Einstein Relation for Reversible Diffusions in a Random Environment

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Abstract

We consider reversible diffusions in a random environment and prove the Einstein relation for this model. It says that the derivative at 0 of the effective velocity under an additional local drift equals the diffusivity of the model without drift. The Einstein relation is conjectured to hold for a variety of models but so far it has only been proved in particular cases. Our proof makes use of homogenization arguments, the Girsanov transform, and a refinement of the regeneration times introduced by Shen. © 2011 Wiley Periodicals, Inc.

1 Introduction

The present paper deals with diffusions in a random stationary environment, a model for the dynamics of particles in a disordered medium at thermal equilibrium. This subject has been the object of intense research over the past 30 years. In spite of much recent progress (see [27], for instance), many questions regarding the long-time behavior of these processes, such as laws of large numbers or central limit theorems, are still challenging open problems.

We shall only be concerned here with reversible dynamics. In this context the idea of the environment seen from the particle, as discussed in [4, 14], provides a powerful tool to adapt the "corrector approach" from homogenization theory and eventually prove invariance principles. One then shows that the trajectory of a particle evolving in such an environment, in a large-time scale, behaves like a Brownian motion with mean square displacement proportional to time, the proportionality being expressed by the asymptotic covariance or *effective diffusivity* matrix Σ. A good understanding of Σ is thus of primary interest.

From reversibility follows a variational formula for the effective diffusivity (see [12, 18]) for the discrete and continuous cases, respectively. Many works in PDEs

and theoretical physics address the question of estimating the effective diffusivity; see [7, 8], for instance. Here we provide a completely different interpretation of Σ as the so-called *mobility*.

In the series of papers A. Einstein devoted to Brownian motion at the beginning of the past century [5], along with the diffusivity matrix, the great physicist introduced another important quantity called the *mobility*. The mobility measures the response of the diffusing particle to a constant exterior force. Let us consider the perturbed process obtained by imposing a constant drift of strength λ in some fixed direction. One would expect the perturbed process to satisfy a law of large numbers with effective drift $\ell(\lambda)$. The mobility can then be interpreted as the derivative of $\ell(\lambda)$ as λ tends to 0. Einstein claimed that the mobility and the diffusivity of a particle coincide.

This "diffusivity vs. mobility" relation played a central role in Einstein's theory of molecular diffusivity because it was amenable to experimental verification and eventually led to evidence that matter is molecular. Since then, Einstein's relation opened the way to important developments both in experimental physics, with J. Perrin's Nobel prize, and theoretical physics, with connections to the fluctuation dissipation theorem or the Green-Kubo relation. Authors in these fields usually assume the Einstein relation, on the basis of heuristics, without actually being able to prove it; see [8], for instance. Indeed, only very few rigorous papers investigate the mathematical contents of the Einstein relation, and finding a general strategy for proving it remained an open mathematical problem for years. Note that the mere existence of the mobility—i.e., the fact that $\ell(\lambda)$ has a derivative at $\lambda = 0$—is far from obvious.

In the case of periodic environments the Einstein relation can be easily checked by direct computations that are mostly reduced to differentiating the perturbed cell problem on the torus. For time-dependent environments with good mixing properties satisfying a spectral gap assumption, the Einstein relation follows by perturbation theory arguments; see [16].

The first consistent mathematical approach to the Einstein relation for random environments was suggested by J. Lebowitz and H. Rost in [19]. These authors then avoided the difficulty of proving the existence of the mobility by considering exterior forces whose strength vanishes as time goes to infinity in such a way that the perturbed process still has a limit on the diffusive scale. The mobility is then defined as the asymptotic mean displacement of the particle; no derivative is involved, and this weak form of the Einstein relation does not tell us anything about the effective drift induced by a constant exterior force. On the other hand, in this weaker form the Einstein relation holds in the general ergodic reversible case. We shall discuss the Lebowitz-Rost theory in more detail later; see the end of Section 2.4.

More recently, in [15], T. Komorowski and S. Olla proved the Einstein relation for random walks with random conductances in dimensions higher than 3 and with

the extra restriction that conductances are only allowed to take two different positive values. Their approach, which is an adaptation of [21] to the random-walks-in-a-random-environment (RWRE) case, is quite different from ours, and it is not clear how it can be adapted to more general models, as for instance the elliptic random conductances model or random walks on percolation clusters; see [2, 26].

In the present paper, we shall prove the Einstein relation for symmetric diffusions driven by a random environment with bounded potential and short-range correlations. A more detailed definition of the model and precise assumptions are given in Section 2. As discussed above, the invariance principle, and therefore the existence of the diffusivity, was already known for such models in the 1980s. The law of large numbers for the perturbed dynamics, i.e., the existence of $\ell(\lambda)$, is ensured by a more general theorem of L. Shen in [25] on renewal properties of drifted diffusions in random environments. The key issue in proving the Einstein relation is then to show that $\ell(\lambda)$ has a derivative at $\lambda = 0$ and to identify this derivative with the effective diffusivity of the nonperturbed diffusion. To this end we compute the mean position of the perturbed diffusion on the critical scale $\lambda^2 t = 1$. This is indeed the scale on which the diffusive and drift components of the perturbed dynamics equilibrate. Thus we obtain two different expressions for the mean displacement in terms of either the diffusivity matrix Σ (see equation (3.1)) or in terms of $\ell(\lambda)$; see (5.1). Identifying both expressions gives the Einstein relation.

The proofs combine different ingredients: homogenization arguments and Girsanov transforms (see Section 3), PDE estimates and a priori bounds on hitting times for perturbed diffusions (see Section 4), and renewal arguments (see Section 5). All these ingredients had already appeared in the literature but, in order to treat the critical scale $\lambda^2 t = 1$, we had to refine many arguments and often introduce alternative strategies as, for instance, with the regeneration times in Section 5.

In Sections 2 through 6, we focus only on smooth environments, an approach that allows us to use stochastic differential equations. In Section 6.1, we relax this smoothness assumption and treat the case of (still bounded) but only measurable environments, relying on Dirichlet form theory.

One might hope that our approach could be adapted to apply to other models of diffusions and random walks in random environments.

2 Model and Statement of the Theorem

2.1 Diffusions in a Random Environment

We shall be dealing with diffusion processes in $\mathbb{R}^d$ whose generators are of the form

$$(2.1) \qquad \mathcal{L}^\omega f(x) = \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x),$$

where a^ω and V^ω are realizations of a random environment with a finite range of dependence.

More precisely, our assumptions are as follows:

Let $(\Omega, \mathcal{A}, \mathbb{Q})$ be a probability space equipped with a group action of $\mathbb{R}^d$ that we denote with $(x, \omega) \to x.\omega$. We also assume that the map $(x, \omega) \to x.\omega$ is $(\mathcal{B}_d \times \mathcal{A}, \mathcal{A})$–measurable, where $\mathcal{B}_d$ is the Borel σ-field on $\mathbb{R}^d$.

Assumption 1. The action $(x, \omega) \to x.\omega$ preserves the measure $\mathbb{Q}$ and is ergodic.

Let V be a measurable real-valued function on Ω, and let σ be a measurable function on Ω taking its values in the set of real $d \times d$ symmetric matrices. Define

$$V^\omega(x) = V(x.\omega), \qquad \sigma^\omega(x) = \sigma(x.\omega).$$

We also introduce the notation

$$a^\omega = (\sigma^\omega)^2 \quad \text{and} \quad b^\omega = \frac{1}{2} \operatorname{div} a^\omega - a^\omega \nabla V^\omega.$$

Observe that both σ^ω and b^ω are then stationary fields; i.e., $\sigma^\omega(x) = \sigma(x.\omega)$ and $b^\omega(x) = b(x.\omega)$ for some functions σ and b.

Assumption 2. For any environment ω, the functions $x \to V^\omega(x)$ and $x \to \sigma^\omega(x)$ are smooth. To avoid triviality, we also assume that at least one of them is not constant.

Assumption 3. V is bounded and a^ω is uniformly elliptic; namely, there exists a constant κ such that, for all ω, x, and y,

$$(2.2) \qquad \kappa |y|^2 \leq |\sigma^\omega(x) y|^2 \leq \kappa^{-1} |y|^2.$$

For a Borel subset $F \subset \mathbb{R}^d$, we define the σ-field

$$\mathcal{H}_F = \sigma\{V(x.\omega), \sigma(x.\omega) : x \in F\},$$

and we assume the following independence condition:

Assumption 4. There exists R such that for any Borel subsets F and G such that $d(F, G) > R$ (where $d(F, G) = \inf\{|x - y| : x \in F, \ y \in G\}$ is the distance between F and G), then

$$(2.3) \qquad \mathcal{H}_F \text{ and } \mathcal{H}_G \text{ are independent.}$$

Let $(W_t : t \geq 0)$ be a Brownian motion defined on some probability space $(\mathcal{W}, \mathcal{F}, P)$. We denote expectation with respect to P by E. By *diffusion in the environment* ω we mean the solution of the stochastic differential equation

$$(2.4) \qquad dX_x^\omega(t) = b^\omega(X_x^\omega(t))dt + \sigma^\omega(X_x^\omega(t))dW_t; \quad X_x^\omega(0) = x.$$

Then X_x^ω is indeed the Markov process generated by the operator $\mathcal{L}^\omega$ in equation (2.1). We shall denote by P_x^ω the law of X^ω on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$. It is usually referred to as the *quenched* law of the diffusion in a random environment. We will also need the so-called *annealed* law:

$$(2.5) \qquad \mathbb{P}_x[A] := \int d\mathbb{Q}(\omega) \int dP_x^\omega(w) \mathbb{1}_A(\omega, w)$$

for any measurable subset $A \subset \Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$.

Expectation with respect to P_x^ω will be denoted by E_x^ω, and expectation with respect to $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x$.

We use the notation $X(t)$ for the coordinate process on path space $C(\mathbb{R}_+, \mathbb{R}^d)$.

2.2 Effective Diffusivity

DEFINITION 2.1. Let Σ be the *effective diffusivity* matrix defined by

$$(2.6) \qquad e \cdot \Sigma e := \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_0[(e \cdot X(t))^2],$$

where e is any vector in $\mathbb{R}^d$ and $x \cdot y$ denotes the scalar product of the two vectors x and y.

The fact that the limit in (2.6) exists is (almost) a consequence of the central limit theorem for the process X under $\mathbb{P}_0$. More is actually known: X satisfies a full invariance principle. Namely, for almost any realization of the environment ω, the laws of the sequence of rescaled processes $(X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2); t \geq 0)$ under P_0^ω weakly converge as ε goes to 0 to the law of a Brownian motion with covariance matrix Σ. References on this theorem include [4, 14, 17, 22, 23] among others. The convergence of the variance of the process to Σ is explicitly stated in [4, formula (2.44)].

The invariance principle also has a PDE counterpart in terms of homogenization theory; see, for instance, the book [12]. The generator of the process X^ε under P_x^ω is the rescaled elliptic operator with rapidly oscillating coefficients

$$\frac{1}{2} a^\omega\left(\frac{\cdot}{\varepsilon}\right)\Delta + \frac{1}{2\varepsilon} b^\omega\left(\frac{\cdot}{\varepsilon}\right)\nabla.$$

Its limit, in the sense of homogenization theory, is the elliptic operator with constant coefficient

$$\frac{1}{2} \operatorname{div}(\Sigma\nabla),$$

where Σ is the same matrix as in (2.6).

The effective diffusivity Σ is a symmetric matrix. As a consequence of Assumption 1 on ergodicity, Σ is deterministic (i.e., Σ does not depend on ω). Furthermore, due to the ellipticity assumption, Assumption 3, Σ is also known to be positive definite.

In general, there is no simple expression of Σ in terms of σ or V. (For instance, Σ is by no means the average of σ^2!) The proof of the homogenization theorem actually provides an expression of Σ as a function of the solution of a Poisson equation—the so-called *corrector approach*. Since the operator $\mathcal{L}^\omega$ is self-adjoint with respect to the measure $e^{-2V^\omega(x)}dx$, there is also a variational formulation of the Poisson equation and therefore a variational formula for Σ itself. We will not need it in this paper. Our main theorem actually gives a quite different interpretation of Σ as the *mobility* of X^ω; see below.

2.3 Perturbed Diffusions

We shall now consider perturbations of the process X obtained by inserting a local drift in equation (2.4).

We use the following notation: Let e_1 be a nonzero vector with $|e_1| = 1$ and $\lambda > 0$. We define $\hat{\lambda}$ to be the vector $\hat{\lambda} = \lambda e_1$. We think of e_1 as being fixed while λ is due to tend to 0. We assume throughout the whole paper that $\lambda \leq 1$.

Let us consider the perturbed stochastic differential equation:

$$
(2.7) \quad
\begin{aligned}
dX_x^{\lambda,\omega}(t) &= b^\omega(X_x^{\lambda,\omega}(t))dt + a^\omega(X_x^{\lambda,\omega}(t))\hat{\lambda}\, dt \\
&\quad + \sigma^\omega(X_x^{\lambda,\omega}(t))dW_t, \quad X_x^{\lambda,\omega}(0) = x.
\end{aligned}
$$

The process $X_x^{\lambda,\omega}$ is now a Markov process with generator

$$
\begin{aligned}
\mathcal{L}^{\lambda,\omega} f(x) &= \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x) + a^\omega(x)\hat{\lambda} \cdot \nabla f(x) \\
(2.8) \qquad &= \frac{1}{2} e^{2V^{\lambda,\omega}(x)} \operatorname{div}(e^{-2V^{\lambda,\omega}} a^\omega \nabla f)(x),
\end{aligned}
$$

where $V^{\lambda,\omega}(x) = V^\omega(x) - \hat{\lambda} \cdot x$. We shall use the notation $P_x^{\lambda,\omega}$ for the law of $X_x^{\lambda,\omega}$, $E_x^{\lambda,\omega}$ for the corresponding expectation, as well as $\mathbb{P}_x^\lambda$ and $\mathbb{E}_x^\lambda$ for the annealed probability and expectation defined analogously to (2.5).

Our model is a special case of diffusions with drifts considered by L. Shen in [25] for which the author proved a law of large numbers: for almost any environment ω, the ratio $X(t)/t$ has an almost sure limit under $P_0^{\lambda,\omega}$, say $\ell(\lambda)$. The convergence also holds in $L^1(\mathbb{P}_0^\lambda)$. Moreover, $\ell(\lambda)$ is deterministic and $\hat{\lambda} \cdot \ell(\lambda) > 0$. Note that the proof strongly relies on the independence property Assumption 4. We thus define the effective velocity:

DEFINITION 2.2. Let $\lambda > 0$. Let $\ell(\lambda)$ be the *effective drift* vector defined by

$$
(2.9) \qquad \ell(\lambda) = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_0^\lambda[X(t)].
$$

By convention $\ell(0) = 0$.

2.4 The Einstein Relation

Our main result is the following theorem:

THEOREM 2.3. *The function $\lambda \to \ell(\lambda)$ has a derivative at $\lambda = 0$ that satisfies*

$$
(2.10) \qquad \lim_{\lambda \to 0} \frac{1}{\lambda} \ell(\lambda) = \Sigma e_1.
$$

This justifies the following definition:

DEFINITION 2.4. The derivative at $\lambda = 0$ of the velocity $e_2 \cdot \ell(\lambda)$ is called *mobility* in the direction e_2.

Theorem 2.3 and Definition 2.4 can be compared with the main result of [19], where the mobility is defined as the mean position of the process X^ε under $\mathbb{P}_0^\varepsilon$. The authors prove that, as ε tends to 0, the law of X^ε under $\mathbb{P}_0^\varepsilon$ converges to the law of Brownian motion with drift v given by $v = \Sigma e_1$. These results are consequences of the invariance principle under $\mathbb{P}_0$ and do not require any information on the asymptotic behavior of the process under $\mathbb{P}_0^\lambda$ for a fixed λ (and indeed the law of large numbers of L. Shen was not known at the time [19] was written).

3 Girsanov Transforms

The aim of this section is to establish Proposition 3.1 below. In this part of the paper we only use Assumptions 1, 2, and 3.

PROPOSITION 3.1. *Let $\alpha \geq 1$. Then*

$$(3.1) \qquad \lim_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \mathbb{E}_0^\lambda \left[\frac{X(t)}{\lambda t} \right] = \Sigma e_1.$$

Also,

$$(3.2) \qquad \sup_{\alpha \geq 1} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \mathbb{E}_0^\lambda \left[\frac{\max_{s \leq t} |X(s)|^2}{\lambda^2 t^2} \right] < \infty.$$

Remark 3.2. Observe that (3.2) directly follows from Lemma 4.5 in the next section of the paper.

PROOF OF PROPOSITION 3.1 FOR THE CASE $V = 0$. We start by proving Proposition 3.1 for the case where there is no potential; i.e., we start assuming that $V = 0$.

We use Girsanov transforms pretty much as in [19], i.e., the explicit expression of the Radon-Nikodym derivative of $P_0^{\lambda,\omega}$ with respect to P_0^ω.

Let us first recall Girsanov transforms; see [24, chap. VIII]. Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be solutions of stochastic differential equations of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dW_t; \ X(0) = x,$$
$$dY(t) = c(Y(t))dt + \sigma(Y(t))dW_t; \ Y(0) = x,$$

where $(W_t : t \geq 0)$ is a Brownian motion and the coefficients σ, b, and c are subject to smoothness and ellipticity assumptions as in Assumptions 2 and 3. Let P^X and P^Y be the laws of the processes X and Y on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$. Let $\mathcal{F}_t = \sigma\{X(s) : s \leq t\}$ be the filtration generated by the coordinate process up to time t. Then the restriction of P^Y to $\mathcal{F}_t$ is absolutely continuous with respect to the restriction of P^X to $\mathcal{F}_t$, and the Radon-Nikodym derivative is given by the Girsanov formula

$$E[F(Y([0,t]))] = E\left[F(X([0,t]))e^{M(t) - \frac{1}{2}\langle M \rangle(t)}\right]$$

for any time t, for any bounded continuous function F on $C([0, t], \mathbb{R}^d)$, and where M is the martingale

$$M(t) = \int_0^t \phi(X(s)) \cdot dW_s,$$

$\langle M \rangle$ is its bracket

$$\langle M \rangle(t) = \int_0^t |\phi(X(s))|^2 \, ds,$$

and $\phi(x) = \sigma^{-1}(x)(c(x) - b(x))$.

In the next discussion we use the expression "Brownian motion with covariance σ^2 and drift c" to denote any process whose law is the same as the law of $(\sigma W_t + ct : t \geq 0)$.

If we choose σ and c constant and $b = 0$ above, then Y is a Brownian motion with covariance σ^2 and drift c, and (X, M) is a centered Brownian motion (in dimension $d + 1$) whose covariance satisfies the following: the covariance of X is σ^2; $E[M(t)X(t)] = ct$. Thus the Girsanov formula then has the following corollary: Let Y be a Brownian motion with covariance matrix σ^2 and drift c; then

$$(3.3) \qquad E[F(Y([0, t]))] = E\left[F(X([0, t]))e^{M(t) - \frac{1}{2}\langle M \rangle(t)}\right]$$

for any time t, for any bounded continuous function F on $C([0, t], \mathbb{R}^d)$, and for any random process M such that (X, M) is a centered Brownian motion, X has covariance σ^2, and the covariance of X and M is $E[M(t)X(t)] = ct$.

Applying the Girsanov formula to the processes X^ω and $X^{\lambda,\omega}$, we get that, for any ω,

$$(3.4) \qquad E[F(X_0^{\lambda,\omega}([0, t]))] = E\left[F(X_0^\omega([0, t]))e^{\lambda \bar{B}(t) - \frac{\lambda^2}{2}\langle \bar{B} \rangle(t)}\right],$$

where $\bar{B}$ is the martingale

$$\bar{B}(t) = \int_0^t \sigma^\omega(X_0^\omega(s))e_1 \cdot dW_s$$

and $\langle \bar{B} \rangle$ is its bracket

$$\langle \bar{B} \rangle(t) = \int_0^t |\sigma^\omega(X_0^\omega(s))e_1|^2 \, ds.$$

In particular, in the range $\lambda^2 t = \alpha$, we have

$$E[F(X_0^{\lambda,\omega}([0, t]))] = E\left[F(X_0^\omega([0, t]))e^{\lambda \bar{B}(\frac{\alpha}{\lambda^2}) - \frac{\lambda^2}{2}\langle \bar{B} \rangle(\frac{\alpha}{\lambda^2})}\right].$$

We shall need the following easy statement:

LEMMA 3.3. *For all $\alpha \geq 1$ and $\mu > 1$, we have*

$$(3.5) \qquad \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} E\left[e^{\mu\lambda\bar{B}(\frac{\alpha}{\lambda^2}) - \frac{\mu\lambda^2}{2}\langle \bar{B} \rangle(\frac{\alpha}{\lambda^2})}\right] < \infty \quad \text{uniformly in } \omega.$$

PROOF OF (3.5). Assumption 3 on the ellipticity of a^ω implies that $\langle \bar{B} \rangle(t) \leq \kappa^{-1}t$. Therefore

$$E\left[e^{\mu\lambda\bar{B}(\frac{\alpha}{\lambda^2}) - \frac{\mu\lambda^2}{2}\langle\bar{B}\rangle(\frac{\alpha}{\lambda^2})}\right] \leq E\left[e^{\mu\lambda\bar{B}(\frac{\alpha}{\lambda^2}) - \frac{\mu^2\lambda^2}{2}\langle\bar{B}\rangle(\frac{\alpha}{\lambda^2})}\right] e^{(\mu-1)\frac{\mu\lambda^2}{2}\frac{\alpha}{\kappa\lambda^2}}$$

$$= e^{(\mu-1)\frac{\mu}{2}\frac{\alpha}{\kappa}},$$

and (3.5) is thus proved. □

Next we apply a (joint) invariance principle for the process $(X_0^{\omega,\varepsilon}, \bar{B}^\varepsilon)$ where $X_0^{\omega,\varepsilon}(t) = \varepsilon X_0^\omega(t/\varepsilon^2)$ and $\bar{B}^\varepsilon(t) = \varepsilon\bar{B}(t/\varepsilon^2)$.

Let us recall some of the ideas of [4, 14].

The process of the environment seen from the particle is as follows: $(\omega(t) = X_0^\omega(t).\omega); t \geq 0)$ is a Markov process under the annealed law, with values in Ω. It is not difficult to check that the measure $\mathbb{Q}$ is invariant, ergodic, and reversible for this process. (Recall that $V = 0$ for now!)

Given the state of the environment at times 0 and t, say $\omega(0)$ and $\omega(t)$, one retrieves the position of the particle itself by solving the equation $z.\omega(0) = \omega(t)$. Note that Assumptions 1 (either V or a is not constant) and 4 (independence property) imply that there cannot be more than one solution. It also follows from the equality $(X_0^\omega(t) - X_0^\omega(s)).\omega(s) = \omega(t)$ that $X_0^\omega(t)$ is an antisymmetric additive functional of the process $\omega(\cdot)$. (Antisymmetric means that reversing time amounts to changing the sign of X_0^ω.)

The process $\bar{B}$ is also an additive functional of $\omega(\cdot)$ since it can be written as the difference

$$\bar{B}(t) - \bar{B}(s) = e_1 \cdot (X_0^\omega(t) - X_0^\omega(s)) - \int_s^t e_1 \cdot b^\omega(X_0^\omega(u))du$$

$$= e_1 \cdot (X_0^\omega(t) - X_0^\omega(s)) - \int_s^t e_1 \cdot b(\omega(u))du;$$

see (2.4).

Sufficient conditions for invariance principles for additive functionals of reversible Markov processes that can be applied to X_0^ω or $\bar{B}$ are given in [4, 14]. They yield a joint invariance principle for $(X_0^{\omega,\varepsilon}, \bar{B}^\varepsilon)$.

In order to compute the covariance matrix, note that $X_0^\omega(t)$ is antisymmetric whereas $\int_0^t e_1 \cdot b(\omega(s))ds$ is a symmetric functional of the environment. Thus they are orthogonal under the annealed measure. Therefore

$$\int E[\bar{B}(t)X_0^\omega(t)]d\mathbb{Q} = \int E[(e_1 \cdot X_0^\omega(t))X_0^\omega(t)]d\mathbb{Q},$$

and thus the asymptotic covariance of $X_0^{\omega,\varepsilon}(1)$ and $\bar{B}^\varepsilon(1)$ coincides with the asymptotic covariance of $X_0^{\omega,\varepsilon}(1)$ and $e_1 \cdot X_0^{\omega,\varepsilon}(1)$ and equals Σe_1.

Applying first the Girsanov formula and then the invariance principle, we get that, for any $\alpha > 0$, as $\lambda \to 0$ and $t \to \infty$ with $\lambda^2 t = \alpha$, we have

(3.6)
$$\int E\left[F\left(\frac{\lambda}{\alpha} X_0^{\lambda,\omega}\left(\frac{\alpha}{\lambda^2} s\right); 0 \leq s \leq 1\right)\right] d\mathbb{Q}$$
$$= \int E\left[F\left(\frac{\lambda}{\alpha} X_0^\omega\left(\frac{\alpha}{\lambda^2} s\right); 0 \leq s \leq 1\right) e^{\lambda \bar{B}(\frac{\alpha}{\lambda^2}) - \frac{\lambda^2}{2}\langle \bar{B}\rangle(\frac{\alpha}{\lambda^2})}\right] d\mathbb{Q}$$
$$\to E\left[F\left(\frac{1}{\sqrt{\alpha}} N(s); 0 \leq s \leq 1\right) e^{\sqrt{\alpha} Z(1) - \frac{\alpha}{2} E(Z(1)^2)}\right],$$

where F is a bounded continuous functional on $C([0,1], \mathbb{R}^d)$ and (N, Z) is a centered Brownian motion of dimension $d + 1$ with N having Σ as covariance matrix, and $E[Z(t)N(t)] = \Sigma e_1 t$. We refer to Lemma 3.3 for the full justification of the passing to the limit in (3.6). Using now formula (3.3), we have

$$E\left[F\left(\frac{1}{\sqrt{\alpha}} N(s); 0 \leq s \leq 1\right) e^{\sqrt{\alpha} Z(1) - \frac{\alpha}{2} E(Z(1)^2)}\right] =$$
$$E\left[F\left(\frac{1}{\sqrt{\alpha}} N(s) + \Sigma e_1 s; 0 \leq s \leq 1\right)\right].$$

We thus conclude that

$$\mathbb{E}_0^\lambda\left[F\left(\frac{\lambda}{\alpha} X\left(\frac{\alpha}{\lambda^2} s\right); 0 \leq s \leq 1\right)\right] = \int E\left[F\left(\frac{\lambda}{\alpha} X_0^{\lambda,\omega}\left(\frac{\alpha}{\lambda^2} s\right); 0 \leq s \leq 1\right)\right] d\mathbb{Q}$$
$$\to E\left[F\left(\frac{1}{\sqrt{\alpha}} N(s) + \Sigma e_1 s; 0 \leq s \leq 1\right)\right];$$

i.e., when $\lambda^2 t = \alpha$, the law of $((\lambda t)^{-1} X(ts); 0 \leq s \leq 1)$ under $\mathbb{P}_0^\lambda$ converges to the law of $(\frac{1}{\sqrt{\alpha}} N(s) + \Sigma e_1 s; 0 \leq s \leq 1)$.

To finish the proof of the proposition, we need a priori bounds on the moments of $|X(t)|$ under $\mathbb{P}_0^\lambda$. We shall prove in Lemma 4.5 that

$$\limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} E_0^{\lambda,\omega}\left[\frac{\max_{s \leq t} |X(s)|^p}{\lambda^p t^p}\right] < \infty$$

uniformly in ω and for all $p \geq 1$ and all $\alpha \geq 1$. Therefore

(3.7)
$$\limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \mathbb{E}_0^\lambda\left[\frac{\max_{s \leq t} |X(s)|^p}{\lambda^p t^p}\right] < \infty$$

for all $p \geq 1$ and all $\alpha \geq 1$, and we observe that (3.7) together with the convergence of the law of $((\lambda t)^{-1} X(ts); 0 \leq s \leq 1)$ under $\mathbb{P}_0^\lambda$ to the law of $(\frac{1}{\sqrt{\alpha}} N(s) +$

$\Sigma e_1 s; 0 \le s \le 1$) implies (3.1) and (3.2). Indeed, we have

$$\frac{\max_{s \le t} |X(s)|^2}{\lambda^2 t^2} = \max_{s \le 1} \left| \frac{\lambda}{\alpha} X\left(\frac{\alpha}{\lambda^2} s\right) \right|^2,$$

and therefore, with the notation above,

$$\mathbb{E}_0^\lambda \left[\frac{\max_{s \le t} |X(s)|^2}{\lambda^2 t^2} \right] \to E \left[\max_{s \le 1} \left| \frac{1}{\sqrt{\alpha}} N(s) + \Sigma e_1 s \right|^2 \right],$$

where we used the function $F(w(s); 0 \le s \le 1) = \max_{s \le 1} |w(s)|^2$ and inequality (3.7) to justify the passing to the limit.

Finally, it is easy to check that

$$\sup_{\alpha \ge 1} E \left[\max_{s \le 1} \left| \frac{1}{\sqrt{\alpha}} N(s) + \Sigma e_1 s \right|^2 \right] < \infty.$$

This last line ends the justification of (3.2). Equation (3.1) is proved the same way by using the function $F(w(s); 0 \le s \le 1) = w(1)$.

The proof of Proposition 3.1 is now complete in the case $V = 0$. $\qquad \square$

PROOF OF PROPOSITION 3.1 FOR THE CASE $V \ne 0$. We do not assume anymore that $V = 0$.

Define Y^ω and $Y^{\lambda,\omega}$ to be the solutions of the stochastic differential equations

$$(3.8) \quad \begin{aligned} dY^\omega(t) &= e^{-2V^\omega(Y^\omega(t))} b^\omega(Y^\omega(t)) dt + e^{-V^\omega(Y^\omega(t))} \sigma^\omega(Y^\omega(t)) dW_t; \\ Y^\omega(0) &= 0 \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} dY^{\lambda,\omega}(t) &= e^{-2V^\omega(Y^{\lambda,\omega}(t))} b^\omega(Y^{\lambda,\omega}(t)) dt + e^{-2V^\omega(Y^{\lambda,\omega}(t))} a^\omega(Y^{\lambda,\omega}(t)) \hat{\lambda} \, dt \\ &\quad + e^{-V^\omega(Y^{\lambda,\omega}(t))} \sigma^\omega(Y^{\lambda,\omega}(t)) dW_t; \ Y^{\lambda,\omega}(0) = 0, \end{aligned}$$

so that the generators of Y^ω and $Y^{\lambda,\omega}$ are the operators

$$\mathcal{M}^\omega f(x) = \frac{1}{2} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x)$$

and

$$\mathcal{M}^{\lambda,\omega} f(x) = \frac{1}{2} e^{\hat{\lambda} \cdot x} \operatorname{div}(e^{-2V^{\lambda,\omega}} a^\omega \nabla f)(x),$$

where $V^{\lambda,\omega}(x) = V^\omega(x) - \hat{\lambda} \cdot x$.

Note that these operators are of the same form as $\mathcal{L}^\omega$ and $\mathcal{L}^{\lambda,\omega}$ with V^ω being replaced by 0 and a^ω being replaced by $\exp(-2V^\omega) a^\omega$. Thus we may apply the results obtained in the special case of a vanishing potential to the processes Y^ω

and $Y^{\lambda,\omega}$; in particular, Y^ω satisfies the invariance principle with some asymptotic diffusivity Σ^Y, and $Y^{\lambda,\omega}$ satisfies

$$(3.10) \qquad \lim_{\substack{\lambda\to 0 \\ t\to+\infty \\ \lambda^2 t=\alpha}} \int_\Omega E\left[\frac{Y^{\lambda,\omega}(t)}{\lambda t}\right]d\mathbb{Q} = \Sigma^Y e_1$$

and

$$(3.11) \qquad \sup_{\alpha\geq 1}\limsup_{\substack{\lambda\to 0 \\ t\to+\infty \\ \lambda^2 t=\alpha}} \int_\Omega E\left[\frac{\max_{s\leq t}|Y^{\lambda,\omega}(s)|^2}{\lambda^2 t^2}\right]d\mathbb{Q} < \infty.$$

Fix ω and set

$$A^\omega(t) = \int_0^t e^{-2V^\omega(Y^\omega(s))}ds = \int_0^t e^{-2V(Y^\omega(s).\omega)}ds.$$

Then X_0^ω has the same law as the time-changed process $Y^\omega((A^\omega)^{-1})$. Similarly, if we let

$$A^{\lambda,\omega}(t) = \int_0^t e^{-2V^\omega(Y^{\lambda,\omega}(s))}ds,$$

then $X_0^{\lambda,\omega}$ has the same law as the time-changed process $Y^{\lambda,\omega}((A^{\lambda,\omega})^{-1})$.

From Assumption 3, we know that V is bounded and therefore

$$(3.12) \qquad A^{\lambda,\omega}(t) \geq \underline{c}t$$

for some constant $\underline{c} \leq 1$. Observe that (3.2) immediately follows from (3.12) and (3.11).

PROOF OF (3.1). Let

$$\gamma = \int d\mathbb{Q}(\omega)e^{-2V(\omega)}.$$

The ergodic theorem for the process $Y^\omega.\omega$ implies that $A^\omega(t)/t$ almost surely converges to γ. We need a similar statement for $A^{\lambda,\omega}$:

LEMMA 3.4. *For fixed $\alpha > 0$ and any positive η, we have $\mathbb{Q}$-a.s.*

$$(3.13) \quad P\left[\left|\frac{A^{\lambda,\omega}(t)}{t} - \gamma\right| \geq \eta\right] \to 0 \quad \text{for } \lambda\to 0,\, t\to+\infty,\, \lambda^2 t = \alpha.$$

PROOF. Use the Girsanov formula (3.4) to see that

$$P\left[\left|\frac{A^{\lambda,\omega}(t)}{t} - \gamma\right| \geq \eta\right] = E\left[\mathbb{1}\left(\left|\frac{A^\omega(t)}{t} - \gamma\right| \geq \eta\right)e^{\lambda\bar{B}(t)-\frac{\lambda^2}{2}\langle\bar{B}\rangle(t)}\right].$$

The convergence in (3.13) then follows from Hölder's inequality, the bound in Lemma 3.3, and the fact that $P[|\frac{A^\omega(t)}{t} - \gamma| \geq \eta] \to 0$. $\square$

Back to the proof of (3.1), we start with the equality

$$\mathbb{E}_0^\lambda\left[\frac{X(t)}{\lambda t}\right] = \int_\Omega E\left[\frac{Y^{\lambda,\omega}((A^{\lambda,\omega})^{-1}(t))}{\lambda t}\right]d\mathbb{Q}.$$

We have

$$\left|\int_\Omega E\left[\frac{Y^{\lambda,\omega}((A^{\lambda,\omega})^{-1}(t))}{\lambda t}\right]d\mathbb{Q} - \int_\Omega E\left[\frac{Y^{\lambda,\omega}(t/\gamma)}{\lambda t}\right]d\mathbb{Q}\right|$$

$$\leq \int_\Omega E\left[\frac{\max_{|s-t/\gamma|\leq\eta t}|Y^{\lambda,\omega}(s) - Y^{\lambda,\omega}(t/\gamma)|}{\lambda t}\right]d\mathbb{Q}$$

$$+ \int_\Omega E\left[\frac{\max_{s\leq t/\underline{c}}|Y^{\lambda,\omega}(s)|}{\lambda t}\mathbb{1}\big(|(A^{\lambda,\omega})^{-1}(t) - t/\gamma| \geq \eta t\big)\right]d\mathbb{Q}$$

$$= \mathrm{I} + \mathrm{II}.$$

($\underline{c}$ is the same constant as in (3.12). Note that $\underline{c} \leq \gamma$.)

By the Markov property, we have

$$\mathrm{I} \leq 2\int_\Omega E\left[\frac{\max_{s\leq 2\eta t}|Y^{\lambda,\omega}(s)|}{\lambda t}\right]d\mathbb{Q}.$$

Now (3.11) implies that, if we let λ tend to 0 and then η tend to 0, then the contribution of I vanishes. Hölder's inequality, (3.11) again, and Lemma 3.4 imply that, for any $\eta > 0$, then II also converges to 0 as t tends to $+\infty$. We conclude that

$$\lim_{\substack{\lambda\to 0 \\ t\to+\infty \\ \lambda^2 t=\alpha}} \int_\Omega \left(E\left[\frac{Y^{\lambda,\omega}((A^{\lambda,\omega})^{-1}(t))}{\lambda t}\right] - E\left[\frac{Y^{\lambda,\omega}(t/\gamma)}{\lambda t}\right]\right)d\mathbb{Q} = 0,$$

and, using (3.10),

$$\lim_{\substack{\lambda\to 0 \\ t\to+\infty \\ \lambda^2 t=\alpha}} \mathbb{E}_0^\lambda\left[\frac{X(t)}{\lambda t}\right] = \frac{1}{\gamma}\Sigma^Y e_1.$$

The last piece of information missing is the equality $\Sigma^Y = \gamma\Sigma$. It comes as follows: since $\frac{1}{\sqrt{t}}Y^\omega(t)$ converges in law to a Gaussian random variable with covariance Σ^Y and since $\frac{1}{t}A^\omega(t)$ almost surely converges to γ, then

$$\frac{1}{\sqrt{t}}X_0^\omega(t) = \frac{1}{\sqrt{t}}Y^\omega((A^\omega)^{-1}(t))$$

converges in law to a Gaussian random variable with covariance $\Sigma = \frac{1}{\gamma}\Sigma^Y$.

This completes the proof of (3.1) and of Proposition 3.1. $\qquad\square$

4 A Priori Estimates

In this section, we prove some a priori estimates on exit times that quantify the fact that the process $X^{\lambda,\omega}$ is transient in the direction e_1.

For a given realization of the environment, the local drift of the process X^ω equals $b^\omega(X^\omega(t))$. Its mean under the annealed law vanishes. The drift of $X^{\lambda,\omega}$ has an extra $a^\omega(X^{\lambda,\omega}(t))\widehat{\lambda}$ term. Since, by Assumption 3 (uniform ellipticity), we have $e_1 \cdot a^\omega(x)e_1 \geq \kappa|e_1|^2 \neq 0$ for any ω and x, one would expect $X^{\lambda,\omega}$ to be transient in the direction e_1, and this turns out to be the case, but we also need more quantitative statements on the tendency of the diffusion to go in the direction e_1.

Roughly speaking, we may think of $e_1 \cdot X^{\lambda,\omega}(t)$ as the sum of a centered term of order $\sqrt{t}$ and a drift term of order λt. Thus the shortest scale on which we may hope the drift term to dominate is $\lambda^2 t \geq 1$ or, in terms of space scale, $\lambda L \geq 1$. Up to the value of the constants κ_1, κ_2, c, and C, our estimates in Lemma 4.1 and Lemma 4.2 are therefore optimal.

In the following lemmata, inequalities (4.1) and (4.2) hold true for any environment ω satisfying Assumptions 2 and 3. Assumptions 1 and 4 are not relevant in this section.

We use $T_L = \inf\{t : e_1 \cdot X(t) = L\}$ to denote the hitting time of the hyperplane $\{x : e_1 \cdot x = L\}$, $L \in \mathbb{R}$.

LEMMA 4.1. *There exist constants $c > 0$ and $\kappa_1 > 0$ that depend on the dimension, the ellipticity constant κ, and the L^∞-bound on V such that for all L, $\lambda \leq 1$, and any environment ω,*

$$(4.1) \qquad\qquad P_0^{\lambda,\omega}[T_{-L} < \infty] \leq ce^{-\kappa_1\lambda L}.$$

LEMMA 4.2. *There exist constants C and $\kappa_2 > 0$ that depend on the dimension, the ellipticity constant κ, and the L^∞-bound on V such that for all L, $\lambda \leq 1$, $t \geq 0$, and any environment ω,*

$$(4.2) \qquad\qquad P_0^{\lambda,\omega}[T_L \geq t] \leq Ce^{-\kappa_2\lambda^2t+\lambda L}.$$

PROOF OF LEMMA 4.1. Let $u(x) = u_{\lambda,\omega,L}(x) := P_x^{\lambda,\omega}[T_L < T_{-L}]$. Then u solves the elliptic boundary value problem

$$(4.3) \qquad\qquad \mathcal{L}^{\lambda,\omega}u(x) = 0, \quad -L \leq e_1 \cdot x \leq L,$$

where the generator $\mathcal{L}^{\lambda,\omega}$ was defined in (2.8), with boundary values

$$(4.4) \qquad\qquad u(x) = \begin{cases} 0 & \text{if } e_1 \cdot x = -L, \\ 1 & \text{if } e_1 \cdot x = L. \end{cases}$$

We first need the following lemma, whose proof is deferred.

LEMMA 4.3. *Assume $\lambda = 1$ and u is the solution of the boundary value problem above. Then, under Assumption 3, there exists $L_0 > 0$ depending only on κ and on*

the dimension such that, for all $L \geq L_0$ and all ω,

$$(4.5) \qquad u(x) \geq \frac{2}{3} \quad \text{for all } x \text{ with } e_1 \cdot x = 0.$$

Let $\widetilde{X}(t) = \lambda X(t/\lambda^2)$, $t \geq 0$. Then $(\widetilde{X}(t))_{t \geq 0}$ is a Markov process with generator

$$\widetilde{\mathcal{L}}^{\lambda,\omega} f(x) = \frac{1}{2} e^{2V^\omega(x/\lambda)} \operatorname{div}(e^{-2V^\omega(x/\lambda)} a^\omega(x/\lambda)\nabla f)(x) + a^\omega(x/\lambda)e_1 \cdot \nabla f(x).$$

(The advantage of $(\widetilde{X}(t))_{t \geq 0}$ is that we scaled away the drift and will be able to apply Lemma 4.3.)

Let us consider exit times for $(\widetilde{X}(t))_{t \geq 0}$,

$$\widetilde{T}_L = \inf\{t : e_1 \cdot \widetilde{X}(t) = L\},$$
$$\widetilde{T}_{-L} = \inf\{t : e_1 \cdot \widetilde{X}(t) = -L\},$$
$$\widetilde{T}_{\pm L} = \inf\{t : |e_1 \cdot \widetilde{X}(t)| = L\}.$$

Then

$$P_0^{\lambda,\omega}[T_{-L} < \infty] = P_0^{\lambda,\omega}[\widetilde{T}_{-\lambda L} < \infty].$$

Hence it suffices to show that for all λ, L, and ω, we have $P_0^{\lambda,\omega}[\widetilde{T}_{-\lambda L} < \infty] \leq ce^{-\kappa_1 \lambda L}$, which in turn is equivalent to

$$(4.6) \qquad P_0^{\lambda,\omega}[\widetilde{T}_{-L} < \infty] \leq ce^{-\kappa_1 L}$$

for all λ, L, and ω. Let L_0 be as in Lemma 4.3. We consider the embedded random walk defined as follows: Let $t_1 := \widetilde{T}_{\pm L_0}$, $t_{i+1} = \inf\{t \geq t_i : |e_1 \cdot (\widetilde{X}(t) - \widetilde{X}(t_i))| = L_0\}$, and $S_i = \widetilde{X}_{t_i}$, $i = 1, 2, \ldots$ (and $S_0 = 0$). Due to Lemma 4.3 and the strong Markov property of $(\widetilde{X}(t))_{t \geq 0}$, we have

$$P_0^{\lambda,\omega}[S_{i+1} = S_i + L_0] \geq \frac{2}{3}.$$

Hence we can couple $(S_i)_{i=0,1,2,\ldots}$ with a standard random walk that has drift $(\overline{S}_i)_{i=0,1,2,\ldots}$ on $L_0 \mathbb{Z}$ with i.i.d. increments, satisfying $P[\overline{S}_{i+1} = \overline{S}_i + L_0] = \frac{2}{3} = 1 - P[\overline{S}_{i+1} = \overline{S}_i - L_0]$ and the coupling is such that $S_i \geq \overline{S}_i$ for all i. Explicit calculation yields

$$P[\overline{S}_i > -mL_0 \ \forall i] = 1 - \frac{1}{2^m},$$

and we obtain

$$P_0^{\lambda,\omega}[S_i > -mL_0 \ \forall i] \geq 1 - \frac{1}{2^m},$$

which implies (4.6) and completes the proof of Lemma 4.1. □

PROOF OF LEMMA 4.2. Inequality (4.2) is equivalent to the following state-
ment in terms of $(\widetilde{X}(t))_{t \geq 0}$: There exist constants C and $\kappa_2 > 0$ that depend
on the dimension, the ellipticity constant κ, and the L^∞-bound on V such that for
all L and t and for any environment ω,

$$(4.7) \qquad P_0^{\lambda,\omega}[\widetilde{T}_L \geq t] \leq C e^{-\kappa_2 t + L}.$$

Indeed, note that $\widetilde{T}_{\lambda L} = \lambda^2 T_L$.

Take L_0 as in Lemma 4.3. Then Aronson's estimate (see [1]) yields that for all
$x \in \Pi_{L_0} := \{x : -L_0 \leq e_1 \cdot x \leq L_0\}$,

$$(4.8) \qquad P_x^{\lambda,\omega}[|e_1 \cdot \widetilde{X}(1)| > L_0] \geq \gamma > 0,$$

where γ depends only on L_0, κ, and the dimension. Indeed, according to [1], the
fundamental solution $\mathcal{P}(x, y, t)$ of the parabolic operator

$$\partial_t - \mathrm{div}(a(x)\nabla) - a(x)e_1 \cdot \nabla$$

(or, equivalently, transition probability density of the corresponding diffusion) sat-
isfies the lower bound

$$\mathcal{P}(x, y, 1) \geq k \exp(-K|x - y|^2)$$

with constants $k > 0$ and $K > 0$, which depend only on κ and d. The estimate (4.8)
is an immediate consequence of this lower bound. Due to the Markov property of
$(\widetilde{X}(t))_{t \geq 0}$, estimate (4.8) implies that for some constant g_0 (which depends only
on L_0, κ, and the dimension),

$$(4.9) \qquad P_x^{\lambda,\omega}[\widetilde{T}_{\pm L_0} \geq g_0] \leq \frac{1}{15} \quad \text{for all } x \in \Pi_{L_0}.$$

Define the stopping time $\widetilde{t}$ as follows:

$$\widetilde{t} := \inf\{t \geq 0 : |e_1 \cdot (\widetilde{X}(t) - \widetilde{X}(0))| = L_0\} \wedge g_0.$$

Combining (4.9) with Lemma 4.3, we obtain

$$P_x^{\lambda,\omega}[e_1 \cdot \widetilde{X}(\widetilde{t}) = e_1 \cdot x + L_0] \geq \frac{3}{5}$$

(with probability at least $\frac{14}{15}$, the layer $\{y : |e_1 \cdot y - e_1 \cdot x| < L_0\}$ has been left by time
g_0, and with probability at least $\frac{2}{3}$, the exit happens at $\{y : e_1 \cdot y = e_1 \cdot x + L_0\}$).
We consider the embedded random walk defined as follows. Let $t_1 := \widetilde{t}$, $t_{i+1} =
\widetilde{t} \circ \theta_{t_i} + t_i$, $S_i = e_1 \cdot \widetilde{X}(t_i)$, $i = 1, 2, \ldots$, and $S_0 = 0$. (Here θ denotes the shift
operator on the path space.) Hence we can couple $(S_i)_{i=0,1,2,\ldots}$ with a standard
random walk with drift $(\overline{S}_i)_{i=0,1,2,\ldots}$ on $L_0 \mathbb{Z}$ with i.i.d. increments, satisfying
$P[\overline{S}_{i+1} = \overline{S}_i + L_0] = \frac{3}{5} = 1 - P[\overline{S}_{i+1} = \overline{S}_i - L_0]$, and the coupling is such
that $S_i \geq \overline{S}_i$ for all i. It is straightforward to check that there are constants $\kappa_3 > 0$
and $c > 0$ such that for all $n \in \mathbb{N}$ and $L \in \mathbb{R}$,

$$P[\overline{S}_n \leq L L_0] \leq c e^{-\kappa_3 n + L}.$$

We conclude, by comparison, that we have for all $L \in \mathbb{R}$ and $n \in \mathbb{N}$

$$P_0^{\lambda,\omega}[\tilde{T}_{LL_0} \geq ng_0] \leq P_0^{\lambda,\omega}[S_n \leq LL_0] \leq P[\bar{S}_n \leq LL_0] \leq ce^{-\kappa_3 n + L},$$

and this implies (4.7). □

PROOF OF LEMMA 4.3. Without loss of generality we can assume that $V = 0$. Indeed, multiplying (4.3) by $\exp(-2V^\omega(x))$ and denoting

$$a(x) = \exp(-2V^\omega(x))a^\omega(x),$$

one can rewrite equations (4.3)–(4.4) in the form

(4.10)
$$\operatorname{div}(a(x)\nabla u) + a(x)e_1 \cdot \nabla u = 0 \quad \text{in } \Pi,$$
$$u(-L, z) = 0, \quad u(L, z) = 1,$$

with $x = (x_1, z)$ and $\Pi = \Pi_L = \{x \in \mathbb{R}^d : -L \leq x_1 \leq L\}$. Under Assumption 3 the matrix $a(x)$ is symmetric and satisfies the following elliptic estimates:

(4.11)
$$\kappa|y|^2 \leq a(x)y \cdot y \leq \kappa^{-1}|y|^2 \quad \forall x, y \in \mathbb{R}^d.$$

Therefore, it suffices to prove the following statement:

LEMMA 4.4. *There is $L_0 = L_0(\kappa, d)$ such that for all $L \geq L_0$, we have*

$$u(0, z) \geq \frac{2}{3}, \quad z \in \mathbb{R}^{d-1}.$$

PROOF. Suppose that $u(0, z) < \frac{2}{3}$ for some $z \in \mathbb{R}^{d-1}$ and some L. Without loss of generality we assume that $z = 0$. The function $1 - u(x)$ is a nonnegative solution of the equation

$$\operatorname{div}(a(x)\nabla(1 - u)) + a(x)e_1 \cdot \nabla(1 - u) = 0;$$

therefore, by the Harnack inequality (see, for instance, [10]) there is a constant $C = C(\kappa, d)$ such that

$$1 - u(0, z) \geq C(\kappa, d)(1 - u(0)) \geq \frac{1}{3} C(\kappa, d), \quad z \in [-1, 1]^{d-1}.$$

For all $z \in [-1, 1]^{d-1}$ this implies the estimate

(4.12)
$$\frac{1}{9} C^2(\kappa, d) \leq \left(\int_0^L \frac{\partial(1 - u(x_1, z))}{\partial x_1} dx_1 \right)^2$$
$$\leq L \int_0^L \left(\frac{\partial u(x_1, z)}{\partial x_1} \right)^2 dx_1$$
$$\leq L \int_0^N |\nabla u(x_1, z)|^2 dx_1 \leq L \int_{-L}^L e^{x_1} |\nabla u(x_1, z)|^2 dx_1,$$

where we used Jensen's inequality for the second inequality. Integrating over $[-1, 1]^{d-1}$, we obtain

$$(4.13) \qquad \int_{-L}^{L} dx_1 \int_{[-1,1]^{d-1}} e^{x_1} |\nabla u(x_1, z)|^2 \, dz \geq \frac{1}{L} C_1(\kappa, d)$$

with a constant $C_1(\kappa, d) > 0$ that depends only on κ and d. For arbitrary sets $G \subseteq \Pi$ and functions $v(x)$, we write $\mathcal{E}(v, G)$ for the energy of v on G:

$$\mathcal{E}(v, G) = \int_G e^{x_1} a(x) \nabla v(x) \cdot \nabla v(x) dx.$$

Hence (4.13) can be rewritten, by setting $G_0 = [-L, L] \times [-1, 1]^{d-1}$, as follows:

$$(4.14) \qquad \mathcal{E}(u, G_0) \geq \frac{1}{L} C_1(\kappa, d).$$

(The value of $C_1(\kappa, d)$ changes from equation (4.13) to equation (4.14).) We will give an upper bound for $\mathcal{E}(u, G_0)$ that contradicts (4.14) when L is too large. We introduce the following subsets of Π:

$$B_0 = [-e^{L/d}, e^{L/d}]^{d-1}, \qquad\qquad \Pi_0 = [-L, L] \times B_0,$$
$$B_j = 2^{-j} B_0 = 2^{-j} [-e^{L/d}, e^{L/d}]^{d-1}, \quad \Pi_j = [-L, L] \times B_j.$$

Also, denote

$$\bar{u}(x) = \bar{u}(x_1) = \begin{cases} 1 & \text{if } x_1 \geq -L + 1, \\ x + L & \text{if } -L \leq x_1 \leq -L + 1, \end{cases}$$

and, writing again $x = (x_1, z)$,

$$v_j(x) = \begin{cases} (1 - d_j(z)) u(x) + d_j(z) \bar{u}(x) & \text{if } d_j(z) \leq 1, \\ \bar{u}(x) & \text{otherwise,} \end{cases}$$

where

$$d_j(z) = 2^{j+1} e^{-L/d} \text{dist}_{\mathbb{R}^{d-1}}(z, \partial B_j), \quad j = 0, 1, \ldots, 2d.$$

Note that $v_j = \bar{u}$ in the domain Π_{j+1}. Since $v_0 = u$ on $\partial \Pi_0$, we know that

$$\mathcal{E}(u, \Pi_0) \leq \mathcal{E}(v_0, \Pi_0).$$

Indeed, the functional $\{\mathcal{E}(v, \Pi_0) : v \in H^1(\Pi_0), \ v|_{\partial \Pi_0} = u|_{\partial \Pi_0}\}$ attains its minimum at the unique solution of the equation

$$\text{div}(a(x) \nabla v) + a(x) e_1 \nabla v = 0 \text{ in } \Pi_0, \quad v\big|_{\partial \Pi_0} = u\big|_{\partial \Pi_0}.$$

Clearly, the function u solves this equation, and the required inequality follows.

For $x \in \Pi_0 \setminus \Pi_1$ we have

$$e^{x_1} a(x) \nabla v_0(x) \cdot \nabla v_0(x)$$
$$= e^{x_1} a(x) \{(1 - d_0(x)) \nabla u(x) + d_0(x) \nabla \overline{u}(x)\}$$
$$\cdot \{(1 - d_0(x)) \nabla u(x) + d_0(x) \nabla \overline{u}(x)\}$$
$$+ 2 e^{x_1} a(x) (\nabla d_0(x)) (\overline{u}(x) - u(x)) \cdot \{(1 - d_0(x)) \nabla u(x) + d_0(x) \nabla \overline{u}(x)\}$$
$$+ e^{x_1} a(x) (\nabla d_0(x)) (\overline{u}(x) - u(x)) \cdot (\nabla d_0(x)) (\overline{u}(x) - u(x))$$

Using the convexity of the scalar product, this is

$$\leq (1 - d_0(x)) e^{x_1} a(x) \nabla u(x) \cdot \nabla u(x) + d_0(x) e^{x_1} a(x) \nabla \overline{u}(x) \cdot \nabla \overline{u}(x)$$
$$+ 2 e^{x_1} \kappa^{-1} |\nabla d_0(x)| \, |\overline{u}(x) - u(x)| (|\nabla u(x)| + |\nabla \overline{u}(x)|)$$
$$+ e^{x_1} a(x) (\nabla d_0(x)) (\overline{u}(x) - u(x)) \cdot (\nabla d_0(x)) (\overline{u}(x) - u(x)).$$

For $x \in \Pi_1$, we have

$$e^{x_1} a(x) \nabla v_0(x) \cdot \nabla v_0(x) = e^{x_1} a(x) \nabla \overline{u}(x) \cdot \nabla \overline{u}(x).$$

After integrating the former inequality over $\Pi_0 \setminus \Pi_1$ and the latter over Π_1, and summing up, we get

$$\mathcal{E}(u, \Pi_0) \leq \mathcal{E}(v_0, \Pi_0)$$

$$\leq \mathcal{E}(u, \Pi_0 \setminus \Pi_1) + \mathcal{E}(\overline{u}, \Pi_0)$$

$$+ 4 \kappa^{-1} e^{-L/d} \int_{\Pi_0 \setminus \Pi_1} e^{x_1} |\overline{u}(x) - u(x)| (|\nabla u(x)| + |\nabla \overline{u}(x)|) dx$$
$$+ 4 \kappa^{-1} e^{-2L/d} \int_{\Pi_0 \setminus \Pi_1} e^{x_1} (\overline{u}(x) - u(x))^2 \, dx.$$

After simple rearrangements this yields, by using Hölder's inequality,

$$(4.15) \quad \begin{aligned} \mathcal{E}(u, \Pi_1) &\leq \mathcal{E}(\overline{u}, \Pi_0) + 4 \kappa^{-1} e^{-2L/d} \int_{\Pi_0 \setminus \Pi_1} e^{x_1} (\overline{u}(x) - u(x))^2 \, dx \\ &\quad + 4 \kappa^{-1} e^{-L/d} \left(\int_{\Pi_0 \setminus \Pi_1} e^{x_1} (\overline{u}(x) - u(x))^2 \, dx \right)^{1/2} \\ &\quad \times \left(\int_{\Pi_0 \setminus \Pi_1} e^{x_1} (|\nabla \overline{u}(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2}. \end{aligned}$$

Our next aim is to estimate the integrals on the right-hand side of the last inequality in terms of the energies of u and $\bar{u}$. First, we estimate

$$\int_{\Pi_0\backslash\Pi_1} e^{x_1}((\bar{u}(x) - u(x))^2 \, dx \leq \int_{\Pi_0\backslash\Pi_1} e^{x_1}(u(x) - 1)^2 \, dx$$

$$+ \int_{\Pi_0\backslash\Pi_1} e^{x_1}(1 - \bar{u}(x))^2 \, dx.$$

Since $(u - 1) = 0$ for $x_1 = L$, we have

$$(u(x) - 1)^2 = \left(\int_{x_1}^L \frac{\partial u}{\partial x_1}(y, z) dy \right)^2 \leq 2L \int_{x_1}^L |\nabla u(y, z)|^2 \, dy,$$

where we used Jensen's inequality. Multiplying this bound by e^{x_1} and integrating over the set $\Pi_0 \setminus \Pi_1$, we obtain

$$\int_{\Pi_0\backslash\Pi_1} e^{x_1}(1 - u(x))^2 \, dx \leq \int_{\Pi_0\backslash\Pi_1} e^{x_1} 2L \int_{x_1}^L |\nabla u(y, z)|^2 \, dy \, dx$$

(4.16)
$$\leq \int_{\Pi_0\backslash\Pi_1} 2L \int_{x_1}^L e^y |\nabla u(y, z)|^2 \, dy \, dz$$

$$\leq 4L^2 \int_{\Pi_0\backslash\Pi_1} e^{x_1} |\nabla u(x)|^2 \, dx.$$

Similarly, taking into account that $\bar{u} \neq 1$ only for $x_1 \leq -L + 1$, we obtain

$$\int_{\Pi_0\backslash\Pi_1} e^{x_1}(1 - \bar{u}(x))^2 \, dx \leq \int_{\Pi_0\backslash\Pi_1\cap\{x_1\leq-L+1\}} e^{x_1} \, dx$$

$$\leq (e - 1)2^{d-1}e^{-L}e^{(d-1)\frac{L}{d}} = a_d e^{-\frac{L}{d}},$$

where $a_d = (e - 1)2^{d-1}$. Combining the latter bound with (4.16) yields

$$\int_{\Pi_0\backslash\Pi_1} e^{x_1}((\bar{u}(x) - u(x))^2 \, dx$$

(4.17)
$$\leq a_d e^{-\frac{L}{d}} + 4L^2 \int_{\Pi_0\backslash\Pi_1} e^{x_1} |\nabla u(x)|^2 \, dx$$

$$\leq a_d e^{-\frac{L}{d}} + 4L^2\kappa^{-1} \int_{\Pi_0\backslash\Pi_1} e^{x_1} a(x)\nabla u(x) \cdot \nabla u(x) dx$$

$$= a_d e^{-\frac{L}{d}} + 4L^2\kappa^{-1} E(u, \Pi_0 \setminus \Pi_1).$$

In the same way, using $|\nabla \bar{u}(x)| = \mathbb{1}_{\{x_1 \le -L+1\}}$, for the last integral on the right-hand side of (4.15) we have

(4.18) $\quad \int\limits_{\Pi_0 \setminus \Pi_1} e^{x_1}(|\nabla \bar{u}(x)|^2 + |\nabla u(x)|^2) dx \le a_d e^{-\frac{L}{d}} + \Lambda^{-1} E(u, \Pi_0 \setminus \Pi_1).$

The following bound for $\mathcal{E}(\bar{u}, \Pi_0)$ is straightforward:

$$\mathcal{E}(\bar{u}, \Pi_0) \le \kappa^{-1} \int\limits_{\Pi_0} e^{x_1} |\nabla \bar{u}(x)|^2 \, dx \le \kappa^{-1} a_d e^{-\frac{L}{d}}.$$

From (4.15), (4.17), (4.18), and the last bound, we derive the inequality

$$\begin{aligned}
\mathcal{E}(u, \Pi_1) &\le \kappa^{-1} a_d e^{-\frac{L}{d}} + \kappa^{-1} e^{-\frac{2L}{d}} \big(a_d e^{-\frac{L}{d}} + 4L^2 \kappa^{-1} \mathcal{E}(u, \Pi_0 \setminus \Pi_1)\big) \\
&\quad + 4\kappa^{-1} e^{-L/d} \big(a_d e^{-\frac{L}{d}} + 4L^2 \kappa^{-1} \mathcal{E}(u, \Pi_0 \setminus \Pi_1)\big)^{\frac{1}{2}} \\
&\qquad \cdot \big(a_d e^{-\frac{L}{d}} + \kappa^{-1} \mathcal{E}(u, \Pi_0 \setminus \Pi_1)\big)^{\frac{1}{2}} \\
&\le C_{\kappa,d}\big(e^{-\frac{L}{d}} + L^2 e^{-\frac{L}{d}} \mathcal{E}(u, \Pi_0 \setminus \Pi_1)\big) \\
&\le C_{\kappa,d}\big(e^{-\frac{L}{d}} + L^2 e^{-\frac{L}{d}} \mathcal{E}(u, \Pi_0)\big)
\end{aligned}$$

(4.19)

Let us now estimate the energy $\mathcal{E}(u, \Pi_0)$. To this end we denote $G(r, x) = x + [-r, r]^d$, and notice that, by the standard elliptic estimates (see [10]),

$$\|\nabla u\|_{L^2(G(1,x) \cap \Pi)} \le C_1(\kappa, d) \|u\|_{L^2(G(2,x) \cap \Pi)} \le C_2(\kappa, d)$$

with a constant $C_2(\kappa, d)$ that depends only on κ and the dimension. This implies the bound

$$\mathcal{E}(u, \Pi_0) \le C_4(\kappa, d) e^L e^{(d-1)\frac{L}{d}} = C_4(\kappa, d) e^{(2d-1)\frac{L}{d}}.$$

Together with (4.19) this gives

(4.20) $\qquad\qquad \mathcal{E}(u, \Pi_1) \le C_5(\kappa, d) L^2 e^{(2d-2)\frac{L}{d}}.$

In exactly the same way as (4.19), we obtain (for $L^2 > C_5(\kappa, d)$)

$$\mathcal{E}(u, \Pi_2) \le C_5(\kappa, d)\big(L^2 e^{-\frac{L}{d}} E(u, \Pi_1) + e^{-\frac{L}{d}}\big) \le C_5(\kappa, d) L^4 e^{(2d-3)\frac{L}{d}}.$$

Iterating this procedure $2d$ times, we finally get

$$\mathcal{E}(u, \Pi_{2d}) \le C_5(\kappa, d)\big(L^{4d-2} e^{-\frac{L}{d}} + e^{-\frac{L}{d}}\big) \le C_5(\kappa, d) L^{4d} e^{-\frac{L}{d}}.$$

If L is sufficiently large, then this estimate implies the bound

(4.21) $\qquad\qquad \mathcal{E}(u, \Pi_{2d}) \le e^{-\frac{L}{2d}}.$

Clearly, for all sufficiently large L we have $[-L, L] \times [-1, +1]^{d-1} \subseteq \Pi_{2d}$, and (4.21) contradicts the lower bound (4.14). We conclude that there is L_0 such that for all $L \ge L_0$, we have

$$u(0, z) \ge \frac{2}{3} \quad \forall z \in \mathbb{R}^{d-1}.$$

Our arguments also ensure that the constant L_0 depends only on κ and d, and Lemma 4.4 is proved. □

This completes the proof of Lemma 4.3. □

LEMMA 4.5. *For any $p \geq 1$ there exists a constant C_0 that depends only on p, the dimension, the ellipticity constant κ, and the L^∞-bound on V such that for all $\lambda \leq 1$ and $t \geq 1/\lambda^2$ and for any environment ω,*

$$(4.22) \qquad E_0^{\lambda,\omega}[\max_{0 \leq s \leq t} |X(s)|^p] \leq C_0 \lambda^p t^p.$$

PROOF. First assume that $V = 0$. Letting $\widetilde{X}(t) = \lambda X(t/\lambda^2)$, $t \geq 0$, we reduce (4.22) to the following inequality: for all $t \geq 1$ and for all $\lambda \leq 1$,

$$(4.23) \qquad E_0^{\lambda,\omega}[\max_{0 \leq s \leq t} |\widetilde{X}(s)|^p] \leq C_0 t^p.$$

Denote $\widetilde{T}_r = \inf\{s > 0 : |\widetilde{X}(s)| = r\}$.

We now rely on Aronson's lower bound for the Green function $\mathcal{G}(x, y, t)$ of the parabolic problem

$$2\partial_t U - \text{div}(a(x)\nabla U) - a(x)e_1 \cdot \nabla U = 0, \quad U\big|_{|x|=1} = 0.$$

According to theorems 8 and 9 in [1], for $t \leq 1$ and x with $|x| \leq \frac{1}{2}$ we have

$$\mathcal{G}(x, y, t) \geq k t^{-\frac{d}{2}} \exp(-K|x - y|^2/t)$$

with constants $k > 0$ and $K > 0$, which depend only on κ and d. This implies the bound

$$P_0^{\lambda,\omega}[\widetilde{T}_1 \geq 1] \geq \delta_0 > 0,$$

which, in turn, yields

$$(4.24) \qquad E_0^{\lambda,\omega}[e^{-\widetilde{T}_1}] \leq 1 - \varepsilon_0$$

for some $\varepsilon_0 > 0$, which depends only on κ and d.

We have

$$E_0^{\lambda,\omega}[\max_{0 \leq s \leq t} |\widetilde{X}(s)|^p] = \int_0^\infty p r^{p-1} P_0^{\lambda,\omega}[\widetilde{T}_r \leq t] dr$$

$$= \int_0^\infty p r^{p-1} P_0^{\lambda,\omega}[e^{-\widetilde{T}_r} \geq e^{-t}] dr$$

$$\leq e^t \int_0^\infty p r^{p-1} E_0^{\lambda,\omega}[e^{-\widetilde{T}_r}] dr.$$

Considering the inequality

$$\widetilde{T}_r \geq \widetilde{T}_1 + \widetilde{T}_1 \circ \theta_{\widetilde{T}_1} + \widetilde{T}_1 \circ \theta_{\widetilde{T}_2} + \cdots + \widetilde{T}_1 \circ \theta_{\widetilde{T}_{\lfloor r-1 \rfloor}},$$

by the Markov property and (4.24), for all $t \le 1$ we have

$$E_0^{\lambda,\omega}\Big[\max_{0\le s\le t}|\widetilde{X}(s)|^p\Big] \le e^t \int_0^\infty pr^{p-1} E_0^{\lambda,\omega}\big[e^{-\widetilde{T}_r}\big]dr$$

$$\le e \int_0^\infty pr^{p-1}(1-\varepsilon_0)^{\lfloor r\rfloor}dr \le C(p,\kappa,d).$$

Using Jensen's inequality and the Markov property again, for all $t \ge 1$ we obtain

$$E_0^{\lambda,\omega}\Big[\max_{0\le s\le t}|\widetilde{X}(s)|^p\Big]$$

$$\le E_0^{\lambda,\omega}\Big[\Big(\max_{0\le s\le 1}|\widetilde{X}(s)| + \max_{1\le s\le 2}|\widetilde{X}(s)-\widetilde{X}(1)|$$

$$+ \cdots + \max_{\lfloor t\rfloor\le s\le t}|\widetilde{X}(s)-\widetilde{X}(\lfloor t\rfloor)|\Big)^p\Big]$$

$$\le (t+1)^{p-1}\Big(E_0^{\lambda,\omega}\Big[\max_{0\le s\le 1}|\widetilde{X}(s)|^p\Big] + E_0^{\lambda,\omega}\Big[\max_{1\le s\le 2}|\widetilde{X}(s)-\widetilde{X}(1)|^p\Big]$$

$$+ \cdots + E_0^{\lambda,\omega}\Big[\max_{\lfloor t\rfloor\le s\le t}|\widetilde{X}(s)-\widetilde{X}(\lfloor t\rfloor)|^p\Big]\Big)$$

$$\le C(p,\kappa,d)(t+1)^p \le C_1(p,\kappa,d)t^p.$$

Recalling the definition of $\widetilde{X}$, we see that this is equivalent to (4.22). Hence the proof is complete in the case $V = 0$.

To extend the statement to the case $V \ne 0$, we use the time change arguments from the last part of the proof of Proposition 3.1. We observe that as in the $V = 0$ case, the process $Y^{\lambda,\omega}$ satisfies estimate (4.22), and due to (3.12), a similar bound also holds for the process $X^{\lambda,\omega}$. This completes the proof of the lemma. $\qquad\square$

Lemma 4.5 implies the following bound on the effective drift:

$$(4.25) \qquad\qquad |\ell(\lambda)| \le C_0\lambda.$$

5 Renewal Structure

As already mentioned in the introduction, X satisfies a law of large numbers under $\mathbb{P}_0^\lambda$. The next proposition is a quantitative version of this convergence.

PROPOSITION 5.1.

$$(5.1) \qquad\qquad \lim_{\alpha\to+\infty} \limsup_{\substack{\lambda\to 0 \\ t\to+\infty \\ \lambda^2 t=\alpha}} \left|\mathbb{E}_0^\lambda\left[\frac{X(t)}{\lambda t}\right] - \frac{\ell(\lambda)}{\lambda}\right| = 0.$$

The proof of Proposition 5.1 is based on a renewal argument. More precisely, (5.1) follows if we can construct a renewal structure such that the interval between two successive renewal times is of order $1/\lambda^2$.

Our definition of regeneration times is a variant of that in [25] where the construction depends on λ, whereas in [25] λ was fixed.

We shall also rely heavily on the PDE estimates proved in the previous section of the paper.

The first issue we have to address is to check that the approach developed in [25] applies to our model and does yield the law of large numbers. In addition, in the proof of Proposition 5.1 we need sharp estimates on the regeneration times.

5.1 Construction of Regeneration Times

We recall that λ is chosen small enough. In particular, we assume that $\frac{1}{\lambda}$ is much larger than the range R in Assumption 4. We shall also need a constant $l \geq 1$ chosen so that $ce^{-\kappa_1 l} \leq \frac{1}{2}$ where c and κ_1 are the constants appearing in Lemma 4.1.

We set $R(\lambda) := \frac{l}{\lambda}$. We now follow the construction of [25], replacing R in this construction with $R(\lambda)$. For details of this construction and proofs (and for pictures), we refer to [25]. We first have to enlarge the probability space by adding an auxiliary sequence $(Y_k)_{k \geq 0}$ of i.i.d. Bernoulli random variables. Denote by $B_R(x)$ the ball with center x and radius R. Let $U^x := B_{6R(\lambda)}(x + 5R(\lambda)e_1)$ and $B^x := B_{R(\lambda)}(x + 9R(\lambda)e_1)$, and let

$$(5.2) \qquad\qquad T_{\text{exit},U^x} := \inf\{s \geq 0 : X(s) \notin U^x\}$$

be the exit time from U^x. We consider the corresponding transition density, designated by $p_{\lambda,\omega,U^x}(s,x,y)$, which is defined by

$$P_x^{\lambda,\omega}[X(s) \in G, T_{\text{exit},U^x} > s] = \int_G p_{\lambda,\omega,U^x}(s,x,y)\,dy$$

for all open sets $G \subseteq U^x$. We will need the following bound for this transition density.

LEMMA 5.2. *There is some $\delta > 0$ (depending on V, σ, and d) such that*

$$(5.3) \quad p_{\lambda,\omega,U^x}(1/\lambda^2, x, y) \geq \frac{2\delta}{|B_{R(\lambda)}|} \quad \text{for all } x \in \mathbb{R}^d, \ y \in B^x, \text{ and } \lambda \leq 1.$$

PROOF. Again we begin with the case $V = 0$. After rescaling $t/\lambda^2 \to t$, $(x/\lambda, y/\lambda) \to (x, y)$, the required bound is an immediate consequence of theorems 8 and 9 in [1].

If $V \neq 0$, then the desired lower bound is an immediate consequence of the following statement:

LEMMA 5.3. *Let a function $\rho(x)$ and a symmetric matrix $\{\alpha_{ij}(x)\}$ satisfy the estimates*

$$\varkappa \leq \rho \leq \varkappa^{-1}, \quad \varkappa I \leq \alpha \leq \varkappa^{-1}I, \quad \varkappa > 0,$$

and denote by $G_B(t,x,y)$ a solution to the following parabolic problem:

$$(5.4) \quad \begin{aligned} \rho(x)\partial_t G_B(t,x,y) - \text{div}_x(\alpha(x)\nabla_x G_B(t,x,y)) &= 0, \\ G_B\big|_{x \in \partial B} = 0, \quad G_B(0,x,y) &= \delta(x-y), \end{aligned}$$

with B being an open ball centered at the origin, and $y \in B$. Then for any ball B_0 such that $\overline{B}_0 \subset B$, the following inequality holds:

$$(5.5) \qquad\qquad G_B(1, x, 0) \geq C \quad \text{for } x \in B_0,$$

with a constant C that depends only on $\varkappa$, d, B, and B_0.

As in the case $V = 0$ estimate (5.3) can be obtained from (5.5) by scaling $t/\lambda^2 \to t$, $(x/\lambda, y/\lambda) \to (x, y)$.

PROOF OF LEMMA 5.3. Enlarging the ball B_0 if necessary, we assume without loss of generality that B_0 contains the origin.

Consider an auxiliary spectral problem in B

$$-\operatorname{div}(\alpha(x)\nabla\Psi(x)) = \rho(x)\nu\Psi(x), \quad \Psi\big|_{\partial B} = 0.$$

By means of the minimax principle one can check that the principal eigenvalue ν_1 satisfies the estimate $0 < \nu_1 < C_1$. The principal eigenfunction Ψ_1 is known to be positive in B. Assuming the normalization

$$\int_B \Psi_1(x)dx = 1,$$

by the Harnack inequality and Hölder continuity arguments (see [10]) we conclude that

$$(5.6) \qquad\qquad \Psi_1(x) \begin{cases} \leq C_2 & \text{in } B, \\ \geq C_3 & \text{in } B_0, \end{cases}$$

where the constants C_1 and C_2 depend only on $\varkappa$, d, and B, and C_3 also depends on B_0. Clearly, the function $e^{-\nu_1 t}\Psi_1(x)$ solves problem (5.4) with the initial condition Ψ_1. Therefore,

$$e^{-\nu_1/2}\Psi_1(x) = \int_B G_B(\tfrac{1}{2}, x, y)\Psi_1(y)dy.$$

Making use of (5.6) and the upper bound for ν_1 we derive the inequality

$$\int_B G_B(\tfrac{1}{2}, 0, y)dy \geq C_4.$$

Considering the symmetry of the operator with respect to the weighted measure $\rho\, dx$, we have $\rho(x)G_B(t, x, y) = \rho(y)G_B(t, y, x)$. It readily follows from the results of [11] that the function $G(\tfrac{1}{2}, y, 0)$ satisfies the upper bound $G_B(\tfrac{1}{2}, y, 0) \leq C_5$ in B with a constant C_5 that depends only on κ and d. Consequently, there is a smaller ball B_1 centered at the origin, $\overline{B}_1 \subset B$, such that

$$\int_{B_1} G_B(\tfrac{1}{2}, y, 0)dy \geq \frac{C_4}{2};$$

the radius of B_1 depends only on B, κ, and d. This yields

$$(5.7) \qquad\qquad \sup_{y \in B_1} G_B(\tfrac{1}{2}, y, 0) \geq C_6.$$

Without loss of generality we suppose that $B_0 \subset B_1$.

According to [11, theorem 1-5], the function $G_B(1, y, 0)$ satisfies the following version of the Harnack inequality in B_1:

$$\inf_{y \in B_1} G_B(1, y, 0) \geq C_7(\varkappa, d) \sup_{y \in B_1} G_B(\tfrac{1}{2}, y, 0).$$

This estimate combined with (5.7) yields (5.5) and we have completed the proof of Lemma 5.3.   □

We have also completed the proof of Lemma 5.2.   □

Due to (5.3), we can give the following coupling construction. Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $(X(t))_{t \geq 0}$ and $\mathcal{S}_m := \sigma(Y_0, \ldots, Y_m)$. We denote by θ_m^λ the rescaled shift operator, defined as

$$\theta_m^\lambda((X(s)_{s \geq 0}) = (X(\lambda^{-2}m + s))_{s \geq 0}.$$

These shift operators θ_m^λ, $m \in \mathbb{N}$, are extended in the obvious way:

$$\theta_m^\lambda((X(s))_{s \geq 0}, (Y_k)_{k \geq 0}) = ((X(\lambda^{-2}m + s))_{s \geq 0}, (Y_{m+k})_{k \geq 0}).$$

PROPOSITION 5.4. *There exists, for every λ, ω, and x, a probability measure $\widehat{P}_x^{\lambda,\omega}$ on the enlarged probability space such that, with δ from (5.3),*

(i) *The law of $(X(t))_{t \geq 0}$ under $\widehat{P}_x^{\lambda,\omega}$ is $P_x^{\lambda,\omega}$, and the sequence $(Y_k)_{k \geq 0}$ is a sequence of i.i.d. Bernoulli variables with success probability δ under $\widehat{P}_x^{\lambda,\omega}$.*

(ii) *Under $\widehat{P}_x^{\lambda,\omega}$, $(Y_n)_{n \geq m}$ is independent of $\mathcal{F}_{\lambda^{-2}m} \times \mathcal{S}_{m-1}$ and conditioned on $\mathcal{F}_{\lambda^{-2}m} \times \mathcal{S}_m$, and $X \circ \theta_m^\lambda$ has the same law as X under $\widehat{P}_{X(\lambda^{-2}m),Y_m}^{\lambda,\omega}$, where $\widehat{P}_{x,y}^{\lambda,\omega}$ denotes the conditioned law $\widehat{P}_x^{\lambda,\omega}[\cdot \mid Y_0 = y]$ (for $y \in \{0, 1\}$).*

(iii) *$\widehat{P}_{x,1}^{\lambda,\omega}$-almost surely, $X(t) \in U^x$ for $t \in [0, \lambda^{-2}]$, and the distribution of $X(\lambda^{-2})$ under $\widehat{P}_{x,1}^{\lambda,\omega}$ is the uniform distribution on B^x.*

We refer to [25] for the proof.

We will now introduce random times $N_k \in \lambda^{-2}\mathbb{Z}^+$ for which $Y_{\lambda^2 N_k} = 1$ and for which the process $(e_1 \cdot X(t))_{t \geq 0}$ essentially reaches a local maximum (within a variation of $R(\lambda)$). The first regeneration time τ_1 will be the first time $N_k + \lambda^{-2}$, $k \geq 1$, such that $(e_1 \cdot X(t))_{t \geq 0}$ never goes below $e_1 \cdot X(N_k + \lambda^{-2}) - R(\lambda)$ after $N_k + \lambda^{-2}$. In order to define N_k, we will first consider stopping times $\widetilde{N}_k \in \lambda^{-2}\mathbb{Z}^+$, which are essentially the times when $(e_1 \cdot X(t))_{t \geq 0}$ reaches local maxima (also within a variation of $R(\lambda)$). Then, N_1 will be the first $\widetilde{N}_k$ with $Y_{\lambda^2 \widetilde{N}_k} = 1$.

Let

$$(5.8) \qquad\qquad M(t) := \sup\{e_1 \cdot (X(s) - X(0)) : 0 \leq s \leq t\}.$$

For $a > 0$, define the stopping times $V_k^\lambda(a)$, $k \geq 1$, as follows. Recall that $T_L = \inf\{t : e_1 \cdot X(t) = L\}$, and define

$$(5.9) \qquad V_0^\lambda(a) := T_{\lambda^{-1}a}, \quad V_{k+1}^\lambda(a) := T_{M(\lceil V_k^\lambda(a)\rceil_\lambda) + R(\lambda)}, \quad k \geq 1;$$

here and later on $\lceil r \rceil_\lambda$ stands for the $\min\{n \in \lambda^{-2}\mathbb{Z} : r \leq n\}$. Then

$$(5.10) \qquad \widetilde{N}_1^\lambda(a) := \inf\Big\{\lceil V_k^\lambda(a)\rceil_\lambda : k \geq 0,$$
$$\sup_{s \in [V_k^\lambda(a), \lceil V_k^\lambda(a)\rceil_\lambda]} \big|e_1 \cdot \big(X(s) - X(V_k^\lambda(a))\big)\big| \leq \frac{R(\lambda)}{2}\Big\},$$

$$(5.11) \quad \widetilde{N}_{k+1}^\lambda(a) := \widetilde{N}_1^\lambda(3\lambda R(\lambda)) \circ \theta_{\lambda^2 \widetilde{N}_k^\lambda(a)}^\lambda + \widetilde{N}_k^\lambda(a), \quad k \geq 1,$$

$$(5.12) \qquad N_1^\lambda(a) := \inf\{\widetilde{N}_k^\lambda(a) : k \geq 1, \ Y_{\lambda^2 \widetilde{N}_k^\lambda(a)} = 1\}$$

(we will see later that $\widetilde{N}_k^\lambda(a) < \infty$ for all k). The random times $\lambda^2 \widetilde{N}_k^\lambda(a)$ are integer valued and $\sup_{s \leq \widetilde{N}_k^\lambda(a)} e_1 \cdot (X(s) - X(\widetilde{N}_k^\lambda(a))) \leq R(\lambda)$.

We next define random times S_1, J_1, and R_1 as follows:

$$(5.13) \qquad S_1^\lambda := N_1^\lambda(3\lambda R(\lambda)) + \lambda^{-2}, \quad J_1^\lambda := S_1^\lambda + T_{-R(\lambda)} \circ \theta_{\lambda^2 S_1^\lambda}^\lambda,$$
$$R_1^\lambda := \lceil J_1^\lambda \rceil_\lambda = S_1^\lambda + D \circ \theta_{\lambda^2 S_1^\lambda}^\lambda,$$

where

$$(5.14) \qquad\qquad\qquad D := \lceil T_{-R(\lambda)} \rceil_\lambda.$$

Now we proceed recursively:

$$(5.15) \qquad\qquad N_{k+1}^\lambda = R_k^\lambda + N_1^\lambda(a_k) \circ \theta_{\lambda^2 R_k^\lambda}^\lambda$$

with

$$a_k = \lambda\big(M\big(R_k^\lambda\big) - e_1 \cdot \big(X\big(R_k^\lambda\big) - X(0)\big) + R(\lambda)\big)$$

and

$$S_{k+1}^\lambda := N_{k+1}^\lambda + \lambda^{-2}, \quad J_{k+1}^\lambda := S_{k+1}^\lambda + T_{R(\lambda)} \circ \theta_{\lambda^2 S_{k+1}^\lambda}^\lambda,$$
$$R_{k+1}^\lambda := \lceil J_{k+1}^\lambda \rceil_\lambda = S_{k+1}^\lambda + D \circ \theta_{\lambda^2 S_{k+1}^\lambda}^\lambda.$$

Note that for all k, the $\mathcal{F}_t \times \mathcal{S}_{\lambda^2 \lceil t \rceil_\lambda}$ stopping times $\lambda^2 N_k^\lambda$, $\lambda^2 S_k^\lambda$, and $\lambda^2 R_k^\lambda$ are integer-valued (the value $+\infty$ is possible). By definition, we have

$$\lambda^{-2} \leq N_1^\lambda \leq S_1^\lambda \leq J_1^\lambda \leq R_1^\lambda \leq N_2^\lambda \leq S_2^\lambda \leq J_2^\lambda \leq R_2^\lambda \leq N_3^\lambda \cdots \leq \infty.$$

The first regeneration time τ_1 is defined as

$$(5.16) \qquad\qquad \tau_1 := \inf\{S_k^\lambda : S_k^\lambda < \infty, \ R_k^\lambda = \infty\} \leq \infty.$$

By definition, $\lambda^2\tau_1$ is integer-valued and $\tau_1 \geq 2\lambda^{-2}$ (since $N_1^\lambda \geq \lambda^{-2}$). We see that on the event $\tau_1 < \infty$ it holds that

$$e_1 \cdot X(s) \leq e_1 \cdot X(\tau_1 - \lambda^{-2}) + R(\lambda)$$

$$\leq e_1 \cdot X(\tau_1) - 7R(\lambda) \quad \text{for } s \leq \tau_1 - \lambda^{-2}, \ \widehat{P}_x^{\lambda,\omega} - \text{a.s.};$$

see also Proposition 5.4, i.e., $(X(s))_{s \leq \tau_1 - \lambda^{-2}}$ remains in the half-space $\{z \in \mathbb{R}^d : e_1 \cdot z \leq e_1 \cdot X(\tau_1) - 7R(\lambda)\}$. On the other hand, since the process $(e_1 \cdot X(t))_{t \geq 0}$ never goes below $e_1 \cdot X(\tau_1) - R(\lambda)$ after τ_1, $\widehat{P}_x^{\lambda,\omega}$-a.s., $(X(t))_{t > \tau_1}$ remains in the half-space $\{z \in \mathbb{R}^d : e_1 \cdot z \geq e_1 \cdot X(\tau_1) - R(\lambda)\}$.

In [25] it is proved that $\tau_1 < \infty$ if and only if the process is transient in direction e_1. More precisely, define the annealed law

$$(5.17) \qquad \widehat{\mathbb{P}}_x^\lambda[A] := \int d\mathbb{Q}(\omega) \int d\widehat{P}_x^{\lambda,\omega}(w) \mathbb{1}_A(\omega, w).$$

PROPOSITION 5.5. *The following statements are equivalent*:

   (i) $\tau_1 < \infty$, $\widehat{\mathbb{P}}_0^\lambda$-*a.s.*
   (ii) $e_1 \cdot X(t) \to \infty$, $\mathbb{P}_0^\lambda$-*a.s.*
   (iii) $\mathbb{P}_0^\lambda[D = \infty] > 0$.

For the proof, we refer to [25, lemma 2.3, prop. 2.7]. Later we will need stronger results than those of Proposition 5.5 involving sharp bounds on τ_1 as $\lambda \to 0$.

We set $\tau_0 = 0$ for convenience. The next theorem, whose statement is essentially the same as theorem 2.5 in [25], gives the renewal structure, which is crucial to establish (for fixed values of λ) the law of large numbers and the central limit theorem.

THEOREM 5.6. *Assume that* $\tau_1 < \infty$, $\widehat{\mathbb{P}}_0^\lambda$-*a.s. Then, under the measure* $\widehat{\mathbb{P}}_0^\lambda$, *the random variables*

$$Z_k := \big((X((\tau_k + t) \wedge (\tau_{k+1} - \lambda^{-2})) - X(\tau_k))_{t \geq 0},$$

$$X(\tau_{k+1}) - X(\tau_k), \tau_{k+1} - \tau_k\big), \quad k \geq 0,$$

are independent. Furthermore, the random variables Z_k, $k \geq 1$, *are i.i.d. under* $\widehat{\mathbb{P}}_0^\lambda$ *and have the same law as* Z_0 *under* $\widehat{\mathbb{P}}_0^\lambda[\cdot \mid D = \infty]$.

Note that the renewal structure is proved for the trajectory between the times τ_k and $\tau_{k+1} - \lambda^{-2}$, but we have a good control over the trajectory between the times $\tau_{k+1} - \lambda^{-2}$ and τ_{k+1}: since $Y_{\lambda^2\tau_{k+1}-1} = 1$, then $X(s) \in U^{X_{\tau_{k+1}-1/\lambda^2}}$ for all $s \in [\tau_{k+1} - \lambda^{-2}, \tau_{k+1}]$; i.e., the trajectory remains in a ball of radius $6R(\lambda)$.

Let

$$(5.18) \qquad K = \inf\{k \geq 1 : S_k^\lambda < \infty \text{ and } R_k^\lambda = \infty\}.$$

Then $\tau_1 = S_K^\lambda$. The points $X(S_1^\lambda)$, $X(S_2^\lambda)$, ..., are *ladder points* of the process. The idea of such a decomposition of the path goes back to [13] and was first proved effective for multidimensional random walks in random environments in [28].

LEMMA 5.7. *The following statements hold*:

(i) S_1^λ is $\widehat{\mathbb{P}}_0^\lambda$-*almost surely finite and, for all k, S_k^λ is $\widehat{\mathbb{P}}_0^\lambda$-almost surely finite on the event $R_{k-1}^\lambda < \infty$.*

(ii) τ_k is $\widehat{\mathbb{P}}_0^\lambda$-*almost surely finite for all $k \geq 1$.*

(iii) $\widehat{\mathbb{E}}_0^\lambda[\tau_k] < \infty$ *for all $k \geq 1$.*

PROOF.

(i) Due to Lemma 4.2, $V_k^\lambda(a) < \infty$, $\widehat{\mathbb{P}}_x^\lambda$-a.s for all k and all x. We have (as in Lemma 5.2 this is a consequence of the Aronson-Nash lower bounds for the Green function of a parabolic equation)

$$(5.19) \qquad \inf_{\lambda \leq 1} \inf_{x \in \mathbb{R}^d} \inf_\omega \widehat{P}_x^{\lambda,\omega} \left[\max_{s \leq 1/\lambda^2} |e_1 \cdot X(s)| \leq \frac{R(\lambda)}{2} \right] = \tilde{\delta} > 0,$$

and this implies that $\tilde{N}_k^\lambda(a) < \infty$ for all a and k, $\widehat{\mathbb{P}}_x^\lambda$-a.s for all x. Due to Proposition 5.4, we conclude that $N_k^\lambda(a) < \infty$ for all a and k, $\widehat{\mathbb{P}}_x^\lambda$-a.s for all x.

(ii) It follows from Lemma 4.1 that under the proper choice of l in the definition of $R(\lambda)$, the following bound holds:

$$(5.20) \qquad \widehat{\mathbb{P}}_0^\lambda[K = 1] = \widehat{\mathbb{P}}_0^\lambda[R_1^\lambda = \infty] \geq 1 - ce^{-\kappa_1 l} \geq \frac{1}{2}.$$

Lemma 4.1 and the Markov property applied at time S_{k+1}^λ also imply that

$$\widehat{\mathbb{P}}_0^\lambda[R_{k+1}^\lambda = \infty \mid R_k^\lambda < \infty] \geq \frac{1}{2}.$$

Thus we get that

$$(5.21) \qquad \widehat{\mathbb{P}}_0^\lambda[K \geq k] \leq 2^{-k+1}$$

and

$$\widehat{\mathbb{P}}_0^\lambda[K = \infty] = 0.$$

Together with part (i), this implies part (ii).

(iii) The next lemma gives a bound for the tail of the random variable τ_1, which will be sufficient to guarantee that τ_1 has finite expectation under $\widehat{\mathbb{P}}_0^\lambda$.

LEMMA 5.8. *There exist constants $c_3 > 0$ and $c_4 > 0$ such that*

$$(5.22) \qquad \text{for all } \lambda \leq 1 \text{ and } t > 0, \quad \widehat{\mathbb{P}}_0^\lambda[\tau_1 \geq \lambda^{-2}t] \leq c_3 \epsilon^{-c_4 t}.$$

The same tail bound holds for the differences $\tau_{k+1} - \tau_k$ for all $k \geq 1$:

$$(5.23) \qquad \text{for all } \lambda \leq 1 \text{ and } t > 0, \quad \widehat{\mathbb{P}}_0^\lambda[\tau_{k+1} - \tau_k \geq \lambda^{-2}t] \leq c_3 \epsilon^{-c_4 t}.$$

PROOF. It suffices to show (5.22). Then (5.23) follows since

$$\widehat{\mathbb{P}}_0^\lambda[\tau_{k+1} - \tau_k \geq \lambda^{-2}t] = \widehat{\mathbb{P}}_0^\lambda[\tau_1 \geq \lambda^{-2}t \mid D = \infty]$$

(see Theorem 5.6) and $\widehat{\mathbb{P}}_0^\lambda[D = \infty] \geq \frac{1}{2}$; see (5.20). To show (5.22), we claim that the following stronger statement holds: There exist constants $c_3 > 0$ and $c_4 > 0$ such that for all $\lambda \leq 1$, $t > 0$, and all ω,

$$(5.24) \qquad \widehat{P}_0^{\lambda,\omega}[\tau_1 \geq \lambda^{-2}t] \leq c_3 \epsilon^{-c_4 t}.$$

Since one can follow the proof of corollary 4.10 in [25], we give only a sketch of the proof of (5.24).

Step 1. (Corresponds to proposition 4.7 in [25].) Recall (5.8). There exists a constant $c_5 > 0$ such that

$$(5.25) \qquad \sup_\omega \widehat{E}_0^{\lambda,\omega}[\exp(c_5 \lambda e_1 \cdot M(T_{-R(\lambda)}))\mathbb{1}(T_{-R(\lambda)} < \infty)] < \infty.$$

PROOF OF (5.25). Due to Lemma 4.1, $\widehat{P}_0^{\lambda,\omega}[\lambda e_1 \cdot M(T_{-R(\lambda)}) > t, T_{-R(\lambda)} < \infty] \leq ce^{-\kappa_1(l+t)}$, and this implies (5.25). $\qquad\square$

Step 2. Follow the proof of theorem 4.9 in [25] to obtain that there is a constant $c_6 > 0$ such that for all $\lambda \leq 1$,

$$(5.26) \qquad \sup_\omega \widehat{E}_0^{\lambda,\omega}[\exp(c_6 \lambda e_1 \cdot X_{\tau_1})] < \infty.$$

Step 3. Take $t > \frac{6l}{\kappa_2}$ and $u = \frac{\kappa_2}{2}t$ where κ_2 is the constant from Lemma 4.2. Then,

$$(5.27) \qquad \begin{aligned} \widehat{P}_0^{\lambda,\omega}[\tau_1 \geq \lambda^{-2}t] &\leq \widehat{P}_0^{\lambda,\omega}[\tau_1 \geq \lambda^{-2}t, e_1 \cdot X_{\tau_1} < \lambda^{-1}u - 3R(\lambda)] \\ &\quad + \widehat{P}_0^{\lambda,\omega}[e_1 \cdot X_{\tau_1} \geq \lambda^{-1}u - 3R(\lambda)]. \end{aligned}$$

Due to Step 2, the second term on the right-hand side of (5.27) is $\leq e^{-c_7 u}$ for some $c_7 > 0$. Turning to the first term in (5.27), note that since

$$\sup_{s \leq \tau_1} e_1 \cdot (X_s - X_{\tau_1}) < 3R(\lambda)$$

(see the definition of U^x and B^x before (5.2)),

$$\begin{aligned} \widehat{P}_0^{\lambda,\omega}[\tau_1 \geq \lambda^{-2}t, e_1 \cdot X_{\tau_1} < \lambda^{-1}u - 3R(\lambda)] &\leq P_0^{\lambda,\omega}[T_{\lambda^{-1}u} \geq \lambda^{-2}t] \\ &\leq Ce^{-\kappa_2 t + u}, \end{aligned}$$

where the last inequality comes from Lemma 4.2. $\qquad\square$

As mentioned before, the regeneration structure implies a law of large numbers for fixed λ.

PROPOSITION 5.9. *We have, for each $\lambda > 0$,*

$$(5.28) \qquad \lim_{t \to \infty} \frac{X(t)}{t} = \frac{\widehat{\mathbb{E}}_0^\lambda[X(\tau_2) - X(\tau_1)]}{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]} \quad \mathbb{P}_0^\lambda\text{- a.s.}$$

As a consequence,

$$(5.29) \qquad \widehat{\mathbb{E}}_0^\lambda[X(\tau_2) - X(\tau_1)] = \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]\ell(\lambda).$$

PROOF. Theorem 5.6 implies, with the ergodic theorem, that if $\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]$ is finite, then $\lim_{t \to \infty} \frac{X(t)}{t}$ exists, $\mathbb{P}_0^\lambda$-a.s., and (5.28) holds true. Once again, we refer to [25] for the details. □

□

5.2 Estimates on the Regeneration Times

We now show that under $\widehat{P}_0^\lambda$, τ_1 and $\tau_2 - \tau_1$ are of order λ^{-2}. More precisely, we have the following:

LEMMA 5.10. *We have*

$$(5.30) \quad \limsup_{\lambda \to 0} \lambda^4 \widehat{\mathbb{E}}_0^\lambda[\tau_1^2] < \infty \quad \text{and} \quad \limsup_{\lambda \to 0} \lambda^4 \widehat{\mathbb{E}}_0^\lambda[(\tau_2 - \tau_1)^2] < \infty.$$

As a consequence,

$$(5.31) \quad \limsup_{\lambda \to 0} \lambda^2 \widehat{\mathbb{E}}_0^\lambda[\tau_1] < \infty \quad \text{and} \quad \limsup_{\lambda \to 0} \lambda^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] < \infty$$

and

$$(5.32) \quad \limsup_{\lambda \to 0} \frac{\widehat{\mathbb{E}}_0^\lambda[\tau_1^2]}{\widehat{\mathbb{E}}_0^\lambda[\tau_1]^2} < \infty \quad \text{and} \quad \limsup_{\lambda \to 0} \frac{\widehat{\mathbb{E}}_0^\lambda[(\tau_2 - \tau_1)^2]}{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2} < \infty.$$

PROOF. Observe that (5.31) directly follows from (5.30) and Hölder's inequality.

To deduce (5.32) from (5.30), it suffices to prove a lower bound on $\widehat{\mathbb{E}}_0^\lambda[\tau_1]$ and $\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]$ of the order λ^{-2}. But note that, since $\tau_1 \geq T_{l/\lambda}$, we have

$$\lambda^2 \widehat{\mathbb{E}}_0^\lambda[\tau_1] \geq \widehat{\mathbb{P}}_0^\lambda[\tau_1 \geq \lambda^{-2}] \geq \mathbb{P}_0^\lambda[T_{l/\lambda} \geq \lambda^{-2}].$$

We next use the Girsanov formula (3.4):

$$\mathbb{P}_0^\lambda[T_{l/\lambda} \geq \lambda^{-2}] = \int E\left[\mathbb{1}(T_{l/\lambda} \geq \lambda^{-2})e^{\lambda \bar{B}(\frac{1}{\lambda^2}) - \frac{\lambda^2}{2}\langle \bar{B}\rangle(\frac{1}{\lambda^2})}\right] d\mathbb{Q}.$$

Thus the invariance principle implies that this last quantity has a positive limit: namely, it converges to $E[\mathbb{1}(T_l \geq 1)e^{Z - (1/2)E(Z^2)}]$, where Z is some Gaussian random variable and T_l is the hitting time of level l by some Brownian motion. This last expectation cannot be 0. The same argument applies to $\tau_2 - \tau_1$.

To prove (5.30), due to (5.22) we have

$$\widehat{\mathbb{P}}_0^\lambda[\tau_1^2 \geq t\lambda^{-4}] = \widehat{\mathbb{P}}_0^\lambda[\tau_1 \geq \sqrt{t}\lambda^{-2}] \leq c_3 e^{-c_4\sqrt{t}}.$$

In the same way, due to (5.23), we have

$$\widehat{\mathbb{P}}_0^\lambda[(\tau_2 - \tau_1)^2 \geq t\lambda^{-4}] \leq c_3 e^{-c_4\sqrt{t}},$$

and (5.30) follows.   □

We can now show the corresponding bounds for the regeneration distances.

LEMMA 5.11. *We have*

$$(5.33) \qquad\qquad \limsup_{\lambda \to 0} \lambda^2 \widehat{\mathbb{E}}_0^\lambda[|X(\tau_1)|^2] < \infty$$

and

$$(5.34) \qquad\qquad \limsup_{\lambda \to 0} \lambda^2 \widehat{\mathbb{E}}_0^\lambda[|X(\tau_2) - X(\tau_1)|^2] < \infty.$$

We note for further reference that, as a consequence of (5.33) and (5.34),

$$(5.35) \qquad\qquad \sup_k \frac{1}{k^2} \limsup_{\lambda \to 0} \lambda^2 \widehat{\mathbb{E}}_0^\lambda[|X(\tau_k)|^2] < \infty.$$

Moreover, note that due to (4.25),

$$(5.36) \qquad\qquad \limsup_{\lambda \to 0} \frac{|\ell(\lambda)|}{\lambda} < \infty.$$

(This also follows from (5.34) together with (5.29), (5.31), and (5.32).)

PROOF OF LEMMA 5.11. To show (5.33), note that

$$\begin{aligned}
\widehat{\mathbb{E}}_0^\lambda[X(\tau_1)^2] &= \sum_{k=0}^{\infty} \widehat{\mathbb{E}}_0^\lambda[X(\tau_1)^2 \mathbb{1}(k \leq \lambda^2\tau_1 < k+1)] \\
&\leq \sum_{k=0}^{\infty} \widehat{\mathbb{E}}_0^\lambda\Big[\max_{t \leq (k+1)/\lambda^2} |X(t)|^2 \mathbb{1}(k \leq \lambda^2\tau_1)\Big] \\
&\leq 2 \sum_{k=0}^{\infty} \big(\mathbb{E}_0^\lambda\big[\max_{t \leq (k+1)/\lambda^2} |X(t)|^4\big]\big)^{1/2} (\widehat{\mathbb{P}}_0^\lambda[\lambda^2\tau_1 \geq k])^{1/2} \\
&\leq 2 \sum_{k=0}^{\infty} \Big(\lambda \frac{k+1}{\lambda^2}\Big)^2 (c_3 e^{-c_4 k})^{1/2} \leq c_8 \lambda^{-2}
\end{aligned}$$

for some constant $c_8 > 0$, where we used (4.22) and (5.22). Now, (5.34) follows from (5.33), since

$$\widehat{\mathbb{E}}_0^\lambda[|X(\tau_2) - X(\tau_1)|^2] = \widehat{\mathbb{E}}_0^\lambda[|X(\tau_1)|^2 | D = \infty]$$

(see Theorem 5.6) and $\widehat{\mathbb{P}}_0^\lambda[D = \infty] \geq \frac{1}{2}$ for all λ; see (5.20).   □

5.3 Proof of Proposition 5.1

Let

$$n(t) := \left\lfloor \frac{t}{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]} \right\rfloor.$$

We write

$$(5.37) \qquad X(t) = X(\tau_{n(t)}) + (X(t) - X(\tau_{n(t)})).$$

We will show that

$$(5.38) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(\tau_{n(t)})] - \frac{\ell(\lambda)}{\lambda} \right| = 0$$

and that

$$(5.39) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(t) - X(\tau_{n(t)})] \right| = 0.$$

Equations (5.38) and (5.39) then imply

$$(5.40) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \mathbb{E}_0^\lambda \left[\frac{X(t)}{\lambda t} \right] - \frac{\ell(\lambda)}{\lambda} \right| = 0,$$

i.e., Proposition 5.1.

To show (5.38), note that (recalling $\tau_0 = 0$),

$$(5.41) \quad \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(\tau_{n(t)})] - \frac{\ell(\lambda)}{\lambda} \right| = \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda \left[\sum_{k=1}^{n(t)} (X(\tau_k) - X(\tau_{k-1})) \right] - \frac{\ell(\lambda)}{\lambda} \right|.$$

Using formula (5.29) in Proposition 5.9, we rewrite and estimate the right-hand side of (5.41) as

$$(5.42) \quad \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(\tau_1)] + \frac{1}{\lambda t}(n(t) - 1)\widehat{\mathbb{E}}_0^\lambda[X(\tau_2) - X(\tau_1)] - \frac{\ell(\lambda)}{\lambda} \right| \leq$$

$$\left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(\tau_1)] \right| + \left| \frac{\ell(\lambda)}{\lambda} \left(\frac{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]}{t} \left(\left\lfloor \frac{t}{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]} \right\rfloor - 1 \right) - 1 \right) \right|.$$

Due to (5.33), the first term in the right-hand side of (5.42) is of order α^{-1} and therefore satisfies

$$\lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[X(\tau_1)] \right| = 0.$$

Now consider the second term in (5.42). From (5.31), we know that $\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]/t$ is of order α^{-1}. Therefore

$$\lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left(\frac{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]}{t} \left(\left\lfloor \frac{t}{\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]} \right\rfloor - 1 \right) - 1 \right) = 0.$$

We also proved in (5.36) that $|\ell(\lambda)|/\lambda$ remains bounded for λ tending to 0. Hence we see that the second term in the right-hand side of (5.42) also tends to 0, thus proving that (5.38) holds true.

To show (5.39), we need the following lemma, whose proof is deferred.

LEMMA 5.12. *We have, for each $\varepsilon > 0$,*

$$(5.43) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \widehat{\mathbb{P}}_0^\lambda[|\tau_{n(t)} - t| \geq \varepsilon t] = 0.$$

We will split the integration in (5.39) according to the partition in the two events $\{|\tau_{n(t)} - t| \geq \varepsilon t\}$ and $\{(1 - \varepsilon)t < \tau_{n(t)} < (1 + \varepsilon)t\}$. We will show that, for each $\varepsilon > 0$,

$$(5.44) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda[(X(t) - X(\tau_{n(t)})\mathbb{1}(|\tau_{n(t)} - t| \geq \varepsilon t)] \right| = 0$$

and that

$$(5.45) \qquad \lim_{\substack{\varepsilon \to 0}} \limsup_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \left| \frac{1}{\lambda t} \widehat{\mathbb{E}}_0^\lambda \left[(X(t) - X(\tau_{n(t)})\mathbb{1}((1 - \varepsilon)t < \tau_{n(t)} < (1 + \varepsilon)t) \right] \right| = 0.$$

To show (5.44), we use Hölder's inequality to get that

$$\left| \widehat{\mathbb{E}}_0^\lambda[(X(t) - X(\tau_{n(t)})\mathbb{1}(|\tau_{n(t)} - t| \geq \varepsilon t)] \right|$$

$$\leq \widehat{\mathbb{E}}_0^\lambda[|X(t) - X(\tau_{n(t)})|^2]^{\frac{1}{2}} \widehat{\mathbb{P}}_0^\lambda[|\tau_{n(t)} - t| \geq \varepsilon t]^{\frac{1}{2}}$$

$$\leq 2 \big(\widehat{\mathbb{E}}_0^\lambda[|X(t)|^2] + \widehat{\mathbb{E}}_0^\lambda[|X(\tau_{n(t)})|^2] \big)^{\frac{1}{2}} \widehat{\mathbb{P}}_0^\lambda[|\tau_{n(t)} - t| \geq \varepsilon t]^{\frac{1}{2}}$$

(for the last inequality, we used the formula $(x - y)^2 \leq 2(x^2 + y^2)$).

Now we conclude with (3.2), (5.35), and Lemma 5.12.

For (5.45), note that

$$\left| \widehat{\mathbb{E}}_0^\lambda[(X(t) - X(\tau_{n(t)})\mathbb{1}((1 - \varepsilon)t < \tau_{n(t)} < (1 + \varepsilon)t)] \right|$$

$$\leq \mathbb{E}_0^\lambda \Big[\max_{(1-\varepsilon)t \leq u < s \leq (1+\varepsilon)t} |X(u) - X(s)| \Big]$$

$$\leq 2\mathbb{E}_0^\lambda \Big[\max_{(1-\varepsilon)t \leq u \leq (1+\varepsilon)t} |X(u) - X((1 - \varepsilon)t)| \Big].$$

Using the Markov property, we see that the last term equals

$$g(\lambda, \varepsilon, t) := 2\mathbb{E}_0^\lambda \Big[\max_{u \leq 2\varepsilon t} |X(u)| \Big].$$

Due to (3.2), for each ε,

$$\sup_{\alpha\varepsilon \geq 1} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 \varepsilon t = \alpha}} \frac{g(\lambda, \varepsilon, t)}{\lambda \varepsilon t} < \infty,$$

and this proves (5.45).

PROOF OF LEMMA 5.12. It remains to prove Lemma 5.12. We first show the following:

LEMMA 5.13. *For all $\varepsilon > 0$,*

$$(5.46) \qquad \widehat{\mathbb{P}}_0^\lambda \big[|\tau_k - k\mathbb{E}_0^\lambda[\tau_2 - \tau_1]| \geq \varepsilon k \mathbb{E}_0^\lambda[\tau_2 - \tau_1] \big] \to 0 \quad for\ k \to \infty$$

uniformly for $\lambda \leq 1$.

PROOF. We have

$$\widehat{\mathbb{P}}_0^\lambda \left[\left| \frac{\tau_k}{k} - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] \right| \geq \varepsilon \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] \right]$$

$$\leq \frac{1}{\varepsilon^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2} \widehat{\mathbb{E}}_0^\lambda \left[\left(\frac{\tau_k}{k} - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] \right)^2 \right]$$

$$\leq \frac{1}{k^2 \varepsilon^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2}$$

$$\times \widehat{\mathbb{E}}_0^\lambda \left[\left(\tau_1 - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] + \sum_{j=2}^k (\tau_j - \tau_{j-1} - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]) \right)^2 \right]$$

$$= \frac{1}{k^2 \varepsilon^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2}$$

$$\times \left(\widehat{\mathbb{E}}_0^\lambda \big[(\tau_1 - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1])^2 \big] + \sum_{j=2}^k \widehat{\mathbb{E}}_0^\lambda \big[(\tau_j - \tau_{j-1} - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1])^2 \big] \right)$$

where we used the independence property stated in Theorem 5.6. But the last term equals

$$\frac{1}{k^2 \varepsilon^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2} \big(\widehat{\mathbb{E}}_0^\lambda[\tau_1^2] - 2\widehat{\mathbb{E}}_0^\lambda[\tau_1]\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] + \widehat{\mathbb{E}}_0^\lambda[(\tau_2 - \tau_1)^2] \big)$$

$$+ \frac{k-1}{k^2 \varepsilon^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]^2} \widehat{\mathbb{E}}_0^\lambda[(\tau_2 - \tau_1 - \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1])^2],$$

and we conclude the proof by using Lemma 5.10. $\square$

Finally, Lemma 5.12 follows from Lemma 5.13 by taking $k = n(t)$: due to (5.31),

$$\lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} n(t) = \infty$$

and

$$(5.47) \qquad \lim_{\substack{\alpha \to +\infty}} \limsup_{\substack{\lambda \to 0 \\ t \to +\infty \\ \lambda^2 t = \alpha}} \frac{n(t)\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]}{t} = 1.$$

To see that (5.47) holds true, note that

$$t\left(1 - \frac{\lambda^2 \widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1]}{\lambda^2 t}\right) \leq n(t)\widehat{\mathbb{E}}_0^\lambda[\tau_2 - \tau_1] \leq t$$

and use (5.31). $\qquad \square$

6 Proof of Theorem 2.3 and Extension to Measurable Coefficients

PROOF OF THEOREM 2.3. Combine (5.1) with Proposition 3.1. $\qquad \square$

6.1 The Case of Measurable Coefficients

Here we explain how the approach developed in the previous sections of the paper can be extended to deal with measurable coefficients. Thus the assumptions in force in this section are Assumptions 1, 3, and 4, which remain unchanged, and a modified Assumption 2, which is replaced by the following, weaker statement:

Assumption 2′. For any environment ω, the functions $x \to V^\omega(x)$ and $x \to \sigma^\omega(x)$ are measurable.

In such generality, it is not possible to use stochastic differential equations to define the processes X^ω or $X^{\lambda,\omega}$ anymore, so our first task is to give an alternative construction.

In the following discussion we fix an environment ω satisfying Assumptions 1, 2′, 3, and 4. The case $\lambda = 0$ is included.

Let $p^{\lambda,\omega}(t, x, y)$ be the kernel associated to the operator $\mathcal{L}^{\lambda,\omega}$ in equation (2.8) or, in the case $\lambda = 0$, to the operator $\mathcal{L}^\omega$ in equation (2.1), now understood in the weak distributional sense. Also, let $T_t^{\lambda,\omega}$ be the corresponding semigroup

$$T_t^{\lambda,\omega} f(x) = \int p^{\lambda,\omega}(t, x, y) f(y) dy.$$

It follows from Aronson's estimate (see [1]) and Harnack's inequality (see [10]) that $T_t^{\lambda,\omega}$ maps continuous functions vanishing at infinity to continuous functions vanishing at infinity. For such functions we also have $\lim_{t\to 0} T_t^{\lambda,\omega} f(x) = f(x)$ for all x. Thus $T_t^{\lambda,\omega}$ is a Feller semigroup, and it follows from [6, theorem 2.7] that there exists a conservative Hunt process with continuous paths whose semigroup

is $T^{\lambda,\omega}$. We denote its law on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$ with $P_x^{\lambda,\omega}$ and $E_x^{\lambda,\omega}$ for the corresponding expectation. Observe that $P_x^{\lambda,\omega}[X(0) = x] = 1$ for all x.

It is proved in [20, prop. 1] that, for almost all ω's, under P_0^ω, the canonical process satisfies an invariance principle with some effective diffusivity matrix Σ. It follows from Aronson's estimate that Σ is also the asymptotic covariance so that Definition 2.1 goes through.

On the other hand, all statements in Section 4 as well as Lemma 5.2 in Section 5 were proved under Assumption 2′ only. Thus the construction of regeneration times we gave in Section 5 and all the upper bounds on τ_1 are still valid with measurable coefficients. Therefore we see that the law of large numbers is satisfied $\mathbb{Q}$ almost surely under $P_0^{\lambda,\omega}$ for all positive λ, and Definition 2.2 can still be used as the definition of the effective drift.

Having defined the effective diffusivity matrix and the effective drift, we claim that the Einstein relation stated in Theorem 2.3 holds true with Assumption 2 replaced by Assumption 2′.

The main difficulties in extending the proofs of the previous sections to measurable coefficients appear in justifying the Girsanov transform and time change arguments from Section 3. Following [20], in order to do it we shall appeal to Dirichlet form theory, as expounded in [9], and related stochastic calculus for Dirichlet processes. Observe that a direct application of Dirichlet form theory a priori only provides information under $P_x^{\lambda,\omega}$ for all ω but only for (Lebesgue) almost all x (in fact, for quasi all x but we won't use fine topological notions here) and therefore, as a consequence of the translation invariance of $\mathbb{Q}$, under $P_0^{\lambda,\omega}$ for almost all ω. Therefore most claims in Section 3, such as Lemma 3.3 or formula (3.4), should now be understood "for $\mathbb{Q}$ almost all ω's." We let the reader convince herself that this does not affect the proofs.

We use the notation $\rho_\lambda(x) = e^{\hat{\lambda} \cdot x}$ for $\lambda \geq 0$. Let $L^2(\rho_\lambda^2)$ be the space of square-integrable functions with respect to the measure $\rho_\lambda^2(x)dx$. Define $H_1(\rho_\lambda^2)$ to be the space of functions in $L^2(\rho_\lambda^2)$ whose gradient is also square-integrable with respect to the measure $\rho_\lambda^2(x)dx$. Let

$$(6.1) \qquad \mathcal{E}^{\lambda,\omega}(f,f) := \frac{1}{2} \int |\sigma^\omega(x) \nabla f(x)|^2 e^{-2V^\omega(x)} \rho_\lambda^2(x)dx.$$

Then $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$ is a regular Dirichlet form. We claim the following:

LEMMA 6.1. *The Dirichlet form of the semigroup $T^{\lambda,\omega}$ on $L^2(e^{2\hat{\lambda} \cdot x - 2V^\omega(x)} dx)$ is $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$.*

(Note that this fact is already used in [20] but without justification.)

PROOF. We first observe that $T^{\lambda,\omega}$ is indeed a strongly continuous symmetric semigroup on $L^2(\rho_\lambda^2)$.

Let $t > 0$ and define the approximating bilinear forms

$$(6.2) \quad \mathcal{E}^{t,\lambda,\omega}(f, f) := \frac{1}{t} \int \left(f(x) - T_t^{\lambda,\omega} f(x) \right) f(x) e^{-2V^\omega(x)} \rho_\lambda^2(x) dx.$$

A function f belongs to the domain of the Dirichlet form associated to the semigroup $T^{\lambda,\omega}$ if and only if $\mathcal{E}^{t,\lambda,\omega}(f, f)$ is bounded in t and the limit as t tends to 0 is then the value of the form; see [9, lemma 1.3.4].

As t tends to 0, a straightforward integration by parts shows that $\mathcal{E}^{t,\lambda,\omega}(f, f)$ converges to $\mathcal{E}^{\lambda,\omega}(f, f)$ on the L^2-domain of the generator $\mathcal{L}^{\lambda,\omega}$, say $\mathcal{D}^{\lambda,\omega}$. Since the function $t \to \mathcal{E}^{t,\lambda,\omega}(f, f)$ is decreasing, it implies that

$$\mathcal{E}^{t,\lambda,\omega}(f, f) \le \mathcal{E}^{\lambda,\omega}(f, f) \quad \text{for } f \in \mathcal{D}^{\lambda,\omega}.$$

This inequality extends by density to all functions in $H_1(\rho_\lambda^2)$. Thus we have proved that the Dirichlet form of the semigroup $T^{\lambda,\omega}$ is well-defined and coincides with $\mathcal{E}^{\lambda,\omega}$ on $H_1(\rho_\lambda^2)$ or, in other words, that it is an extension of the form $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$. But since $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$ is its own maximal Markovian extension (see [9, theorem 3.3.1]), both forms coincide. □

From now on we will drop the superscript λ from the notation when $\lambda = 0$. We now consider properties of the canonical process X for a fixed environment ω and under P_x^ω for almost every starting point x. The function $\phi(x) = e_1 \cdot x$ locally belongs to the domain of the Dirichlet form $\mathcal{E}^\omega$. From theorem 5.5.1 in [9] we deduce that $e_1 \cdot X(t) - e_1 \cdot X(0)$ is a local Dirichlet process under P_x^ω for almost every starting point x. Thus $e_1 \cdot X(t) - e_1 \cdot X(0)$ admits a unique Fukushima decomposition as the sum of a local martingale, say $\bar{B}$, and a process of locally vanishing quadratic variation. The bracket of $\bar{B}$ is given by theorem 5.5.2 in [9] and satisfies

$$(6.3) \qquad \langle \bar{B} \rangle(t) = \int_0^t |\sigma^\omega(X(s)) e_1|^2 \, ds.$$

By Assumption 3, we have $\langle \bar{B} \rangle(t) \le \kappa^{-1} t$. Thus we see that in fact $\bar{B}$ is a square-integrable martingale. By the same argument, the exponential local martingale $e^{\mu \bar{B}(t) - (\mu^2/2)\langle \bar{B} \rangle(t)}$ is also seen to be a martingale for all μ, and one proves as in Lemma 3.3 that

$$(6.4) \qquad E\left[e^{\mu\lambda \bar{B}(t) - \frac{\mu\lambda^2}{2} \langle \bar{B} \rangle(t)} \right] \le e^{(\mu-1)\frac{\mu\lambda^2}{2}\frac{t}{\kappa}}$$

for every ω and almost all x. Note that the translation invariance of $\mathbb{Q}$ then implies that (6.4) also holds for almost all ω with $x = 0$.

We now justify the Girsanov formula:

PROPOSITION 6.2. *For any environment ω, for almost any x, any t, and any continuous bounded function F, we have*

$$E_x^{\lambda,\omega}[F(X([0, t]))] = E_x^\omega \left[F(X([0, t])) e^{\lambda \bar{B}(t) - \frac{\lambda^2}{2}\langle \bar{B} \rangle(t)} \right].$$

The translation invariance of $\mathbb{Q}$ then implies that for almost all environments

$$E_0^{\lambda,\omega}[F(X([0,t]))] = E_0^{\omega}\big[F(X([0,t]))e^{\lambda\bar{B}(t)-\frac{\lambda^2}{2}\langle\bar{B}\rangle(t)}\big]$$

for any continuous and bounded function F, and we have obtained the almost sure version of the Girsanov formula (3.4) that is sufficient to proceed through the proofs of the previous sections.

PROOF. We would like to invoke theorem 3.1 of [3], but unfortunately ϕ does not belong to L^2 so that some work is needed.

Let G_n be the ball centered at the origin with radius n, and let ζ_n be the exit time from G_n.

We shall first prove that

$$(6.5) \quad E_x^{\lambda,\omega}[F(X([0,t])); t < \zeta_n] = E_x^{\omega}\big[F(X([0,t]))e^{\lambda\bar{B}(t)-\frac{\lambda^2}{2}\langle\bar{B}\rangle(t)}; t < \zeta_n\big].$$

Using the bound (6.4), it is then possible to let n tend to infinity and deduce Proposition 6.2 from (6.5).

Choose a function ϕ_n that coincides with ϕ on G_n, is smooth, and has compact support. Let M_n be the martingale part of the process $\phi_n(X(t)) - \phi_n(X(0))$ in its Fukushima decomposition under P_x^{ω}, and let $Z_n(t) := e^{\lambda M_n(t)-(\lambda^2/2)\langle M_n\rangle(t)}$. Define

$$Q_t^{n,\lambda,\omega}f(x) := E_x^{\omega}[f(X(t))Z_n(t)].$$

Then $Q_t^{n,\lambda,\omega}$ defines a strongly continuous Markovian semigroup on $L^2(dx)$. We need the following:

LEMMA 6.3. *The Dirichlet form of the semigroup $Q_t^{n,\lambda,\omega}$ acting on*

$$L^2(e^{2\phi_n(x)-2V^{\omega}(x)}dx)$$

is

$$\frac{1}{2}\int|\sigma^{\omega}(x)\nabla f(x)|^2 e^{2\phi_n(x)-2V^{\omega}(x)}dx$$

with domain $H_1(dx)$.

PROOF. The Revuz measure of the positive continuous additive functional $\langle M_n\rangle$ is $|\nabla\sigma^{\omega}(x)\phi_n(x)|^2\,dx$, which is easily seen to belong to the Hardy class since the gradient of ϕ_n is uniformly bounded. In addition, condition (3.8) in [3] is fulfilled whenever $\nabla\phi_n$ is uniformly bounded. Thus theorem 3.1 of [3] applies. We get that the quadratic form of the semigroup $Q_t^{n,\lambda,\omega}$ acting on $L^2(e^{-2V^{\omega}(x)}dx)$ is

$$(6.6) \quad \begin{aligned} \mathcal{Q}_n(f,g) &:= \frac{1}{2}\int\sigma^{\omega}(x)\nabla f(x)\cdot\sigma^{\omega}(x)\nabla g(x)e^{-2V^{\omega}(x)}dx \\ &\quad - \int g(x)\sigma^{\omega}(x)\nabla f(x)\cdot\sigma^{\omega}(x)\nabla\phi_n(x)e^{-2V^{\omega}(x)}dx; \end{aligned}$$

see formula (3.3) in [3].

We now use the same approximating sequence as in (6.2); see also the explanation in [3, p. 242]. We then know that for all functions $f, g \in H_1(dx)$

$$Q_n(f, g) = \lim_{t \to 0} \frac{1}{t} \int (f(x) - Q_t^{n,\lambda,\omega} f(x)) g(x) e^{-2V^\omega(x)} dx.$$

Applying this formula to the function $ge^{2\phi_n}$ and using (6.6), we deduce that the Dirichlet form of the semigroup $Q_t^{n,\lambda,\omega}$ acting on $L^2(e^{2\phi_n(x) - 2V^\omega(x)} dx)$ is

$$\lim_{t \to 0} \frac{1}{t} \int (f(x) - Q_t^{n,\lambda,\omega} f(x)) g(x) e^{2\phi_n(x)} e^{-2V^\omega(x)} dx$$

$$= \frac{1}{2} \int \sigma^\omega(x) \nabla f(x) \cdot \sigma^\omega(x) \nabla (g(x) e^{2\phi_n(x)}) e^{-2V^\omega(x)} dx$$

$$- \int g(x) \sigma^\omega(x) \nabla f(x) \cdot \sigma^\omega(x) \nabla \phi_n(x) e^{2\phi_n(x) - 2V^\omega(x)} dx$$

$$= \frac{1}{2} \int \sigma^\omega(x) \nabla f(x) \cdot \sigma^\omega(x) \nabla g(x) e^{2\phi_n(x) - 2V^\omega(x)} dx.$$

$\square$

We can now conclude the proof of Proposition 6.2. Comparing the expression of the Dirichlet form generated by $Q_t^{n,\lambda,\omega}$ we just obtained with formula (6.1) for $\mathcal{E}^{\lambda,\omega}$ and observing that $\phi_n = \phi$ on G_n, we see that the parts of both these Dirichlet forms on G_n coincide. Clearly $\bar{B}$ and M_n also coincide up to time ζ_n. Thus we obtain (6.5). $\square$

Finally, we should say a word about the time change argument used in the proof of Proposition 3.1 in the case $V \neq 0$. Define $Y^{\lambda,\omega}$ to be the Hunt process with Dirichlet form $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$ with reference measure $\rho_\lambda^2(x) dx$. Theorem 6.2.1 in [9] implies that the process obtained by time-changing $Y^{\lambda,\omega}$ through the additive functional $A^{\lambda,\omega}(t) = \int_0^t e^{-2V^\omega(Y^{\lambda,\omega}(s))} ds$ admits as Dirichlet form $(\mathcal{E}^{\lambda,\omega}, H_1(\rho_\lambda^2))$ with reference measure $e^{-2V^\omega(x)} \rho_\lambda^2(x) dx = e^{2\hat{\lambda} \cdot x - 2V^\omega(x)}$ so that, for almost any initial point x, the law of $Y^{\lambda,\omega} \circ (A^{\lambda,\omega})^{-1}$ coincides with $P_x^{\lambda,\omega}$.

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