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UNIFORM SPECTRAL ASYMPTOTICS FOR SINGULARLY PERTURBED LOCALLY PERIODIC OPERATORS

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ABSTRACT

We consider the homogenization of the spectral problem for a singularly perturbed diffusion equation in a periodic medium. Denoting by ε the period, the diffusion coefficients are scaled as ε^2 and vary both on the macroscopic scale and on the periodic microscopic scale. We make a structural hypothesis on the first cell eigenvalue, which is assumed to admit a unique minimum in the domain with non-degenerate quadratic behavior. We then prove an exponential localization phenomena at this minimum point. Namely, the k-th original eigenfunction is shown to be asymptotically given by the product of the first cell eigenfunction (at the ε scale) times the k-th eigenfunction of an homogenized problem (at the $\sqrt{\varepsilon}$ scale). The homogenized problem is a diffusion equation with quadratic potential in the whole space. We first perform asymptotic expansions, and then prove convergence by using a factorization strategy.

1. INTRODUCTION

We study the spectral asymptotics of a singularly perturbed second order elliptic operator with locally periodic rapidly oscillating coefficients of the form

$$\mathcal{A}^\varepsilon = -\varepsilon^2 \frac{\partial}{\partial x_i}\left(a^{ij}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right) + c\left(x, \frac{x}{\varepsilon}\right), \tag{1}$$

defined in a bounded open set G of $\mathbb{R}^n$. We assume that the coefficients $a^{ij}(x, z)$ and $c(x, z)$ are real sufficiently smooth (at least of class C^2) functions defined on $G \times \mathbb{T}^n$ where $\mathbb{T}^n$ is the unit torus. Equivalently, the coefficients can be seen as periodic functions with respect to z with period 1 in all the coordinate directions. Furthermore, the matrix $\{a^{ij}(x, z)\}$ is symmetric, uniformly positive definite. We consider the following eigenvalue problem

$$\mathcal{A}^\varepsilon p^\varepsilon = \lambda^\varepsilon p^\varepsilon \quad \text{in } G, \qquad p^\varepsilon = 0 \quad \text{on } \partial G. \tag{2}$$

As is well known, for each fixed $\varepsilon > 0$ this problem is self-adjoint in $L^2(G)$ and admits a discrete spectrum $\lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \ldots$, where $\lambda_k^\varepsilon \to \infty$ as $k \to \infty$, with corresponding eigenvector p_k^ε, normalized by $\|p_k^\varepsilon\|_{L^2(G)} = 1$. Moreover, by the Krein-Rutman theorem, λ_1^ε is of multiplicity one and the corresponding eigenfunction p_1^ε can be chosen positive in G.

The ground state asymptotics (i.e., characterizing the limit of the first eigenpair as ε goes to 0) plays an important role when studying the long time behaviour of solutions of the corresponding parabolic equation. Namely, the first eigenvalue governs the rate of decay (or growth) of solutions while the limit profile of the solutions can be determined in terms of the first eigenfunction. Other motivations for studying the limit of (2) are its link with semi-classical analysis of Schrödinger-type equations, or the uniform controllability of the wave equation (see e.g., (11)), or the modelling of the so-called criticality problem for the one-group neutron diffusion equation (which allows to compute the power distribution in a nuclear reactor core, see e.g., (2)).

The general study of the homogenization of (2) is far from being complete. When the coefficients are not rapidly oscillating, i.e., $a^{ij}(x, z) = a^{ij}(x)$ and $c(x, z) = c(x)$, it is a problem of singular perturbation (without homogenization) which is quite well understood now although the asymptotic behaviour of p_1^ε is rather complex. For instance, if $c(x)$ has a unique global minimum point $x_0 \in G$ then $p_1^\varepsilon(x)$ is exponentially small everywhere except at x_0, and the logarithmic asymptotics of p_1^ε is given by the following formula

$$\lim_{\varepsilon\to 0} \varepsilon \log p_1^\varepsilon(x) = -\mathrm{dist}_{(c(x_0)-c(x))b_{ij}(x)}(x, x_0),$$

where the distance is taken in the metric $[c(x_0) - c(x)]b_{ij}(x)$ and $\{b_{ij}\} = \{a^{ij}\}^{-1}$ (see (12) where more general nonself-adjoint operators have also been studied). A similar logarithmic asymptotics of the ground state for an operator with locally periodic coefficients of the type (1) was obtained in (13). The limit of the entire spectrum of (2) was studied in (4), but with no precise asymptotics of the eigenvectors.

When the coefficients are purely periodically oscillating functions, i.e., $a^{ij}(x,z) = a^{ij}(z)$ and $c(x,z) = c(z)$, problem (2) is also quite well understood, and more precise results are obtained. This problem, as well as similar ones for nonself-adjoint operators or systems with periodic coefficients, were studied in (2,6,9,10). These works rely on a factorization principle first introduced in the earlier works (14) and (17). In the case of the scalar self-adjoint problem (2), all these previous results boils down to the following theorem.

Theorem 1.1. *Assume that $a^{ij}(x,z) = a^{ij}(z)$ and $c(x,z) = c(z)$. The kth eigenpair $(\lambda_k^\varepsilon, p_k^\varepsilon)$ of (2) satisfies*

$$p_k^\varepsilon(x) = u_k^\varepsilon(x) p_1\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \lambda_k^\varepsilon = \lambda_1 + \varepsilon^2 \nu_k + o(\varepsilon^2),$$

where $(\lambda_1, p_1(z))$ is the first eigenpair of the cell eigenproblem (3) and, up to a subsequence, the sequence u_k^ε converges weakly in $H_0^1(G)$ to u_k such that (ν_k, u_k) is a kth eigenpair for the homogenized problem

$$-\frac{\partial}{\partial x_i}\left(a_{\mathrm{eff}}^{ij} \frac{\partial u}{\partial x_j}\right) = \nu u \quad \text{in } G, \qquad u = 0 \quad \text{on } \partial G.$$

The homogenized coefficients are given by formula (22).

The presence of both "slow" and "rapid" arguments in the coefficients drastically changes the asymptotic behavior of the eigenfunctions and eigenvalues of (2). In the present paper we formulate a simple sufficient condition (see hypothesis **H1** and **H2** in Section 2) for asymptotic localization of p_k^ε in a $\sqrt{\varepsilon}$-neighbourhood of an interior point of the domain, and then construct the leading terms of the asymptotics of p_1^ε in this neighbourhood. This allows to improve the logarithmic asymptotics mentioned above in the vicinity of the localization point, and to approximate p_1^ε in the metric of uniform convergence. Our main results are Theorems 4.1 and 5.3. Remark that they still hold true if the domain G were a compact manifold, or the whole space $\mathbb{R}^n$, under some localization condition for ground state in the latter setting.

The case of non self-adjoint operators is much more complicated, and its study is the focus of a next paper (5). The assumption of smooth coefficients is crucial since in the case of discontinuous coefficients completely different results are obtained in 1-D (3). Finally, the content of the paper is the following. In Section 2 we introduce notations and detail our main assumptions. Section 3 is devoted to formal asymptotic expansions, while Section 4 furnishes a rigorous proof of convergence. Lastly, Section 5 gives an error estimate. Throughout this paper we use the Einstein summation convention for repeated indices and C stands for a generic constant, independent of ε.

2. NOTATIONS AND ASSUMPTIONS

In order to formulate our conditions on the operator $\mathcal{A}^\varepsilon$ we introduce an auxiliary eigenvalue problem (cell eigenproblem) in the space of periodic functions (or equivalently on the unit torus $\mathbb{T}^n$) as follows

$$A(x)p \equiv -\frac{\partial}{\partial z_i}\left(a^{ij}(x,z)\frac{\partial p}{\partial z_j}\right) + c(x,z)p = \lambda p \quad \text{for } z \in \mathbb{T}^n. \tag{3}$$

In the sequel, for any $p \in H^1(\mathbb{T}^n)$, we use the notation

$$(A(x)p, p) = \int_{\mathbb{T}^n}\left(a^{ij}(x,z)\frac{\partial p}{\partial z_j}\frac{\partial p}{\partial z_i} + c(x,z)p^2\right)dz.$$

In (3) the variable $x \in G$ is just a parameter. Recall that the matrix $\{a^{ij}(x,z)\}$ is symmetric and uniformly coercive. As is well-known, $A(x)$ is a self-adjoint operator in $L^2(\mathbb{T}^n)$ which admits a discrete spectrum $\lambda_1(x) < \lambda_2(x) \leq \lambda_3(x) \leq \ldots$ with corresponding eigenfunctions $p_1(x,z)$, $p_2(x,z)$, $p_3(x,z), \ldots,$ normalized by $\|p_k(x,\cdot)\|_{L^2(\mathbb{T}^n)} = 1$. By the Krein-Rutman theorem, $\lambda_1(x)$ is of multiplicity one and $p_1(x,z)$ can be chosen positive in $\mathbb{T}^n$. Combined with the smoothness of the eigenfunctions, this implies that there exists a positive constant C such that $p_1(x,z) > C > 0$ uniformly in $z \in \mathbb{T}^n$ and $x \in \bar{G}$. Another consequence of the simplicity of $\lambda_1(x)$ is that the first eigenvalue and normalized eigenfunction have the same differentiability property as the coefficients with respect to x. Our main assumptions are

Hypothesis H1. The function $\lambda_1(x)$ has a unique global minimum point x_0 in the interior of G.

Hypothesis H2. The coefficients $a^{ij}(x,z)$ and $c(x,z)$ are of class C^2 in $\bar{G} \times \mathbb{T}^n$, and the Taylor series for $\lambda_1(x)$ about x_0 has non-degenerate (positive definite) quadratic form

$$\lambda_1(x) = \lambda_1(x_0) + D_{ij}(x-x_0)_i(x-x_0)_j + o(|x-x_0|^2), \quad D_{ij}\zeta_i\zeta_j \geq C|\zeta|^2 \tag{4}$$

for any vector $\zeta \in \mathbb{R}^n$, where D_{ij} stands for $(1/2)(\partial^2\lambda_1(x_0))/(\partial x_i \partial x_j)$ and $C > 0$.

Hypothesis H2′. The coefficients $a^{ij}(x,z)$ and $c(x,z)$ are of class C^3 in $\bar{G} \times \mathbb{T}^n$, and the Taylor series for $\lambda_1(x)$ about x_0 has non-degenerate (positive definite) quadratic form

$$\lambda_1(x) = \lambda_1(x_0) + D_{ij}(x-x_0)_i(x-x_0)_j + O(|x-x_0|^3),$$

with the same positive definite matrix $D = \{D_{ij}\}$ as in **H2**.
Without loss of generality we shall assume in the sequel that $x_0 = 0$.

Remark 2.1. Hypothesis **H1** ensures the concentration of p_1^ε in the neighbourhood of x_0 while Hypothesis **H2** allows to characterize, in the vicinity of x_0, the asymptotic behaviour of its profile.

Assumption **H2′** is a little stronger than **H2** and gives a more precise remainder term in the Taylor series (4). The proof of Theorem 5.3 requires C^3-smoothness of the coefficients, while the convergence results of Theorem 4.1 remain valid for C^2 coefficients.

3. FORMAL EXPANSION

In this section we construct the leading terms of a formal asymptotic expansion of $p_1^\varepsilon(x)$ in the vicinity of the point $x_0 = 0$. To this end we reduce the locally periodic problem (2) to a series of "purely periodic" problems, i.e., problems that do not depend on the slow variable x but merely on the fast periodic variable z.

First, using assumption **H2**$'$, we write down Taylor series in the x variable for the coefficients $a^{ij}(x,z)$ and $c(x,z)$ about 0; this gives

$$
\begin{aligned}
a^{ij}(x,z) &= a^{ij}(0,z) + x_k \frac{\partial}{\partial x_k} a^{ij}(0,z) + \frac{1}{2} x_k x_l \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l} a^{ij}(0,z) + O(|x|^3)\\
&\equiv a_0^{ij}(z) + x_k a_{1,k}^{ij}(z) + x_k x_l a_{2,kl}^{ij}(z) + O(|x|^3),\\
c(x,z) &= c(0,z) + x_k \frac{\partial}{\partial x_k} c(0,z) + \frac{1}{2} x_k x_l \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l} c(0, z + O(|x|^3))\\
&\equiv c_0(z) + x_k c_{1,k}(z) + x_k x_l c_{2,kl}(z) + O(|x|^3).
\end{aligned} \tag{5}
$$

Then we write the following ansatz for the first eigenfunction of (2)

$$
\begin{aligned}
p_1^\varepsilon &= \frac{q_1^\varepsilon}{\|q_1^\varepsilon\|_{L^2(G)}} + r_\varepsilon\\
q_1^\varepsilon &= \left[p_0\left(\frac{x}{\varepsilon}\right) + x_k p_{1,k}\left(\frac{x}{\varepsilon}\right) + x_k x_l p_{2,kl}\left(\frac{x}{\varepsilon}\right) + \varepsilon q_0\left(\frac{x}{\varepsilon}\right)\right]\exp\left(-\frac{Mx\cdot x}{2\varepsilon}\right),
\end{aligned} \tag{6}
$$

where r_ε is (hopefully) a small remainder, $p_0(z)$, $p_{1,k}(z)$, $p_{2,kl}(z)$, $q_0(z)$ are periodic functions and $M = \{M_{ij}\}$ is a positive definite matrix, that are to be determined. Remark that, by symmetry, we have $p_{2,kl} = p_{2,lk}$. The corresponding asymptotics for the first eigenvalue in (2) is

$$
\lambda_1^\varepsilon = \lambda_1(0) + \varepsilon\mu_1 + o(\varepsilon), \tag{7}
$$

where μ_1 has also to be determined. Since M is positive definite, an easy computation shows that, for any power $1 \le \alpha < +\infty$ and for any norm-exponent $1 \le m \le +\infty$, we have

$$
\frac{\left\| x^\alpha \exp\left(-\frac{Mx\cdot x}{2\varepsilon}\right)\right\|_{L^m(G)}}{\left\|\exp\left(-\frac{Mx\cdot x}{2\varepsilon}\right)\right\|_{L^m(G)}} = O(\varepsilon^{\alpha/2}). \tag{8}
$$

Remark that (8) holds true also in the case $m = +\infty$, which means that $x^\alpha \exp(-Mx\cdot x/2\varepsilon)$ is uniformly of order $\varepsilon^{\alpha/2}$ in G. Therefore, in the right hand side of (6), if the first term is normalized to be of order 1, the second term $x_k p_{1,k}(x/\varepsilon)\exp(-Mx\cdot x/2\varepsilon)$ is of order $\sqrt{\varepsilon}$, the third term $x_k x_l p_{2,kl}(x/\varepsilon)\exp(-Mx\cdot x/2\varepsilon)$ is of order ε, as well as the fourth one. In the sequel we neglect any other higher-order terms.

Now we substitute (5)–(7) in (2) and we find a cascade of equations according to the various powers of ε and of x. This gives

$$
0 = (\mathcal{A}^\varepsilon - \lambda_1^\varepsilon)p_1^\varepsilon = (\mathcal{A}^\varepsilon - (\lambda_1(0) + \varepsilon\mu_1))\frac{q_1^\varepsilon}{\|q_1^\varepsilon\|_{L^2(G)}} + \tilde{r}_\varepsilon
$$

where $\tilde{r}_\varepsilon = (\mathcal{A}^\varepsilon - \lambda_1^\varepsilon) r_\varepsilon + (\lambda_1(0) + \varepsilon\mu_1 - \lambda_1^\varepsilon) q_1^\varepsilon / \|q_1^\varepsilon\|_{L^2(G)}$ is hopefully small and

$$\begin{aligned}
&(\mathcal{A}^\varepsilon - \lambda_1(0) - \varepsilon\mu_1) q_1^\varepsilon \\
&\quad = \Bigg\{ -\varepsilon^2 \frac{\partial}{\partial x_i}\left(\left[a_0^{ij}\left(\frac{x}{\varepsilon}\right) + x_k a_{1,k}^{ij}\left(\frac{x}{\varepsilon}\right) + x_k x_l a_{2,kl}^{ij}\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial}{\partial x_i}\right) \\
&\qquad + \left(c_0\left(\frac{x}{\varepsilon}\right) + x_k c_{1,k}\left(\frac{x}{\varepsilon}\right) + x_k x_l c_{2,kl}\left(\frac{x}{\varepsilon}\right) - \lambda_1(0) - \varepsilon\mu_1\right)\Bigg\} \\
&\qquad \times \Bigg\{\left[p_0\left(\frac{x}{\varepsilon}\right) + x_k p_{1,k}\left(\frac{x}{\varepsilon}\right) + x_k x_l p_{2,kl}\left(\frac{x}{\varepsilon}\right)\right. \\
&\qquad \left. + \varepsilon q_0\left(\frac{x}{\varepsilon}\right)\right] \exp\left(-\frac{Mx\cdot x}{2\varepsilon}\right)\Bigg\} + r'_\varepsilon
\end{aligned}$$

where r'_ε stands for higher order terms which are small according to (8). For brevity we introduce the notation

$$\begin{aligned}
\mathbb{A}^0 &= -\frac{\partial}{\partial z_i}\left(a_0^{ij}(z)\frac{\partial}{\partial z_j}\right) + c_0(z) - \lambda_1(0) \\
\mathbb{A}_k^1 &= -\frac{\partial}{\partial z_i}\left(a_{1,k}^{ij}(z)\frac{\partial}{\partial z_j}\right) + c_{1,k}(z) \\
\mathbb{A}_{kl}^2 &= -\frac{\partial}{\partial z_i}\left(a_{2,kl}^{ij}(z)\frac{\partial}{\partial z_j}\right) + c_{2,kl}(z) \\
\mathbb{B}^{0,k} &= -a_0^{ki}(z)\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_i}\left(a_0^{ik}(z)\cdot\right) \\
\mathbb{B}_l^{1,k} &= -a_{1,l}^{ki}(z)\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_i}\left(a_{1,l}^{ik}(z)\cdot\right)
\end{aligned}$$

Differentiating all terms, including the exponential, and replacing x/ε by z, we get

$$\begin{aligned}
&(\mathcal{A}^\varepsilon - \lambda_1(0) - \varepsilon\mu_1) q_1^\varepsilon \\
&\quad = \Bigg\{\mathbb{A}^0 p_0(z) + x_k\left[\mathbb{A}^0 p_{1,k}(z) + \mathbb{A}_k^1 p_0(z) - M_{kl}\mathbb{B}^{0,l} p_0(z)\right] \\
&\qquad + x_k x_l\Big[\mathbb{A}^0 p_{2,kl}(z) + \mathbb{A}_k^1 p_{1,l}(z) + \mathbb{A}_{kl}^2 p_0(z) \\
&\qquad - M_{kj}\mathbb{B}^{0,j} p_{1,l}(z) - M_{kj}\mathbb{B}_l^{1,j} p_0(z) - M_{kj} a_0^{ij}(z) M_{il} p_0(z)\Big] \\
&\qquad + \varepsilon\Big[\mathbb{A}^0 q_0(z) + M_{ij} a_0^{ij} p_0(z) - a_{1,i}^{ij}\frac{\partial}{\partial z_j} p_0(z) \\
&\qquad + \mathbb{B}^{0,j} p_{1,j}(z) - \mu_1 p_0(z)\Big]\Bigg\}\Bigg|_{z=x/\varepsilon} \exp\left(-\frac{Mx\cdot x}{2\varepsilon}\right) + r''_\varepsilon
\end{aligned} \tag{9}$$

where r''_ε is another small remainder thanks to (8).

Equating to zero the corresponding expressions on the r.h.s. of (9), we derive the sequence of auxiliary problems which allow us to determine all the unknown elements in the above expansion. The equation for the leading term (or order 1) of the asymptotics reads

$$\mathbb{A}^0 p_0(z) = 0. \tag{10}$$

This equation is solvable in the space of periodic functions $L^2(\mathbb{T}^n)$ and has a unique (up to a multiplicative constant) solution $p_0(z) = p_1(0, z)$. Since the coefficients of the operator $\mathbb{A}^0$ are smooth, the solution p_0 belongs, at least, to $H^2(\mathbb{T}^n)$. For definiteness we impose the normalization condition

$$\int_{T^n} p_0^2(z)\, dz = 1.$$

At the next step, we collect all terms which are of order x and we obtain n equations

$$\mathbb{A}^0 p_{1,k}(z) = -\mathbb{A}_k^1 p_0(z) + M_{kl}\mathbb{B}^{0,l} p_0(z), \quad k = 1, 2, \ldots, n.$$

Due to the presence of the coefficients M_{kl} here, it is natural to represent $p_{1,k}(z)$ as the linear combination $\tilde{p}_{1,k}(z) + M_{kl}\tilde{\tilde{p}}_1^l(z)$, and to consider the following two equations separately

$$\mathbb{A}^0 \tilde{p}_{1,k}(z) = -\mathbb{A}_k^1 p_0(z), \tag{11}$$

and

$$\mathbb{A}^0 \tilde{\tilde{p}}_1^l(z) = \mathbb{B}^{0,l} p_0(z). \tag{12}$$

According to the Fredholm alternative, these equations admit solutions if and only if their right hand sides are orthogonal to the function p_0 that spans the kernel of $\mathbb{A}^0$ (orthogonality with respect to the usual scalar product in $L^2(\mathbb{T}^n)$). Eq. (12) is evidently solvable since $\mathbb{B}^{0,l}$ is a skew-symmetric operator. Indeed, it suffices to multiply the right hand side of this equation by $p_0(z)$ and integrate by parts. To show that the solvability condition is satisfied in (11), we use the fact that $x_0 = 0$ is a minimum point of $\lambda_1(x)$. Recalling the definition of $A(x)$, $p_0(z)$ and $p_1(x, z)$, we have

$$\begin{aligned}(A(x)p_0, p_0) &\geq (A(x)p_1(x, \cdot), p_1(x, \cdot)) = \lambda_1(x) \\ &\geq \lambda_1(0) = (A(0)p_1(0, \cdot), p_1(0, \cdot)) = (A(0)p_0, p_0),\end{aligned}$$

that is the function $(A(x)p_0, p_0)$ assumes its minimum at the point $x_0 = 0$. Taking the derivatives in x of the said function at $x_0 = 0$ gives

$$\int_{T^n} \Big(a^{ij}_{1,k} \frac{\partial}{\partial z_i} p_0(z) \frac{\partial}{\partial z_j} p_0(z) + c_{1,k} p_0^2(z) \Big)\, dz = (\mathbb{A}^1_k p_0, p_0)_{L^2(T^n)} = 0$$

for any $k = 1, 2, \ldots, n$; this implies the desired solvability condition.

The next equation involves all the quadratic in x terms of (9). It reads

$$\begin{aligned} &\mathbb{A}^0 p_{2,kl} + \mathbb{A}^1_k p_{1,l} + \mathbb{A}^2_{kl} p_0 - M_{kj} \mathbb{B}^{0,j} p_{1,l} - M_{kj} \mathbb{B}^{1,j}_l p_0 - M_{kj} a_0^{ij} M_{il} p_0 \\ &\quad = \mathbb{A}^0 p_{2,kl} + \mathbb{A}^1_k \tilde{p}_{1,l} + \mathbb{A}^1_k M_{lm} \tilde{\tilde{p}}{}^m_1 + \mathbb{A}^2_{kl} p_0 - M_{kj} \mathbb{B}^{0,j} \tilde{p}_{1,l} \\ &\quad\quad - M_{kj} \mathbb{B}^{0,j} M_{lm} \tilde{\tilde{p}}{}^m_1 - M_{kj} \mathbb{B}^{1,j}_l p_0 - M_{kj} a_0^{ij} M_{il} p_0 = 0, \\ &\quad\quad k, l = 1, 2, \ldots, n. \end{aligned} \tag{13}$$

In truth, Eq. (13) should be symmetrized with respect to k, l since $p_{2,kl}$ and $x_k x_l$ are symmetric. The solvability condition of this equation requires special considerations. There are two unknowns in the equation, namely the matrix-function $\{p_{2,kl}(z)\}$ and the constant matrix M_{ij}. Our goal is to choose M_{ij} so that the above equation has a solution $\{p_{2,kl}(z)\}$ in the space of periodic functions.

First of all let us show that the linear in M_{ij} terms do not make any difficulty. Indeed, by (11) and (12) we have

$$\tilde{\tilde{p}}{}^m_1(z) = \big(\mathbb{A}^0\big)^{-1} \mathbb{B}^{0,m} p_0(z) \quad \text{and} \quad \tilde{p}_{1,k}(z) = -\big(\mathbb{A}^0\big)^{-1} \mathbb{A}^1_k\, p_0(z).$$

Thus

$$\begin{aligned} &\int_{T^n} \Big(\mathbb{A}^1_k \tilde{\tilde{p}}{}^m_1(z) - \mathbb{B}^{0,m} \tilde{p}_{1,k}(z) \Big) p_0(z)\, dz \\ &\quad = \int_{T^n} \Big\{ \mathbb{A}^1_k \big(\mathbb{A}^0\big)^{-1} \mathbb{B}^{0,m} p_0(z) + \mathbb{B}^{0,m} \big(\mathbb{A}^0\big)^{-1} \mathbb{A}^1_k\, p_0(z) \Big\} p_0(z)\, dz = 0 \end{aligned}$$

since $\mathbb{A}^1_k$ and $(\mathbb{A}^0)^{-1}$ are symmetric operators while $\mathbb{B}^{0,m}$ is skew-symmetric.

Thus, the solvability condition in (13) is satisfied if and only if the following relation holds for all $k, l = 1, 2, \ldots, n$

$$\begin{aligned} \int_{T^n} \Big\{ &p_0(z) \mathbb{A}^2_{kl}\, p_0(z) + p_0(z) \mathbb{A}^1_k\, \tilde{p}_{1,l}(z) - p_0(z) M_{km} \mathbb{B}^{0,m} \tilde{\tilde{p}}{}^j_1(z) M_{jl} \\ &- p_0^2(z) M_{ki} a_0^{ij}(z) M_{jl} \Big\}\, dz = 0. \end{aligned} \tag{14}$$

Introducing a matrix $\mathcal{X}$ defined by its entries

$$\mathcal{X}_{ij} = \int_{T^n} \Big\{ p_0(z) \mathbb{B}^{0,i} \tilde{p}_1^{\,j}(z) + p_0^2(z) a_0^{ij}(z) \Big\}\, dz, \tag{15}$$

and a matrix $\mathcal{Y}$ defined by its entries

$$\mathcal{Y}_{kl} = \int_{T^n} \big(p_0(z) \mathbb{A}^2_{kl}\, p_0(z) + p_0 \mathbb{A}^1_k\, \tilde{p}_{1,l}(z) \big)\, dz, \tag{16}$$

Equation (14) is equivalent to

$$M \mathcal{X} M = \mathcal{Y}.$$

Let us check that this equation determines the matrix M. If $\mathcal{X}$ and $\mathcal{Y}$ are symmetric positive definite, it is a classical result that there exists a unique positive solution M given by

$$M = \mathcal{X}^{-1/2} \big(\mathcal{X}^{1/2} \mathcal{Y} \mathcal{X}^{1/2} \big)^{1/2} \mathcal{X}^{-1/2}.$$

We first prove the positive definiteness of the matrix $\mathcal{X}$.

Lemma 3.1. *The matrix $\mathcal{X}$ defined by* (15) *is symmetric positive definite. Furthermore, it coincides with the homogenized matrix for the periodic coefficients $p_0^2(z) a_0^{ij}(z)$.*

Proof. By virtue of (12) and of the skew-symmetric character of $\mathbb{B}^{0,i}$, the matrix $\mathcal{X}$ is equivalently given by

$$\mathcal{X}_{ij} = \int_{T^n} \Big\{ -(\mathbb{B}^{0,i} p_0(z)) \, (\mathbb{A}^0)^{-1} \mathbb{B}^{0,j} p_0(z) + p_0^2(z) a_0^{ij}(z) \Big\}\, dz,$$

which implies it is symmetric. Next for any smooth function φ, we have

$$p_0(z) \mathbb{A}^0 (p_0(z) \varphi(z)) = -\frac{\partial}{\partial z_i} \left(p_0^2(z) a_0^{ij}(z) \frac{\partial \varphi}{\partial z_j} \right). \tag{17}$$

The matrix $p_0^2(z) a_0^{ij}(z)$ is uniformly positive definite. Therefore, homogenization theory applies to the operator $\partial/\partial x_i (p_0^2(x/\varepsilon) a_0^{ij}(x/\varepsilon) \partial/\partial x_j)$ (see, for instance, (8)) which admits the following effective matrix

$$a^{ij}_{\mathrm{eff}} = \int_{T^n} p_0^2(z) a_0^{ik}(z) \left(\delta_{kj} + \frac{\partial}{\partial z_k} \chi^j(z) \right) dz$$

where δ_{kj} is the Kronecker symbol and $\chi^j(z)$ is the solution in $H^1(\mathbb{T}^n)$ of the following cell problem

$$-\frac{\partial}{\partial z_i}\left(p_0^2(z)a_0^{ij}(z)\frac{\partial}{\partial z_j}\chi^k(z)\right)=\frac{\partial}{\partial z_i}\left(p_0^2(z)a_0^{ik}(z)\right)$$

or, equivalently, by (17)

$$\begin{aligned}p_0\mathbb{A}^0(p_0\chi^k)&=-\frac{\partial}{\partial z_i}\left(p_0^2(z)a_0^{ik}(z)\right)\equiv-\left(p_0\frac{\partial}{\partial z_i}\left(p_0a_0^{ik}\right)+p_0a_0^{ik}\frac{\partial}{\partial z_i}p_0\right)\\&\equiv p_0\mathbb{B}^{0,k}p_0\end{aligned}$$

This yields a new expression for χ^k since the solution of this equation is

$$\chi^k=\frac{1}{p_0}(\mathbb{A}^0)^{-1}\mathbb{B}^{0,k}p_0. \tag{18}$$

Finally, considering the above relations, we derive

$$\begin{aligned}\mathcal{X}_{kl}&=\int_{T^n}\left(p_0^2a_0^{kl}+p_0\mathbb{B}^{0,k}\tilde{p}_1^l\right)dz=\int_{T^n}\left(p_0^2a_0^{kl}+p_0\mathbb{B}^{0,k}(\mathbb{A}^0)^{-1}\mathbb{B}^{0,l}p_0\right)dz\\&=\int_{T^n}\left(p_0^2a_0^{kl}+p_0\mathbb{B}^{0,k}(p_0\chi^l)\right)dz=\int_{T^n}\left(p_0^2a_0^{kl}-\chi^lp_0\mathbb{B}^{0,k}p_0\right)dz\\&=\int_{T^n}\left(p_0^2a_0^{kl}-\chi^l\frac{\partial}{\partial z_i}(p_0^2a_0^{ik})\right)dz=\int_{T^n}\left(p_0^2a_0^{kl}+p_0^2a_0^{ik}\frac{\partial}{\partial z_i}\chi^l\right)dz=a_{\text{eff}}^{kl},\end{aligned}$$

which is the desired result since the matrix a_{eff}^{kl} is known to be positive definite. □

Our next aim is to prove the positive definiteness of the matrix $\mathcal{Y}$.

Lemma 3.2. *Under Hypothesis* **H1** *the matrix* $\mathcal{Y}$ *is positive semidefinite. If, in addition, Hypothesis* **H2** *holds then* $\mathcal{Y}=D=(1/2)(\partial^2\lambda_1(0)/\partial x_i\partial x_j)$ *is positive definite.*

Proof. The three first terms of the Taylor series of $p_1(x,z)$ in the x variable around $x_0=0$ are

$$\begin{aligned}p_1(x,z)&=p_1(0,z)+x_k\frac{\partial}{\partial x_k}p_1(0,z)+\frac{1}{2}x_kx_l\frac{\partial^2}{\partial x_k\partial x_l}p_1(0,z)\\&\equiv p_0(z)+x_k\hat{p}_{1,k}(z)+x_kx_l\hat{p}_{2,kl}(z).\end{aligned}$$

Inserting this, (5) and (4) in (3) and collecting powers of x we obtain

$$\begin{aligned}&\mathbb{A}^0 p_0 + x_k(\mathbb{A}^0 \hat{p}_{1,k} + \mathbb{A}_k^1 p_0) + x_k x_l(\mathbb{A}^0 \hat{p}_{2,kl} + \mathbb{A}_k^1 \hat{p}_{1,l} + \mathbb{A}_{kl}^2 p_0)\\ &\quad = D_{kl} x_k x_l p_0 + O(x^3).\end{aligned}$$

Therefore,

$$\hat{p}_{1,k} = -(\mathbb{A}^0)^{-1}\mathbb{A}_k^1 p_0 = \tilde{p}_{1,k}$$

and

$$D_{kl} = \int_{T^n} p_0^2 D_{kl}\, dz = \int_{T^n} \left\{p_0 \mathbb{A}^0 \hat{p}_{2,kl} + p_0 \mathbb{A}_k^1 \hat{p}_{1,l} + p_0 \mathbb{A}_{kl}^2 p_0\right\} dz.$$

Integrating by parts and since $\mathbb{A}^0 p_0 = 0$, we get

$$D_{kl} = \int_{T^n} \left\{p_0 \mathbb{A}_k^1 \tilde{p}_{1,l} + p_0 \mathbb{A}_{kl}^2 p_0\right\} dz = \mathcal{Y}_{kl},$$

which is the desired result. □

Remark 3.3. As a byproduct of Lemma 3.2, we obtained that the derivative $\partial/\partial x_k p_1(0,z)$ is equal to $\tilde{p}_{1,k}$ and not to $p_{1,k}$.

The last equation related to the ansatz (9) collects all terms of the first order in ε. It reads

$$\mathbb{A}^0 q_0 = -p_0 M_{ij} a_0^{ij} - \mathbb{B}^{0,j} p_{1,j} + a_{1,i}^{ij} \frac{\partial}{\partial z_j} p_0 + \mu_1 p_0.$$

Writing down the solvability condition for this equation we find

$$\mu_1 = M_{ij} \int_{T^n} p_0^2 a_0^{ij}\, dz + \int_{T^n} \left(p_0 \mathbb{B}^{0,j} p_{1,j} - p_0 a_{1,i}^{ij} \frac{\partial}{\partial z_j} p_0\right) dz$$

This equation gives the value of the corrector μ_1 in the asymptotic expansion (7). Thus, we determined all the unknown elements in the asymptotic expansions (6) and (7). This shows that our ansatz is viable and one can safely hope to prove that it indeed holds true.

More precisely, collecting the above results and remarking that, by virtue of (8), the remainder term in (9) is actually small, the conclusion of this section is the following lemma.

Lemma 3.4. *The approximation* q_1^ε *of the first eigenfunction satisfies the estimate*

$$\left\| (\mathcal{A}^\varepsilon - (\lambda_1(0) + \varepsilon\mu_1)) \frac{q_1^\varepsilon}{\|q_1^\varepsilon\|} \right\|_{L^2(G)} \le c\varepsilon^{3/2}. \tag{19}$$

The proof of this bound is an immediate consequence of the fact that the neglected terms are proportional to x^3, εx or higher order terms. Remark that a similar result holds true in any $L^m(G)$-norm. In the sequel, it remains to prove that $q_1^\varepsilon / \|q_1^\varepsilon\|_{L^2(G)}$ is indeed close to the true first eigenfunction p_1^ε. In theory we could continue the ansatz and compute further correctors, but the algebra becomes soon formidable and anyway we are able only to prove the correctness of the first term of the ansatz of q_1^ε.

4. VARIATIONAL PROOF OF THE CONVERGENCE

In this section we develop the analysis of the bottom spectrum of eigenproblem (1), which relies on a factorization principle in the neighbourhood of the concentration point of the ground state, and on homogenization technique. In particular, this allows to justify the first two terms of the asymptotics of the leading eigenvalues in (1) and to obtain a lower bound for the spectral gap.

We first introduce the following homogenized problem in $\mathbb{R}^n$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(a_{\text{eff}}^{ij} \dfrac{\partial u}{\partial y_j} \right) + \left(c_{\text{eff}} + D_{ij} y_i y_j \right) u = \mu u \quad \text{in } \mathbb{R}^n, \\ u \in L^2(\mathbb{R}^n), \end{cases} \tag{20}$$

where $D = \{D_{ij}\}$ is the Hessian matrix $(1/2)\nabla_x \nabla_x \lambda_1(0)$. The homogenized coefficients are given by

$$c_{\text{eff}} = -\int_{T^n} p_1(0,z) \left(\frac{\partial a^{ij}}{\partial x_i} \frac{\partial p_1}{\partial z_j} + a^{ij} \frac{\partial^2 p_1}{\partial z_j \partial x_i} + \frac{\partial}{\partial z_i}\left(a^{ij} \frac{\partial p_1}{\partial x_j} \right) \right)(0,z)\, dz \tag{21}$$

and

$$a_{\text{eff}}^{ij} = \int_{T^n} p_1^2(0,z) \left(a^{ij}(0,z) + a^{ki}(0,z) \frac{\partial \chi^j}{\partial z_k}(z) \right) dz \tag{22}$$

where the functions $(\chi^k)_{1\leq k\leq n}$ are the solutions in $H^1(\mathbb{T}^n)$ of

$$-\frac{\partial}{\partial z_i}\left(p_1^2(0,z)a^{ij}(0,z)\frac{\partial \chi^k}{\partial z_j}(z)\right)=\frac{\partial}{\partial z_i}\Big(p_1^2(0,z)a^{ik}(0,z)\Big) \tag{23}$$

The homogenized eigenvalue problem is well-posed in $H^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n;|y|^2)$ and defines a self-adjoint compact operator in $L^2(\mathbb{R}^n)$.

Theorem 4.1. *Let $p_1(x,z)$ and $\lambda_1(x)$ be the first eigenvector and eigenvalue of the cell problem (3) normalized by $\|p_1(x,\cdot)\|_{L^2(\mathbb{T}^n)}=1$. Assume that assumptions* **H1** *and* **H2** *hold true. For $k\geq 1$, let λ_k^ε and p_k^ε be the* kth *eigenvalue and normalized eigenvector of (1). Then,*

$$p_k^\varepsilon(x)=u_k^\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right)p_1\left(x,\frac{x}{\varepsilon}\right),\qquad \lambda_k^\varepsilon=\lambda_1(0)+\varepsilon\mu_k+o(\varepsilon), \tag{24}$$

where, up to a subsequence, the sequence $u_k^\varepsilon(y)/\|u_k^\varepsilon\|_{L^2(\mathbb{R}^n)}$ converges weakly in $H^1(\mathbb{R}^n)$, and strongly in $L^2(\mathbb{R}^n)$, to $u_k(y)$, and (μ_k,u_k) is the kth *eigenvalue and eigenvector of the homogenized problem (20).*

Remark 4.2. In order to see the connection between Theorem 4.1 and the results of the formal asymptotic expansion, we can rewrite the homogenized coefficients with the notation of Section 3. Recall first that

$$p_1(0,z)\equiv p_0(z),\qquad \frac{\partial p_1}{\partial x_j}(0,z)\equiv \tilde{p}_{1,j}(z),$$

$$a^{ij}(0,z)\equiv a_0^{ij}(z),\quad\text{and}\quad \frac{\partial a^{ij}}{\partial x_i}(0,z)\equiv a_{1,i}^{ij}(z).$$

Thus, we obtain $a_{\text{eff}}^{ij}=\mathcal{X}_{ij}$ and

$$c_{\text{eff}}=\int_{T^n}\left(p_0\mathbb{B}^{0,j}\tilde{p}_{1,j}-p_0a_{1,i}^{ij}\frac{\partial p_0}{\partial z_j}\right)dz.$$

The eigenvalues and eigenfunctions of the homogenized problem (20) can be computed explicitely (see e.g., (15)). Therefore, we recover the result of the formal asymptotic expansion. In particular, the first eigenpair of (20) is

$$\mu_1=c_{\text{eff}}+\operatorname{tr}(M\mathcal{X}),\quad\text{and}\quad u_1(y)=\exp\left(-\frac{My\cdot y}{2}\right),$$

with $M=\mathcal{X}^{-1/2}(\mathcal{X}^{1/2}\mathcal{Y}\mathcal{X}^{1/2})^{1/2}\mathcal{X}^{-1/2}$.

Remark 4.3. Since the eigenfunction p_k^ε is normalized in $L^2(G)$, the norm $\|u_k^\varepsilon\|_{L^2(\mathbb{R}^n)}$ is of the order of $\varepsilon^{-n/4}$. Furthermore, the exponential character of $u_k(y)$ implies that the factorization (24) can be localized around the origin. In other words, as a corollary of Theorem 4.1 we have

$$\lim_{\varepsilon\to 0}\left\| p_k^\varepsilon - \varepsilon^{-n/4} p_1\Big(0, \frac{x}{\varepsilon}\Big) u_k\Big(\frac{x}{\sqrt{\varepsilon}}\Big)\right\|_{L^2(G)} = 0.$$

Proof. Let $(\lambda^\varepsilon, p^\varepsilon)$ be an eigenpair of

$$\begin{cases} -\varepsilon^2 \dfrac{\partial}{\partial x_i}\Big(a^{ij}\Big(x, \dfrac{x}{\varepsilon}\Big)\dfrac{\partial p^\varepsilon}{\partial x_j}\Big) + c\Big(x, \dfrac{x}{\varepsilon}\Big)p^\varepsilon = \lambda^\varepsilon p^\varepsilon & \text{in } G, \\ p^\varepsilon = 0 & \text{on } \partial G. \end{cases} \tag{25}$$

We perform the following change of unknown

$$v^\varepsilon(x) = \frac{p^\varepsilon(x)}{p_1(x, x/\varepsilon)}, \tag{26}$$

which, according to Proposition 3.6 in (2), defines an invertible and bicontinuous change of variables in $H_0^1(G)$. We replace p^ε by v^ε in (25), and we recall that $p_1(x,z)$ is the first eigenfunction of (3). After a little algebra and using the following identity (identical to (17))

$$p_1 \frac{\partial}{\partial x_i}\Big(a^{ij}\frac{\partial(p_1 v^\varepsilon)}{\partial x_j}\Big) = \frac{\partial}{\partial x_i}\Big(p_1^2 a^{ij}\frac{\partial v^\varepsilon}{\partial x_j}\Big) + p_1 v^\varepsilon \frac{\partial}{\partial x_i}\Big(a^{ij}\frac{\partial p_1}{\partial x_j}\Big),$$

we obtain that (25) is equivalent to

$$\begin{cases} -\varepsilon \dfrac{\partial}{\partial x_i}\Big(p_1^2 a^{ij}\dfrac{\partial v^\varepsilon}{\partial x_j}\Big) + \Big(\Sigma^\varepsilon(x) + \dfrac{\lambda_1(x) - \lambda(0)}{\varepsilon} p_1^2\Big) v^\varepsilon = \mu^\varepsilon p_1^2 v^\varepsilon & \text{in } G, \\ v^\varepsilon = 0 & \text{on } \partial G, \end{cases} \tag{27}$$

where the coefficients p_1^2 and a^{ij} are evaluated at $(x, x/\varepsilon)$, with $\mu^\varepsilon = \varepsilon^{-1}(\lambda^\varepsilon - \lambda_1(0))$ and

$$\Sigma^\varepsilon(x) = -\Big\{p_1\Big[\frac{\partial}{\partial z_i}\Big(a^{ij}\frac{\partial p_1}{\partial x_j}\Big) + \frac{\partial}{\partial x_i}\Big(a^{ij}\frac{\partial p_1}{\partial z_j}\Big) + \varepsilon\frac{\partial}{\partial x_i}\Big(a^{ij}\frac{\partial p_1}{\partial x_j}\Big)\Big]\Big\}\Big(x, \frac{x}{\varepsilon}\Big).$$

In order to eliminate the ε scaling in front of the second-order operator in (27), we rescale the space variable by introducing

$$y = \frac{x}{\sqrt{\varepsilon}} \in G^\varepsilon = \varepsilon^{-1/2} G \quad \text{and} \quad u^\varepsilon(y) = v^\varepsilon(x).$$

This yields

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(\tilde{a}^{ij}_\varepsilon \dfrac{\partial u^\varepsilon}{\partial y_j}\right) + \left(\tilde{\Sigma}^\varepsilon(y) + \dfrac{\lambda_1(\sqrt{\varepsilon}y) - \lambda(0)}{\varepsilon}\tilde{p}^2_{1,\varepsilon}\right)u^\varepsilon = \mu^\varepsilon \tilde{p}^2_{1,\varepsilon} u^\varepsilon \quad \text{in } G^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial G^\varepsilon, \end{cases} \tag{28}$$

with

$$\tilde{a}^{ij}_\varepsilon(y) = \{p_1^2 a^{ij}\}(\sqrt{\varepsilon}y, y/\sqrt{\varepsilon}), \quad \tilde{p}^2_{1,\varepsilon}(y) = p_1^2(\sqrt{\varepsilon}y, y/\sqrt{\varepsilon}),$$
$$\tilde{\Sigma}^\varepsilon(y) = \Sigma^\varepsilon(\sqrt{\varepsilon}y),$$

and

$$\frac{\lambda_1(\sqrt{\varepsilon}y) - \lambda(0)}{\varepsilon} = \frac{1}{2}\nabla_x\nabla_x\lambda_1(0)y \cdot y + o(1).$$

Equation (28) is a combined problem of homogenization and singular perturbations: the coefficients are oscillating with a period $\sqrt{\varepsilon}$, and they concentrate to 0 with respect to their first macroscopic argument. Remark also that the domain G^ε is converging to $\mathbb{R}^n$. Therefore, we expect that the limit problem of (28) is precisely the homogenized problem (20). To prove this statement and study the spectral asymptotics of (28), we follow the methodology of (2,4). We introduce the corresponding Green operator

$$\begin{aligned} S^\varepsilon : L^2(G^\varepsilon) &\to L^2(G^\varepsilon) \\ f &\to U^\varepsilon \end{aligned} \tag{29}$$

where U^ε is the unique solution in $H^1_0(G^\varepsilon)$ of

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(\tilde{a}^{ij}_\varepsilon \dfrac{\partial U^\varepsilon}{\partial y_j}\right) + \left(\tilde{\Sigma}^\varepsilon(y) + \dfrac{\lambda_1(\sqrt{\varepsilon}y) - \lambda(0)}{\varepsilon}\tilde{p}^2_{1,\varepsilon}\right)U^\varepsilon = \tilde{p}^2_{1,\varepsilon} f \quad \text{in } G^\varepsilon, \\ U^\varepsilon = 0 \quad \text{on } \partial G^\varepsilon. \end{cases} \tag{30}$$

Remark that, under the assumed smoothness of the coefficients, the function $\tilde{\Sigma}^\varepsilon(y)$ is uniformly bounded in $\mathbb{R}^n$. Thus, adding to it $C\tilde{p}^2_{1,\varepsilon}(y)$, with C positive and sufficiently large, will make it positive too and will have the effect of simply shifting the entire spectrum by this constant C.

Therefore, we shall assume without loss of generality that $\tilde{\Sigma}^{\varepsilon}(y)$ is positive. In the sequel we shall consider that S^{ε} is an operator defined in $L^2(\mathbb{R}^n)$ by simply taking f as the restriction to G^{ε} of a function of $L^2(\mathbb{R}^n)$ and extending by zero outside G^{ε} the solution $U^{\varepsilon} = S^{\varepsilon}f$. The homogenization of (29) is quite standard. We introduce the limit Green operator

$$\begin{aligned} S\colon L^2(\mathbb{R}^n) &\rightarrow L^2(\mathbb{R}^n) \\ f &\rightarrow U \text{ unique solution in } \mathrm{H}^1(\mathbb{R}^n) \text{ of} \\ &-\frac{\partial}{\partial y_i}\left(a_{\mathrm{eff}}^{ij}\frac{\partial U}{\partial y_j}\right) + (c_{\mathrm{eff}} + Dy\cdot y)U = f \quad \text{in } \mathbb{R}^n, \end{aligned}$$

which is a compact operator (see e.g., (15)) whose spectrum can be explicitly computed. Then, we obtain the following convergence result which completes the proof.

Lemma 4.4. *The sequence of operators S^{ε} compactly converges to the limit operator S in the sense that* (*see e.g.*, (7))

(*i*) *for any $f \in L^2(\mathbb{R}^n)$, $\lim_{\varepsilon\to 0} \|S^{\varepsilon}(f) - S(f)\|_{L^2(\mathbb{R}^n)} = 0$,*
(*ii*) *the set $\{S^{\varepsilon}(f) : \|f\|_{L^2(\mathbb{R}^n)} \le 1,\ \varepsilon \ge 0\}$ is sequentially compact.*

Proof. The proof is quite classical (see e.g., (2,4) for similar examples), so we simply indicate the main ingredients. First, we multiply (30) by U_{ε} and integrate by parts to obtain a priori estimates. Since by assumptions **H1** and **H2** there exists a positive constant $C > 0$ such that

$$\frac{\lambda_1(\sqrt{\varepsilon}y) - \lambda(0)}{\varepsilon} \ge C|y|^2,$$

we get

$$\|\nabla U^{\varepsilon}\|_{L^2(\mathbb{R}^n)} + \|yU^{\varepsilon}(y)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{32}$$

This implies that the sequence U^{ε} is not only pre-compact in $H^1(\mathbb{R}^n)$-weak but also pre-compact in $L^2(\mathbb{R}^n)$-strong. Second, we pass to the limit in (30) by using the two-scale convergence (1). We multiply (30) by a test function $\varphi(y) + \varepsilon\varphi_1(y, y/\sqrt{\varepsilon})$ where φ, φ_1 are smooth functions with compact support with respect to the first variable y and periodic with respect to the second variable $z = y/\sqrt{\varepsilon}$. Since this test function has compact support (fixed with respect to ε), the effect of the non-periodic modulation in the coefficients is negligible. Indeed, on any fixed bounded domain, the values of the coefficients, depending on $(\sqrt{\varepsilon}y, y/\sqrt{\varepsilon})$ are uniformly close to their values at $(0, y/\sqrt{\varepsilon})$. Now, this is a standard matter in the theory of two-scale convergence to deduce that any converging subsequence of U_ε converges weakly in $H^1(\mathbb{R}^n)$ to U which is the unique solution of (31). The homogenized coefficients in (31) are thus obtained by considering the cell problems with the frozen macroscopic variable $x = 0$ (remark that the weak limit of $\tilde{p}^2_{1,\varepsilon}(y)$ is precisely $\int_{\mathbb{T}^n} p_1^2(0,z)\,dz$ which is equal to 1 by our normalization condition). By uniqueness of the limit, the entire sequence U_ε converges. Furthermore, estimate (32) shows that U_ε does also converge strongly in $L^2(\mathbb{R}^n)$. This proves statement (*i*) of the lemma. To prove statement (*ii*) we simply remark that estimate (32) as well as the strong $L^2(\mathbb{R}^n)$ convergence of U_ε is still valid if the right hand side f is replaced by a bounded sequence f_ε in $L^2(\mathbb{R}^n)$. This shows that S^ε compactly converges to S. □

To finish the proof of Theorem 4.1, it remains to check that the operator convergence furnished by Lemma 4.4 yields the desired convergence of the spectrum, as stated in Theorem 4.1. This is indeed true by a classical result on the operator compact convergence (see (7)) that we recall.

Lemma 4.5. (7) *If a sequence of compact self-adjoint operators S^ε compactly converges to a limit compact self-adjoint operator S in $L^2(\mathbb{R}^n)$, then the spectrum of S^ε converges to that of S in the sense that the* kth *eigenvalue of S^ε converges to the* kth *one of S and, up to a subsequence, the* kth *normalized eigenvector of S_ε converges strongly in $L^2(\mathbb{R}^n)$ to a* kth *eigenvector of S.*

Remark 4.6. Lemma 4.5 would be obvious if the sequence S^ε were to converge uniformly to S. However, this is not the case because the right hand side coefficient $\tilde{p}^2_{1,\varepsilon}(y)$ converges merely weakly to its limit value $\int_{\mathbb{T}^n} p_1^2(0,z)dz = 1$. Lemma 4.5 extends to the case of non self-adjoint operators.

Corollary 4.7. *In the statement of Theorem* 4.1 *the whole sequence $u_1^\varepsilon(x/\sqrt{\varepsilon})$ associated to the ground state $p_1^\varepsilon(x)$, does converge, as $\varepsilon \to 0$. Thus, the asymptotics of the ground state is uniquely defined.*

Proof. This is an immediate consequence of the fact that the principal eigenvalue of the homogenized problem (20) is simple. □

5. ERROR ESTIMATE FOR THE GROUND STATE ASYMPTOTICS

In this section we show that, under hypotheses **H1–H2**$'$, the remainders in (6) and (7) admit qualified upper bounds. To this end we combine the formal asymptotics built above with the estimates proved in the preceding section.

The statement below is an obvious consequence of Theorem 4.1.

Lemma 5.1. *Under hypotheses* **H1** *and* **H2** *there exists a positive constant* $C > 0$, *independent of* ε, *such that*

$$\lambda_1(0) - C\varepsilon \le \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_1(0) + C\varepsilon, \tag{33}$$

and

$$\lambda_2^\varepsilon - \lambda_1^\varepsilon \ge C\varepsilon. \tag{34}$$

Remark 5.2. We derive the statement of Lemma 5.1 as a consequence of the homogenization results of Theorem 4.1. Another, direct way to prove this statement would be to use the min–max principle and a properly chosen ansatz of the form

$$\Big(q_0\Big(\frac{x}{\varepsilon}\Big) + x_i q_{1,i}\Big(\frac{x}{\varepsilon}\Big) + x_i x_j q_{2,ij}\Big(\frac{x}{\varepsilon}\Big)\Big)\exp\Big(-\frac{|x|^2}{\varepsilon}\Big).$$

Combining the bounds of Lemmas 5.1 and 3.4 with (24), we obtain the main estimates of this work. Let p_1^ε be the leading normalized eigenfunction of problem (2) and λ_1^ε the corresponding eigenvalue.

Theorem 5.3. *Under Hypotheses* **H1** *and* **H2**$'$ *there exists a positive constant* $C > 0$, *independent of* ε, *such that*

$$|\lambda_1^\varepsilon - \lambda_1(0) - \varepsilon\mu_1| \le C\varepsilon^{3/2}$$
$$\Big\| p_1^\varepsilon - \frac{q_1^\varepsilon}{\|q_1^\varepsilon\|}\Big\|_{L^2(G)} \le C\varepsilon^{1/2}.$$

Proof. We write down the Fourier series of the function $(q_1^\varepsilon/\|q_1^\varepsilon\|)$ with respect to the eigenbasis $\{p_i^\varepsilon\}_{i=1}^\infty$

$$\frac{q_1^\varepsilon}{\|q_1^\varepsilon\|_{L^2(G)}} = \sum_{i=1}^\infty \alpha_i p_i^\varepsilon, \qquad \sum_{i=1}^\infty \alpha_i^2 = 1.$$

Substituting this series in (19) we get

$$\Big\|(\mathcal{A}^\varepsilon - (\lambda_1(0) + \varepsilon\mu_1))\frac{q_1^\varepsilon}{\|q_1^\varepsilon\|}\Big\|^2_{L^2(G)}$$
$$= \alpha_1^2(\lambda_1^\varepsilon - \lambda_1(0) - \varepsilon\mu_1)^2 + \sum_{i=2}^\infty \alpha_i^2(\lambda_i^\varepsilon - \lambda_1(0) - \varepsilon\mu_1)^2 \le C\varepsilon^3.$$

By Theorem 4.1 or Lemma 5.1, we have for all $i \geq 2$

$$(\lambda_i^\varepsilon - \lambda_1(0) - \varepsilon\mu_1)^2 \geq C\varepsilon^2.$$

Therefore,

$$\sum_{i=2}^{\infty} \alpha_i^2 \leq C\varepsilon$$

and the second inequality of Theorem 5.3 follows. To justify the first one it suffices to note that α_1 tends to 1 as ε goes to 0.

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