COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **9**, Number **5**, September **2010**

doi:10.3934/cpaa.2010.9.1295

pp. **1295–1310**

ON THE EFFECTIVE INTERFACIAL RESISTANCE THROUGH ROUGH SURFACES

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Abstract. The paper deals with homogenization of an elliptic boundary value problem stated in a domain which consists of two connected components separated by a rapidly oscillating interface with a periodic microstructure, the interface being situated in a small neighbourhood of a hyperplane. At the interface we suppose the following transmission conditions: (i) the flux is continuous, (ii) the jump of a solution at the interface is proportional to the flux through the interface.

We derive the homogenized problem and effective transmission condition for different values of the ratio between the microstructure period and the amplitude of the interface oscillations, as well as for the different values of the mentioned proportionality coefficient.

1. **Introduction.** In this paper we study the asymptotic behaviour of the stationary heat diffusion in a medium which is composed of two connected composite components. The components are separated by a rough interface, represented as the graph of a rapidly oscillating periodic function defined on a hyperplane, and giving rise to an imperfect contact between the two components. As derived in [5], an imperfect contact between two materials can be modeled by a jump of the solution of the diffusion equation, which is proportional to the flux through the interface.

Throughout the paper we assume that the coefficients of the diffusion matrix field are ε-periodic in each coordinate direction, and that the function representing the interface is ε-periodic in the first $n-1$ variables, ε being a small positive parameter. We suppose that the amplitude of the interface oscillations is of order ε^κ, with $\kappa > 0$. That is, the interface approaches a flat surface denoted by Γ_0, as $\varepsilon \to 0$.

The proportionality coefficient appearing in the transmission conditions is of order ε^γ, with $\gamma \in \mathbb{R}$. The value of γ plays a crucial role in the asymptotic behaviour of solutions.

2000 *Mathematics Subject Classification.* 35B27, 35L05.

Key words and phrases. Homogenization, rough surfaces, interface conditions.

The corresponding boundary-value problem reads

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) = f \qquad \text{in } Q \setminus \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)^- \cdot n_\varepsilon = (A^\varepsilon \nabla u_\varepsilon)^+ \cdot n_\varepsilon \qquad \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)^+ \cdot n_\varepsilon = \varepsilon^\gamma h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-), \qquad \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 \qquad \text{on } \partial Q, \end{cases}$$

where $Q = \omega \times]-l, l[$ with ω being a regular bounded domain in $\mathbb{R}^{n-1}$, Γ_ε is the oscillating interface, and n_ε is the unit normal on Γ_ε. We suppose that $A^\varepsilon = A(\frac{x}{\varepsilon})$ with A periodic bounded and uniformly elliptic matrix field, and $h^\varepsilon = h(\frac{x'}{\varepsilon})$, $x' \in \omega$, with h being a bounded and positive periodic function in $\mathbb{R}^{n-1}$.

Our aim is to study the asymptotic behaviour, as $\varepsilon \to 0$, of solutions of this problem for the different values of the parameters κ and γ.

Previously, the homogenization problem for elliptic equation with the same kind of interface conditions on the boundary of bulk periodic perforations, has been considered in the pioneer work [1] and then in [13], [9], [10], [17] (see also the references therein).

For homogenization problems in domains with an oscillating interface having a fixed amplitude of oscillations we refer to [3], [12], [15] and [16].

The homogenization problems in domains perforated in a small neighbourhood of a hyper-surface, has been addressed in [14], the effective transmission conditions for processes in domains separated by a thin heterogeneous layer have been derived in [18]. More references can be found in the recent book [7].

To our knowledge, the homogenization of elliptic problems with the resistances at the oscillating interface situated in the vicinity of a hyperplane, have not been discussed in the existing literature.

For classical homogenization results we refer to the book [2], [19] and [8].

The paper is organized as follows. Section 2 is devoted to problem setup and basic compactness results adapted to the interface geometry. In Section 3 we define the homogenized matrix and consider the asymptotic behaviour of fluxes. In Section 4 we present the homogenization result in the self-similar case where the interface is obtained by a homothetic dilatation of a fixed periodic profile. In this case $\kappa = 1$. We prove in Theorem 4.1 that if $\gamma = 0$, then the limit problem consists of the homogenized diffusion equation and the effective transmission condition with a jump on the limit interface Γ_0, which is proportional to the co-normal derivative of a solution of the homogenized operator on Γ_0. The corresponding proportionality coefficient depends on the function describing the geometry of the interface, and on the function h. If $\gamma > 0$, one obtains two independent homogenized problems in the upper and lower parts of the domain, with a Neumann boundary condition on Γ_0. If $\gamma < 0$, then the homogenized problem is the same as in the case of a domain without interface. The case $\kappa > 1$ is studied in Section 5. We prove that in this case the limit problem is the same as in the case where the oscillating surface is replaced by the flat interface Γ_0. Finally, in Section 6 we consider the case of highly oscillating interface (that is the case $0 < \kappa < 1$). Here, the effective interface condition also depends on whether the value of γ is greater than, or less than, or equal to the critical value which depends on κ and is not anymore equal to zero. In all the cases our analysis relies on the suitable compactness results for function having a jump on the oscillating interface, proved in Section 2.

2. **The problem setup.** Let Q be an open bounded cylinder in $\mathbb{R}^n$ given by $Q = \omega\times] - l, l[$, with $l > 0$, ω being a smooth bounded domain of $\mathbb{R}^{n-1}$, and ε a small positive parameter converging to zero.

We suppose that Q is divided in two parts, separated by an oscillating interface Γ_ε, defined by

$$\Gamma_\varepsilon = \big\{x \in Q, x_n = \varepsilon^\kappa g(\frac{x'}{\varepsilon})\big\}, \tag{2.1}$$

where $\kappa > 0$, $x' = (x_1, \dots, x_{n-1})$ and $g :]0, 1[^{n-1} \to \mathbb{R}$ is a periodic positive Lipschitz continuous function, and we set $\overline{g} = \max g$.

Remark 2.1. Instead of a cylindrical domain we can consider an arbitrary smooth domain Q such that for any point of $\partial Q \cap \{x : x_n = 0\}$ the normal to ∂Q is not parallel to the n-th coordinate vector.

By construction, the set $\omega \times [0, \varepsilon^\kappa \overline{g}]$ contains the oscillating interface, and the measure of this set goes to zero as $\varepsilon \to 0$. The subdomains

$$Q_\varepsilon^+ = \{x \in Q, x_n > \varepsilon^\kappa g(\tfrac{x'}{\varepsilon})\}, \tag{2.2}$$
$$Q_\varepsilon^- = \{x \in Q, x_n < \varepsilon^\kappa g(\tfrac{x'}{\varepsilon})\} \tag{2.3}$$

are called the upper and the lower parts of Q, respectively (see Figure 1).

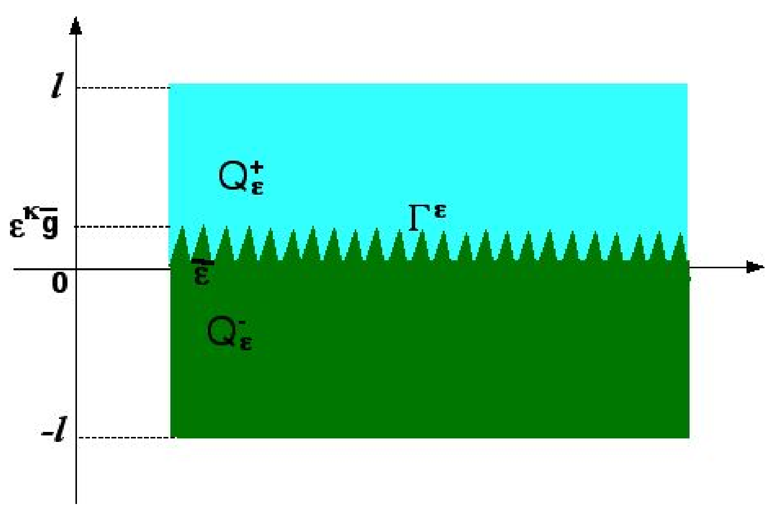

Figure 1: The upper and the lower parts of Q and the interface.

In the rest of the paper, we will also use the following decomposition of $\omega\times]0, \varepsilon^\kappa \overline{g}[$ as (see Figure 2)

$$\omega\times]0, \varepsilon^\kappa \overline{g}[= B_\varepsilon^+ \cup B_\varepsilon^- \cup \Gamma_\varepsilon, \tag{2.4}$$

where

$$B_\varepsilon^+ = \omega\times]0, \varepsilon^\kappa \overline{g}[\cap Q_\varepsilon^+, \qquad B_\varepsilon^- = \omega\times]0, \varepsilon^\kappa \overline{g}[\cap Q_\varepsilon^-. \tag{2.5}$$

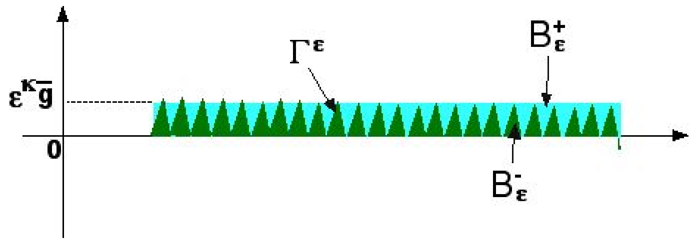

Figure 2 : The set $\omega \times [0, \varepsilon^\kappa \overline{g}]$.

We denote by $Y =]0,1[^n$ the volume reference cell and by $Y' =]0,1[^{n-1}$ the surface reference cell. We suppose that A is a Y-periodic matrix field satisfying

$$(A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \qquad |A(y)\lambda| \leq \beta\lambda, \text{ a.e. in } Y \text{ and for any } \lambda \in \mathbb{R}^n, \tag{2.6}$$

with $0 < \alpha < \beta$, and that h is a Y'-periodic function such that

$$h \in L^\infty(\Gamma), \text{ and } 0 < h_0 < h(y'), \text{ a.e. on } \Gamma, \tag{2.7}$$

for some $h_0 \in \mathbb{R}^*_+$, where

$$\Gamma = \{y_n = g(y'),\ y' \in Y'\}. \tag{2.8}$$

We set, for any $\varepsilon > 0$,

$$A^\varepsilon(x) = A(x/\varepsilon), \qquad h^\varepsilon(x') = h\Big(\frac{x'}{\varepsilon}\Big). \tag{2.9}$$

Our aim is to study, for $\gamma \in \mathbb{R}$, the asymptotic behaviour of solutions of the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) = f \qquad \text{in } Q \setminus \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)^- \cdot n_\varepsilon = (A^\varepsilon \nabla u_\varepsilon)^+ \cdot n_\varepsilon \qquad \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)^+ \cdot n_\varepsilon = \varepsilon^\gamma h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-), \qquad \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 \qquad \text{on } \partial Q, \end{cases} \tag{2.10}$$

where for any function v defined on Q we set

$$v_\varepsilon^+ = v_{|Q_\varepsilon^+} \qquad v_\varepsilon^- = v_{|Q_\varepsilon^-} \tag{2.11}$$

and n_ε stands for the unit outward normal to Q_ε^+.

Remark 2.2. The results of the paper remain valid for the case of distinct diffusion matrices in the upper and lower parts of the domain, as well as for the case of locally periodic diffusion matrix and interface conditions. More precisely, one can assume that $A^\varepsilon(x) = A\big(x, \frac{x}{\varepsilon}\big)$, $h^\varepsilon(x') = h\big(x', \frac{x'}{\varepsilon}\big)$ and $g^\varepsilon(x') = g\big(x', \frac{x'}{\varepsilon}\big)$, where $A(x,y)$ and $h(x',y')$ are periodic in the second argument Caratheodory functions and $g(x',y')$ is a Lipschitz continuous function periodic in y'.

We deal in the paper with the pure periodic case just for presentation simplicity.

In what follows, we use the notation:
- $\tilde{v}$ stands for the zero extension of a function v defined on a subset of Q,
- χ_E, the characteristic function of any set $E \subset \mathbb{R}^n$,
- $m_{Y'}(v) = \frac{1}{|Y'|}\int_{Y'} v\, dy'$, the average on Y' of any function $v \in L^1(Y')$.

We introduce the spaces W_0^ε and W_0^0 defined by

$$W_0^\varepsilon := \{v \mid v_\varepsilon^+ \in H^1(Q_\varepsilon^+),\ v_\varepsilon^- \in H^1(Q_\varepsilon^-) \quad \text{and} \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$\|v\|_{W_0^\varepsilon} := \|\nabla v\|_{L^2(Q \setminus \Gamma_\varepsilon)}, \tag{2.12}$$

where

$$\nabla v = \chi_{Q_\varepsilon^+} \nabla v_\varepsilon^+ + \chi_{Q_\varepsilon^-} \nabla v_\varepsilon^-,$$

that is, we identify ∇v with the absolutely continuous part of the gradient of v, and

$$W_0^0 := \{v \mid v^+ \in H^1(Q^+),\ v^- \in H^1(Q^-) \quad \text{and} \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$\|v\|_{W_0^0} := \|\nabla v\|_{L^2(Q\setminus\Gamma_0)}, \tag{2.13}$$

where (see Figure 3)

$$Q^+ = \{x \in Q \,:\, x_n > 0\}, \quad Q^- = \{x \in Q \,:\, x_n < 0\}, \quad \Gamma_0 = \{x \in Q \,:\, x_n = 0\}, \tag{2.14}$$

and

$$v^+ = v_{|Q^+} \qquad v^- = v_{|Q^-}. \tag{2.15}$$

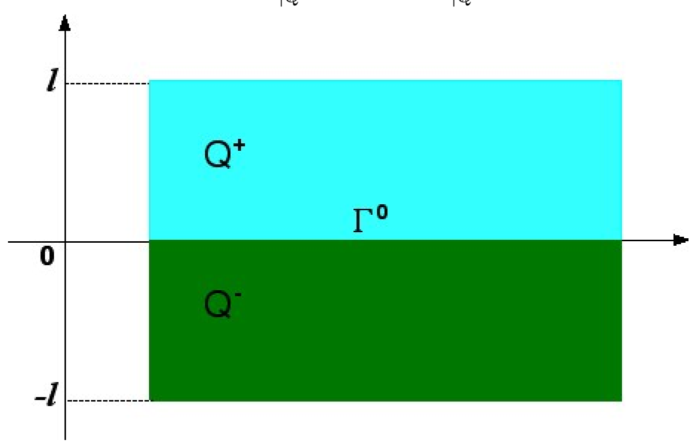

Figure 3: The limit domain.

With the above definitions of $Q_\varepsilon^\pm$ and $B_\varepsilon^\pm$ we have

$$Q^+ = Q_\varepsilon^+ \cup B_\varepsilon^-, \qquad Q^- = Q_\varepsilon^- \setminus B_\varepsilon^-. \tag{2.16}$$

Let us observe that (2.12) is a norm, due to the following Poincaré inequality: there exists a constant c (independent of ε) such that, for any $v \in W_0^\varepsilon$

$$\|v\|_{L^2(Q)} \le c\|\nabla v\|_{L^2(Q)}. \tag{2.17}$$

Then, problem (2.10) has the following variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in W_0^\varepsilon \text{ such that} \\ \displaystyle\int_{Q\setminus\Gamma_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi\, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\, d\sigma = \int_Q f\varphi\, dx, \\ \text{for every } \varphi \in W_0^\varepsilon. \end{cases} \tag{2.18}$$

Remark 2.3. Notice that in the coordinates x' the boundary integral in the variational formulation reads

$$\begin{aligned} &\varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\, d\sigma \\ &= \varepsilon^\gamma \int_\omega h\Big(\frac{x'}{\varepsilon}\Big) \Big(u_\varepsilon^+\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - u_\varepsilon^-\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\Big) \\ &\times \Big(\varphi^+\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - \varphi^-\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\Big) \Big(1 + \varepsilon^{2(\kappa-1)} (|\nabla_{y'} g(y')|^2)\big|_{y'=x'/\varepsilon}\Big)^{1/2} dx'. \end{aligned}$$

It is easy to check that due to (2.17), the bilinear form associated to (2.18) is coercive, so that by the Lax-Milgram Theorem, problem (2.18) has a unique solution in W_0^ε and that the following a priori estimate holds:

$$\|u_\varepsilon\|_{W_0^\varepsilon} + \varepsilon^{\frac{\gamma}{2}} \|u_\varepsilon^+ - u_\varepsilon^-\|_{L^2(\Gamma_\varepsilon)} \le c\|f\|_{L^2(Q)}. \tag{2.19}$$

Our purpose is to study the asymptotic behaviour of problem (2.18).

To this end, in the following proposition we state a suitable compactness result, adapted to the geometry of the problem, which plays an essential role in the sequel.

Proposition 2.4. *Let Γ_ε be defined by (2.1) with $\kappa > 0$ and suppose that $\{v_\varepsilon\}_\varepsilon$ is a family of functions $v_\varepsilon \in W_0^\varepsilon$ such that*

$$\|v_\varepsilon\|_{W_0^\varepsilon} \le c, \tag{2.20}$$

with c independent of ε. Then, the family $\{v_\varepsilon\}_\varepsilon$ is compact in $L^2(Q)$ and the families $\{\chi_{Q_\varepsilon^+} \nabla v_\varepsilon\}_\varepsilon$ and $\{\chi_{Q_\varepsilon^-} \nabla v_\varepsilon\}_\varepsilon$ are weakly compact in $L^2(Q)$.

In particular, there exist a subsequence (still denoted $\{\varepsilon\}$) and a function $v(x)$ in W_0^0 such that

$$\begin{cases} i) \ v_\varepsilon \to v, & \textit{strongly in } L^2(Q), \\ ii) \ \chi_{Q_\varepsilon^+} \nabla v_\varepsilon \rightharpoonup \chi_{Q^+} \nabla v, & \textit{weakly in } (L^2(Q))^n, \\ iii) \ \chi_{Q_\varepsilon^-} \nabla v_\varepsilon \rightharpoonup \chi_{Q^-} \nabla v, & \textit{weakly in } (L^2(Q))^n. \end{cases} \tag{2.21}$$

Proof. Observe first that

$$\text{the family } \{v_\varepsilon\}_\varepsilon \text{ is weakly compact in } H^1(\mathcal{A}) \text{ and compact in } L^2(\mathcal{A}), \tag{2.22}$$

for every open set $\mathcal{A} \subset\subset Q^+$. Indeed, since $\kappa > 0$, from (2.1) it follows that for every $\mathcal{A} \subset\subset Q^+$ there exists ε_0 such that for every $\varepsilon \le \varepsilon_0$, $\mathcal{A} \subset Q_\varepsilon^+$. Hence, (2.22) follows from (2.20) and the compact Sobolev embedding theorem applied to any open set $\mathcal{A}_1$ with a smooth boundary, such that $\mathcal{A} \subset \mathcal{A}_1 \subset\subset Q^+$.

Similarly,

$$\text{the family } \{v_\varepsilon\}_\varepsilon \text{ is weakly compact in } H^1(\mathcal{A}) \text{ and compact in } L^2(\mathcal{A}), \tag{2.23}$$

for every open set $\mathcal{A} \subset\subset Q^-$.

Consequently, to prove the compactness of $\{v_\varepsilon\}_\varepsilon$ in $L^2(Q)$ it is sufficient to prove that

$$\lim_{\varepsilon \to 0} \int_{\omega \times]0, \varepsilon^\kappa \bar{g}[} v_\varepsilon^2(x)\, dx = 0. \tag{2.24}$$

Let us first show that

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon^+} v_\varepsilon^2(x)\, dx = 0, \tag{2.25}$$

where B_ε^+ is given by (2.4)-(2.5). We have

$$v(x + \varepsilon^\kappa \bar{g} e_n) - v(x) = \int_x^{x + e^\kappa \bar{g} e_n} \frac{\partial v_\varepsilon}{\partial x_n}\, dx_n, \qquad \text{for a.e. } x \in B_\varepsilon^+,$$

where $e_n = (0, \ldots, 0, 1)$. Hence,

$$\begin{aligned}\int_{B_\varepsilon^+} v_\varepsilon^2(x)\,dx &= \int_{B_\varepsilon^+}\Big(v_\varepsilon(x+\varepsilon^\kappa \bar g e_n) - \int_{x_n}^{x_n+\varepsilon^\kappa\bar g}\frac{\partial v_\varepsilon(x',s)}{\partial s}\,ds\Big)^2 dx\\
&\le 2\int_{B_\varepsilon^+ + \varepsilon^\kappa \bar g e_n} v_\varepsilon^2(x)\,dx + 2\int_{B_\varepsilon^+}\Big(\int_{x_n}^{x_n+\varepsilon^\kappa\bar g}\frac{\partial v_\varepsilon(x',s)}{\partial s}\,ds\Big)^2 dx\\
&\le 2\int_{\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa \bar g[} v_\varepsilon^2(x)\,dx + 2\varepsilon^\kappa\bar g\int_{B_\varepsilon^+}\int_{x_n}^{x_n+\varepsilon^\kappa\bar g}\Big|\frac{\partial v_\varepsilon(x',s)}{\partial s}\Big|^2 ds\,dx\\
&\le 2\Big(\int_{\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa \bar g[} v_\varepsilon^2(x)\,dx + (\varepsilon^\kappa\bar g)^2\int_\omega\int_0^{2\varepsilon^\kappa\bar g}\Big|\frac{\partial v_\varepsilon}{\partial x_n}\Big|^2 dx\Big)\\
&\le 2\Big(\int_{\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa \bar g[} v_\varepsilon^2(x)\,dx + (\varepsilon^\kappa\bar g)^2\|v_\varepsilon\|^2_{W_0^\varepsilon}\Big).\end{aligned} \tag{2.26}$$

By the Sobolev embedding theorem, the embedding operator from $H^1(\omega\times]\varepsilon^\kappa\bar g, l[)$ into $L^{2^*}(\omega\times]\varepsilon^\kappa\bar g, l[)$ is bounded uniformly in ε, where $2^* = \dfrac{2n}{n-2}$ if $n>2$; if $n=2$ then the embedding $H^1(\omega\times]\varepsilon^\kappa\bar g, l[) \subset L^p(\omega\times]\varepsilon^\kappa\bar g, l[)$ is continuous for any $p<\infty$. Therefore, using (2.20) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_{\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa \bar g[} v_\varepsilon^2(x)\,dx &\le \|v_\varepsilon\|^{(n-2)/n}_{L^{2^*}(\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa\bar g[)}\ \mathrm{meas}(\omega\times]\varepsilon^\kappa\bar g, 2\varepsilon^\kappa\bar g[)^{\frac{2}{n}}\\
&\le C\|v_\varepsilon\|^{(n-2)/n}_{L^{2^*}(\omega\times]\varepsilon^\kappa\bar g, l[)}\varepsilon^{2\kappa/n} \le c\varepsilon^{\frac{2\kappa}{n}} \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

This, together with (2.20) and (2.26) proves (2.25). A similar argument shows that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon^-} v_\varepsilon^2(x)\,dx = 0. \tag{2.27}$$

This completes the proof of compactness of the family $\{v_\varepsilon\}_\varepsilon$ in $L^2(Q)$. Consequently, there exists a subsequence (still denoted ε) and a function v in $L^2(Q)$ such that (2.21) i) holds.

Let us prove that v belongs to W_0^0 and that (2.21) ii)–iii) holds true.

For $\varphi \in L^2(Q)$, we have (see Figures 1 and 2)

$$\int_Q \chi_{Q_\varepsilon^+}\nabla v_\varepsilon\,\varphi\,dx = \int_{Q_\varepsilon^+}\nabla v_\varepsilon\,\varphi\,dx = \int_{Q^+\setminus(\omega\times]0,\varepsilon^\kappa\bar g[)}\nabla v_\varepsilon\,\varphi\,dx + \int_{B_\varepsilon^+}\nabla v_\varepsilon\,\varphi\,dx. \tag{2.28}$$

But, from (2.22) and (2.21)i) one has (for a subsequence)

$$\int_{Q^+\setminus(\omega\times]0,\varepsilon^\kappa\bar g[)}\nabla v_\varepsilon\,\varphi\,dx \to \int_{Q^+}\nabla v\,\varphi\,dx, \tag{2.29}$$

for any φ in $\mathcal{D}(Q^+)$ and also, by (2.20) and a density argument, for any φ in $L^2(Q)$. On the other hand, using again (2.20) we obtain

$$\Big|\int_{B_\varepsilon^+}\nabla v_\varepsilon\,\varphi\,dx\Big| \le \|\nabla v_\varepsilon\|_{L^2(Q^+)}\|\varphi\|_{L^2(B_\varepsilon^+)} \to 0, \tag{2.30}$$

since the measure of B_ε^+ goes to zero.

Hence, using (2.29) and (2.30) in (2.28) gives

$$\int_Q \chi_{Q_\varepsilon^+} \nabla v_\varepsilon \, \varphi \, dx \to \int_{Q^+} \nabla v \, \varphi \, dx,$$

for every φ in $L^2(Q)$, which gives convergence (2.21) ii).

Similarly, for proving convergence (2.21) iii) we write

$$\int_Q \chi_{Q_\varepsilon^-} \nabla v_\varepsilon \, \varphi \, dx = \int_{Q_\varepsilon^-} \nabla v_\varepsilon \, \varphi \, dx = \int_{Q^-} \nabla v_\varepsilon \, \varphi \, dx + \int_{B_\varepsilon^-} \nabla v_\varepsilon \, \varphi \, dx,$$

for φ in $L^2(Q)$.

Arguing as above (using (2.23) instead of (2.22)) we deduce that

$$\int_Q \chi_{Q_\varepsilon^-} \nabla v_\varepsilon \, \varphi \, dx \to \int_{Q^-} \nabla v \, \varphi \, dx,$$

for every φ in $\mathcal{D}(Q^-)$, which gives convergence (2.21) iii) and ends the proof. $\square$

As a consequence of Proposition 2.4, from (2.19) we have

Corollary 2.5. *Let u_ε be a solution, for every ε, of problem (2.18) and assume that Γ_ε is defined by (2.1) for $\kappa > 0$. Then, there exists a subsequence (still denoted u_ε) and a function u in W_0^0 such that*

$$\begin{cases} i) \ u_\varepsilon \to u, & \text{strongly in } L^2(Q), \\ ii) \ \chi_{Q_\varepsilon^+} \nabla u_\varepsilon \rightharpoonup \chi_{Q^+} \nabla u, & \text{weakly in } (L^2(Q))^n, \\ iii) \ \chi_{Q_\varepsilon^-} \nabla u_\varepsilon \rightharpoonup \chi_{Q^-} \nabla u, & \text{weakly in } (L^2(Q))^n. \end{cases} \tag{2.31}$$

When $\kappa \geq 1$, the results below complete these convergences.

Proposition 2.6. *If $\kappa \geq 1$ in (2.1) then, there exists two families of linear continuous extensions operators $P_\varepsilon^+ : H^1(Q_\varepsilon^+) \to H^1(Q)$ and $P_\varepsilon^- : H^1(Q_\varepsilon^-) \to H^1(Q)$ which are bounded uniformly in ε, that is*

$$\|P_\varepsilon^+ v\|_{H^1(Q)} \leq c\|v\|_{H^1(Q_\varepsilon^+)}, \qquad \text{for every } v \in H^1(Q_\varepsilon^+),$$
$$\|P_\varepsilon^- v\|_{H^1(Q)} \leq c\|v\|_{H^1(Q_\varepsilon^-)}, \qquad \text{for every } v \in H^1(Q_\varepsilon^-),$$

with c independent of ε.

Proof. Observe that if $\kappa \geq 1$, the function defining the interface Γ^ε is uniformly Lipschitz continuous. Then, the result follows from the usual extension theorem in Sobolev spaces, since the norm of the extension operators depend on the Lipschitz constant of the function defining the boundary. $\square$

Corollary 2.7. *If $\kappa \geq 1$ in (2.1) then, then there exists a subsequence (still denoted ε) and two functions U^+ and U^- in $H^1(Q)$ such that (2.31) holds true and*

$$\begin{cases} i) \ P_\varepsilon^+(u_\varepsilon^+) \rightharpoonup U^+, & \text{weakly in } H^1(Q), \\ ii) \ P_\varepsilon^-(u_\varepsilon^-) \rightharpoonup U^-, & \text{weakly in } H^1(Q), \end{cases} \tag{2.32}$$

with

$$U^+_{|Q^+} = u^+, \qquad U^-_{|Q^-} = u^-. \tag{2.33}$$

Proof. It follows from (2.19) and Proposition 2.6 that $\|P_\varepsilon^\pm(u_\varepsilon^\pm)\|_{H^1(Q)} \leq c$. This implies (2.32) for a subsequence. Moreover, choosing a subsequence, we can also assume that convergences (2.31) hold. We have

$$P_\varepsilon^+(u_\varepsilon^+)\chi_{Q^+} = P_\varepsilon^+(u_\varepsilon^+)\chi_{Q_\varepsilon^+} + P_\varepsilon^+(u_\varepsilon^+)(\chi_{Q^+} - \chi_{Q_\varepsilon^+}) = u_\varepsilon^+\chi_{Q_\varepsilon^+} + P_\varepsilon^+(u_\varepsilon^+)(\chi_{Q^+} - \chi_{Q_\varepsilon^+}).$$

Since $\chi_{Q_\varepsilon^+}$ converges a.e. to χ_{Q^+}, then Proposition 2.6 implies that

$$P_\varepsilon^+(u_\varepsilon^+) \to u^+, \qquad \text{strongly in } L^2(Q^+),$$

which gives $U^+_{|Q^+} = u^+$. Similarly, one proves the equality $U^-_{|Q^-} = u^-$. □

Remark 2.8. *Since $\chi_{Q_\varepsilon^+}$ and $\chi_{Q_\varepsilon^-}$ converge a.e. to χ_{Q^+} and χ_{Q^-} respectively, under the notation of Corollary 2.5 one has for any $\kappa > 0$,*

$$\begin{cases} i)\ \chi_{Q_\varepsilon^+} u_\varepsilon \longrightarrow \chi_{Q^+} u^+, & \textit{strongly in } L^2(Q), \\ ii)\ \chi_{Q_\varepsilon^-} u_\varepsilon \longrightarrow \chi_{Q^-} u^-, & \textit{strongly in } L^2(Q). \end{cases} \tag{2.34}$$

Our aim is to identify u. This is done in the following sections.

3. **The homogenized tensor.** We want to pass to the limit in the integral identity (2.18). To do that, denote by A^0 the homogenized tensor (see [2]) defined by

$$A^0\lambda = m_Y(A\nabla w_\lambda)$$

with $w_\lambda \in H^1(Y)$ being a solution, for any $\lambda \in \mathbb{R}^n$, of

$$\begin{cases} \text{-div } (A\nabla w_\lambda) = 0 & \text{in } Y, \\ w_\lambda - \lambda \cdot y & Y\text{-periodic.} \end{cases}$$

Proposition 3.1. *Let Γ_ε be defined by (2.1) with $\kappa > 0$ and u_ε be the solution, for every ε, of problem (2.18). Then, for every φ in $H_0^1(Q)$,*

$$\lim_{\varepsilon\to 0} \int_Q A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx = \int_Q A^0 \nabla u \nabla \varphi\, dx. \tag{3.1}$$

Moreover,

$$\begin{cases} i)\ \chi_{Q_\varepsilon^+} A^\varepsilon \nabla u^\varepsilon \rightharpoonup \chi_{Q^+} A^0\ \nabla u, & \textit{weakly in } (L^2(Q))^n, \\ ii)\ \chi_{Q_\varepsilon^-} A^\varepsilon \nabla u_\varepsilon \rightharpoonup \chi_{Q^-} A^0 \nabla u, & \textit{weakly in } (L^2(Q))^n. \end{cases} \tag{3.2}$$

Proof. For a fixed $\delta > 0$ denote $Q_\delta^+ = \{x \in Q : x_n > \delta\}$ and $\Pi_\delta = \{x \in Q : 0 \le x_n \le \delta\}$. Clearly, for all small enough $\varepsilon > 0$ we have $Q_\delta^+ \subset Q_\varepsilon^+$. Then, exploiting the classical result on convergence of fluxes for arbitrary solutions, we conclude that

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u \quad \text{weakly in } L^2(Q_\delta^+), \tag{3.3}$$

as $\varepsilon \to 0$. Therefore, for any $\varphi \in H_0^1(Q)$ we have

$$\lim_{\varepsilon\to 0} \int_{Q_\delta^+} A^\varepsilon \nabla u^\varepsilon \nabla \varphi\, dx = \int_{Q_\delta^+} A^0 \nabla u \nabla \varphi\, dx. \tag{3.4}$$

Due to a priori estimate (2.19) the contribution of Π_δ can be estimated as follow

$$\begin{aligned} \Big| \int_{\Pi_\delta \setminus \Gamma_\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla \varphi\, dx \Big| &\le C\|A^\varepsilon \nabla u^\varepsilon\|_{L^2(\Pi_\delta\setminus\Gamma_\varepsilon)} \|\nabla\varphi\|_{L^2(\Pi_\delta)} \le C\|u^\varepsilon\|_{W_0^\varepsilon} \|\nabla\varphi\|_{L^2(\Pi_\delta)} \\ &\le C\|f\|_{L^2(Q)} \|\nabla \varphi\|_{L^2(\Pi_\delta)}. \end{aligned}$$

Similarly,

$$\Big| \int_{\Pi_\delta} A^\varepsilon \nabla u^0 \nabla\varphi\, dx \Big| \le C\|f\|_{L^2(Q)} \|\nabla\varphi\|_{L^2(\Pi_\delta)}.$$

Therefore, from (3.4)

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\Big|\int\limits_{Q^+} A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx - \int\limits_{Q^+} A^0 \nabla u \nabla\varphi\, dx\Big| \\
&\le \limsup_{\varepsilon\to 0}\Big|\int\limits_{Q^+_\delta} A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx - \int\limits_{Q^+_\delta} A^0 \nabla u \nabla\varphi\, dx\Big| \\
&\quad + \limsup_{\varepsilon\to 0}\Big|\int\limits_{\Pi_\delta\setminus\Gamma_\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx - \int\limits_{\Pi_\delta} A^0 \nabla u \nabla\varphi\, dx\Big| \\
&\le 2C\|f\|_{L^2(Q)}\|\nabla\varphi\|_{L^2(\Pi_\delta)}.
\end{aligned}$$

Since the left hand side in the last inequality does not depend on δ, and $\|\nabla\varphi\|_{L^2(\Pi_\delta)}$ tends to zero as $\delta\to 0$, we conclude that

$$\lim_{\varepsilon\to 0}\int\limits_{Q^+} A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx = \int\limits_{Q^+} A^0 \nabla u \nabla\varphi\, dx.$$

In the same way one can show that

$$\lim_{\varepsilon\to 0}\int\limits_{Q^-} A^\varepsilon \nabla u^\varepsilon \nabla\varphi\, dx = \int\limits_{Q^-} A^0 \nabla u \nabla\varphi\, dx$$

and conclude the proof of (3.1).

Observe now that from (2.19) and (2.6) it follows that the sequence $\{\chi_{Q^+} A^\varepsilon\nabla u^\varepsilon\}$ is weakly compact in $L^2(Q)$. Then, convergence (3.2) i) follows from (3.3) by the same arguments as those used to prove convergences (2.21) i) and (2.25). Similarly, one proves (3.2) ii). □

In order to pass to the limit in the surface term, we need the following Lemma, which was proved in [6]:

Lemma 3.2. *For any $v\in H^1(Q)$ the inequalities hold*

$$\|v\Big(x',\varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\Big) - v(x',0)\|_{L^2(\omega)} \le c\sqrt{\varepsilon^\kappa}\|v\|_{H^1(Q)}, \tag{3.5}$$

$$\|v\Big(x',\varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\Big) - v(x',\varepsilon^\kappa \bar g)\|_{L^2(\omega)} \le c\sqrt{\varepsilon^\kappa}\|v\|_{H^1(Q)}, \tag{3.6}$$

where c is independent of ε.

4. **Self-similar case.**

In this section we consider the case of $\kappa=1$ and any real γ. We call this geometry self-similar because for $\kappa=1$ the profile of interface Γ_ε can be obtained by homothetic dilatation of the fixed function $y_n=g(y')$ in $\mathbb{R}^n$.

We have the following result

Theorem 4.1. *Suppose that $\kappa=1$.*

- *If $\gamma=0$, then the solution u_ε of problem (2.18) converges, as $\varepsilon\to 0$, in $L^2(Q)$ towards a unique solution*

$$u=\begin{cases} u^+(x), & x\in Q^+\\ u^-(x), & x\in Q^-\end{cases}$$

of the problem

$$\begin{cases} -\mathrm{div}(A^0\nabla u) = f \qquad \text{in } Q\setminus\Gamma_0, \\ (A^0\nabla u)^-\cdot n = (A^0\nabla u)^+\cdot n \qquad \text{on } \Gamma_0, \\ (A^0\nabla u)^+\cdot n = \bar{G}(u^+ - u^-), \qquad \text{on } \Gamma_0, \\ u = 0 \qquad \text{on } \partial Q, \end{cases} \tag{4.1}$$

where

$$\bar{G} = m_{Y'}\Big(h(1+(|\nabla g|^2)^{1/2}\Big). \tag{4.2}$$

- *If $\kappa = 1$ and $\gamma > 0$, then u_ε converges in $L^2(Q)$ to the unique solution $u = (u^+, u^-)$ of the problem*

$$\begin{cases} -\mathrm{div}(A^0\nabla u) = f \qquad \text{in } Q\setminus\Gamma_0, \\ (A^0\nabla u)^-\cdot n = (A^0\nabla u)^+\cdot n = 0 \quad \text{on } \Gamma_0, \\ u = 0 \qquad \text{on } \partial Q, \end{cases} \tag{4.3}$$

- *If $\kappa = 1$ and $\gamma < 0$, then u_ε converges in $L^2(Q)$ to a function $u \in H_0^1(Q)$ being the unique solution in $H_0^1(Q)$ of the problem*

$$\begin{cases} -\mathrm{div}(A^0\nabla u) = f \qquad \text{in } Q, \\ u = 0 \qquad \text{on } \partial Q. \end{cases} \tag{4.4}$$

Moreover, for any $\gamma \in \mathbb{R}$ one has

$$\begin{cases} \chi_{Q_\varepsilon^+}\nabla u_\varepsilon \rightharpoonup \chi_{Q^+}\nabla u, \qquad \text{weakly in } (L^2(Q))^n, \\ \chi_{Q_\varepsilon^-}\nabla u_\varepsilon \rightharpoonup \chi_{Q^-}\nabla u, \qquad \text{weakly in } (L^2(Q))^n. \end{cases} \tag{4.5}$$

Proof. Applying Lemma 3.2 to the functions $P_\varepsilon^\pm(u_\varepsilon^\pm)$ given by Corollary 2.7 and to $\varphi^\pm$ and considering the above a priori estimates, we obtain (see Remark 2.3)

$$\begin{aligned} &\int_{\Gamma_\varepsilon} h^\varepsilon(u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\,d\sigma \\ &= \int_\omega h\Big(\frac{x'}{\varepsilon}\Big)\big(P_\varepsilon^+u_\varepsilon^+(x',0) - P_\varepsilon^-u_\varepsilon^-(x',0)\big)(\varphi^+(x',0) - \varphi^-(x',0)) \\ &\times\Big(1 + (|\nabla g(y')|^2)\big|_{y'=x'/\varepsilon}\Big)^{1/2}\,dx' + O(\sqrt{\varepsilon}). \end{aligned}$$

Taking into account the compactness of the family $P_\varepsilon^\pm(u_\varepsilon^\pm)|_{\Gamma_0}$ in $L^2(\omega)$ insured by (2.32), and using (2.33), we conclude that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} h^\varepsilon(u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\,d\sigma = \bar{G}\int_\omega \big(u^+(x',0) - u^-(x',0)\big)(\varphi^+(x',0) - \varphi^-(x',0))\,dx',$$

where $\bar{G}$ is given by (4.2). Therefore, for $\gamma > 0$ we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\,d\sigma = 0.$$

For $\gamma = 0$ we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\,d\sigma \\ &= \bar{G}\int_\omega \big(u^+(x',0) - u^-(x',0)\big)(\varphi^+(x',0) - \varphi^-(x',0))\,dx'. \end{aligned}$$

Finally, let us show that if $\gamma < 0$, then $u^+_{|\Gamma_0} = u^-_{|\Gamma_0}$, so that u is in $H^1_0(Q)$. To this end, we derive from Corollary 2.7 and inequality (3.5) of Lemma 3.2 that

$$\int_{\Gamma_\varepsilon} (u^+_\varepsilon - u^-_\varepsilon)^2 \, d\sigma = \int_{\Gamma_\varepsilon} (P^+_\varepsilon u^+_\varepsilon - P^-_\varepsilon u^-_\varepsilon)^2 \, d\sigma \to \bar{G}_1 \int_\omega \Big(u^+(x',0) - u^-(x',0)\Big)^2 dx',$$

where $\bar{G}_1 = m_{Y'}\Big(1 + (|\nabla g|^2)^{1/2}\Big)$.

Also, from (2.19) if follows that

$$\int_{\Gamma_\varepsilon} (u^+_\varepsilon - u^-_\varepsilon)^2 \, d\sigma \le c\varepsilon^{-\gamma} \to 0,$$

since $\gamma < 0$, which yields the desired coincidence of the traces. Here we have also used the fact that $\bar{G}_1 > 0$. The above results, together with Proposition 3.1, allow to pass to the limit in (2.18) and give the limit problems for the different values of γ. Moreover, since these problems have a unique solution, all the sequences in (2.31) converge. □

Remark 4.2. The shape of g contributes in the limit problems only for the case $\gamma = 0$, where it describes the jump of the homogenized solution on Γ_0. Observe also that problem (4.3) is equivalent to the following two (independent) Neumann problems solved by u^+ and u^- respectively:

$$\begin{cases} -\mathrm{div}(A^0 \nabla u^+) = f & \text{in } Q^+, \\ A^0 \nabla u^+ \cdot n = 0 & \text{on } \Gamma_0, \\ u = 0 & \text{on } \partial Q^+ \setminus \Gamma_0, \end{cases} \qquad \begin{cases} -\mathrm{div}(A^0 \nabla u^-) = f & \text{in } Q^-, \\ A^0 \nabla u^- \cdot n = 0 & \text{on } \Gamma_0, \\ u = 0 & \text{on } \partial Q^- \setminus \Gamma_0. \end{cases} \tag{4.6}$$

That is, the problems in the two components are split at the limit, and Γ_0 represent then an isolating interface.

In the third case, the presence of the interface is neglectful, and the homogenized problem is the same as that of the case without any interface.

5. **The flat case.** We consider now the case of $\kappa > 1$ and any real γ, for which we have the following result:

Theorem 5.1. *Suppose that $\kappa > 1$.*

- *If $\gamma = 0$, then the solution u_ε of problem (2.18) converges, as $\varepsilon \to 0$, in $L^2(Q)$ towards a unique solution*

  $$u = \begin{cases} u^+(x), & x \in Q^+ \\ u^-(x), & x \in Q^- \end{cases}$$

  of the problem

  $$\begin{cases} -\mathrm{div}(A^0 \nabla u) = f & \text{in } Q \setminus \Gamma_0, \\ (A^0 \nabla u)^- \cdot n = (A^0 \nabla u)^+ \cdot n & \text{on } \Gamma_0, \\ (A^0 \nabla u)^+ \cdot n = m_{Y'}(h)(u^+ - u^-), & \text{on } \Gamma_0, \\ u = 0 & \text{on } \partial Q. \end{cases} \tag{5.1}$$

- *If $\gamma > 0$, then u_ε converges in $L^2(Q)$ towards the unique solution of problem (4.3).*
- *If $\gamma < 0$, then u_ε converges in $L^2(Q)$ towards the unique solution of problem (4.4).*

Moreover, for any $\gamma \in \mathbb{R}$ the convergence (4.5) still holds true.

Proof. Since Proposition 2.6 and Corollary 2.7 still hold in this case, the proof follows the line of Theorem 4.1. The only difference is that here

$$\begin{aligned}
&\int_{\Gamma_\varepsilon} h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\, d\sigma \\
&= \int_\omega h\Big(\frac{x'}{\varepsilon}\Big)\big(P_\varepsilon^+ u_\varepsilon^+(x',0) - P_\varepsilon^- u_\varepsilon^-(x',0)\big)\big(\varphi^+(x',0) - \varphi^-(x',0)\big) \\
&\quad \times \Big(1 + \varepsilon^{\kappa-1}(|\nabla_{y'} g(y')|^2)\big|_{y'=x'/\varepsilon}\Big)^{1/2} dx',
\end{aligned}$$

so that in the jump condition of problem (5.1) we obtain $m_{Y'}(h)$ instead of $\bar{G}$. $\square$

Remark 5.2. Since in the case considered in this section $\varepsilon^\kappa \ll \varepsilon$, then the surface measure on Γ_ε converges, as $\varepsilon \to 0$, to the surface measure on Γ_0, and the shape of g does not contribute to the limit problem for all values of γ. In this case the principal term of the asymptotics of a solution to problem (2.10) does not change if one replaces the interface Γ_ε with the flat interface Γ_0.

6. **Highly oscillating interface.** This section deals with the case $0 < \kappa < 1$. As in the previous section, we are going to characterize the limit $u = (u^+, u^-)$ of the family $\{u_\varepsilon\}$.

In contrast with the case $\kappa = 1$, the critical value of γ giving rise to a nontrivial effective transmission condition, is not equal to zero in the present case. We show that this critical value is equal to $1-\kappa$. As above, in the supercritical mode $\gamma > 1-\kappa$ the limit problem consists of two separate problems in Q^+ and Q^-, each of them having homogeneous Neumann condition at $\Gamma_0 = \{x \in Q : x_n = 0\}$. In the subcritical regime the limit function u does not have a discontinuity at Γ_0 so that the limit problem happens to be a usual Dirichlet problem in the whole domain Q.

Let us mention that for $0 < \kappa < 1$ we cannot construct any bounded family of linear extension operator as done in Proposition 2.6. Then, we have to give here a different and direct proof.

In order to formulate the main result of this section we introduce the following notation:

$$\bar{G}_2 = m_{Y'}\Big(h|\nabla g)|\Big).$$

Theorem 6.1. *Suppose that $0 < \kappa < 1$.*

- *If $\gamma = 1-\kappa$, then the solution u_ε of problem (2.18) converges, as $\varepsilon \to 0$, in $L^2(Q)$ towards a unique solution*

$$u = \begin{cases} u^+(x), & x \in Q^+ \\ u^-(x), & x \in Q^- \end{cases}$$

of the problem

$$\begin{cases} -\mathrm{div}(A^0 \nabla u) = f & \text{in } Q \setminus \Gamma_0, \\ (A^0\nabla u)^- \cdot n = (A^0 \nabla u)^+ \cdot n & \text{on } \Gamma_0, \\ (A^0 \nabla u)^+ \cdot n = \bar{G}_2(u^+ - u^-), & \text{on } \Gamma_0, \\ u = 0 & \text{on } \partial Q. \end{cases} \tag{6.1}$$

- *If $\gamma > 1-\kappa$, then u_ε converges in $L^2(Q)$ towards the unique solution of problem (4.6).*
- *If $\gamma < 1-\kappa$, then u_ε converges in $L^2(Q)$ towards the solution of problem (4.4).*

Moreover, convergence (4.5) is still valid.

Proof. Our aim is to pass to the limit in the integral identity (2.18). To this end we consider two arbitrary functions φ^+ and φ^- in $\mathcal{D}(Q)$ and set

$$\varphi_\varepsilon = \chi_{Q_\varepsilon^+}\varphi^+ + \chi_{Q_\varepsilon^-}\varphi^-, \qquad \varphi = \chi_{Q^+}\varphi^+ + \chi_{Q^-}\varphi^-, \qquad \Phi_\varepsilon^\pm(x') = \varphi^\pm(x', \varepsilon^\kappa g(\frac{x'}{\varepsilon})).$$

Clearly, φ_ε belongs to W_0^ε. Substituting φ_ε as a test function in (2.18), passing to the limit and considering (3.2) and (2.31), we obtain

$$\lim_{\varepsilon\to 0}\int_{Q\setminus\Gamma_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi_\varepsilon \, dx = \int_{Q\setminus\Gamma_0} A^0\nabla u\nabla\varphi\, dx, \qquad \lim_{\varepsilon\to 0}\int_Q f u_\varepsilon\, dx = \int_Q fu\, dx. \tag{6.2}$$

In order to pass to the limit in the surface integral in (2.18) we restrict the function u_ε^+ to the set $\omega\times]\varepsilon^\kappa\bar g, l[$ and then extend the resulting function to the set Q^+ in such a way that

$$\|\check u_\varepsilon^+\|_{H^1(Q^+)} \le 2\|u_\varepsilon^+\|_{H^1(\omega\times]\varepsilon^\kappa \bar g, l[)};$$

here $\check u_\varepsilon^+$ stands for the extended function. Since (see (2.16))

$$\check u_\varepsilon^+ = \chi_{Q_\varepsilon^+} u_\varepsilon^+ + \chi_{B_\varepsilon^-}\check u_\varepsilon^+$$

and the measure of B_ε^- goes to zero, from (2.34) and the fact that $\check u_\varepsilon^+ \in L^{2^*}(Q^+)$ it follows that $\check u_\varepsilon^+$ converges to u^+ in $L^2(Q^+)$ as $\varepsilon \to 0$. Since $\|\check u_\varepsilon^+\|_{H^1(Q^+)} \le c$, this yields

$$\|u_\varepsilon^+(\cdot, \varepsilon^\kappa\bar g) - u^+(\cdot,\varepsilon^\kappa \bar g)\|_{L^2(\omega)} = \|\check u_\varepsilon^+(\cdot,\varepsilon^\kappa\bar g) - u^+(\cdot,\varepsilon^\kappa\bar g)\|_{L^2(\omega)} \longrightarrow 0. \tag{6.3}$$

Indeed,

$$\begin{aligned} \|\check u_\varepsilon^+(\cdot,\varepsilon^\kappa\bar g) - u^+(\cdot,\varepsilon^\kappa\bar g)\|_{L^2(\omega)} &\le \|\check u_\varepsilon^+(\cdot,0) - u^+(\cdot,0)\|_{L^2(\omega)} \\ &+ \|\check u_\varepsilon^+(\cdot,\varepsilon^\kappa\bar g) - u^+(\cdot,\varepsilon^\kappa\bar g) - \check u_\varepsilon^+(\cdot,0) + u^+(\cdot,0)\|_{L^2(\omega)}. \end{aligned} \tag{6.4}$$

Considering the fact that $(u^+ - \check u_\varepsilon^+)$ converges to zero weakly in $H^1(Q^+)$ as $\varepsilon \to 0$, we conclude that the first term on the right-hand side of (6.4) tends to 0. The second term goes to 0 by Lemma 3.2, and (6.3) follows.

By the continuity of trace arguments we have

$$\|u^+(\cdot,\varepsilon^\kappa\bar g) - u^+(\cdot,0)\|_{L^2(\omega)} \longrightarrow 0. \tag{6.5}$$

Finally, combining (6.3), (6.5) and statement (3.6) of Lemma 3.2, we conclude that

$$\|u_\varepsilon^+\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - u^+(x',0)\|_{L^2(\omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Similarly,

$$\|u_\varepsilon^-\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - u^-(x',0)\|_{L^2(\omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

If $\gamma = 1-\kappa$, this yields

$$\varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_\varepsilon^+ - u_\varepsilon^-)(\varphi^+ - \varphi^-)\, d\sigma$$

$$= \varepsilon^\gamma \int\limits_{\Gamma_0} h\big(\frac{x'}{\varepsilon}\big)\varepsilon^{\kappa-1}\left(\varepsilon^{2(1-\kappa)} + \big|\nabla_{y'} g(y')\big|^2_{y'=x'/\varepsilon}\right)^{\frac{1}{2}}$$

$$\big\{u_\varepsilon^+\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - \big(u_\varepsilon^-\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\big\}(\Phi_\varepsilon^+ - \Phi_\varepsilon^-)dx'$$

$$= \int\limits_{\Gamma_0} h\big(\frac{x'}{\varepsilon}\big)\big|\nabla_{y'} g(y')\big|_{y'=x'/\varepsilon}\big\{u_\varepsilon^+\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)$$

$$-\big(u_\varepsilon^-\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\big\}(\Phi_\varepsilon^+ - \Phi_\varepsilon^-)dx' \ + o(1)$$

$$= \int\limits_{\Gamma_0} h\big(\frac{x'}{\varepsilon}\big)\big|\nabla_{y'} g(y')\big|_{y'=x'/\varepsilon}\{u^+(x',0) - (u^-(x',0)\}$$

$$(\varphi^+(x',0) - \varphi^-(x',0))dx' \ + o(1)$$

$$\underset{\varepsilon\to 0}{\longrightarrow} (\bar{G}_2)\int\limits_{\Gamma_0}\{u^+(x',0) - (u^-(x',0)\}(\varphi^+(x',0) - \varphi^-(x',0))dx';$$

here $o(1)$ stands for sequences which tend to 0 as $\varepsilon \to 0$. From (6.2) and the last relation it follows that $u = (u^+, u^-)$ satisfies the following integral identity

$$\int\limits_{G\setminus\Gamma_0} A^0 \nabla u \cdot \nabla\varphi\, dx + (\bar{G}_2)\int\limits_{\Gamma_0}\{u^+(x',0) - (u^-(x',0)\}(\varphi^+(x',0) - \varphi^-(x',0))dx'$$

$$= \int_G f u\, dx$$

for any $\varphi^+,\ \varphi^- \in \mathcal{D}(G)$. By density arguments it also holds for arbitrary $\varphi \in W_0^0$. This completes the proof of the first statement of the theorem. The other two statements can be proved in exactly the same way as in the case $\kappa = 1$ studied in the previous section, if we replace the extension of function u_ε^+ with that of function $\breve{u}_\varepsilon^+$. □

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Received June 2009; revised November 2009.

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