$ with a constant A_2 independent of ε. Then for any finite interval $(0, T)$ problem (1) has a unique solution $u^\varepsilon(t) = u^\varepsilon(t, x)$ such that*

$$\big\|u^\varepsilon(t)\big\|^2_{p,\Omega} + \big\|u^\varepsilon_t\big\|^2_{2,\Omega_T} + \big\|\sqrt{a^\varepsilon(x)}\,\big|\nabla u^\varepsilon(t)\big|\big\|^2_{2,\Omega} \leqslant A_3 \tag{4}$$

$$\big\|u^\varepsilon(t + \Delta t) - u^\varepsilon(t)\big\|^2_{2,\Omega} \leqslant A_4\Delta t \tag{5}$$

where A_3, A_4 are constants independent of ε and t.

3. Statement of the main result

We consider the boundary value problem (1) and study the asymptotic behaviour of the solutions $u^\varepsilon = u^\varepsilon(t, x)$ as $\varepsilon \to 0$.

The following homogenization theorem states that the homogenized problem is a double permeability reaction–diffusion model.

Theorem 3.1. *Let h^ε and U^ε satisfy the conditions of Theorem* 2.1. *Suppose also that the sequences of functions h^ε and U^ε two-scale converge to $h = h(x, y)$ and $U = U(x, y)$, $\|h^\varepsilon\|_{2,\Omega}$ and $\|U^\varepsilon\|_{2,\Omega}$ converge to $\|h\|_{2,\Omega\times Y}$ and $\|U\|_{2,\Omega\times Y}$, respectively. Then $U(x, y) = U_f(x)$ if $y \in \mathcal{F}$, and we have*:

(i) *If $\kappa(\varepsilon)/\varepsilon^2 \to \theta < +\infty$ then, for any $t \in (0, T)$, the solutions $u^\varepsilon(t) = u^\varepsilon(t, x)$ of* (1)*, converge in two-scale sense in $L^p(\Omega)$ to $u_f(t, x) + u_m(t, x, y)$, where $u_m(t, x, y) = 0$ if $y \in \mathcal{F}$. The couple (u_f, u_m) is the unique solution of*

$$\begin{cases} |\mathcal{F}|\big(\partial_t u_f + g(u_f)\big) - \nabla_x \cdot \big(A^*(x)\nabla_x u_f\big) = S_\theta(x, u_m) & \text{in } \Omega_T \\ A^*\nabla_x u_f \cdot \vec{\nu} = 0 & \text{on } (0, T) \times \partial\Omega \\ u_f(0, x) = U_f(x) & \text{in } \Omega \\ \partial_t(u_m + u_f) - \theta\nabla_y \cdot \big(a(x, y)\nabla_y u_m\big) + g(u_m + u_f) = h(x, y) & \text{in } \Omega_T \times \mathcal{M} \\ u_m(t, x, y) = 0 & \text{on } \Omega_T \times \partial\mathcal{M} \\ u_m(0, x, y) = U(x, y) - U_f(x) & \text{in } \Omega \times \mathcal{M} \end{cases} \tag{6}$$

where $A^* = \{a^*_{ij}\}$ is the homogenized permeability tensor defined by:

$$a^*_{ij} = \int_{\mathcal{F}} a(x, y)(\vec{e}_i + \nabla_y w_i)(\vec{e}_j + \nabla_y w_j)\,\mathrm{d}y \tag{7}$$

with $\{\vec{e}_1,\dots,\vec{e}_n\}$ *the canonical basis of* $\mathbf{R}^n$ *and* w_i *being the unique solution in* $H^1_\#(\mathcal{F})\setminus\mathbf{R}$ *of*

$$\begin{cases} -\nabla_y\cdot\big(a(x,y)(\vec{e}_i+\nabla_y w_i)\big)=0 & \text{in } \mathcal{F}\\ (\vec{e}_i+\nabla_y w_i)\cdot\vec{\nu}=0 & \text{on } \partial\mathcal{M}\\ y\to w_i(x,y) & Y\text{-periodic}\end{cases} \tag{8}$$

the effective source term $S_\theta(x,u_m)$ *is given by*

$$S_\theta(x,u_m)=\int_{\mathcal{F}} h(x,y)\,\mathrm{d}y-\theta\int_{\partial\mathcal{M}} a(x,y)(\nabla_y u_m\cdot\vec{\nu})\,\mathrm{d}s_y \tag{9}$$

(ii) *If* $\kappa(\varepsilon)/\varepsilon^2\to+\infty$ (*while* $\kappa(\varepsilon)\to 0$) *then the solutions* $u^\varepsilon(t)=u^\varepsilon(t,x)$ *of* (1), *converge in two-scale sense in* $L^p(\Omega)$ *to a unique solution* $u=u(t,x)$ *of*

$$\begin{cases} \partial_t u-\nabla_x\cdot\big(A^*(x)\nabla_x u\big)+g(u)=\int_Y h(x,y)\,\mathrm{d}y & \text{in } \Omega_T\\ A^*\nabla u\cdot\vec{\nu}=0 & \text{on } (0,T)\times\partial\Omega\\ u(0,x)=U_f(x) & \text{in } \Omega\end{cases} \tag{10}$$

where $A^*=\{a^*_{ij}\}$ *is still defined through* (7), (8).

Moreover, $\|u^\varepsilon\|_{p,\Omega}$ *converges to* $\|u_f+u_m\|_{p,\Omega\times Y}$.

Remark 1. The main result of this Note remains true (with evident modifications) when the weakly permeable inclusions $\mathcal{M}^{(\varepsilon)}$ form a connected set in Ω, provided that the fissures set $\mathcal{F}^{(\varepsilon)}$ is also connected. In this case the boundary condition for u_m in (6) becomes $u_m=0$ when $y\in\partial\mathcal{M}\setminus\partial Y$ and $u_m(t,x,y)$ is Y-periodic in y.

Remark 2. As far as the convergence in time is concerned, the uniform estimate (4) implies that u^ε and $\partial_t u^\varepsilon$ are bounded in $L^2(\Omega_T)$. From this bound and the pointwise convergence in t one easily deduces that

$$\int_0^T \mathrm{d}t\int_\Omega u^\varepsilon(t,x)\varphi\Big(x,\frac{x}{\varepsilon}\Big)\psi^\varepsilon(t)\,\mathrm{d}x\to\int_0^T\mathrm{d}t\int_{\Omega\times Y}(u_f+u_m)\varphi(x,y)\psi(t)\,\mathrm{d}x\,\mathrm{d}y \tag{11}$$

for any $\varphi\in C(\overline{\Omega};L^2_\#(Y))$ and any sequence $\{\psi^\varepsilon\}$ which converges to ψ weakly in $L^2(0,T)$.

Remark 3. It follows from the structure of problem (6) that for $\theta\in(0,+\infty)$ the function $S_\theta(x,u_m(t))$ given by (9), can be expressed as a functional of x and $u_f(\cdot,x)$. An important property of this functional is the fact that it not only depends on the value of u_f at the current time t but on the whole trajectory $\{u_f(s,x)\colon 0\leqslant s\leqslant t\}$. Thus the term on the RHS of the first equation in (6) is nonlocal in time. This represents the memory phenomenon in the effective model.

4. Sketch of the proof of Theorem 3.1

For the sake of brevity we only consider the case $\theta=1$ and $p=2$ in (3). First of all we establish the following preliminary compactness result.

Lemma 4.1. *Let* $\{u^\varepsilon=u^\varepsilon(t,x)\}$ *be the sequence of solutions of problem* (1). *Then there exist a subsequence* (*still denoted by* $\{u^\varepsilon\}$) *and functions* $u_f(t,x)$, $u_m(t,x,y)$ *such that*

(a) $u_f\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))$, $u_m\in H^1(0,T;L^2(\Omega\times Y))\cap L^\infty(0,T;L^2(\Omega;H^1_{\mathcal{M}}(Y))$, *where* $H^1_{\mathcal{M}}(Y))=\{u\in H^1_\#(Y)\colon u=0 \text{ in } \mathcal{F}\}$;

(b) *for any* $t \in [0, T]$, u^ε *two-scale converges to* $u_f + u_m$;
(c) *for any* $\varphi \in L^\infty(0, T; L^2(\Omega; C_\#(Y)))$,

$$\int_0^T \mathrm{d}t \int_\Omega \partial_t u^\varepsilon \varphi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to \int_0^T \mathrm{d}t \int_{\Omega\times Y} \partial_t (u_f + u_m)\varphi(t, x, y)\, \mathrm{d}x\, \mathrm{d}y$$

(d) *for almost all* $t \in (0, T)$, *up to a subsequence*, $\sqrt{a^\varepsilon}\nabla u^\varepsilon$ *two-scale converges to*

$$\sqrt{a(x, y)}\,(\nabla_x u_f + \nabla_y v_f)\mathbf{I}_{\mathcal{F}}(y) + \sqrt{a(x, y)}\nabla_y u_m(t)\mathbf{I}_{\mathcal{M}}(y)$$

where $v_f \in L^2(\Omega; H^1_\#(\mathcal{F}) \setminus \mathbf{R})$.

The proof of the Lemma is based on the *a priori* information on the family $\{u^\varepsilon\}_{\varepsilon>0}$ provided by Theorem 2.1 along with the extension result [8], and makes use of arguments similar to those in [9]. In what follows $\{u^\varepsilon\}$ stands for a sequence satisfying the conditions of Lemma 4.1, if the contrary is not indicated.

It is well known, that the solution u^ε of problem (1) verifies, for almost all $t \in (0, T)$,

$$\begin{cases} -\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) + \Lambda_1 u^\varepsilon + g(u^\varepsilon) = H^\varepsilon & \text{in } \Omega \\ \nabla u^\varepsilon \cdot \vec{\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

where $H^\varepsilon = h^\varepsilon - \partial_t u^\varepsilon + \Lambda_1 u^\varepsilon$ is considered as a given function and $\Lambda_1 > \Lambda$. Thus, for any $\Delta_t \subset [0, T]$, u^ε minimizes the functional

$$I^{(\varepsilon)}[u] = \int_{\Delta_t} \mathrm{d}t \int_\Omega \left\{\frac{1}{2}a^\varepsilon|\nabla u|^2 + G(u) - H^\varepsilon u\right\} \mathrm{d}x, \quad \text{where } G(u) = \int_0^u g(\xi)\, \mathrm{d}\xi + \frac{\Lambda_1}{2}u^2 \tag{13}$$

over $u \in L^\infty(0, T; H^1(\Omega))$. In the following two steps we study the problem of minimization of (13) in the limit of small ε and obtain a limiting (homogenized) functional.

Step I (upper bound). Let $\phi_f \in L^\infty(0, T; C^1(\overline{\Omega}))$, $\phi_m \in L^\infty(0, T; C^1(\overline{\Omega}; H^1_{\mathcal{M}}(Y) \cap C^1_\#(Y)))$ and $\zeta \in L^\infty(0, T; C^1(\Omega; C^1_\#(Y)))$. We introduce the test function $w^\varepsilon = \phi_f(t, x) + \phi_m(t, x, \frac{x}{\varepsilon}) + \varepsilon\zeta(t, x, \frac{x}{\varepsilon})$. By using Lemma 4.1 we get that $\limsup_{\varepsilon\to 0} I^{(\varepsilon)}[u^\varepsilon] \leqslant \lim_{\varepsilon\to 0} I^{(\varepsilon)}[w^\varepsilon] = J[\phi_f, \phi_m, \zeta]$, where

$$J[\phi_f, \phi_m, \zeta] = \int_{\Delta_t} \mathrm{d}t \left\{ \int_{\Omega\times\mathcal{F}} \frac{1}{2}a(x, y)|\nabla_x\phi_f + \nabla_y\zeta|^2\, \mathrm{d}x\, \mathrm{d}y + \int_{\Omega\times\mathcal{M}} \frac{1}{2}a(x, y)|\nabla_y\phi_m|^2\, \mathrm{d}x\, \mathrm{d}y \right.$$
$$\left. + \int_{\Omega\times Y} \big(G(\phi_f + \phi_m) - (\phi_f + \phi_m)H\big)\, \mathrm{d}x\, \mathrm{d}y \right\} \tag{14}$$

with $H = h(x, y) - \partial_t(u_f + u_m) + \Lambda_1(u_f + u_m)$. Minimizing $J[\phi_f, \phi_m, \zeta]$ with respect to ζ yields

$$\limsup_{\varepsilon\to 0} I^{(\varepsilon)}[u^\varepsilon] \leqslant I_{hom}[\phi_f, \phi_m] \tag{15}$$

where

$$I_{hom}[\phi_f, \phi_m] = \int_{\Delta_t} \mathrm{d}t \left\{ \int_\Omega \frac{1}{2}\big(A^*(x)\nabla_x\phi_f \cdot \nabla_x\phi_f\big)\, \mathrm{d}x + \int_{\Omega\times\mathcal{M}} \frac{1}{2}a(x, y)|\nabla_y\phi_m|^2\, \mathrm{d}x\, \mathrm{d}y \right.$$
$$\left. + \int_{\Omega\times Y} \big(G(\phi_f + \phi_m) - (\phi_f + \phi_m)H\big)\, \mathrm{d}x\, \mathrm{d}y \right\} \tag{16}$$

It is clear that (15) remains true for any couple $(\phi_f, \phi_m) \in L^\infty(0, T; H^1(\Omega)) \times L^\infty(0, T; L^2(\Omega; H^1_{\mathcal{M}}(Y)))$.

Step II (lower bound). Lemma 4.1 and the lower semicontinuity property of convex functionals with respect to the two-scale convergence (see, e.g., [9]) imply that

$$\liminf_{\varepsilon\to 0}\int_{\Delta_t} \mathrm{d}t \int_{\Omega} \left\{\frac{1}{2}a^\varepsilon|\nabla u^\varepsilon|^2 + G(u^\varepsilon) - h^\varepsilon u^\varepsilon\right\}\mathrm{d}x \geqslant \int_{\Delta_t}\mathrm{d}t\left\{\int_{\Omega}\frac{1}{2}(A^*\nabla_x u_f\cdot\nabla_x u_f)\,\mathrm{d}x + \int_{\Omega\times\mathcal{M}}\frac{1}{2}a(x,y)|\nabla_y u_m|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y}\big(G(u_f+u_m)-(u_f+u_m)h\big)\,\mathrm{d}x\,\mathrm{d}y\right\} \tag{17}$$

Therefore, (15)–(17) yield

$$\limsup_{\varepsilon\to 0}\int_{\Delta_t}\mathrm{d}t\int_{\Omega}\big(\partial_t u^\varepsilon u^\varepsilon - \Lambda_1|u^\varepsilon|^2\big)\,\mathrm{d}x \leqslant \int_{\Delta_t}\mathrm{d}t\int_{\Omega\times Y} V(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{18}$$

where $V(t,x,y) = (u_f+u_m)\partial_t(u_f+u_m) - \Lambda_1(u_f+u_m)^2$. On the other hand we have

$$\lim_{\varepsilon\to 0}\int_0^T\mathrm{d}t\int_{\Omega}\mathrm{e}^{-2\Lambda_1 t}\big(u_t^\varepsilon u^\varepsilon - \Lambda_1|u^\varepsilon|^2\big)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{2}\big(\mathrm{e}^{-2\Lambda_1 T}\left\|u^\varepsilon(T)\right\|_{2,\Omega}^2 - \left\|U^\varepsilon\right\|_{2,\Omega}^2\big)$$
$$\geqslant \frac{1}{2}\big(\mathrm{e}^{-2\Lambda_1 T}\left\|u_f(T)+u_m(T)\right\|_{2,\Omega\times Y}^2 - \|U\|_{2,\Omega\times Y}^2\big) = \int_0^T\mathrm{d}t\int_{\Omega\times Y}\mathrm{e}^{-2\Lambda_1 t}V(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{19}$$

This implies that the strict inequality in (18) is impossible. Thus the couple (u_f, u_m) minimizes the functional I_{hom} in the class $L^\infty(0,T;H^1(\Omega))\times L^\infty(0,T;L^2(\Omega;H^1_{\mathcal{M}}(Y))$.

Now, in order to complete the proof of Theorem 3.1 we minimize the energy functional (16) with respect to (ϕ_f,ϕ_m). In a standard way (see, e.g., [10]), we obtain the homogenized problem (6). This problem is clearly well posed. Therefore the convergence holds for the whole family $\{u^\varepsilon\}_{\varepsilon>0}$.

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References

[1] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, New York, 1981.
[2] T. Arbogast, J. Douglas, U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Appl. Math. 21 (1990) 823–826.
[3] A. Bourgeat, A. Mikelic, A. Piatnitski, Modèle de double porosité aléatoire, C. R. Acad. Sci. Paris, Sér. I 327 (1998) 99–104.
[4] A. Bourgeat, M. Goncharenko, M. Panfilov, L. Pankratov, A general double porosity model, C. R. Acad. Sci. Paris, Sér. IIb 327 (1999) 1245–1250.
[5] Homogenization and Porous Media, in: U. Hornung (Ed.), Interdisciplinary Appl. Math., Vol. 6, Springer-Verlag, New York, 1997.
[6] L. Pankratov, A. Piatnitski, Nonlinear "double porosity" type model, C. R. Acad. Sci. Paris, Sér. I 334 (2002) 435–440.
[7] A.V. Babin, M.J. Vishik, Attractors of Evolutionary Equations, North-Holland, Amsterdam, 1992.
[8] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percival, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Anal. 18 (1992) 481–496.
[9] G. Allaire, Homogenization and two scale convergence, SIAM J. Math. Anal. 23 (6) (1992) 1482–1518.
[10] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1973.