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Averaging of nonlinear random parabolic operators

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Dedicated to Alain Bensoussan at the occasion of his 60th birthday.

1 Introduction

The aim of this work is to study the limit as $\varepsilon \to 0$ of the solution u^ε of the second order semilinear parabolic PDE

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathbf{R}^n \quad u^\varepsilon(0,x) = u_0(x).$$

The main assumptions are the periodicity (of period one in each direction) of a_{ij} and g with respect to their first variable, the fact that $\{\xi_t,\, t \geq 0\}$ is a d–dimensional ergodic diffusion process with a unique invariant measure π, and a centering condition for g :

$$\int_{T^n} \int_{\mathbf{R}^d} g(z,y,u) dz\, \pi(dy) = 0, \quad \forall u \in \mathbf{R}.$$

Our equation is a particular model of random homogenization, where the stochastic perturbation fluctuates as time evolves, in contradiction with the more traditional model where the coefficients are time invariant stationary random fields. Note also that the equation is nonlinear and that the nonlinear term is highly oscillating. For the basic results on homogenization of periodic and random equations, we refer respectively to Bensoussan, Lions, Papanicolaou [1], and Jikov, Kozlov, Oleinik [4].

The same problem, with g linear, has been considered by Campillo, Kleptsina, Piatnitski [3]. Note also that the same problem, whithout the appearance of the process $\{\xi_t\}$, has been studied by Pardoux [5], and without the dependance upon x/ε, by Bouc, Pardoux [2]. It follows clearly from the last quoted work that the limit of u^ε as $\varepsilon \to 0$ should satisfy a stochastic partial differential equation. This is our main result. Note however that our approach is not just a combination of the techniques in [5] and [2]. We need to introduce new types of correctors, which depend on the whole trajectory of the process $\{\xi_t\}$ after time t/ε.

2 Setup and preliminaries

This work is aimed at averaging the following Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathbf{R}^n \quad u^\varepsilon(0,x) = u_0(x) \in L^2(\mathbf{R}^n). \tag{1}$$

where $\varepsilon > 0$ is a small parameter, and ξ_s a stationary diffusion with values in $\mathbf{R}^d$. We denote the infinitesimal generator of ξ by L,

$$L = q_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + b_i(y)\frac{\partial}{\partial y_i},$$

and impose the following conditions on the coefficients of (1) and on the generator of the process ξ :

C1 The functions $a_{ij}(z,y)$ and $g(z,y,u)$ are sufficiently regular, periodic in z of period 1 in all the coordinate directions; the matrix $\{a_{ij}(z,y)\}$ is uniformly positive definite:

$$0 < c\mathbf{I} \le a_{ij}(z,y) \le c^{-1}\mathbf{I};$$

moreover, the gradient of a_{ij} is uniformly bounded:

$$|\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le c \tag{2}$$

C2 $g(z,y,u)$ satisfies the estimates

$$|g(z,y,u)| \le c(1+|y|)^{\mu_0}(1+|u|), \tag{3}$$

$$|g'_u(z,y,u)| \le c(1+|y|)^{\mu_0}, \tag{4}$$

$$|ug''_{uu}(z,y,u)| \le c(1+|y|)^{\mu_0}; \tag{5}$$

with some $\mu_0 < \alpha - 1$ (α is defined in the next assumption).

C3 The following bounds hold

$$0 < c\mathbf{I} \le q_{ij}(z,y) \le c^{-1}\mathbf{I},$$

$$|\nabla q_{ij}(y)| \le c, \quad |b(y)| + |\nabla b(y)| \le c(1+|y|)^{\mu_1},$$

$$\frac{(b(y)\cdot y)}{|y|} \le |y|^{\alpha}, \qquad \alpha > -1, \tag{6}$$

here $(b(y)\cdot y)$ stands for the inner product in $\mathbf{R}^d$.
Under these bounds the process ξ possesses a unique invariant probability measure $\pi(dy) = p(y)dy$ whose density decays at the infinity faster than any negative power of $|y|$ (see [8]).

C4 The relation

$$\int_{T^n}\int_{R^d} g(z,y,u)p(y)dzdy = 0 \tag{7}$$

holds for any $u \in \mathbf{R}$.

It is convenient to decompose $g(z,y,u)$ as follows

$$g(z,y,u) = \tilde{g}(z,y,u) + \bar{g}(y,u)$$

where

$$\bar{g}(y,u) = \int_{T^n} g(z,y,u)dz,$$

so that

$$\int_{T^n} \tilde{g}(z,y,u)dz = 0,\ \forall y \in \mathbf{R}^d,\ u \in \mathbf{R}; \quad \int_{\mathbf{R}^d} \bar{g}(y,u)p(y)dy = 0,\ \forall u \in \mathbf{R}. \tag{8}$$

The first relation here implies in a standard way the existence of a vector function $\tilde{G}(z,y,u)$ such that

$$\tilde{g}(z,y,u) = \mathrm{div}_z \tilde{G}(z,y,u)$$

For any $u(t,x)$ we have now

$$\mathrm{div}_x \tilde{G}(\frac{x}{\varepsilon}, y, u(t,x)) = \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon}, y, u(t,x)) + \tilde{G}'_u(\frac{x}{\varepsilon}, y, u(t,x))\nabla_x u(t,x) \tag{9}$$

According to [8], under the assumptions **C3** and **C4** the second relation in (8) ensures the solvability of the Poisson equation

$$L\bar{G}(y,u) + \bar{g}(y,u) = 0, \qquad \forall u \in \mathbf{R} \tag{10}$$

in the space of functions of polynomial growth in $|y|$; the solution is unique up to an additive constant, for definiteness we assume that it has zero mean w.r.t. the invariant probability π.

3 A priori estimates and tightness

In this section we obtain uniform a priori estimates for the solution u^ε and then use them to show the tightness of the distributions of u^ε.

First, considering (8) and (9) one can rewrite the term $g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t,x))$ on the right hand side of (1) in the form

$$\begin{aligned} &\frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t,x)) = \mathrm{div}_x \tilde{G}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(x,t)) - \\ &\tilde{G}'_u(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(x,t))\nabla_x u^\varepsilon(t,x) + \frac{1}{\varepsilon}\bar{g}(\xi_{t/\varepsilon^2}, u^\varepsilon(x,t)) \end{aligned} \tag{11}$$

For $u \in L^2(\mathbf{R}^n)$ and $y \in \mathbf{R}^d$ denote

$$\Psi^\varepsilon(u,y) = \frac{1}{2}\|u\|^2_{L^2} + \varepsilon(u, \bar{G}(y,u)).$$

From Ito's formula, using (1) and (11), we get

$$\begin{aligned} &d\Psi^\varepsilon(u^\varepsilon(t), \xi_{t/\varepsilon^2}) = (A^\varepsilon u^\varepsilon(t), u^\varepsilon(t))dt - (\nabla_x u^\varepsilon(t), \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t)))dt - \\ &-(\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t))\nabla_x u^\varepsilon(t), u^\varepsilon(t))dt + \frac{1}{\varepsilon}(u^\varepsilon(t), \bar{g}(\xi_{t/\varepsilon^2}, u^\varepsilon(t))dt + \\ &+\frac{1}{\varepsilon}(u^\varepsilon(t), L\bar{G}(\xi_{t/\varepsilon^2}, u^\varepsilon(t)))dt + (u^\varepsilon(t), \nabla_y \bar{G}(\xi_{t/\varepsilon^2}, u^\varepsilon(t)))dw^\varepsilon_t + \\ &+\varepsilon(A^\varepsilon u^\varepsilon(t), \bar{G}(\xi_{t/\varepsilon^2}, u^\varepsilon(t)))dt + (g(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t)), \bar{G}(\xi_{t/\varepsilon^2}, u^\varepsilon(t)))dt + \\ &+\varepsilon(A^\varepsilon u^\varepsilon(t), \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(t))u^\varepsilon(t))dt \\ &+(g(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t)), \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(t))u^\varepsilon(t))dt, \end{aligned} \tag{12}$$

where A^ε stands for $\frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}) \frac{\partial}{\partial x_j}$. Taking the expectation in the last formula, considering (10) and integrating by parts, one gets

$$\mathbf{E}\Psi^\varepsilon_\cdot(u^\varepsilon(t), \xi_{t/\varepsilon^2}) + \mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2})\nabla_x u^\varepsilon(s), \nabla_x u^\varepsilon(s))ds =$$

$$= -\mathbf{E}\int_0^t (\nabla_x u^\varepsilon(s), \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)))ds -$$

$$-\mathbf{E}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s), u^\varepsilon(s))ds -$$

$$-\varepsilon\mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2})\nabla_x u^\varepsilon(s), \bar{G}'_u(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds +$$

$$+\mathbf{E}\int_0^t (g(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)), \{\bar{G}(\xi_{s/\varepsilon^2}, u^\varepsilon(s)) + \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(t))u^\varepsilon(t)\})ds +$$

$$-\varepsilon\mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2})\nabla_x u^\varepsilon(s), \bar{G}'_u(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s)) -$$

$$-\varepsilon\mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2})\nabla_x u^\varepsilon(s), u^\varepsilon(s)\bar{G}''_{uu}(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds.$$

According to Theorem 2 from [8], under condition **C2** the functions $\bar{G}(z,y,u)$, $\bar{G}'_u(z,y,u)$ and $\bar{G}''_{uu}(z,y,u)$ admit the following bounds

$$|\bar{G}(z,y,u)| \le c(1+|u|), \qquad |\bar{G}'_u(z,y,u)| \le c, \qquad |(1+|u|)\bar{G}''_{uu}(z,y,u)| \le c.$$

Thus, the first two terms on the r.h.s. of the above relation can be estimated as follows

$$\Big|\mathbf{E}\int_0^t (\nabla_x u^\varepsilon(s), \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)))ds -$$

$$-\mathbf{E}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s), u^\varepsilon(s))ds\Big| \le$$

$$\le \mathbf{E}\int_0^t \|u^\varepsilon(s)\|\,\|\nabla_x u^\varepsilon(s)\|ds \le \frac{c}{\gamma}\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds + c\gamma\mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2 ds,$$

while all the terms involving the factor ε are dominated by (the second term on) the l.h.s. Finally, taking sufficiently small γ we have by the Gronwall lemma

$$\mathbf{E}\|u^\varepsilon(t)\|^2 + \mathbf{E}\int_0^t |\nabla_x u^\varepsilon(s)\|^2 ds \le C, \qquad t \le T. \tag{13}$$

To obtain tightness we should also estimate, for any $\varphi \in C_0^\infty(\mathbf{R}^n)$, the modulus of continuity of the family $(u^\varepsilon(t), \varphi)$. Namely, we are going to show that for any $\gamma > 0$ there is a compact set K^γ in $C(0,T)$ such that

$$\mathbf{P}\{(u^\varepsilon, \varphi) \notin K^\gamma\} \le \gamma. \tag{14}$$

To this end we consider the expression

$$\Phi^{\varepsilon,\varphi} = (u^\varepsilon(t), \varphi) + \varepsilon(\bar{G}(\xi_{t/\varepsilon^2}, u^\varepsilon(t)), \varphi)$$

By the Ito formula

$$d\Phi^{\varepsilon,\varphi} = -(a(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2})\nabla_x u^\varepsilon(t), \nabla_x \varphi)dt - (\nabla_x \varphi, \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(t)))dt -$$

$$-(\tilde{G}'_u(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t))\nabla_x u^\varepsilon(t),\varphi)ds+(\varphi,\nabla_y\bar{G}(\xi_{t/\varepsilon^2},u^\varepsilon(t)))dw_t^\varepsilon-$$
$$-\varepsilon(a(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2})\nabla_x\varphi,\bar{G}'_u(\xi_{t/\varepsilon^2},u^\varepsilon(t))\nabla_x u^\varepsilon(t))dt-$$
$$-\varepsilon(a(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2})\nabla_x u^\varepsilon(t),\varphi\bar{G}''_{uu}(\xi_{t/\varepsilon^2},u^\varepsilon(t))\nabla_x u^\varepsilon(t))dt+$$
$$+(g(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t)),\bar{G}'_u(\xi_{t/\varepsilon^2},u^\varepsilon(t))\varphi)dt$$

Denote by τ_N the Markov time $\inf\{t>0:\|u^\varepsilon(t)\|\geq N\}$ and by u_N^ε the function $u(t\wedge\tau_N)$. For each $N>0$ the family

$$I_N(t)=\int_0^t(\varphi,\nabla_y\bar{G}(\xi_{s/\varepsilon^2},u_N^\varepsilon(s)))dw_s^\varepsilon$$

is tight in $C([0,T])$ by the Kolmogorov criterion. Also, from (13) we get

$$\lim_{N\to\infty}\mathbf{P}\{\|u_N^\varepsilon(\cdot)-u^\varepsilon(\cdot)\|_{C([0,T])}\neq 0\}=0.$$

Thus, the family

$$I(t)=\int_0^t(\varphi,\nabla_y\bar{G}(\xi_{s/\varepsilon^2},u^\varepsilon(s)))dw_s^\varepsilon$$

is also tight. All the absolutly continuous terms in the above formula do not make any difficulties and the required tightness in $C([0,T])$ follows.

We summarize this in the following statement.

Proposition 1 *For the family of solutions of problem (1) the estimate holds*

$$\mathbf{E}\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2_{L^2(\mathbf{R}^n)}+\int_0^T\|\nabla_x u^\varepsilon(t)\|^2_{L^2(\mathbf{R}^n)}dt\right)\leq C, \tag{15}$$

and for any $\varphi\in C_0^\infty(\mathbf{R}^n)$ and any $\gamma>0$ there is a compact subset K^γ of $C(0,T)$ such that (14) takes place.

Proof. The second statement has just been proved. The first one follows from (13) and (12) and the Burkholder–Davis–Gundy inequality (see [6]).

According to [10] the bounds of the above proposition imply the tightness of the distributions of $\{u^\varepsilon,\ \varepsilon>0\}$ in the functional space

$$V=\left[L^2(0,T;H^1(\mathbf{R}^n))\right]_w\cap C([0,T];L^2_w(\mathbf{R}^n)),$$

where the index w means that the corresponding space is equipped with its weak topology. Note that this space V has first been introduced by Viot [10].

4 Passage to the limit

The aim of this section is to pass to the limit, as $\varepsilon\to 0$, in the family of laws of $\{u^\varepsilon\}$ and to determine the limiting problem. In view of the tightness results of the preceding section it is sufficient to find the limit distributions of the inner products (φ,u^ε) with $\varphi\in C_0^\infty(\mathbf{R}^n)$, see [10]. To this end we introduce the following two auxiliary parabolic equations

$$\frac{\partial}{\partial\tau}+\frac{\partial}{\partial z_i}a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\chi^k(z,\tau)=-\frac{\partial}{\partial z_i}a_{ik}(z,\xi_\tau),\quad (z,\tau)\in T^n\times(-\infty,+\infty), \tag{16}$$

and

$$\frac{\partial}{\partial \tau} + \frac{\partial}{\partial z_i} a_{ij}(z, \xi_\tau) \frac{\partial}{\partial z_j} \Psi(z, \tau, u) = -\tilde{g}(z, \xi_\tau, u), \quad (z, \tau) \in T^n \times (-\infty, +\infty), \tag{17}$$

where u is a parameter. The functions $\chi^k(z, \tau)$ and $\Psi(z, \tau, u)$ are now defined as stationary solutions to these equations.

Lemma 1 *There exist stationary solutions to (16) and (17), these solutions are ergodic and unique up to an additive constant. Moreover, under the normalization* $\int_{T^n} \chi(z, \tau) dz = 0$ *and* $\int_{T^n} \Psi(z, \tau, u) dz = 0$ *the following estimates hold*

$$\int_t^{t+1} \|\chi(\cdot, s)\|^2_{H^1(T^n)} ds \leq C \tag{18}$$

$$\int_t^{t+1} \|\Psi(\cdot, s, u)\|^2_{H^1(T^n)} ds \leq C(1 + |u|) \tag{19}$$

$$\int_t^{t+1} \|\Psi'_u(\cdot, s, u)\|^2_{H^1(T^n)} ds \leq C \tag{20}$$

Proof The existence and uniqueness of stationary ergodic solution as well as the bound (18) have been proved in [7]. The other two estimates (19) and (20) follow easily from **C2** and the definition of $\tilde{g}$.

Remark 1 *In contrast with* $\bar{G}(y, u)$*, the functions* $\chi(z, \tau)$ *and* $\Psi(z, \tau, u)$ *depend not only on the value of* ξ *at a given time* τ*, but on the whole halftrajectory* $\{\xi_s,\ s \geq \tau\}$.

Having defined $\bar{G}(y, u)$, $\chi(z, \tau)$ and $\Psi(z, \tau, u)$, for an arbitrary function $\varphi \in C_0^\infty$, we consider the expression

$$\Phi^\varepsilon = (u^\varepsilon(t), \varphi) + \varepsilon(\chi^\varepsilon(t) u^\varepsilon(t), \nabla_x \varphi) + \varepsilon(\Psi^\varepsilon(t, u^\varepsilon(t)), \varphi) + \varepsilon(\bar{G}(\xi_t^\varepsilon, u^\varepsilon(t)), \varphi),$$

where $\chi^\varepsilon(t)$, $\Psi^\varepsilon(t, u)$ and ξ_t^ε stand for $\chi(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^2})$, $\Psi(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^2}, u)$ and ξ_{t/ε^2} respectively.

By the Itô formula:

$$d\Phi^\varepsilon = \{(u_t^\varepsilon(t), \varphi) + \varepsilon^{-1}(\chi_\tau^\varepsilon(t) u_t^\varepsilon(t), \nabla_x \varphi) + \varepsilon(\chi^\varepsilon(t) u_t^\varepsilon(t), \nabla_x \varphi) +$$
$$+\varepsilon^{-1}(\Psi_\tau^\varepsilon(t, u^\varepsilon(t)), \varphi) + \varepsilon(\Psi_u^\varepsilon(t, u^\varepsilon(t)) u_t^\varepsilon(t), \varphi) + \varepsilon^{-1}(L\bar{G}(\xi_t^\varepsilon, u^\varepsilon(t)), \varphi) +$$
$$+\varepsilon(\bar{G}_u(\xi_t^\varepsilon, u^\varepsilon(t)) u_t^\varepsilon(t), \varphi)\} dt + (q(\xi_t^\varepsilon) \nabla_y \bar{G}(\xi_t^\varepsilon, u^\varepsilon(t)), \varphi). dw_t^\varepsilon$$

Considering (1), after multiple integration by parts and simple rearrangements, we obtain

$$d\Phi^\varepsilon = \{(u^\varepsilon(t), a^\varepsilon \nabla_x \nabla_x \varphi) + \varepsilon^{-1}(u^\varepsilon(t), \nabla_z a^\varepsilon \nabla_x \varphi) +$$
$$+\varepsilon^{-1}(\bar{g}(\xi_t^\varepsilon, u^\varepsilon(t)), \varphi) + \varepsilon^{-1}(\tilde{g}(\xi_t^\varepsilon, u^\varepsilon(t)), \varphi) + \varepsilon^{-1}(\chi_\tau^\varepsilon(t) u^\varepsilon(t), \nabla_x \varphi) +$$
$$+\varepsilon^{-1}(\mathcal{A}\chi^\varepsilon(t) u^\varepsilon(t), \nabla_x \varphi) + ([a^\varepsilon \nabla_z \chi^\varepsilon(t) + \nabla_z(a^\varepsilon \chi^\varepsilon(t))], u^\varepsilon(t) \nabla_x \nabla_x \varphi) +$$
$$+\varepsilon(a^\varepsilon \chi^\varepsilon(t), u^\varepsilon(t) \nabla_x \nabla_x \nabla_x \varphi) + (\chi^\varepsilon(t) g^\varepsilon(t, u^\varepsilon(t)), \nabla_x \varphi) + \varepsilon^{-1}(\Psi_\tau^\varepsilon(t, u^\varepsilon(t)), \varphi) -$$
$$-(\nabla_z \Psi_u^\varepsilon(t, u^\varepsilon(t)) a^\varepsilon \nabla_x u^\varepsilon(t), \varphi) - \varepsilon(\Psi_{uu}^\varepsilon a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi) -$$
$$-\varepsilon(\Psi_u^\varepsilon a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi) + (\Psi_u^\varepsilon g^\varepsilon, \varphi) + \varepsilon^{-1}(L\bar{G}^\varepsilon, \varphi)\} dt + (\sigma^\varepsilon \nabla_y \bar{G}, \varphi) dw_t^\varepsilon +$$
$$+\{-\varepsilon(\bar{G}_{uu}^\varepsilon a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi) - \varepsilon(\bar{G}_u^\varepsilon a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi) + (\bar{G}_u^\varepsilon g^\varepsilon, \varphi)\} dt.$$

In view of (10), (16), (17) and the obvious relation

$$(a^\varepsilon \nabla_z \Psi^\varepsilon_u \nabla_x u^\varepsilon, \varphi) = -(a^\varepsilon \nabla_z \Psi^\varepsilon, \nabla_x \varphi) - \varepsilon^{-1}(\mathcal{A}\Psi^\varepsilon, \varphi),$$

the above expression can be simplified further as follows

$$\begin{aligned} d\Phi^\varepsilon = \{(u^\varepsilon(t)), a^\varepsilon \nabla_x \nabla_x \varphi) + ([a^\varepsilon \nabla_z \chi^\varepsilon(t) + \nabla_z(a^\varepsilon \chi^\varepsilon(t))], u^\varepsilon(t) \nabla_x \nabla_x \varphi) + \\ +(\chi^\varepsilon(t) g^\varepsilon(t, u^\varepsilon(t)), \nabla_x \varphi) - (a^\varepsilon \nabla_z \Psi^\varepsilon, \nabla_x \varphi) + (\Psi^\varepsilon_u g^\varepsilon, \varphi) + (\bar{G}^\varepsilon_u g^\varepsilon, \varphi)\} dt + \\ +(\sigma^\varepsilon \nabla_y \bar{G}, \varphi) dw^\varepsilon_t + \varepsilon\{a^\varepsilon \chi^\varepsilon(t) u^\varepsilon(t), \nabla_x \nabla_x \nabla_x \varphi) - \qquad (21) \\ -(\Psi^\varepsilon_{uu} a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi) - (\Psi^\varepsilon_u a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi) - (\bar{G}^\varepsilon_u a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi) - \\ -(\bar{G}^\varepsilon_{uu} a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi)\} dt. \end{aligned}$$

The following statements will allow us to pass to the limit, as $\varepsilon \to 0$, in the laws of Φ^ε and thus to obtain the desired limiting distribution of (u^ε, φ).

Proposition 2 *Let $v^\varepsilon(x,t)$ converge to $v^0(x,t)$ in V. Then $v^\varepsilon(x,t)$ converges towards $v^0(x,t)$ in $L^2_{\text{loc}}(\mathbf{R}^n \times (0,T))$. In other words V is continuously embedded in $L^2_{\text{loc}}(\mathbf{R}^n \times (0,T))$.*

Proposition 3 *Let the family of laws of u^ε be tight in V, and suppose that $\theta(z,\tau,u)$ is periodic in z, stationary and ergodic in τ. Assume, furthermore, that the following bounds hold:*

$$\|\theta(\cdot,\tau,u)\|_{C(T^n)} \le c\eta(\tau)(1+|u|), \qquad (22)$$

$$\|\theta(\cdot,\tau,u_1) - \theta(\cdot,\tau,u_2)\|_{C(T^n)} \le c\eta(\tau)|u_1 - u_2|, \qquad (23)$$

with a stationary process $\eta(\tau)$ subject to the estimate $\mathbf{E}|\eta(\tau)|^p \le c(p)$ for each $p > 1$. Then for any $\varphi \in C_0^\infty$ one has

$$\mathbf{P} - \lim_{\varepsilon\to 0} \sup_{t\le T} \left| \int_0^t (\theta(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u^\varepsilon) - <\theta>(u^\varepsilon), \varphi) dt \right| = 0$$

where $<\theta>(u) = \mathbf{E}\int_{T^n} \theta(z,\tau,u)dz$.

Proposition 4 *Let u^ε converge in law towards u_0 in the space V, and suppose $\Theta : \mathbf{R} \to \mathbf{R}$ is a uniformly continuous function satisfying the estimate $|\Theta(u)| \le c(1+|u|)$. Then for any $\varphi \in C_0^\infty$ the family $\{\int_0^t (\Theta(u^\varepsilon), \varphi),\ 0 \le t \le T\}$ converges in law to $\{\int_0^t (\Theta(u^0), \varphi),\ 0 \le t \le T\}$ in $C([0,T])$.*

The next statement will allow us to deal with the stochastic term.

Proposition 5 *Suppose the family of laws of u^ε is tight in V. If a continuous function $H : \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}^d$ satisfies the estimates*

$$|H(y,u)| \le c(1+|y|)^\mu (1+|u|), \quad |H(y,u_1) - H(y,u_2)| \le c(1+|y|)^\mu (|u_1 - u_2|)$$

then for any $\varphi \in C_0^\infty$ the following quantity tends to zero in probability :

$$\sup_{t\le T} \left| \int_0^t < q(\xi_{t/\varepsilon^2})(H(\xi_{t/\varepsilon^2}, u^\varepsilon), \varphi), (H(\xi_{t/\varepsilon^2}, u^\varepsilon), \varphi) > - (R(u^\varepsilon)\varphi, \varphi) dt \right|,$$

where

$$(R(u)\varphi,\varphi) = \int_{R^d}\int_{R^d} < q(y)H(y,u(x')), H(y,u(x'')) > \varphi(x')\varphi(x'')\pi(dy)dx'dx.''$$

Note that below H will be identified with $\nabla_y \bar{G}$. Now it is natural to rewrite (21) as follows

$$(u^\varepsilon(t),\varphi)-(u_0,\varphi)-\int_0^t\{(u^\varepsilon(s),<a(\mathbf{I}+\nabla_z\chi)>,\nabla_x\nabla_x\varphi)\}ds-$$
$$-\int_0^t\{(<\chi g>(u^\varepsilon(s)),\nabla_x\varphi)-(<a\nabla_x\Psi>(u^\varepsilon(s)),\nabla_x\varphi)\}ds-$$
$$-\int_0^t\{(<\Psi_u g>(u^\varepsilon(s)),\varphi)+(<\bar{G}_u g>(u^\varepsilon(s)),\varphi)\}ds= \tag{24}$$
$$=\int_0^t(\sigma^\varepsilon\nabla_y\bar{G},\varphi)dw_s^\varepsilon+K^\varepsilon(t)$$

where $K^\varepsilon(t)\to 0$ in probability, as $\varepsilon\to 0$.

Hence, if we denote by Q^0 an accumulating point of the family of laws of u^ε in V, as $\varepsilon\to 0$, then the functional

$$F_\varphi(u)\equiv(u(t),\varphi)-(u(0),\varphi)-\int_0^t\{(u(s),<a(\mathbf{I}+\nabla_z\chi)>,\nabla_x\nabla_x\varphi)\}ds-$$
$$-\int_0^t\{(<\chi g>(u(s)),\nabla_x\varphi)-(<a\nabla_x\Psi>(u(s)),\nabla_x\varphi)\}ds-$$
$$-\int_0^t\{(<\Psi_u g>(u(s)),\varphi)+(<\bar{G}_u g>(u(s)),\varphi)\}ds=$$

is a martingale w.r.t. Q^0 equipped with the natural filtration, whose bracket is given by

$$\ll F_\varphi(u)\gg(t)=$$
$$\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}<q(y)\nabla_y\bar{G}(y,u(s,x')),\nabla_y\bar{G}(y,u(s,x''))>\varphi(x')\varphi(x'')ds\pi(dy)dx'dx''.$$

Theorem 1 *Any accumulation point of the family of laws of the solutions $\{u^\varepsilon\}$ of problem (1), as $\varepsilon\to 0$, is a solution of the martingale problem*

$$du(t)=\hat{A}(u(t))dt+dM(t),\ where$$
$$\ll M\gg_t=\int_0^t R(u(s)ds,$$

$$\hat{A}(u)=\nabla_x\cdot<a(bfI+\nabla_z\chi>\nabla_x u-\nabla_x\cdot<\chi g>(u)+\nabla_x\cdot<a\nabla_x\Psi>(u)+$$
$$+<\Psi_u g>(u)-<\bar{G}_u g>(u),$$

and the covariance operator given by

$$(R(u)\varphi,\varphi)=$$
$$\int_{R^d}\int_{R^n}\int_{R^n}<q(y)\nabla_y\bar{G}(y,u(x')),\nabla_y\bar{G}(y,u(x''))>\varphi(x')\varphi(x'')\pi(dy)dx'dx''$$

Remark 2 *Uniqueness of the limiting martingale problem remains so far an openn problem.*

Remark 3 *The detailed proofs will appear elsewhere.*

References

[1] A. Bensoussan, J.L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, North Holland, 1978.

[2] R. Bouc, E. Pardoux, Moments of semi–linear random evolutions, *SIAM J. Appl. Math.* **41**, 370–399, 1981.

[3] F. Campillo, M. Kleptsina, A. Piatnitski, Homogenization of random parabolic operator with large potential, *INRIA Report 3520*, 1998.

[4] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer 1994.

[5] E. Pardoux, Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients : a probabilistic approach, *J. of Funct. Anal.* **167**, 498–520, 1999.

[6] R.Liptser, A.Shiryaev, *Theory of Martingales*, Kluwer Academic Publishrs, 1989.

[7] M.Kleptsyna, A.Piatnitski, Averaging of non-selfadjoint parabolic equations with random evolution, *INRIA Report 3951*, 2000.

[8] E.Pardoux, A.Veretennikov, On Poisson equation and diffusion approximation. I., *Ann. Prob*, to appear.

[9] E.Pardoux, A.Veretennikov,On Poisson equation and diffusion appoximation II. *LATP/UMR 6632, Preprint*, 2000.

[10] M.Viot, Solutions Faibles d'Equations Dérivées Stochastiques Non Linéaires, *Thèse, Université Paris VI*, 1976.