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Asymptotic Behaviour of the Ground State of Singularly Perturbed Elliptic Equations

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Abstract: The ground state of a singularly perturbed nonselfadjoint elliptic operator

$$\mu^2 \nabla_i a^{ij}(x) \nabla_j + \mu b^i(x) \nabla_i + v(x)$$

defined on a smooth compact Riemannian manifold with metric $a_{ij}(x) = (a^{ij}(x))^{-1}$, is studied. We investigate the limiting behaviour of the first eigenvalue of this operator as μ goes to zero, and find the logarithmic asymptotics of the first eigenfunction everywhere on the manifold. The results are formulated in terms of auxiliary variational problems on the manifold. This approach also allows to study the general singularly perturbed second order elliptic operator on a bounded domain in R^n.

0. Introduction

Let M be an n-dimensional smooth compact Riemannian manifold endowed with metric $a_{ij}(x)$. Consider the following eigenvalue problem

$$\begin{aligned} A^\mu p &= \frac{1}{4}\mu^2 |a(x)|^{-1/2} \frac{\partial}{\partial x^i} a^{ij}(x) |a(x)|^{1/2} \frac{\partial}{\partial x^j} p(x) + \mu b^i(x) \frac{\partial}{\partial x^i} p(x) + v(x) p(x) \\ &= -\lambda p(x) \end{aligned} \tag{0.1}$$

on M, where $\mu > 0$ is a small parameter; $(a^{ij}(x)) = (a_{ij}(x))^{-1}$, $|a(x)| = det\{a_{ij}(x)\}$ and $|a(x)|^{-1/2} \frac{\partial}{\partial x^i} a^{ij}(x) |a(x)|^{1/2} \frac{\partial}{\partial x^j}$ is the Laplace–Beltrami operator on M (see [1], [8] for the relevant definitions). The coefficients $a^{ij}(x), b^i(x), v(x)$ are supposed to be continuously differentiable real functions and the matrix $a_{ij}(x)$ is uniformly positive:

$$a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad \xi \in R^n, c > 0.$$

Thus, we deal with singularly perturbed uniformly elliptic operator.

It is well known (see, for example, [15]) that for any fixed $\mu > 0$ the operator A^μ has a discrete spectrum $\{\lambda_0, \lambda_1, ...\}$, $\Re\lambda_k \to +\infty$ as $k \to \infty$. According to [6], the first eigenvalue (i.e., the eigenvalue with the smallest real part) λ_0 is simple and real, and the first eigenfunction $p_0(x)$ is also real and does not change sign. It then follows from the maximum principle that $p_0(x)$ can be chosen positive (see [4]). We assume, without loss of generality, the following normalizing conditions to be satisfied:

$$\int_M p_0(x)m(dx) = 1, \qquad \int_M m(dx) = 1, \qquad \max_M v(x) = 0, \tag{0.2}$$

where $m(dx) = |a(x)|^{1/2}dx^1...dx^n$ in local coordinates. This paper is aimed at an investigation of the asymptotic behaviour of λ_0 and $p_0(x)$ as $\mu \to 0$.

One of the most important applications of the results obtained here is a study of the large-time behaviour of solutions to the Cauchy problem for singularly perturbed parabolic equations. The growth or decay rate of the solutions, as well as their limiting shape can be described in terms of the corresponding ground state.

Moreover, in the case of a torus, these asymptotics play a significant role in homogenization theory (see [5], [12], [14]). Indeed, as demonstrated in [4], [7], the homogenization problem for A^μ can be reduced to the homogenization problem for a certain operator involving no zero-order term. Then, the standard homogenization technique can be applied. The coefficients of the latter operator depend on the ground state of A^μ and, therefore, in order to investigate the limit behaviour of the effective coefficients of A^μ for small μ, one should know the asymptotics of the ground state.

If the operator A^μ is selfadjoint, i.e. if $b(x) \equiv 0$, one can use the variational technique in order to find the limit of λ_0. In contrast with selfadjoint operators, in the case under consideration the standard variational approach cannot be applied so even the study of the first eigenvalue becomes nontrivial. Both the behaviour of λ_0 and the asymptotics of $p_0(x)$ are described here in terms of auxiliary variational problems on the manifold M. In particular, we give a simple necessary and sufficient condition for convergence of λ_0 to zero (recall that it is always so in the selfadjoint case). We also show that under certain conditions $p_0(x)$ decays exponentially at almost all points of M, and give the corresponding asymptotics in a logarithmic scale.

Previously, closely related eigenvalue problems in a smooth bounded domain in R^n for a singularly perturbed operator of special form

$$\mu a^{ij}(x)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + b^i(x)\frac{\partial}{\partial x^i} \tag{0.3}$$

were considered in [9]–[11], where only the asymptotics of the first eigenvalue were analyzed, but the first eigenfunction was not considered. Note that this operator is, in fact, a special case of (0.1). Indeed, it suffices to divide (0.1) by μ and set $v(x) \equiv 0$. It turns out that the limit behaviour of $\lambda_0(\mu)$ in this special case depends crucially on the geometry of the integral curves of the equation $\dot{x} = b(x)$, especially near the boundary of the domain. If, for instance, $b(x)\cdot\nu > 0$ at the boundary, where ν is the inner normal, i.e. if all the trajectories starting in the domain never leave it, then, as shown in [10], [11], $\lambda_0(\mu)$ decays exponentially as $\mu \to 0$ and, under additional assumptions, its logarithmic asymptotics can be calculated with the help of the rate functional for the corresponding diffusion process. On the other hand, if there is a smooth $\phi(x)$ such that $b(x)\cdot\nabla\phi(x) > 0$ in the entire domain then, according to [9], $\lambda_0(\mu) > c$ for some $c > 0$ (note that our normalization differs from [9]).

Although in some special cases, the methods developed in the cited papers yield precise asymptotics of the first eigenvalue, they only work for operators with $v(x) \equiv 0$, and for domains with non-empty boundary. Then, even in those cases, the asymptotic upper and lower bounds for the lowest eigenvalue need not coincide. Moreover, there are no results on the corresponding eigenfunction.

In the present paper we use another approach which combines various variational methods with the large deviation technique. This approach can also be applied to operators defined in a bounded domain. In particular, the rough asymptotics of $\lambda_0(\mu)$ can be found for the Dirichlet problem for an arbitrary elliptic operator of the form (0.1) or (0.3).

In Sect. 1 we introduce auxiliary variational functionals and study their properties. The basic result here is the existence of the following limit

$$\hat{\lambda} = \lim_{T\to\infty} \inf_{x(\cdot)} \frac{1}{T} \int_0^T \big(a_{ij}(x(t))(\dot{x}^i - b^i(x(t)))(\dot{x}^j - b^j(x(t))) - v(x(t))\big)\, dt,$$

where the infimum is taken over all smooth curves on the manifold.

In Sect. 2 we prove the convergence of the first eigenvalue as $\mu \to 0$. The main assertion here, Theorem 1, states that

$$\lim_{\mu\to 0} \lambda_0(\mu) = \hat{\lambda}.$$

Section 3 is devoted to the investigation of $p_0(x)$. Under an additional condition on the operator (see the definition of recursive operator below), the logarithmic asymptotics of $p_0(x)$, uniform over the manifold, is constructed. This condition concerns the set of accumulating points of trajectories satisfying the relation

$$\sup_T \int_0^T \big(a_{ij}(x(t))(\dot{x}^i - b^i(x(t)))(\dot{x}^j - b^j(x(t))) - v(x(t))\big)\, dt - \hat{\lambda}T \; < \; \infty.$$

Namely, the intersection of all such sets is assumed to be nonempty.

Section 4 is of special interest. Here we consider the operators with "potential" first order terms:

$$b^i(x) = a^{ij}(x)\frac{\partial}{\partial x^j}U(x), \qquad i = 1, 2, \dots, n\,,$$

for a smooth function $U(x)$ on M, i.e. $b(x)$ is the gradient of $U(x)$ in the metric $a_{ij}(x)$. Then, the problem of finding $\hat{\lambda}$ takes the following algebraic form:

$$\hat{\lambda} = \min_{x\in M} \big(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\big).$$

Moreover, an operator with potential first order terms is recursive iff the minimum point of the above expression is unique; therefore, in this case, the recursiveness condition is that of a general position.

Section 5 contains results about selfadjoint operators ($b(x) \equiv 0$). These results have been proved in [12] by other methods and are included in this paper for the sake of completeness. It should be noted that in case of selfadjoint operators the logarithmic asymptotics of $p_0(x)$ admits a simple geometric interpretation:

$$\lim_{\mu\to 0} \mu \ln p_0(x) = -\mathrm{dist}_{|v|a_{ij}}(x, x_0),$$

where x_0 is the unique maximum point of $v(x)$, $v(x_0) = 0$, and the distance is taken in the metric $|v(x)|a_{ij}(x)$.

1. Auxiliary Variational Problems

For absolutely continuous curves $x(t) = (x^1(t), ..., x^n(t))$, $0 \le t \le T$, on M, we define the functional $I(x(\cdot), T)$ as follows:

$$I(x(\cdot), T) = \int_0^T \big(a_{ij}(x(t))(\dot{x}^i - b^i(x(t)))(\dot{x}^j - b^j(x(t))) - v(x(t))\big)\, dt;$$

here $a_{ij}(x)$ is the inverse matrix to $a^{ij}(x)$; recall that a_{ij} is the metric on M. Let us extend $I(x(\cdot), T)$ to the space $C(0, T; M)$ of continuous functions by setting $I(x(\cdot), T) = \infty$ for all other $x(\cdot)$. It is easy to see that for every fixed $T > 0$ the functional $I(x(\cdot), T)$ maps $C(0, T; M)$ into $(0, +\infty)$. In what follows, we also use the functionals

$$S(x, y, T) = \inf_{x(\cdot),\, x(0)=x,\, x(T)=y} I(x(\cdot), T),$$

$$m(T) = \inf_{x,y\in M} S(x, y, T), \qquad M(T) = \inf_{x\in M} S(x, x, T),$$

$$I_0(x(\cdot), T) = I(x(\cdot), T) + \int_0^T v(x(t))dt$$

$$= \int_0^T \big(a_{ij}(x(t))(\dot{x}^i - b^i(x(t)))(\dot{x}^j - b^j(x(t)))\big)\, dt.$$

Taking the curve $x(\cdot) = const$ as a trial function in the definitions of $m(T)$ and $M(T)$ we obtain

$$m(t) \le M(T) \le c_0 T \tag{1.1}$$

with a constant $c_0 \ge 0$. Hence,

$$\limsup_{T\to\infty} \frac{m(T)}{T} \le \limsup_{T\to\infty} \frac{M(T)}{T} \le c_0. \tag{1.2}$$

In fact, the limits $\left(\lim_{T\to\infty} \frac{m(T)}{T}\right)$ and $\left(\lim_{T\to\infty} \frac{M(T)}{T}\right)$ exist and coincide.

Lemma 1. *The functions $m(T)/T$ and $M(T)/T$ converge to the same limit $\hat{\lambda} \ge 0$ as $T \to \infty$. The inequality*

$$m(T) \le \hat{\lambda} T \le M(T) \tag{1.3}$$

holds for all $T > 0$, and there exists a sequence $t_k \to \infty$ such that $\lim_{k\to\infty} |M(t_k) - \hat{\lambda} t_k| = 0$.

Proof. The main idea of the proof is to use sub- and super-additivity of $m(t)$ and $M(t)$, respectively, in combination with the upper bound for $|M(t) - m(t)|$ obtained in Proposition 2 below. Propositions 1 and 3 provide relevant technical estimates and Proposition 4 states that the function $S(x, y, t)$ is Lipschitz continuous in all the variables.

The following relations

$$M(kt) \leq kM(t), \qquad m(kt) \geq km(t) \tag{1.4}$$

hold for any $t > 0$ and any integer $k > 0$. To prove the first one, let us rewrite the definition of $M(t)$ in the form $M(t) = \inf\limits_{\substack{x(\cdot) \\ x(0)=x(t)}} I(x(\cdot), t)$ and note that the infimum in the last relation can be replaced by the minimum, i.e. that $M(t)$ assumes its minimum. Indeed, in view of the positive definiteness of $a_{ij}(x)$ and the compactness of M, any minimizing sequence $\{x_k(\cdot)\}$ for $M(t)$ is uniformly bounded and, therefore, weakly compact in the functional space $H^1(0, t)$ endowed with the norm

$$\|x(\cdot)\|^2 = \int\limits_0^t (x^2(t) + \dot{x}^2(t))dt.$$

Passing to the weak limit as $k \to \infty$ and taking into account the weak semicontinuity of $I(x(\cdot), t)$, we obtain the curve desired.

Then, iterating the closed curve $\bar{x}(\cdot)$ which provides the minimum for $M(t)$ and using the same notation $\bar{x}(\cdot)$ for the curve obtained, we find that

$$M(kt) \leq I(\bar{x}(\cdot), kt) = kI(\bar{x}(\cdot), t) = kM(t)$$

for any positive integer k. To prove the second inequality in (1.4), it suffices to consider the curve $\tilde{x}(\cdot)$ which provides the minimum for $m(kt)$ and to divide an interval $(0, kt)$ into k equal parts:

$$m(kt) = I(\tilde{x}(\cdot), kt) = \sum_{l=0}^{k-1} I(\tilde{x}(\cdot + lt), t) \geq km(t).$$

Set $\overline{\lambda} = \left(\limsup\limits_{T\to\infty} \frac{M(T)}{T}\right)$ and $\underline{\lambda} = \left(\liminf\limits_{T\to\infty} \frac{M(T)}{T}\right)$ and suppose that the difference $\delta = \overline{\lambda} - \underline{\lambda}$ is positive. Then for some sequences $t'_k \to \infty$ and $t''_k \to \infty$ we have

$$\lim_{k\to\infty} \frac{M(t'_k)}{t'_k} = \overline{\lambda}, \qquad \lim_{k\to\infty} \frac{M(t''_k)}{t''_k} = \underline{\lambda}.$$

Let us fix $T > 1$ such that $M(T)/T < \underline{\lambda} + \delta/4$. Then by (1.4) we have $M(kT)/(kT) < \underline{\lambda} + \delta/4$ for all positive integer k. On the other hand, for sufficiently large k, we have

$$M(t'_k)/t'_k > \overline{\lambda}, \qquad \left(t'_k - T[t'_k/T]\right)/t'_k < T/t'_k < \delta/(4c_1), \tag{1.5}$$

where $c_1 = \max\limits_{x\in M} \left(a_{ij}(x)b^i(x)b^j(x) + v(x) + 1\right)$ and $[\cdot]$ means the integral part. Now let us iterate $[t'_k/T]$ times the curve that provides the minimum for $M(T)$ and extend it as constant. Using the curve obtained as a trial function in the definition of $M(t'_k)$ and taking into account (1.5) and the choice of T, we find that

$$\overline{\lambda} - \delta/4 < M(t_k')/t_k' = M([t_k']T + (t_k' - [t_k']T))/t_k'$$

$$\leq [t_k'/T]M(T)/t_k' + c_1(t_k' - [t_k'/T]T)/t_k' \leq M(T)/T + c_1\delta/(4c_1) \leq \underline{\lambda} + \delta/2.$$

This contradicts the fact that $\delta > 0$. Thus $\underline{\lambda} = \overline{\lambda}$ and the limit $\lim\limits_{t\to\infty}(M(t)/t)$ does exist. The existence of $\lim\limits_{t\to\infty}(m(t)/t)$ is a consequence of the following statements.

Proposition 1. *The inequality*

$$S(x, y, T) \leq c\left(T + \frac{1}{T}\right)$$

holds uniformly in $x, y \in M$.

Proof. Let us first note that in view of the compactness of M and the positive definiteness of a_{ij}, we have

$$\inf_{\substack{x(\cdot),\, x(0)=x,\\ x(T)=y}} \int\limits_0^T a_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)dt = \frac{1}{T} \inf_{\substack{x(\cdot),\, x(0)=x,\\ x(1)=y}} \int\limits_0^1 a_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)dt$$

$$\leq \frac{c}{T}\mathrm{dist}(x, y),$$

where $\mathrm{dist}(x, y)$ is the geodesic distance in the metric a_{ij}, and c does not depend on x and y. Then, reducing the set of trial curves in the definition of $S(x, , y, T)$ to the only curve $\hat{x}(\cdot)$ that provides a minimum in the above problem, we obtain

$$S(x, y, T) \leq I(\hat{x}(\cdot), T) = \int\limits_0^T a_{ij}(\hat{x}(t))\dot{\hat{x}}^i(t)\dot{\hat{x}}^j(t)dt - 2\int\limits_0^T a_{ij}(\hat{x}(t))\dot{\hat{x}}^i(t)b^j(\hat{x}(t))dt$$

$$+\int\limits_0^T \left(a_{ij}(\hat{x}(t))b^i(\hat{x}(t))b^j(\hat{x}(t)) - v(\hat{x}(t))\right)dt \leq 2\int\limits_0^T a_{ij}(\hat{x}(t))\dot{\hat{x}}^i(t)\dot{\hat{x}}^j(t)dt$$

$$+\int\limits_0^T \left(2a_{ij}(\hat{x}(t))b^i(\hat{x}(t))b^j(\hat{x}(t)) - v(\hat{x}(t))\right)dt \leq \frac{2c}{T}\mathrm{dist}(x, y) + 2c_1T \leq c\left(T + \frac{1}{T}\right).$$

The last inequality follows from the compactness of M. □

Proposition 2. *The inequality* $m(t) \leq M(t) \leq m(t) + c$ *holds uniformly in* $t > 0$.

Proof. By Proposition 1, the function $S(x, y, 1)$ is bounded uniformly in x and y. Let $\tilde{x}(\cdot)$ be a curve providing the minimum for $m(t)$. Combining $\tilde{x}(\cdot)$ with a curve which provides the minimum for $S(\tilde{x}(t), \tilde{x}(0), 1)$ and using the curve obtained as a trial function in the definition of $M(t + 1)$ we find

$$M(t + 1) \leq m(t) + S(\tilde{x}(t), \tilde{x}(0), 1) \leq m(t + 1) + c_2;$$

here the monotonicity of $m(t)$ has also been used. □

The relation $\lim\limits_{t\to\infty}(m(t)/t) = \lim\limits_{t\to\infty}(M(t)/t)$ easily follows from Proposition 2.

Further, the estimate $m(t) \le \hat{\lambda}t \le M(t)$ follows from (1.4). Indeed, if $M(t_0) < \hat{\lambda}t_0$ for some $t_0 > 0$, then $M(t_0) = (\hat{\lambda} - \delta')t_0$ for some $\delta' > 0$. By (1.4) we have $M(kt_0)/(kt_0) \le \hat{\lambda} - \delta'$ for all positive integer k. Therefore, $\lim\limits_{t\to\infty} \frac{M(t)}{t} \le \hat{\lambda} - \delta'$ in contradiction with the definition of $\hat{\lambda}$. The estimate $m(t) \le \hat{\lambda}t$ can be derived from (1.4) in the same way.

The last assertion of the lemma relies on the following statements.

Proposition 3. *The inequality*

$$\int_0^T a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt \le c_2 I(x(\cdot), T) + c_3 T \tag{1.6}$$

holds uniformly in $T > 0$ and $x(\cdot) \in C(0, T; M)$.

Proof. By the Schwarz inequality,

$$\begin{aligned} I(x(\cdot), T) &= \int_0^T a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt - 2\int_0^T a_{ij}(x(t))\dot{x}^i b^j(x(t))dt \\ &+ \int_0^T \big(a_{ij}(x(t))b^i(x(t))b^j(x(t)) - v(x(t))\big)\,dt \ge \int_0^T a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt \\ &- \frac{1}{2}\int_0^T a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt - c_4\int_0^T a_{ij}(x(t))b^i(x(t))b^j(x(t))dt \\ &+ \int_0^T \big(a_{ij}(x(t))b^i(x(t))b^j(x(t)) - v(x(t))\big)\,dt \ge \frac{1}{2}\int_0^T a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt \\ &- c_5\int_0^T \big(a_{ij}(x(t))b^i(x(t))b^j(x(t)) - v(x(t))\big)\,dt. \end{aligned}$$

In view of the boundness of the coefficients of A^μ this inequality implies (1.6). □

Proposition 4. *For each $T_0 > 0$, the function $S(x, y, t)$ is uniformly Lipschitz continuous on the set $M \times M \times [T_0, \infty)$.*

Proof. First, let us establish the estimate

$$|S(x, y, t') - S(x, y, t'')| \le L|t' - t''| \tag{1.7}$$

with a constant L that depends only on T_0. To this end we note that, after a simple transformation, the difference $(I(x(\cdot), t') - I(x(\frac{t''}{t'}\cdot), t'')$ can be written as

$$I(x(\cdot),t') - I(x(\frac{t''}{t'}\cdot),t'') = \frac{t'-t''}{t'}\int_0^{t'} a_{ij}(x(t))\dot{x}^i\dot{x}^j\,dt$$
$$+\frac{t''-t'}{t''}\int_0^{t'}\big(a_{ij}(x(t))b^i(x(t))b^j(x(t)) - v(x(t))\big)\,dt.$$

From this relation, substituting the curve $x(\cdot)$ that provides the minimum for $S(x,y,t')$ and using Propositions 1 and 3, we get

$$\begin{aligned}S(x,y,t'') - S(x,y,t') &\le I(x(\frac{t''}{t'}\cdot),t'') - (I(x(\cdot),t')\\ &\le \frac{|t'-t''|}{t'}cc_2\left(t' + \frac{1}{t'} + cc_3t'\right) + \frac{|t'-t''|}{t''}ct'\\ &\le c|t'-t''|\left\{\left(1+\frac{1}{T_0^2}\right) + \frac{t'}{t''}\right\}.\end{aligned}$$

If we suppose that $|t'-t''|$ is bounded, say by T_0, then $t'/t'' \le 2$ and the last estimate takes the form

$$S(x,y,t'') - S(x,y,t') \le c|t'-t''|\left(3+\frac{1}{T_0^2}\right).$$

It remains to note that (1.7) for arbitrary $|t'-t''|$ follows from (1.7) for sufficiently small $|t'-t''|$.

Similarly, it suffices to verify the inequality

$$|S(x,y',t) - S(x,y'',t)| \le L\text{dist}(y',y'') \tag{1.8}$$

for sufficiently small dist(y',y''). This allows us to use the same local coordinates for y' and y''. In particular, we can write $|y'-y''|$ instead of dist(y',y''). Let $x(\cdot)$ be a curve minimizing $S(x,y',t)$. Extending this curve to the interval $(t,t+|y'-y''|)$ as the function $x(s) = y' + (y''-y')(s-t)/|y'-y''|$ linear in the local coordinates and considering (1.7), we obtain

$$\begin{aligned}S(x,y',t) + c|y'-y''| &\ge I(x(\cdot),t+|y'-y''|)\\ &\ge S(x,y'',t+|y'-y''|) \ge S(x,y'',t) - c|y'-y''|.\end{aligned}$$

Thus, $S(x,y'',t) - S(x,y',t) \le c|y'-y''| \le c_1\text{dist}(y',y'')$. In view of the symmetry between y' and y'', this implies (1.8). The inequality

$$|S(x',y,t) - S(x'',y,t)| \le L\text{dist}(x',x'')$$

can be proved in the same way. □

To complete the proof of Lemma 1 we have to show that for any $T>0$ and $\delta>0$ there is $t>T$ such that $|M(t)-\hat{\lambda}t| < \delta$. For this purpose, we cover the manifold M by finitely many balls of radius $\delta_1 = \delta/L$, where L is the Lipschitz constant from Proposition 4 corresponding to $T_0 = 1$. Denote by N the number of the balls forming the covering. According to (1.3), for any positive integer k there exists a curve $x(\cdot)$ defined on the interval $(0,k(N+1)T)$ such that

$$I(x(\cdot), k(N+1)T) \leq \hat{\lambda} k(N+1)T. \tag{1.9}$$

Consider the set $\{x(jT)\}_{j=0}^{k(N+1)}$. It is clear that at least one ball in the covering contains $(k+1)$ or more points of this set. Denote these points by $z_1, z_2, ..., z_s, s \geq k+1$, and the corresponding arguments by $t_1, ..., t_s$. Let us suppose that the inequalities

$$I(x(\cdot - t_j), t_{j+1} - t_j) \geq \hat{\lambda}(t_{j+1} - t_j) + \delta/2 \tag{1.10}$$

hold for all $j < s$. According to (1.3) and Proposition 2, the first and the last segments of the curve satisfy the estimates

$$I(x(\cdot), t_1) \geq \hat{\lambda} t_1 - c_0,$$
$$I(x(\cdot - t_s), k(N+1)T - t_s) \geq \hat{\lambda}(k(N+1)T - t_s) - c_0.$$

Taking the sum of the inequalities (1.10) for all $j = 1, 2, ..., s$, and then adding the last two inequalities, we find that

$$I(x(\cdot), k(N+1)T) \geq \hat{\lambda} k(N+1)T + k\delta/2 - 2c_0.$$

For sufficiently large k this relation contradicts (1.9). Hence, for some $j < s$ we get

$$S(z_j, z_{j+1}, t_{j+1} - t_j) \leq I(x(\cdot - t_j), t_{j+1} - t_j) < \hat{\lambda}(t_{j+1} - t_j) + \delta/2.$$

Our construction guarantees that $|z_j - z_{j+1}| \leq \frac{\delta}{2L}$ and $t_{j+1} - t_j \geq T$. Therefore, by (1.3) and Proposition 4, we have

$$\hat{\lambda}(t_{j+1} - t_j) \leq M(t_{j+1} - t_j) \leq S(z_j, z_j, t_{j+1} - t_j)$$
$$\leq S(z_j, z_{j+1}, t_{j+1} - t_j) + L\frac{\delta}{2L} \leq \hat{\lambda}(t_{j+1} - t_j) + \delta,$$

and Lemma 1 is proved. □

Corollary 1. *The inequality*

$$|S(x, y, t) - \hat{\lambda} t| < c \tag{1.11}$$

holds uniformly in $t > T_0$ *and* $x, y \in M$.

2. Convergence of the First Eigenvalue

In this section we study the first eigenvalue λ_0 of problem (0.1). The limit behaviour of λ_0 is described by the following

Theorem 1. *The relation* $\lim_{\mu \to 0} \lambda_0(\mu) = \hat{\lambda}$ *holds.*

First of all we establish some rough estimates for λ_0 and $p_0(x)$. This is the subject of the following two statements.

Proposition 5. *For all* $\mu > 0$,

$$\min_{x \in M} v(x) \leq -\lambda_0(\mu) \leq \max_{x \in M} v(x) = 0.$$

Proof. Due to the assumed normalizing conditions, $p_0(x)$ is a positive function on M. Denote by x_1 a maximum point of p_0. Then, according to the maximum principle, we have $-\lambda_0 p_0(x_1) = A^\mu p(x_1) \le v(x_1)p_0(x_1)$. This implies the upper bound for λ_0. Similarly, writing down Eq. (0.1) at a minimum point of p_0, we obtain the lower bound. □

Proposition 6. *The following inequalities hold uniformly in $x \in M$:*

$$e^{-c(M)/\mu} \le p_0(x) \le \mu^{-n}; \qquad \max_{x\in M} p_0(x) \ge 1. \tag{2.1}$$

Proof. The last inequality in (2.1) obviously follows from the normalizing conditions (0.2). To prove the first one, let us rewrite Eq. (0.1) in the rescaled local coordinates $y = \frac{x}{\mu}$:

$$\frac{1}{4}|a(\mu y)|^{-\frac{1}{2}}\frac{\partial}{\partial y^i}|a(\mu y)|^{\frac{1}{2}}a^{ij}(\mu y)\frac{\partial}{\partial y^j}p_0(\mu y) + b^i(\mu y)\frac{\partial}{\partial y^i}p_0(\mu y) + v(\mu y)p_0(\mu y)$$
$$= -\lambda_0 p_0(\mu y).$$

According to our assumptions and Proposition 5, the coefficients of this equation are continuously differentiable functions bounded uniformly in μ; therefore, by the Harnack inequality (see [3]) we have

$$0 < c_1 < p_0(\mu y_1)/p_0(\mu y_2) < c_2$$

uniformly in $\mu > 0$ and $y_1, y_2 \in M$ satisfying the condition dist$(y_1, y_2) < 1$. Thus, in the coordinates x we have

$$c_1 < p_0(x_1)/p_0(x_2) < c_2 \tag{2.2}$$

for all $x_1, x_2 \in M$ such that dist$(x_1, x_2) < \mu$. Let x^1 be a maximum point of p_0. Since M is compact, it follows that for any $x \in M$ there exists a sequence $x = z_1, z_2, z_3, ..., z_{N-1}, z_N = x_1$ with the following properties: dist$(z_j, z_{j+1}) < \mu$ for all $j < N$; $N \le N_0(M)/\mu$ with $N_0(M)$ independent of μ and x. Iterating (2.2), we find

$$p_0(x)/p_0(x_1) = \prod_{j=1}^{N-1}\frac{p_0(z_j)}{p_0(z_{j+1})} \ge (c_1)^N \ge e^{N_0(M)\ln c_1/\mu}.$$

This yields the lower bound (2.1). To prove the upper bound, let us consider p_0 in the ball $Q_x = \{z|\text{dist}(z, x) < \mu\}$. By (2.2) and (0.2) we have

$$p_0(x) \le c_2 \min_{z\in Q_x} p_0(z) \le c_3\mu^{-n} \min_{z\in Q_x} p_0(z)\int_{Q_x} m(dz) \le c_3\mu^{-n}\int_M p_0(z)m(dz) \le c_3\mu^{-n},$$

and thereby the proposition is proved. □

Remark 1. In fact, the method developed here allows us to obtain the following inequality

$$p_0(x)/p_0(y) \le \exp(c\,\text{dist}(x, y)/\mu). \tag{2.3}$$

[. Proof of Theorem 1.] In order to obtain the proper lower and upper bounds for $\lambda_0(\mu)$, let us consider an auxiliary Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{1}{4}\mu|a(x)|^{-1/2}\frac{\partial}{\partial x^i}|a(x)|^{1/2}a^{ij}(x)\frac{\partial}{\partial x^j}u(x,t) + b^i(x)\frac{\partial}{\partial x^i}u(x,t) + \frac{1}{\mu}v(x)u(x,t)$$
$$u|_{t=0} = p_0(x) \tag{2.4}$$

in the cylinder $M \times (0, +\infty)$. To estimate its solution $u(x,t)$, which is obviously equal to $\exp(-\frac{\lambda_0}{\mu}t)p_0(x)$, we introduce the operator

$$B^\mu = \frac{1}{4}\mu|a(x)|^{-1/2}\frac{\partial}{\partial x^i}|a(x)|^{1/2}a^{ij}(x)\frac{\partial}{\partial x^j} + b^i(x)\frac{\partial}{\partial x^i},$$

and denote by ξ_t^x the corresponding diffusion process on M issuing from the point x. The process ξ_t^x is defined on some probabilistic space $(\Omega, F, \mathbf{P})$ and is assumed to have continuous trajectories. The relevant definitions can be found in [1].

Now the solution of (2.4) has the following probabilistic representation (see [1]):

$$u(x,t) = \mathbf{E}\left\{p_0(\xi_t^x)\exp\left(\frac{1}{\mu}\int_0^t v(\xi_s^x)ds\right)\right\}, \tag{2.5}$$

where $\mathbf{E}$ denotes the expectation of random variables.

Upper bound. Let $\mathrm{dist}_{[0,t]}(x'(\cdot), x''(\cdot))$ stand for the distance $\sup_{0\le s\le t} \mathrm{dist}(x'(s), x''(s))$ in the functional space $C(0,t;M)$. According to [2, Chapter 5, Th.3.2], the process ξ_t^x satisfies the large deviation principle uniformly in $x \in M$, with the rate functional $I_0(x(\cdot),t)$; see the definition of $I_0(x(\cdot),t)$ in the previous section. In particular, for any absolutely continuous curve $x(\cdot)$ and any $\delta > 0$ and $\gamma > 0$, there exists $\mu_0 > 0$ such that for all $\mu < \mu_0$,

$$\mathbf{P}\left\{\mathrm{dist}_{[0,t]}(\xi_\cdot^x, x(\cdot)) < \delta\right\} \ge \exp\left(-\frac{I_0(x(\cdot),t)+\gamma}{\mu}\right). \tag{2.6}$$

Since the function $v(x)$ is smooth, the inequality

$$\left|\int_0^t v(\xi_s^x)ds - \int_0^t v(x(s))ds\right| \le c\delta t \tag{2.7}$$

holds for any trajectory ξ_t^x that satisfies the estimate $\mathrm{dist}(\xi_\cdot^x, x(\cdot)) \le \delta$. Let $\bar{x}(\cdot)$ be the minimizing curve for $M(t)$. By (2.6) and (2.7), for all $\mu < \mu_0$ we have

$$\begin{aligned}
p_0(\bar{x}(0))e^{-\lambda_0 t/\mu} &= u(x,t) = \mathbf{E}\left\{ p_0(\xi_t^{\bar{x}(0)}) \exp\left(\frac{1}{\mu}\int_0^t v(\xi_s^{\bar{x}(0)})ds \right) \right\} \\
&\geq \min_{y\in M} p_0(y)\mathbf{P}\big\{\mathrm{dist}_{[0,t]}(\xi_\cdot^{\bar{x}(0)}, \bar{x}(\cdot)) < \delta\big\} \exp\left(\frac{1}{\mu}\int_0^t v(\bar{x}(s))ds - c\delta t \right) \\
&\geq \min_{y\in M} p_0(y) \exp\left(-\frac{I_0(\bar{x}(\cdot),t)+\gamma}{\mu} \right) \exp\left(\frac{1}{\mu}\int_0^t v(\bar{x}(s))ds - c\delta t \right) \\
&= \min_{y\in M} p_0(y) \exp\left(-\frac{I(\bar{x}(\cdot),t)+\gamma+c\delta t}{\mu} \right);
\end{aligned}$$

here the equality $I(x(\cdot),t) = I_0(x(\cdot),t) - \int_0^t v(\bar{x}(s))ds$ has also been used. According to our choice of $\bar{x}(\cdot)$, we have $I(\bar{x}(\cdot),t) = M(t)$, and therefore,

$$p_0(\bar{x}(0))e^{-\lambda_0 t/\mu} \geq \min_{y\in M} p_0(y) \exp\left(-\frac{M(t)+\gamma+c\delta t}{\mu} \right).$$

Finally, by Proposition 6

$$\begin{aligned}
e^{-\lambda_0 t/\mu} &\geq \exp\left(-\frac{2c(M)}{\mu} \right) \exp\left(-\frac{I(\bar{x}(\cdot),t)+\gamma+c\delta t}{\mu} \right) \\
&= \exp\left(-\frac{I(\bar{x}(\cdot),t)+\gamma+c\delta t+2c(M)}{\mu} \right).
\end{aligned}$$

Therefore,

$$\lambda_0 \leq \frac{M(t)}{t} + c\delta + \frac{\gamma}{t} + \frac{2c(M)}{t}$$

for all sufficiently small μ. Since δ and γ are arbitrary numbers and $c(M)$ does not depend on t, this implies

$$\limsup_{\mu\to 0} \lambda_0 \leq \lim_{t\to\infty} \frac{M(t)}{t} = \hat{\lambda}. \tag{2.8}$$

Lower bound. To estimate λ_0 from below, we consider the following subset of $C(0,t;M)$:

$$\Phi_t^x(\alpha) = \{x(\cdot) | x(0) = x, I_0(x(\cdot),t) \leq \alpha\}.$$

According to [2], this set is compact in $C(0,t;M)$, and for every $\delta > 0$, $\gamma > 0$ and $\alpha > 0$ there exists $\mu_0 > 0$ such that

$$\mathbf{P}\left\{\mathrm{dist}_{[0,t]}(\xi_\cdot^x, \Phi_t^x(\alpha)) > \delta\right\} \leq \exp\left(-\frac{\alpha-\gamma}{\mu} \right) \tag{2.9}$$

for all $\mu < \mu_0$.

Moreover, for any $\alpha_0 > 0$ the estimate (2.9) is uniform in $\alpha < \alpha_0$ and in $x \in M$. Let x be the initial point of the curve that provides the minimum for $m(t)$. Representing Ω as a union of the following two events: $Q_1 = \left\{\mathrm{dist}_{[0,t]}(\xi_\cdot^x, \Phi_t^x(2M(t)) \geq \delta\right\}$,

$Q_2 = \left\{\text{dist}_{[0,t]}(\xi^x_\cdot, \Phi^x_t(2\bar{M}(t)) < \delta\right\}$, $\bar{M}(t) = \max(M(t), 1)$, and denoting by $\chi_\cdot$ the characteristic function of a set, one can rewrite (2.5) in the form

$$\begin{aligned} p_0(x)e^{-\lambda_0 t/\mu} &= \mathbf{E}\left\{\chi_{Q_1} p_0(\xi^x_t)\exp\left(\frac{1}{\mu}\int\limits_0^t v(\xi^x_s)ds\right)\right\} \\ &+ \mathbf{E}\left\{\chi_{Q_2} p_0(\xi^x_t)\exp\left(\frac{1}{\mu}\int\limits_0^t v(\xi^x_s)ds\right)\right\}. \end{aligned} \tag{2.10}$$

It follows from (2.9), Proposition 6 and the negativity of $v(\cdot)$ that the first term on the right hand side satisfies an estimate

$$\mathbf{E}\left\{\chi_{Q_1} p_0(\xi^x_t)\exp\left(\frac{1}{\mu}\int\limits_0^t v(\xi^x_s)ds\right)\right\} \le c\mu^{-n}\mathbf{P}(Q_1) \le c\mu^{-n}\exp\left(-\frac{2\bar{M}(t)-\gamma}{\mu}\right) \tag{2.11}$$

for sufficiently small μ. To estimate the second term, let us represent Q_2 in the form

$$\begin{aligned} Q_2 &= \{\text{dist}_{[0,t]}(\xi^x_\cdot, \Phi^x_t(\delta)) < \delta\} \\ &\cup \bigcup_{k=2}^{[2\bar{M}(t)/\delta]+1} \left(\{\text{dist}_{[0,t]}(\xi^x_\cdot, \Phi^x_t(k\delta)) < \delta\} \cap \{\text{dist}_{[0,t]}(\xi^x_\cdot, \Phi^x_t((k-1)\delta)) \ge \delta\}\right) \\ &= Q_2^1 \cup \bigcup_{k=2}^{[2\bar{M}(t)/\delta]+1} Q_2^k. \end{aligned}$$

It should be noted that some events in this union can be empty. Since $m(t) \le I(x(\cdot), t) = I_0(x(\cdot), t) - \int_0^t v(x(s))ds$, we obtain $-\int_0^t v(x(s))ds \ge m(t) - k\delta$ for any $x(\cdot) \in \Phi^x_t(k\delta)$. Hence, by (2.7) we get

$$-\int\limits_0^t v(\xi^x_s)ds \ge m(t) - k\delta - c\delta t \tag{2.12}$$

for all trajectories from Q_2^k. At the same time, according to (2.9),

$$\mathbf{P}(Q_2^k) \le \mathbf{P}\left\{\text{dist}_{[0,t]}(\xi^x_\cdot, \Phi^x_t((k-1)\delta) \ge \delta\right\} \le \exp\left(-\frac{(k-2)\delta}{\mu}\right).$$

Combining the last two estimates, we find

$$
\begin{aligned}
&\mathbf{E}\left\{\chi_{Q_2} p_0(\xi_t^x) \exp\left(\frac{1}{\mu}\int_0^t v(\xi_s^x)ds\right)\right\}\\
&\quad\le \sum_{k=1}^{[2\bar M(t)/\delta]+1} \mathbf{P}(Q_2^k)\max_{y\in M} p_0(y)\exp\left(-\frac{m(t)-k\delta-c\delta t}{\mu}\right)\\
&\quad\le c\mu^{-n}\sum_{k=1}^{[2\bar M(t)/\delta]+1}\exp\left(-\frac{(k-2)\delta}{\mu}\right)\exp\left(-\frac{m(t)-k\delta-c\delta t}{\mu}\right)\\
&\quad\le c\mu^{-n}\bar M(t)/\delta\exp\left(-\frac{m(t)-2\delta-c\delta t}{\mu}\right).
\end{aligned}
$$

From (2.10), taking into account (2.11) and the last inequality we derive

$$
p_0(x)e^{-\lambda_0 t/\mu}\le c\mu^{-n}\left\{\exp\left(-\frac{2\bar M(t)-\gamma}{\mu}\right)+\frac{3\bar M(t)}{\delta}\exp\left(-\frac{m(t)-2\delta-c\delta t}{\mu}\right)\right\}
$$

or, after simple transformation,

$$
\lambda_0 t\ge c(M)-n\mu\ln\mu(\ln\delta-\ln\bar M(t))+m(t)-\gamma-2\delta-c\delta t-c_1;
$$

here c_1 depends neither on t nor μ. Since δ and γ are arbitrary numbers, this implies

$$
\liminf_{\mu\to 0}\lambda_0\ge\lim_{t\to\infty}\frac{m(t)}{t}=\hat\lambda,
$$

which, in view of (2.8), completes the proof of Theorem 1. □

Corollary 2. *The first eigenvalue λ_0 of problem (0.1) converges to zero as $\mu\to 0$ if and only if the identity $v(x(t))\equiv 0$ holds along at least one solution of the equation $\dot x = b(x)$ on M.*

3. Asymptotics of the First Eigenfunction

In this section the asymptotic behaviour of the first eigenfunction $p_0(x)$ is studied. Some additional assumptions are required in order to ensure the existence of the asymptotics. These assumptions, in turn, involve the following definitions.

Condition A. *A curve $x(\cdot)$ defined on $(0,+\infty)$ satisfies condition A, if for any $\varepsilon>0$ there is $T>0$ such that for all $t>0$ we have*

$$
I(x(\cdot-T),t)=\int_T^{T+t}\left(a_{ij}(x(t))(\dot x^i-b^i(x(t))(\dot x^j-b^j(x(t)))-v(x(t))\right)dt<\hat\lambda t+\varepsilon,
$$

where $\hat\lambda$ is defined in Lemma 1.

Condition B. *A curve $x(\cdot)$ defined on $(0,+\infty)$ satisfies condition B, if the inequality $I(x(\cdot),t)\le\hat\lambda t+c$ holds uniformly in $t>0$.*

First of all, we should answer the question if curves satisfying Conditions A and B do exist. The proof of the following two simple assertions is outlined briefly.

Proposition 7. *The conditions A and B are equivalent.*

Proof. The implication A⇒B is obvious. To derive A from B it suffices to note that the set $\{t \mid I(x(\cdot), t) - \hat{\lambda}t > \sup_s (I(x(\cdot), s) - \hat{\lambda}s) - \varepsilon\}$ is not empty for each $x(\cdot)$ satisfying Condition B, and to take arbitrary T from this set. □

Proposition 8. *A curve satisfying condition B does exist.*

Proof. Thanks to the last statement of Lemma 1 and the definition of $M(t)$, there exist a sequence $t_k \to \infty$ and curves $x_k(t)$, $0 \le t \le t_k$, $x(0) = x(t_k)$ such that

$$\lim_{k\to\infty} |I(x_k(\cdot), t_k) - \hat{\lambda}t_k| = 0.$$

Taking, if necessary, a proper subsequence one can assume that the sequence $x_k(0)$ does converge, and that the inequalities

$$\mathrm{dist}(x_k(0), x_{k+1}(0)) < \exp(-k), \qquad |I(x_k(\cdot), t_k) - \hat{\lambda}t_k| < \exp(-k)$$

hold. Now, combining the curves $x_k(\cdot)$ and segments of geodesics that connect $x_k(t_k)$ and $x_{k+1}(0)$, we obtain the desired curve. □

Next, we introduce the class of operators to be studied.

Definition. *The operator A^μ is recursive if there is at least one point x_0 of M such that for any $\varepsilon > 0, T > 0$ and any $x(\cdot)$ satisfying condition A, the inequality $\mathrm{dist}(x(t), x_0) < \varepsilon$ holds for some $t > T$. The point x_0 is called recurrent for A^μ.*

The following property of recurrent points plays an important role in further considerations.

Proposition 9. *For each recurrent point x_0 of A^μ,*

$$\liminf_{t\to\infty} (S(x_0, x_0, t) - \hat{\lambda}t) = 0.$$

Proof. By Lemma 1,

$$S(x_0, x_0, t) - \hat{\lambda}t \ge M(t) - \hat{\lambda}t \ge 0.$$

Thus, $\liminf_{t\to\infty} (S(x_0, x_0, t) - \hat{\lambda}t) \ge 0$. If we suppose that $\liminf_{t\to\infty} (S(x_0, x_0, t) - \hat{\lambda}t) = c > 0$, then Proposition 4 implies that $S(x, y, t) - \hat{\lambda}t \ge c/2$ for sufficiently large t and x, y close to x_0. Let $x(\cdot)$ satisfy condition B. Since x_0 is a recurrent point of A^μ, one can find a sequence $\{t_k\}_{k=1}^\infty$ such that $(t_{k+1} - t_k) \to \infty$ as $k \to \infty$ and $x(t_k)$ are close to x_0 for all k. Hence,

$$I(x(\cdot), t_{k+1}) - \hat{\lambda}t_{k+1} = (I(x(\cdot), t_1) - \hat{\lambda}t_1) + \sum_{s=1}^{k} (I(x(\cdot - t_s), t_{s+1} - t_s) - \hat{\lambda}(t_{s+1} - t_s)) \ge$$

$$\ge kc/2 - c_1.$$

For sufficiently large k this estimate contradicts the fact that $x(\cdot)$ satisfies condition B. □

For a recursive operator we define the function W_0 as follows

$$W_0(x) = \inf_{t>0} \inf_{\substack{x(\cdot),\, x(0)=x,\\ x(t)=x_0}} (I(x(\cdot), t) - \hat{\lambda} t) = \inf_{t>0} (S(x, x_0, t) - \hat{\lambda} t), \tag{3.1}$$

where x_0 is a recurrent point.

Remark 2. In fact, the infimum over all $t > 0$ in (3.1) can be replaced by that over an arbitrary half-line $t > T_0$, $T_0 \geq 0$. Indeed, let $\{x_k(\cdot)\}$ be a sequence of curves with the following properties:

$$x_k(0) = x_k(t_k) = x_0, \qquad \lim_{k\to\infty} t_k = \infty, \qquad \lim_{k\to\infty} (I(x_k(\cdot), t_k) - \hat{\lambda} t_k) = 0.$$

Proposition 9 guarantees the existence of such a sequence. Now it suffices to extend the curves from an arbitrary minimizing sequence for W_0 as the curves from the sequence just constructed.

In view of Remark 2, the following statement easily follows from Proposition 4.

Proposition 10. *W_0 is a Lipschitz continuous function on M.*

It should be observed that, in general, the function W_0 depends on the choice of the recurrent point x_0. Define the function $W(x)$ on M by the formula

$$W(x) = W_0(x) - \min_{y\in M} W_0(y).$$

A remarkable property of W is its independence of x_0.

Proposition 11. *$W(x)$ is a well-defined function on M; it does not depend on the choice of the recurrent point x_0.*

Proof. Consider two arbitrary recurrent points x_0' and x_0'' of the operator A^μ. The corresponding functions (3.1) will be marked by $'$ and $''$, respectively. Our proof is based on the following relation:

$$W_0'(x_0'') + W_0''(x_0') = 0. \tag{3.2}$$

In order to establish (3.2), let us first assume that $W_0'(x_0'') + W_0''(x_0') = c > 0$. In view of Propositions 4 and 10, this implies the estimate

$$\inf_{t>0} (S(x_1, y_1, t) - \hat{\lambda} t) + \inf_{t>0} (S(y_2, x_2, t) - \hat{\lambda} t) \geq c/2$$

for all x_1, x_2 close to x_0' and y_1, y_2 close to x_0''. Fixing an arbitrary curve $x(\cdot)$ which satisfies condition B and taking into account the properties of x_0' and x_0'', one can easily construct an increasing sequence $\{t_k\}$ such that $x(t_{2k})$ are close to x_0' and $x(t_{2k-1})$ are close to x_0'' for all $k > 0$. Then our assumption leads to the following inequality:

$$\begin{aligned}
I(x(\cdot), t_{2k+1}) - \hat{\lambda} t_{2k+1} &= (I(x(\cdot), t_1) - \hat{\lambda} t_1) \\
&+ \sum_{s=1}^{k} \Big\{ (I(x(\cdot - t_{2s-1}), t_{2s} - t_{2s-1}) - \hat{\lambda}(t_{2s} - t_{2s-1})) \\
&+ (I(x(\cdot - t_{2s}), t_{2s+1} - t_{2s}) - \hat{\lambda}(t_{2s+1} - t_{2s})) \Big\} \geq (I(x(\cdot), t_1) - \hat{\lambda} t_1) \\
&+ \sum_{s=1}^{k} \Big\{ (S(x(t_{2s-1}), x(t_{2s}), t_{2s} - t_{2s-1}) - \hat{\lambda}(t_{2s} - t_{2s-1})) \\
&+ (S(x(t_{2s}), x(t_{2s+1}), t_{2s+1} - t_{2s}) - \hat{\lambda}(t_{2s+1} - t_{2s})) \Big\} \geq c_1 + kc/2,
\end{aligned}$$

which contradicts the fact that $x(\cdot)$ satisfies condition B. Thus, $W_0'(x_0'') + W_0''(x_0') \leq 0$. On the other hand,

$$\begin{aligned} W_0'(x_0'') + W_0''(x_0') &= \inf_{t>0}(S(x_0', x_0'', t) - \hat{\lambda}t) + \inf_{t>0}(S(x_0'', x_0', t) - \hat{\lambda}t) \\ &\geq \inf_{t>0}(S(x_0', x_0', t) - \hat{\lambda}t) \geq \inf_{t>0}(M(t) - \hat{\lambda}t) \geq 0 \end{aligned}$$

and (3.2) follows. Now,

$$\begin{aligned} W_0'(x) = \inf_{t>0}(S(x, x_0', t) - \hat{\lambda}t) &\leq \inf_{t>0}(S(x, x_0'', t) - \hat{\lambda}t) + \inf_{t>0}(S(x_0'', x_0', t) - \hat{\lambda}t) \\ &= W_0''(x) + W_0'(x_0''). \end{aligned}$$

Similarly, $W_0''(x) \leq W_0'(x) + W_0''(x_0')$. In view of (3.2), this means that $W_0''(x) = W_0'(x) + W_0''(x_0')$. In other words, the difference $W_0'(x) - W_0''(x)$ is constant, and therefore, W is well-defined. □

The main result of this section is the following

Theorem 2. *Let operator A^μ be recursive. Then*

$$\lim_{\mu\to 0} \mu \ln p_0(x) = -W(x) \tag{3.3}$$

uniformly in $x \in M$.

Proof. **Lower bound.** Let us fix an arbitrary recurrent point x_0 of the operator A^μ and estimate the ratio $p_0(x)/p_0(x_0)$ from below. According to the definition of W_0, for any $x \in M$ and $\delta > 0$ there is a curve $x(\cdot)$ defined on the interval $(0, T(\delta))$ and such that

$$x(0) = x, \qquad x(T(\delta)) = x_0, \qquad I(x(\cdot), T(\delta)) - \hat{\lambda}T(\delta) < W_0(x) + \delta. \tag{3.4}$$

Moreover, using compactness arguments and Proposition 4 and 10, we can choose $T(\delta)$ bounded by some $T_0(\delta)$ uniformly in $x \in M$. Indeed, if we construct the segment of geodesic curve that connects y and x, combine it with $x(\cdot)$ and denote the obtained curve by $\tilde{x}(\cdot)$, then we get

$$I(\tilde{x}(\cdot), T(\delta) + \text{dist}(y, x)) - \hat{\lambda}(T(\delta) + \text{dist}(y, x)) < W_0(x) + \frac{2\delta}{3} < W_0(y) + \frac{\delta}{3}$$

for all y from a sufficiently small neighbourhood of x.

According to [2], for any $\delta_1 > 0$ there exists $\mu_0 > 0$ such that

$$\mathbf{P}\left\{\text{dist}_{[0,T(\delta)]}(\xi_\cdot^x, x(\cdot)) < \delta_1\right\} \geq \exp\left(-\frac{I_0(x(\cdot), T(\delta)) + \delta_1}{\mu}\right) \tag{3.5}$$

for all $\mu < \mu_0$. From (2.3), (2.7), (3.4) and the last estimate, we get

$$\begin{aligned} p_0(x)e^{-\lambda_0 T(\delta)/\mu} &= \mathbf{E}\left\{p_0(\xi_{T(\delta)}^x)\exp\left(\frac{1}{\mu}\int_0^{T(\delta)} v(\xi_s^x)ds\right)\right\} \\ &\geq \mathbf{P}\{\text{dist}_{[0,T(\delta)]}(\xi_\cdot^x, x(\cdot)) < \delta_1\} \\ &\quad \times p_0(x_0)\exp\left(\frac{-c\delta_1}{\mu}\right)\exp\left\{\frac{1}{\mu}\left(\int_0^{T(\delta)} v(x(s))ds - cT(\delta)\delta_1\right)\right\} \\ &\geq p_0(x_0)\exp\left(-\frac{I(x(\cdot), T(\delta)) + \delta_1 + c\delta_1 + cT(\delta)\delta_1}{\mu}\right). \end{aligned}$$

Finally, using (3.4), we find

$$p_0(x) \geq p_0(x_0) \exp\left(-\frac{W_0(x) + \delta + T(\delta)|\hat{\lambda} - \lambda_0| + (1 + c_1 + cT(\delta))\delta_1}{\mu}\right).$$

For suitably chosen δ, δ_1 and μ_0, the quantity $(\delta + T(\delta)|\hat{\lambda} - \lambda_0| + (1 + c_1 + cT(\delta))\delta_1)$ becomes arbitrary small, and therefore,

$$\liminf_{\mu \to 0} \mu \ln(p_0(x)/p_0(x_0)) \geq -W_0(x). \tag{3.6}$$

Upper bound. The following statement is a direct consequence of the definition of a recurrent point.

Proposition 12. *Under the above conditions, for any* $\delta > 0$ *and* $\overline{c} > 0$, *there exists* $t_0 = t_0(\overline{c}, \delta)$ *such that for all* $t > t_0$ *the inequality* $\min\limits_{0 \leq s \leq t} \operatorname{dist}(x(s), x_0) > \delta$ *implies that*

$$I(x(\cdot), t) \geq \hat{\lambda} t + \overline{c}. \tag{3.7}$$

The constant $\overline{c}$ will be fixed later. Again using compactness arguments, we deduce from Corollary 1 that for any $\delta > 0$ there is $t_1 = t_1(\delta)$ such that

$$\left| \inf_{0 \leq t \leq t_1} (S(x, x_0, t) - \hat{\lambda} t) - \inf_{t > 0} (S(x, x_0, t) - \hat{\lambda} t) \right| < \delta \tag{3.8}$$

uniformly in $x \in M$. Let us denote $\max(t_0(\overline{c}, \delta), t_1(\delta))$ by $\overline{t}$ and fix $\mu_0(\delta)$ such that the estimate

$$|\lambda_0 - \hat{\lambda}| \overline{t} < \delta \tag{3.9}$$

holds for all $\mu < \mu_0(\delta)$. Later on we assume that $\mu < \mu_0(\delta)$.

It is easy to check that the function $\tilde{u}(x, t) = p_0(x) \exp(-(\lambda_0 - \hat{\lambda}) t/\mu)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{\mu} A^\mu - \frac{\hat{\lambda}}{\mu} \right) \tilde{u} = 0, \qquad \tilde{u}|_{t=0} = p_0. \tag{3.10}$$

Moreover, according to our choice of $\mu_0(\delta)$, the relation $\tilde{u}(x, t) = p_0(x) \exp(O(\delta)/\mu)$ takes place for all $t < \overline{t}$ and $\mu < \mu_0(\delta)$.

Let $\tau_{2\delta}^x$ be the exit time for the domain $M \setminus O_{2\delta}(x_0)$, where $O_{2\delta}(x_0)$ is the ball $\{y \in M | \operatorname{dist}(y, x_0) < 2\delta\}$. For our purposes, it is convenient to fix $\delta_0 > 0$ and divide the set Ω into three parts:

$$\Omega_1 = \left\{ \operatorname{dist}_{[0,\bar{t}]}(\Phi_t^x(K), \xi_\cdot^x) \geq \delta_0 \right\},$$

$$\Omega_2 = \left\{ (\tau_{2\delta}^x > \overline{t}) \cap (\operatorname{dist}_{[0,\bar{t}]}(\Phi_t^x(K), \xi_\cdot^x) < \delta_0) \right\},$$

$$\Omega_3 = \left\{ (\tau_{2\delta}^x \leq \overline{t}) \cap (\operatorname{dist}_{[0,\bar{t}]}(\Phi_t^x(K), \xi_\cdot^x) < \delta_0) \right\},$$

where $K = \overline{c} + \overline{t} \max\limits_{y \in M} |v(y)|$ and $\Phi_t^x(K)$ is defined in the previous section. According to [2], for sufficiently small μ we have

$$\mathbf{P}(\Omega_1) \leq \exp\left(-\frac{K - \delta}{\mu}\right). \tag{3.11}$$

To estimate the contribution of Ω_3 into the solution $\tilde{u}(x,t)$ written in a probabilistic form, let us fix an arbitrary positive δ_1 and represent Ω_3 as a finite union of the following events:

$$\Omega_3 = \bigcup_{k=1}^{\bar{t}/\delta_1} \bigcup_{l=1}^{K/\delta} \big(\{k\delta_1 \le \tau_{2\delta}^x < (k+1)\delta_1\} \cap \{\text{dist}_{[0,(k+1)\delta_1]}(\Phi^x_{(k+1)\delta_1}((l+1)\delta), \xi^x_\cdot) < \delta_0\}$$
$$\cap \{\text{dist}_{[0,(k+1)\delta_1]}(\Phi^x_{(k+1)\delta_1}(l\delta), \xi^x_\cdot) \ge \delta_0\}\big) = \bigcup_{k=1}^{\bar{t}/\delta_1} \bigcup_{l=1}^{K/\delta} \Omega_3^{k,l}.$$

We also fix a positive ν and suppose that $x \in M \setminus O_\nu(x_0)$. The opposite case, namely $x \in O_\nu(x_0)$, will be examined later. In what follows we assume that $\delta,\ \delta_0,\ \delta_1$ and ν are sufficiently small and satisfy the relations $\nu \gg \delta > \delta_0$, $\nu \gg \delta_1$. According to [2], there exists $t_2(\nu)$ such that

$$\mathbf{P}\{\tau_{2\delta}^x < t_2(\nu)\} \le \exp(-K/\mu)$$

for all $x \in M \setminus O_\nu(x_0)$. In view of the definition of $\Omega_3^{k,l}$, this implies that $\mathbf{P}(\Omega_3^{k,l}) \le \exp(-K/\mu)$ for all $k < t_2(\nu)/\delta_1$, and therefore it suffices to examine $\Omega_3^{k,l}$ only for k from the interval $t_2(\nu) \le k\delta_1 \le \bar{t}$. According to [2],

$$\mathbf{P}(\Omega_3^{k,l}) \le \mathbf{P}\{\text{dist}_{[0,(k+1)\delta_1]}(\Phi^x_{(k+1)\delta_1}(l\delta), \xi^x_\cdot) \ge \delta_0\} \le \exp\left(-\frac{l\delta - \delta}{\mu}\right) \tag{3.12}$$

for sufficiently small μ. At the same time, it follows from the definition of $\Phi^x_{(k+1)\delta_1}((l+1)\delta)$ that for any $\xi^x_\cdot \in \Omega_3^{k,l}$ there is a curve $x(\cdot)$ satisfying the estimates

$$I_0(x(\cdot),(k+1)\delta_1) \le (l+1)\delta, \qquad \text{dist}_{[0,(k+1)\delta_1]}(x(\cdot), \xi^x_\cdot) < \delta_0. \tag{3.13}$$

In view of the evident relation $|x(\xi^x_{\tau_{2\delta}}) - x_0| = 2\delta$, this implies

$$|x(\tau_{2\delta}^x) - x_0| < 2\delta + \delta_0 \tag{3.14}$$

where the argument $\tau_{2\delta}^x$ is random. To estimate the same difference at a nonrandom point, we apply the following

Proposition 13. *The inequality*

$$\text{dist}(x(s_1), x(s_2)) \le c(K,\bar{t})\sqrt{|s_1 - s_2|}, \quad s_1, s_2 \le t \tag{3.15}$$

holds uniformly in $t \le \bar{t}$ *and* $x(\cdot)$ *satisfying the condition* $I_0(x(\cdot),t) \le K$.

Proof. Proposition 3, the definition of the distance and the Schwarz inequality yield

$$\text{dist}(x(s_1), x(s_2)) \le \int_{s_1}^{s_2} \left(a_{ij}(x(s)\dot{x}^i(s)\dot{x}^j(s)\right)^{1/2} ds$$
$$\le \sqrt{\int_{s_1}^{s_2} a_{ij}(x(s)\dot{x}^i(s)\dot{x}^j(s) ds}\sqrt{|s_1 - s_2|} \le c(K,\bar{t})\sqrt{|s_1 - s_2|}.$$

□

Now, taking into account the inequality $k\delta_1 \le \tau^x_{2\delta} < (k+1)\delta_1$ and Proposition 13, we deduce from (3.14),

$$|x((k+1)\delta_1) - x_0| < (c(K,\overline{t})\sqrt{\delta_1} + 2\delta + \delta_0). \tag{3.16}$$

Note that the constant $c(K,\overline{t})$ in (3.16) does not depend on δ_1 and δ_0. Then, by the definition of $S(x,y,t)$, we get

$$I_0(x(\cdot),(k+1)\delta_1) - \int\limits_0^{(k+1)\delta_1} v(x(t))dt \ge S(x, x((k+1)\delta_1), (k+1)\delta_1).$$

In view of (3.16), Proposition 4 and the inequality $k\delta_1 > t_2(\nu)$, this implies

$$I_0(x(\cdot),(k+1)\delta_1) - \int\limits_0^{(k+1)\delta_1} v(x(t))dt \ge S(x, x_0, (k+1)\delta_1) - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0).$$

Hence by virtue of (3.13),

$$-\int\limits_0^{(k+1)\delta_1} v(x(t))dt \ge S(x, x_0, (k+1)\delta_1) - (l+1)\delta - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0).$$

Finally, by (2.7,)

$$-\int\limits_0^{(k+1)\delta_1} v(\xi^x_t)dt \ge S(x, x_0, (k+1)\delta_1) - (l+1)\delta - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0) - c\overline{t}\delta_0.$$

Using this inequality and (3.12), we obtain

$$\mathbf{E}\left(\chi_{\Omega_3^{k,l}} \exp\left\{\frac{1}{\mu}\int\limits_0^{\tau^x_{2\delta}} (v(\xi^x_s) - \hat\lambda)ds\right\}\right)$$

$$\le \exp\left(\frac{c\delta_1}{\mu}\right)\mathbf{E}\left(\chi_{\Omega_3^{k,l}} \exp\left\{\frac{1}{\mu}\int\limits_0^{(k+1)\delta_1} (v(\xi^x_s) - \hat\lambda)ds\right\}\right) \le \exp\left(\frac{c\delta_1}{\mu}\right)\mathbf{P}(\Omega_3^{k,l})$$

$$\le \exp\Bigl(-\frac{S(x,x_0,(k+1)\delta_1) - (k+1)\delta_1\hat\lambda - (l+1)\delta - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0) - c\overline{t}\delta_0}{\mu}\Bigr) \tag{3.17}$$

$$\le \exp\Bigl(-\frac{l\delta - \delta}{\mu}\Bigr)\exp\Bigl(-\frac{W_0(x) - (l+1)\delta - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0) - c\overline{t}\delta_0 - c\delta_1}{\mu}\Bigr)$$

$$\le \exp\left(-\frac{W_0(x) - 2\delta - c'(\nu)(c(K,\overline{t})\sqrt{\delta_1} + \delta + \delta_0) - c\overline{t}\delta_0 - c\delta_1}{\mu}\right);$$

here we have also used the obvious inequality $W_0(x) \le S(x, x_0, (k+1)\delta_1) - (k+1)\delta_1\hat{\lambda}$. Then, representing Ω_2 in the form

$$\Omega_2 = \bigcup_{l=1}^{K/\delta} \left(\{\mathrm{dist}_{[0,\bar{t}]}(\Phi_t^x((l+1)\delta), \xi_\cdot^x) < \delta_0\} \cap \{\mathrm{dist}_{[0,\bar{t}]}(\Phi_t^x(l\delta), \xi_\cdot^x) \ge \delta_0\}\right) = \bigcup_{l=1}^{K/\delta} \Omega_2^l,$$

and applying the above arguments with obvious simplifications, we find that

$$\mathbf{E}\left(\chi_{\Omega_2^l} \exp\left\{\frac{1}{\mu}\int_0^{\bar{t}} (v(\xi_s^x) - \hat{\lambda})ds\right\}\right) \le \exp\left(-\frac{\bar{c} - \bar{S} - c\delta - c\bar{t}\delta_0}{\mu}\right), \qquad (3.18)$$

where $\bar{S} = \inf_{x,y,t}(S(x, y, t) - \hat{\lambda}t)$. Now let us notice that the solution of the following initial boundary value problem

$$\left(\frac{\partial}{\partial t} - \frac{1}{\mu}A^\mu - \frac{\hat{\lambda}}{\mu}\right)\bar{u} = 0, \qquad x \in M \setminus O_{2\delta}(x_0);$$

$$\bar{u}|_{t=0} = p_0, \qquad \bar{u}|_{\partial O_{2\delta}(x_0)} = \tilde{u}|_{\partial O_{2\delta}(x_0)},$$

coincides with $\tilde{u}(x, t)$ for $x \in M \setminus O_{2\delta}(x_0)$ and can be written in the form (see [1])

$$\tilde{u}(x, t) = \bar{u}(x, t) = \mathbf{E}\left(\tilde{u}(\xi^x_{\bar{\tau}^x_{2\delta}}, \bar{\tau}^x_{2\delta}) \exp\left\{\frac{1}{\mu}\int_0^{\bar{\tau}^x_{2\delta}} (v(\xi_s^x) - \hat{\lambda})ds\right\}\right),$$

where $\bar{\tau}^x_{2\delta} = \min(\tau^x_{2\delta}, \bar{t})$. Using the relation $\tilde{u}(x, t) = \exp(O(\delta)/\mu)p_0(x), t < \bar{t}$, we obtain

$$p_0(x) \le \exp\left(\frac{c\delta}{\mu}\right)\mathbf{E}\left((\chi_{\Omega_1} + \chi_{\Omega_2} + \chi_{\Omega_3})p_0(\xi^x_{\bar{\tau}^x_{2\delta}}) \exp\left\{\frac{1}{\mu}\int_0^{\bar{\tau}^x_{2\delta}} (v(\xi_s^x) - \hat{\lambda})ds\right\}\right).$$

In view of (3.11), Proposition 6 and the choice of K the first term on the right hand side can be estimated as follows

$$\begin{aligned}
&\mathbf{E}\left(\chi_{\Omega_1} p_0(\xi^x_{\bar{\tau}^x_{2\delta}}) \exp\left\{\frac{1}{\mu}\int_0^{\bar{\tau}^x_{2\delta}} (v(\xi_s^x) - \hat{\lambda})ds\right\}\right) \\
&\le \exp\left(-\frac{K - \delta - c(M) - \bar{t}\max|v(y)|}{\mu}\right)p_0(x_0) \\
&\le \exp\left(-\frac{\bar{c} - c(M) - \delta}{\mu}\right)p_0(x_0).
\end{aligned}$$

Then, by the definition of Ω_3, Remark 1 and (3.17), we get

$$\mathbf{E}\left(\chi_{\Omega_3} p_0(\xi^x_{\bar\tau^x_{2\delta}}) \exp\left\{\frac{1}{\mu}\int\limits_0^{\bar\tau^x_{2\delta}} (v(\xi^x_s) - \hat\lambda) ds\right\}\right)$$
$$\leq \frac{\bar t}{\delta_1}\frac{K}{\delta} \exp\left(-\frac{W_0(x) - 2\delta - c'(\nu)(c(K,\bar t)\sqrt{\delta_1} + \delta + \delta_0) - c\bar t\delta_0 - c\delta_1}{\mu}\right)$$
$$\times \exp\left(\frac{c\delta}{\mu}\right) p_0(x_0).$$

Similarly, by (3.18) and Proposition 6, we get

$$\mathbf{E}\left(\chi_{\Omega_2} p_0(\xi^x_{\bar\tau^x_{2\delta}}) \exp\left\{\frac{1}{\mu}\int\limits_0^{\bar\tau^x_{2\delta}} (v(\xi^x_s) - \hat\lambda) ds\right\}\right)$$
$$\leq \frac{K}{\delta} \exp\left(-\frac{\bar c - \bar S - c\delta - c\bar t\delta_0}{\mu}\right) \exp\left(\frac{c(M)}{\mu}\right) p_0(x_0).$$

Combining the last three estimates and choosing $\bar c$, ν, δ, δ_0, δ_1 properly, we find that

$$\limsup_{\mu\to 0} \mu \ln(p_0(x)/p_0(x_0)) \leq -W_0(x)$$

for all $x \in M \setminus O_\nu(x_0)$. In view of (3.6), this yields

$$\lim_{\mu\to 0} \mu \ln(p_0(x)/p_0(x_0)) = -W_0(x)$$

for all $x \in M \setminus O_\nu(x_0)$. Since $\nu > 0$ is arbitrary, the last equality holds for all $x \neq x_0$. But, according to Remark 1, the functions $\mu \ln(p(x)/p(x_0))$ are equicontinuous. Therefore, this equality holds uniformly in $x \in M$. Now, the statement of Theorem 2 follows from our normalizing conditions. □

4. Operators with Potential First Order Terms

In the section we consider operators A^μ with 'potential' first order terms. These operators admit explicit formula, both for the limit of the first eigenvalue and the recurrent point. Moreover, the function $W(x)$ can be expressed in terms of the geodesic distance in a proper auxiliary metric.

Definition. *The operator A^μ has potential first order terms, if there is a function $U(x)$ on M such that*

$$b^i(x) = a^{ij}(x)\frac{\partial}{\partial x^j} U(x), \quad i = 1, 2, ..., n.$$

Theorem 3. *Suppose that the operator A^μ has potential first order terms. Then*

$$\lim_{\mu\to 0} \lambda_0 = \min_{x\in M}\left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right).$$

The operator A^μ is recursive if and only if the function $a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)$ has a unique minimum point on M. This minimum point is the only recurrent point of A^μ.

Proof. Let x_0 be a minimum point of the function $\left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right)$. After simple transformation, we find that

$$I(x(\cdot), t) = \int_0^t \left(a_{ij}(x(s)\dot{x}^i\dot{x}^j + a^{ij}(x(s))\frac{\partial}{\partial x^i}U(x(s))\frac{\partial}{\partial x^j}U(x(s)) - v(x(s))\right) ds$$
$$+ 2(U(x(t)) - U(x(0))) \geq t \min_{x\in M}\left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right)$$
$$+ 2(U(x(t)) - U(x(0)))$$

for any absolutely continuous curve $x(\cdot)$. Since $2(U(x(t)) - U(x(0)))$ is bounded uniformly in t, we have $\hat{\lambda} \geq \min_{x\in M}\left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right)$. On the other hand, taking the curve $x(\cdot)$ identically equal to x_0, we obtain

$$\hat{\lambda} \leq \lim_{t\to\infty}\frac{1}{t}I(x(\cdot), t) = \min_{x\in M}\left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right).$$

The other assertions of the theorem can be proved in the same way. □

Denote

$$V(x) = \left(a^{ij}(x)\frac{\partial}{\partial x^i}U(x)\frac{\partial}{\partial x^j}U(x) - v(x)\right)$$
$$- \min_{y\in M}\left(a^{ij}(y)\frac{\partial}{\partial y^i}U(y)\frac{\partial}{\partial y^j}U(y) - v(y)\right).$$

The next statement provides the geometric interpretation for $W_0(x)$.

Theorem 4. *Let x_0 be the unique minimum point of $V(x)$ on M. Then*

$$W_0(x) = 2\left[U(x_0) - U(x) + dist_{(V(x))a_{ij}(x)}(x, x_0)\right];$$

here $dist_{(V(x))a_{ij}(x)}$ is a distance in the metric $(V(x))a_{ij}(x)$.

The proof is the same as that of Theorem 5 below.

Remark 3. The point x_0 need not belong to the set of minimum points of $W_0(x)$ (and, hence, $W(x)$). Thus, $p_0(x_0)$ might be exponentially small.

5. Selfadjoint Operators

In this section we suppose that $b(x) \equiv 0$, i.e. that the operator A^μ is selfadjoint. Then, the formula of the previous section admit an interesting geometric interpretation. Clearly, for selfadjoint operators $\hat{\lambda} = \min_{x\in M}(-v(x)) = 0$ and Condition B is equivalent to the uniqueness of a minimum point of $-v(x)$. Without loss of generality we suppose that $\min_{x\in M}(-v(x)) = 0$. Denote the minimum point by x_0.

Theorem 5. *Let $b(x) \equiv 0$, and assume that the function $(-v(x))$ has a unique minimum point. Then*

$$\lim_{\mu\to 0}\mu \ln p(x) = -2dist_{(-v(x))a_{ij}(x)}(x, x_0);$$

here $dist_{(-v(x))a_{ij}(x)}$ is a distance in the metric $(-v(x))a_{ij}(x)$.

Remark 4. Under the assumption of Theorem 5 the metric $(-v(x))a_{ij}(x)$ degenerates only at the point x_0.

Proof of Theorem 5.. We will prove the following chain of equalities

$$\lim_{\mu\to 0}\mu\ln p(x) = -\inf_{T>0}\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(T)=x_0}}\int_0^T \big(a_{ij}(x(t))\dot x^i\dot x^j - v(x(t))\big)dt$$
$$= -\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(1)=x_0}} 2\int_0^1 \sqrt{(-v(x(t))a_{ij}(x(t))\dot x^i\dot x^j}\,dt = -2\mathrm{dist}_{(-v(x))a_{ij}(x)}(x,x_0). \tag{4.1}$$

The first equality in (4.1) is a direct consequence of Theorem 3. To obtain the second one let us consider a family of regularized functions $v_\kappa(x) = v(x)-\kappa, \kappa>0$. We have

$$\inf_{T>0}\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(T)=x_0}}\int_0^T \big(a_{ij}(x(t))\dot x^i\dot x^j - v_\kappa(x(t))\big)dt$$
$$= \inf_{T>0}\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(1)=x_0}}\int_0^1 \big(\frac{1}{T}a_{ij}(x(t))\dot x^i\dot x^j - Tv_\kappa(x(t))\big)dt$$
$$\geq 2\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(1)=x_0}}\int_0^1 \sqrt{(-v_\kappa(x(t))a_{ij}(x(t))\dot x^i\dot x^j}\,dt.$$

Now, for any fixed curve $x(t),\ x(0)=x,\ x(1)=x_0$ we consider an equation

$$\dot\tau = T\sqrt{-a_{ij}(x(\tau(t)))\dot x_i\dot x_j\Big/ v_\kappa(x(\tau(t)))},\qquad \tau(0)=0,$$

and choose T in such a way that $\tau(1)=1$. Changing the parametrization $z(t)=x(\tau(t))$ gives

$$\int_0^1 \big(\frac{1}{T}a_{ij}(z(t))\dot z^i\dot z^j - Tv_\kappa(z(t))\big)dt = \int_0^1 \big(\frac{\dot\tau}{T}a_{ij}(x(\tau))\dot x^i\dot x^j - \frac{T}{\dot\tau}v_\kappa(x(\tau))\big)d\tau$$
$$= 2\int_0^1 \sqrt{(-v_\kappa(x(\tau))a_{ij}(x(\tau))\dot x^i\dot x^j}\,d\tau.$$

Thus, the relation

$$\inf_{T>0}\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(1)=x_0}}\int_0^1 \big(\frac{1}{T}a_{ij}(x(t))\dot x^i\dot x^j - Tv_\kappa(x(t))\big)dt$$
$$= 2\inf_{\substack{x(\cdot)\\ x(0)=x,\, x(1)=x_0}}\int_0^1 \sqrt{(-v_\kappa(x(t))a_{ij}(x(t))\dot x^i\dot x^j}\,dt$$

holds. Passing to the limit as $\kappa \to 0$ we obtain (4.1). The theorem is proved. $\square$

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