Asymptotic Analysis 87 (2014) 1–28
DOI 10.3233/ASY-131194
IOS Press

On the rate of convergence of solutions in domain with random multilevel oscillating boundary

G.A. Chechkin [a], C. D'Apice [b], U. De Maio [c] and A.L. Piatnitski [d,*]

[a] *Department of Differential Equations, Faculty of Mechanics and Mathematics, Moscow Lomonosov State University, Moscow 119991, Russia*
E-mail: chechkin@mech.math.msu.su

[b] *Dipartimento di Ingegneria dell'Informazione e Matematica Applicata, Università degli Studi di Salerno, Via Ponte don Melillo, 1, Fisciano (SA) 84084, Italia*
E-mail: dapice@diima.unisa.it

[c] *Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Monte S. Angelo, Via Cintia, 80126 Napoli, Italy*
E-mail: udemaio@unina.it

[d] *P.N. Lebedev Physical Institute RAS, Leninski pr., 53, Moscow 117924, Russia*
and
Narvik University College, Postboks 385, 8505 Narvik, Norway
E-mail: andrey@sci.lebedev.ru

Abstract. In the paper we deal with the homogenization problem for the Poisson equation in a singularly perturbed domain with multilevel oscillating boundary. This domain consists of the body, a large number of thin periodically situated cylinders joining to the body through thin random transmission zone with rapidly oscillating boundary. Inhomogeneous Fourier boundary conditions with perturbed coefficients are set on the boundaries of the thin cylinders and on the boundary of the transmission zone. We prove the homogenization theorems. Moreover we derive estimates of deviation of the solution to initial problem from the solution to the homogenized problem in different cases.

It appears that depending on small parameters in Fourier boundary conditions of initial problem one can obtain Dirichlet, Neumann or Fourier boundary conditions in the homogenized problem. We estimate the convergence of solutions in these three cases.

Keywords: homogenization, estimates of convergence, rapidly oscillating boundary, singular perturbations, random structures

1. Introduction

This paper deals with boundary value problems in domains with multilevel rapidly oscillating boundaries; these problems play important role in various applications. The domains of this type appear naturally when describing some physical materials and biological structures.

*Corresponding author. E-mail: andrey@sci.lebedev.ru.

0921-7134/14/$27.50 © 2014 – IOS Press and the authors. All rights reserved

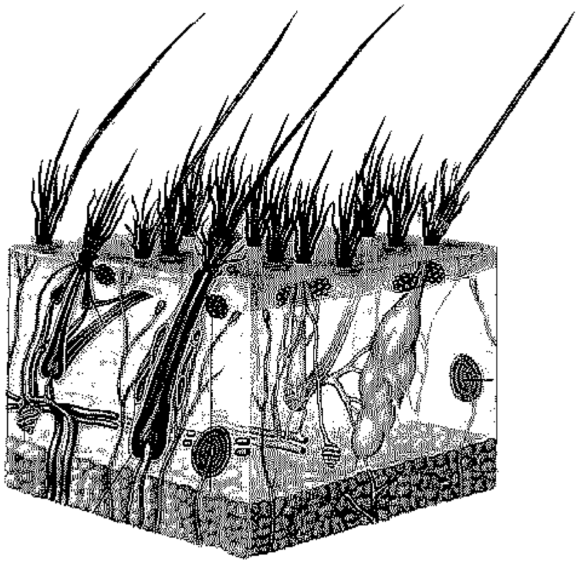

Fig. 1. Skin with hairs of two types.

One of the key examples is the skin of mammals. Typically, it is covered with hair of two types (see Fig. 1). The set of long hair insulates the animal body from the environment influence such as moisture, wind, etc., while short hair (down hair) protect the long hair bulbs from low and high outside temperature. Hence, in the corresponding model problem the coefficients of boundary operator might differ essentially for the parts of the domain boundary representing short and long hair. This reflects the difference in the corresponding heat conductivities.

It can be observed that the long hair is situated quite regularly and thus we assume that its geometry is periodic. The short hair is less regular, and it is natural to use the stochastic framework to describe its geometry.

We consider a domain with singularly perturbed random multilevel boundary. Under natural assumptions on randomness we prove the homogenization theorem and then estimate the difference between solutions to the perturbed (initial) problem and the homogenized problem.

Problems in domains with singularly perturbed boundary attract the attention of many scientists. The presence of a microinhomogeneous rough boundary influences the macroscopic effective boundary conditions. Appropriate tools for studying these problems are methods of asymptotic analysis and boundary homogenization (see, e.g., [5,9,22,23,29,35,43,44,46], and references therein).

The effective behaviour of problems with random microstructures have been widely studied in the existing literature. The first rigorous homogenization results for divergence form elliptic operators with random coefficients have been obtained in the pioneer works [32] and [45]. Then the estimates for the rate of convergence were derived in [49]. For the stochastic two-scale convergence in the mean we refer to [14] and for almost sure two-scale convergence to [50]. The boundary homogenization for elliptic boundary value problem with randomly alternating type of boundary conditions has been studied in [8]; the effective boundary condition in a domain randomly perforated along the boundary, was obtained in [17]. The paper [19] dealt with the homogenization of a thick junction through a thin random transmission zone.

The interest in boundary-value problems in domains with rough boundaries have been increasing recent years due to essential progress in many applied sciences. A number of important models in this fields can be studied mathematically by means of boundary homogenization technique. These models can be found in biology (see, for instance, [16]) in physics (see [7,11,24,28,31,36,38]), in engineering sciences [13,15,47].

Previously, problems in domain with oscillating boundary have been considered in [6,27] and then in several other papers. In [26] one can find rigorous asymptotic analysis of a problem in domain with multiscale oscillating boundary. A rich collection of new results on asymptotic analysis of boundary-value problems in thick multi-structures is presented in the following papers [10–12,18,19,25,30,38–41]. For further results in domains with oscillating boundaries see [2–4,27].

The work [20] is devoted to boundary value problem in domain with periodic multilevel rapidly oscillating boundary.

The present paper deals with problems in domain with multilevel rapidly oscillating boundary. In contrast with [20] here the boundary microstructure combines periodic and random components. The presence of two scales of oscillation of the boundary as well as the randomness of the corresponding geometry is natural in various applications. However it leads to additional mathematical difficulties. We consider a model problem in a domain with random double level oscillating boundary, the first level is random and has the height ε; the second one is of the height ε^{α}, $0 < \alpha < 1$, and is periodic (see Fig. 2).

Here ε is a small parameter, which also characterizes the distance between neighboring thin domains and their thickness.

We study boundary-value problems in a domain with multilevel oscillating boundary. It is assumed that inhomogeneous Fourier boundary condition is stated on the oscillating parts of the boundary, the coefficient of the boundary operator being a rapidly oscillating function. Out goal is to prove the homogenization results and estimate the rate of convergence.

The paper is organized as follows. In Section 2 we define the geometry and set the problem. Section 3 is devoted to the detailed definition of random structure and conditions on the random functions. In Section 4 we formulate main theorems and in Section 6 we prove them. Section 5 contains auxiliary technical statements and their proofs. In Section 7 we study the limit behaviour of eigenvalues and eigenfunctions of the corresponding spectral problems.

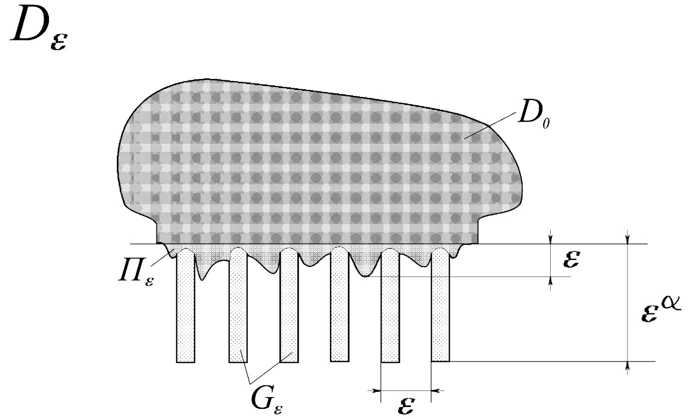

Fig. 2. Domain with multilevel oscillating boundary with random transmission zone.

2. Setting of the problem

Assume that B is a $(d-1)$-dimensional ball centered at the origin and lying in the unit cube

$$\square = \left\{\xi' = (\xi_1, \dots, \xi_{d-1})\colon -\frac{1}{2} < \xi_i < \frac{1}{2}, i = 1, \dots, d-1\right\}.$$

Here and throughout the paper $d \geqslant 2$. A model domain D_ε with multilevel oscillating boundary (see Fig. 2) consists of a body

$$D_0 = \{x \in \mathbb{R}^d\colon x' = (x_1, \dots, x_{d-1}) \in I_0,\ 0 < x_d < \Phi(x')\},$$

where $I_0 = (0, a)^{d-1}$, $\Phi \in C^1(\overline{I_0})$, $\min_{x' \in \overline{I}_0} \Phi(x') = \Phi_0 > 0$, of a large number of the thin cylinders

$$\widehat{G}_{i_1,\dots,i_{d-1}}(\varepsilon) = \{x \in \mathbb{R}^d\colon (\varepsilon^{-1}x_1 - i_1, \dots, \varepsilon^{-1}x_{d-1} - i_{d-1}) \in B,\ x_d \in (-l\varepsilon^\alpha, 0]\},$$

with $i_j = 0, 1, \dots, N-1$, $j = 1, \dots, d-1$, and of the thin oscillating layer

$$\Pi_\varepsilon = \left\{x \in \mathbb{R}^d\colon x' \in I_0, \varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon}, \omega\right) < x_d \leqslant 0\right\},$$

where $\Theta(x')$ is a smooth nonnegative function with $\operatorname{supp}\Theta(x') \subset I_0$, and $F(\xi', \omega)$ is a random statistically homogeneous non-positive function with smooth realizations, ω is an element of a standard probability space $(\Omega, \mathcal{A}, \mu)$ (see the detailed definitions below). Thus, $D_\varepsilon = D_0 \cup \Pi_\varepsilon \cup \widehat{G}_\varepsilon$, where

$$\widehat{G}_\varepsilon = \bigcup_{j=1}^{d-1} \bigcup_{i_j=0}^{N-1} \widehat{G}_{i_1,\dots,i_{d-1}}(\varepsilon).$$

Here N is a big natural number, hence $\varepsilon = a/N$ is a small discrete parameter. We identify the cube I_0 and the set $\{x \in \mathbb{R}^d\colon x' \in I_0, x_d = 0\}$. In what follows we denote

$$J_\varepsilon = \{i = (i_1, \dots, i_{d-1}) \in \mathbb{Z}^{d-1}\colon 0 \leqslant i_j \leqslant N-1\}.$$

Equivalently, $D_\varepsilon = D_0 \cup \Pi_\varepsilon \cup G_\varepsilon$, where $G_\varepsilon = \widehat{G}_\varepsilon \setminus \Pi_\varepsilon$. We denote also

$$B_\varepsilon^0 = \bigcup_{i \in J_\varepsilon} \{x \in \mathbb{R}^d\colon (\varepsilon^{-1}x_1 - i_1, \dots, \varepsilon^{-1}x_{d-1} - i_{d-1}) \in B, x_d = 0\},$$

$$\Gamma_\varepsilon = \left\{x \in \overline{D}_\varepsilon \setminus \widehat{G}_\varepsilon\colon x_d = \varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon}, \omega\right)\right\},$$

$\widehat{\Upsilon}_\varepsilon := \partial\widehat{G}_\varepsilon \setminus \overline{B}_\varepsilon^0$ or $\widehat{\Upsilon}_\varepsilon = \widehat{S}_\varepsilon \cup B_\varepsilon$, where $\widehat{S}_\varepsilon$ is the lateral surface of the set $\widehat{G}_\varepsilon$, and B_ε is the lower surface of $\widehat{G}_\varepsilon$; $\Upsilon_\varepsilon := \partial G_\varepsilon \setminus \partial\Pi_\varepsilon$ and respectively S_ε is the lateral surface of the set G_ε, $\Gamma_1 = \{x\colon x_d =$

$\Phi(x'), x' \in \overline{I_0}\}$, $\gamma = \partial D_\varepsilon \setminus (\varGamma_\varepsilon \cup \varUpsilon_\varepsilon \cup \varGamma_1)$. It is easy to see that γ does not depend on ε and $\gamma = \partial D_0 \setminus (\varGamma_1 \cup I_0)$. In D_ε we consider the following boundary value problem:

$$\begin{cases} -\Delta_x u_\varepsilon(x) = f(x), & x \in D_\varepsilon; \\ \partial_\nu u_\varepsilon(x) + \varepsilon^\tau \theta(x') p\left(\dfrac{x'}{\varepsilon}, \omega\right) u_\varepsilon(x) = \theta(x') q\left(\dfrac{x'}{\varepsilon}, \omega\right), & x \in \varGamma_\varepsilon; \\ \partial_\nu u_\varepsilon(x) + \varepsilon^\mu k_1 u_\varepsilon(x) = \varepsilon^{1-\alpha} k_2, & x \in \varUpsilon_\varepsilon; \\ u_\varepsilon(x) = 0, & x \in \varGamma_1; \\ \partial_\nu u_\varepsilon(x) = 0, & x \in \gamma. \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the derivative with respect to the outer normal; the constants k_1, k_2 are positive; the parameters μ, τ are real; $p(\xi', \omega)$ and $q(\xi', \omega)$ are random statistically homogeneous positive functions, $\theta \in C^\infty(\overline{I}_0)$ with $0 < \theta_- \leqslant \theta(x') \leqslant \theta^+$. Also, we assume that $f \in L_2^{loc}(\mathbb{R}^d)$.

Function $u_\varepsilon \in H^1(D_\varepsilon, \varGamma_1) = \{v \in H^1(D_\varepsilon)\colon v|_{\varGamma_1} = 0\}$ is a solution of the problem (1), if the following integral identity

$$\begin{aligned} &\int_{D_\varepsilon} \nabla u_\varepsilon(x) \nabla v(x)\,\mathrm{d}x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta(x') p\left(\frac{x'}{\varepsilon}, \omega\right) u_\varepsilon(x) v(x)\,\mathrm{d}\sigma_x + \varepsilon^\mu k_1 \int_{\varUpsilon_\varepsilon} u_\varepsilon(x) v(x)\,\mathrm{d}\sigma_x \\ &\quad = \int_{D_\varepsilon} f(x) v(x)\,\mathrm{d}x + \int_{\varGamma_\varepsilon} \theta(x') q\left(\frac{x'}{\varepsilon}, \omega\right) v(x)\,\mathrm{d}\sigma_x + \varepsilon^{1-\alpha} \int_{\varUpsilon_\varepsilon} k_2 v(x)\,\mathrm{d}\sigma_x \end{aligned} \tag{2}$$

holds true for any function $v \in H^1(D_\varepsilon, \varGamma_1)$.

For any fixed $\varepsilon > 0$ there exists a unique solution to problem (1) (see, for instance, [33]).

The main goal of the paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the solution to this problem. It should be noted that the limit behavior of solutions to problem (1) depends crucially on the relation between the parameters α, τ, μ. In the paper for different values of the parameters we derive the effective model and estimate the rate of convergence.

3. The probabilistic framework and main assumptions

In this section we introduce the probabilistic framework. Further details can be found in [29] (see also [22]).

Let $(\varOmega, \mathcal{A}, \mu)$ be a standard probability space.

Definition 3.1. A family of measurable maps

$$T_{x'} : \varOmega \to \varOmega, \quad x' \in \mathbb{R}^{d-1},$$

is called a $(d-1)$-*dynamical system* if the following properties hold true:

- Group property:

$$T_{x'+y'} = T_{x'} T_{y'} \quad \forall x', y' \in \mathbb{R}^{d-1}, \qquad T_0 = \mathit{Id} \quad (\mathit{Id} \text{ is the identical mapping}).$$

- Isometry property:

$$T_{x'}\mathcal{U} \in \mathcal{A}, \qquad \mu(T_{x'}\mathcal{U}) = \mu(\mathcal{U}), \quad \forall x' \in \mathbb{R}^{d-1},\ \forall \mathcal{U} \in \mathcal{A}.$$

- Measurability: for any measurable functions $\phi(\omega)$ on Ω, the function $\phi(T_{x'}\omega)$ is measurable on $\Omega \times \mathbb{R}^{d-1}$, where the space $\mathbb{R}^{d-1}$ is equipped with the Borel σ-algebra $\mathcal{B}$.

Let $L_q(\Omega)$ ($q \geqslant 1$) be the space of measurable functions integrable in the power q with respect to the measure μ. The following assertion is a consequence of the Fubini theorem (see [29] for the proof).

Proposition 3.1. *Assume that $\phi \in L_q(\Omega)$. Then almost all realizations $\phi(T_{x'}\omega)$ belong to $L_q^{loc}(\mathbb{R}^{(d-1)})$.*

If the sequence $\{\phi_k\} \subset L_q(\Omega)$ converges in $L_q(\Omega)$ to the function ϕ, then there exists a subsequence $\{\phi_{\widehat{k}}\}$ such that almost all realizations $\phi_{\widehat{k}}(T_{x'}\omega)$ converge in $L_q^{loc}(\mathbb{R}^{(d-1)})$ to the realization $\phi(T_{x'}\omega)$.

Definition 3.2. A measurable function $\phi(\omega)$ on Ω is called *invariant* if, for any $x' \in \mathbb{R}^{d-1}$, $\phi(T_{x'}\omega) = \phi(\omega)$ almost surely.

Definition 3.3. A dynamical system $T_{x'}$ is said to be *ergodic* if all its invariant functions are almost surely constant.

Definition 3.4. Let $\digamma \in L_1^{loc}(\mathbb{R}^{d-1})$. We say that the function $\digamma$ has a spatial average if the limit

$$M(\digamma) = \lim_{\varepsilon\to 0} \frac{1}{|B|} \int_B \digamma\left(\frac{x'}{\varepsilon}\right) \mathrm{d}x'$$

exists for any bounded Borel set $B \in \mathcal{B}$ with $|B| > 0$, and moreover this limit does not depend on the choice of B. The quantity $M(\digamma)$ is called the *spatial average* of the function $\digamma$.

The proof of the following statement can be found in [22].

Proposition 3.2. *Let a function $\digamma$ have a spatial average in $\mathbb{R}^{d-1}$, and suppose that the family $\{\digamma(\frac{x'}{\varepsilon}), 0 < \varepsilon \leqslant 1\}$ is bounded in $L_q(\mathcal{K})$, for some $q \geqslant 1$, where $\mathcal{K}$ is a compact in $\mathbb{R}^{d-1}$ whose interior is not empty and contains the origin. Then*

$$\digamma\left(\frac{x'}{\varepsilon}\right) \rightharpoonup M(\digamma) \quad \textit{weakly in } L_q^{loc}\big(\mathbb{R}^{d-1}\big),\ \textit{as } \varepsilon \to 0.$$

In what follows we repeatedly use the Birkhoff ergodic theorem in the following particular form (see, for instance, [29] for more details).

Theorem 3.1 (Birkhoff ergodic theorem). *Let $T_{x'}$ be an ergodic $(d-1)$-dynamical system and let $\phi \in L_q(\Omega)$, $q \geqslant 1$. Then, almost surely (i.e. for almost all $\omega \in \Omega$), the realization $\phi(T_{x'}\omega)$ admits a spatial average $M(\phi(T_{x'}\omega))$. Moreover,*

$$M\big(\phi(T_{x'}\omega)\big) = \mathbb{E}(\phi),$$

where $\mathbb{E}(\phi)$ is the mathematical expectation of ϕ.

Definition 3.5. A random field $\zeta(x', \omega)$ ($x' \in \mathbb{R}^{d-1}$, $\omega \in \Omega$) is called *statistically homogeneous* if

$$\zeta(x', \omega) = \widetilde{\zeta}(T_{x'}\omega)$$

for some random variable $\widetilde{\zeta}$ on $(\Omega, \mathcal{A}, \mu)$.

We suppose that the functions $F(\xi', \omega)$, $p(\xi', \omega)$ and $q(\xi', \omega)$ appearing in the formulation of problem (1), are realizations of statistically homogeneous random fields, that is

$$F(\xi', \omega) = \widetilde{F}(T_{\xi'}\omega), \qquad p(\xi', \omega) = \widetilde{p}(T_{\xi'}\omega), \qquad q(\xi', \omega) = \widetilde{q}(T_{\xi'}\omega),$$

for all $\xi' \in \mathbb{R}^{d-1}$, where $\widetilde{F}$, $\widetilde{p}$ and $\widetilde{q}$ are random variables on $(\Omega, \mathcal{A}, \mu)$, and $T_{\xi'}$ is an ergodic $(d-1)$-dynamical system on Ω.

Moreover, we assume that $\widetilde{F}$ has, almost surely, continuously differentiable or locally Lipschitz realizations. We denote

$$\partial_\omega^i \widetilde{F}(\omega) = \partial_{\xi_i} \widetilde{F}(T_{\xi'}\omega)|_{\xi'=0}, \qquad \partial_\omega \widetilde{F}(\omega) = \nabla_{\xi'} \widetilde{F}(T_{\xi'}\omega)|_{\xi'=0}.$$

We have $\nabla_{\xi'} F(\xi', \omega) = \partial_\omega \widetilde{F}(T_{\xi'}\omega)$ (see, for instance, [29]).

Finally, we make the following assumptions on the functions $\widetilde{F}$, $\widetilde{p}$ and $\widetilde{q}$:

(h1) $\widetilde{F} \in L_\infty(\Omega)$, $\widetilde{F}(\omega) \leqslant 0$ a.s.;
(h2) $\partial_\omega \widetilde{F} \in (L_2(\Omega))^{d-1}$;
(h3) $\widetilde{p} \in L_\infty(\Omega)$, $\widetilde{p}(\omega) \geqslant 0$ a.s., $\mu\{\omega\colon \widetilde{p}(\omega) > 0\} > 0$;
(h4) $\widetilde{q} \in L_2(\Omega)$, $\widetilde{q}\partial_\omega \widetilde{F} \in (L_2(\Omega))^{d-1}$.

Also in a number of statements we assume that

(h2′) $\partial_\omega \widetilde{F} \in (L_2(\Omega))^{d-1}$ if $d < 5$; $\partial_\omega \widetilde{F} \in (L_{d/2}(\Omega))^{d-1}$ if $d \geqslant 5$.

Part of the results on the rate of convergence are obtained under the following condition

(h2″) $\partial_\omega \widetilde{F} \in (L_\infty(\Omega))^{d-1}$.

Remark 3.1. Notice that in the lower dimensions $d < 5$ conditions (h2) and (h2′) coincide.

4. Main results

In this section we introduce two possible homogenized problems and formulate the convergence results. The first of these problems takes the form

$$\begin{cases} -\Delta_x u_0(x) = f(x), & x \in D_0, \\ u_0(x) = 0, & x \in \Gamma_1, \\ \partial_\nu u_0(x) = 0, & x \in \gamma, \\ -\partial_{x_d} u_0(x) + \big(\mathbf{1}_{\tau=0}\theta(x')P(x') + \mathbf{1}_{\mu=1-\alpha}\Xi k_1\big)u_0(x) = \theta(x')Q(x') + \Xi k_2, & x \in I_0. \end{cases} \tag{3}$$

Here

$$
\begin{aligned}
&\mathbf{1}_{\varpi=\varsigma} = \begin{cases} 1 & \text{if } \varpi = \varsigma, \\ 0 & \text{otherwise,} \end{cases} \qquad \Xi = l|\partial B|, \\
&P(x') = \mathbb{E}\Big(\widetilde{p}(\omega)\sqrt{1 + \big(\theta(x')\partial_\omega \widetilde{F}(\omega)\big)^2}\Big)\big(1 - |B|\big), \\
&Q(x') = \mathbb{E}\Big(\widetilde{q}(\omega)\sqrt{1 + \big(\theta(x')\partial_\omega \widetilde{F}(\omega)\big)^2}\Big)\big(1 - |B|\big),
\end{aligned}
\tag{4}
$$

and the symbols $|B|$ and $|\partial B|$ stand for the $(d-1)$-dimensional volume of B and $(d-2)$-dimensional volume of the boundary ∂B, respectively. Problem (3) has a solution $u_0 \in H^1(D_0, \varGamma_1) = \{v \in H^1(D_0)\colon v|_{\varGamma_1} = 0\}$. The corresponding integral identity reads

$$
\begin{aligned}
&\int_{D_0} \nabla u_0(x)\nabla v(x)\,\mathrm{d}x + \int_{I_0} \big(\mathbf{1}_{\tau=0}\theta(x')P(x') + \mathbf{1}_{\mu=1-\alpha}\Xi k_1\big)u_0(x)v(x)\,\mathrm{d}x' \\
&\quad = \int_{D_0} f(x)v(x)\,\mathrm{d}x + \int_{I_0} \big(\theta(x')Q(x') + \Xi k_2\big)v(x)\,\mathrm{d}x'
\end{aligned}
\tag{5}
$$

for any function $v \in H^1(D_0, \varGamma_1)$.

The second homogenized problem reads

$$
\begin{cases} -\Delta_x u_0(x) = f(x), & x \in D_0, \\ u_0(x) = 0, & x \in \varGamma_1 \cup I_0, \\ \partial_\nu u_0(x) = 0, & x \in \gamma. \end{cases}
\tag{6}
$$

It has a solution $u_0 \in H^1(D_0, \varGamma_1 \cup I_0) = \{v \in H^1(D_0)\colon v|_{\varGamma_1 \cup I_0} = 0\}$. This solution u_0 satisfies the integral identity

$$
\int_{D_0} \nabla u_0(x)\nabla v(x)\,\mathrm{d}x = \int_{D_0} f(x)v(x)\,\mathrm{d}x
\tag{7}
$$

for any function $v \in H^1(D_0, \varGamma_1 \cup I_0)$.

By the standard regularity results for elliptic equations and thanks to the structure of the junction between I_0 and γ, the solution u_0 of problem (3) (respectively (6)) belongs to the space $H^2(D_0)$.

Since D_ε depends on ε, it is convenient to introduce a domain, say D^+, which contains all the domains D_ε, $\varepsilon \leqslant 1$. The existence of such a domain is assured by condition (h1).

Remark 4.1. Clearly, the function u_0 is not defined in the whole domain D_ε. Applying the technique of symmetric extension (see, e.g., [33,34,37]), allows us to extend u_0 into a larger domain D^+; we keep the same notation u_0 for the extended function. In particular, for all $\varepsilon \in (0, 1]$ we have $\|u_0\|_{H^2(D_\varepsilon)} \leqslant C\|u_0\|_{H^2(D_0)}$, where C does not depend on ε.

Theorem 4.1. *Let conditions* (h1)–(h4) *be fulfilled and assume that* $f \in L_2(D^+)$ *and* $F(\xi', \omega)$ *has, almost surely, continuously differentiable realizations. Then, almost surely for all sufficiently small* $\varepsilon > 0$, *problem* (1) *has a unique solution. Moreover, if* $\tau \geqslant 0$ *and* $\mu \geqslant 1 - \alpha$ *then* u_ε *converges to a solution of* *problem* (3) *and the following limit relations hold true*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u_0\|_{L_2(D_\varepsilon)} = 0 \quad a.s. \tag{8}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}\big(\|u_\varepsilon - u_0\|_{L_2(D_\varepsilon)}\big) = 0. \tag{9}$$

Under assumptions (h1), (h2′), (h3) *and* (h4), *we have a stronger convergence*:

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u_0\|_{H^1(D_\varepsilon)} = 0 \quad a.s. \tag{10}$$

and

$$\mathbb{E}\big(\|u_\varepsilon - u_0\|_{H^1(D_\varepsilon)}\big) \to 0, \tag{11}$$

as $\varepsilon \to 0$.

The rate of convergence of u_ε towards u_0 can be estimated under an additional mixing assumption on the random fields $F(\xi')$, $p(\xi')$ and $q(\xi')$. In order to introduce this assumption we first define the so-called maximum correlation coefficient.

For a bounded set A in $\mathbb{R}^{d-1}$, denote by σ_A the σ-algebra $\sigma\{F(\xi',\cdot), p(\xi',\cdot), q(\xi',\cdot)\colon\ \xi' \in A\}$, i.e. the σ-algebra generated in Ω by $F(\xi',\cdot), p(\xi',\cdot), q(\xi',\cdot)$, for $\xi' \in A$.

Definition 4.1. The *maximum correlation coefficient* $\rho(s)$, $s > 0$, of the random field (F, p, q) is defined by

$$\rho(s) = \sup_{\substack{A_1,A_2\subset\mathbb{R}^{d-1},\\ \mathrm{dist}(A_1,A_2)\geqslant s}}\ \sup_{\substack{\eta_1\in L_2(\Omega,\sigma_{A_1}),\\ \eta_2\in L_2(\Omega,\sigma_{A_2})}} \big|\mathbb{E}(\eta_1\eta_2)\big|,$$

where the second supremum is taken over all σ_{A_1}-measurable η_1 and σ_{A_2}-measurable η_2 such that $\mathbb{E}\eta_j = 0$ and $\mathbb{E}\{(\eta_j)^2\} = 1$, $j = 1, 2$.

We first consider the case $\tau \geqslant 0$ and $\mu \geqslant 1-\alpha$.

Theorem 4.2. *Let* $\tau \geqslant 0$ *and* $\mu \geqslant 1-\alpha$, *and assume that conditions* (h1)–(h4) *are fulfilled*, $F(\xi',\omega)$ *has almost surely continuously differentiable realizations*, *and* $f \in L_2(D^+)$ *with* $\mathrm{dist}(\mathrm{supp}(f), \Gamma_1) > 0$. *If*, *in addition*,

$$\int_0^\infty \rho(s)\,\mathrm{d}s < \infty, \tag{12}$$

then the following estimate holds true

$$\mathbb{E}\big(\|u_0 - u_\varepsilon\|_{H^1(D_\varepsilon)}\big) \leqslant K\big(\mathbf{1}_{\mu=1-\alpha}\varepsilon^{\mu-1+\alpha} + \varepsilon^{\alpha/2} + \varepsilon^{1/4} + \mathbf{1}_{\tau\neq 0}\varepsilon^{\tau}\big), \tag{13}$$

where u_ε *and* u_0 *solve problems* (1) *and* (3), *respectively*, *and the constant* K *does not depend on* ε.

If we replace (h2) *with the stronger condition* (h2″), *then estimate* (13) *holds for any* $f \in L_2(D^+)$.

In the case $\tau < 0$ or $\mu < 1 - \alpha$ the following statement holds.

Theorem 4.3. *Let $\tau < 0$ or $\mu < 1 - \alpha$, and assume that* (h1)–(h4) *are fulfilled, $F(\xi', \omega)$ has almost surely continuously differentiable realizations, and $f \in L_2(D^+)$. Then relations* (8) *and* (9) *remain valid with u_0 being a solution of problem* (6). *Furthermore, for the solutions u_ε and u_0 of problems* (1) *and* (6), *respectively, the following estimates take place*:

- *If $\tau \leqslant \mu - 1 + \alpha$ and $\tilde{p} \geqslant p_- > 0$, then*

$$\|u_\varepsilon - u_0\|_{L_2(D_\varepsilon)} \leqslant C\big(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\big), \tag{14}$$

- *If $\tau > \mu - 1 + \alpha$ and $k_1 > 0$, then*

$$\|u_\varepsilon - u_0\|_{L_2(D_\varepsilon)} \leqslant C\big(\varepsilon^{(1-\mu-\alpha)/2} + \varepsilon^{\alpha/2}\big). \tag{15}$$

Remark 4.2. The condition in (12) is fulfilled if the random field (F, p, q) has finite range of dependence.

5. Auxiliary statements

This section is devoted to various technical assertions, which will be used in the further analysis. Some of these assertions have been proved in [21] and [1]. We omit their proofs.

Lemma 5.1. *Almost surely, the inequalities*

$$\left\| v\left(x', \varepsilon\Theta\big(x'\big)F\left(\frac{x'}{\varepsilon}, \omega\right)\right) - v\big(x', 0\big)\right\|_{L_2(I_0)} \leqslant C_1\sqrt{\varepsilon}\|v\|_{H^1(D_\varepsilon)} \tag{16}$$

and

$$\|v\|_{L_2(\Pi_\varepsilon)} \leqslant C_2\sqrt{\varepsilon}\|v\|_{H^1(D_\varepsilon)} \tag{17}$$

hold for any function $v \in H^1(D_\varepsilon)$, with deterministic constants C_1 and C_2.
If $u \in H^2(D^+)$, then we have, for $d > 2$,

$$\left\| u\left(x', \varepsilon\Theta\big(x'\big)F\left(\frac{x'}{\varepsilon}, \omega\right)\right) - u\big(x', 0\big)\right\|_{L_{(2d/(d-2))}(I_0)} \leqslant C_3\varepsilon^{(d+2)/(2d)}\|u\|_{H^2(D^+)}, \tag{18}$$

with a deterministic constant C_3.

As a consequence of the previous lemma and the trace theorem we have

$$\left\| v\left(x', \varepsilon\Theta\big(x'\big)F\left(\frac{x'}{\varepsilon}, \omega\right)\right)\right\|_{L_2(I_0)} \leqslant C\|v\|_{H^1(D_\varepsilon)}$$

for $v \in H^1(D_\varepsilon)$ and

$$\left\| u\left(x', \varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon}, \omega\right)\right)\right\|_{L_{(2d/(d-2))}(I_0)} \leqslant C\|u\|_{H^2(D^+)} \tag{19}$$

for $u \in H^2(D^+)$ with a deterministic constant C which does not depend on ε.

When computing boundary integrals over Γ_ε, it is convenient to choose the coordinates $x' = (x_1, \dots, x_{d-1})$ on Γ_ε.

Lemma 5.2. *Let* $(\mathrm{d}\sigma_x)$ *be an element of* $(d-1)$*-dimensional volume of* Γ_ε. *Then, almost surely,*

$$\mathrm{d}\sigma_x = \sqrt{1 + \left|\Theta(x')\partial_\omega \widetilde{F}(T_{x'/\varepsilon}\omega)\right|^2}\, \mathrm{d}x'\big(1 + \mathrm{O}(\varepsilon)\big), \tag{20}$$

where $|\mathrm{O}(\varepsilon)| \leqslant C\varepsilon$ *with a deterministic constant* C.

By the Sobolev embedding theorem one has

$$\left|\int_{I_0} uv\, \mathrm{d}x'\right| \leqslant C_4\|u\|_{H^{1/2}(I_0)}\|v\|_{H^{1/2}(I_0)} \tag{21}$$

for all $u, v \in H^{1/2}(I_0)$.

The following result is a consequence of the Birkhoff ergodic theorem.

Lemma 5.3. *Let* $h(\xi', \omega)$, $\xi' \in \mathbb{R}^{d-1}$, *be a random statistically homogeneous function such that* $\|h(0, \cdot)\|_{L_\infty(\Omega)} < \infty$, *and assume that*

$$\mathbb{E}\big(h(\xi', \omega)\big) \equiv 0.$$

Then, almost surely,

$$\int_{I_0} h\left(\frac{x'}{\varepsilon}, \omega\right) u_\varepsilon(x') v_\varepsilon(x')\, \mathrm{d}x' \to 0, \tag{22}$$

as $\varepsilon \to 0$, *for any families* $u_\varepsilon, v_\varepsilon \in H^{1/2}(I_0)$ *such that* $\|u_\varepsilon\|_{H^{1/2}(I_0)} \leqslant C$ *and* $\|v_\varepsilon\|_{H^{1/2}(I_0)} \leqslant C$.

If $\widetilde{h}_0 : \Omega \mapsto \mathbb{R}^k$, $k \geqslant 1$, *is a random vector such that* $\widetilde{h}_0 \in (L_2(\Omega))^k$, *and a function* $\mathcal{R}(x', z) : I_0 \times \mathbb{R}^k \mapsto \mathbb{R}$ *has the following properties*:

$$\mathcal{R} \in C\big(I_0 \times \mathbb{R}^k\big), \quad \big|\mathcal{R}(x', \zeta)\big| \leqslant C\big(1 + |\zeta|\big) \tag{23}$$

for all $x' \in I_0$ *and* $\zeta \in \mathbb{R}^k$, *and*

$$\mathbb{E}\mathcal{R}\big(x', \widetilde{h}_0(\cdot)\big) = 0 \quad \textit{for each } x' \in \overline{I}_0, \tag{24}$$

then a.s.

$$\int_{I_0} \mathcal{R}\big(x', \widetilde{h}_0(T_{x'/\varepsilon}\omega)\big) v^\varepsilon\big(x'\big)\,\mathrm{d}x' \overset{\varepsilon \to 0}{\to} 0 \tag{25}$$

for any family $v^\varepsilon \in H^1(D_\varepsilon)$ *with* $\|v^\varepsilon\|_{H^1(D_\varepsilon)} \leqslant C$.

The proof (22) is similar to the proof of Lemma 5 in [21]. The convergence in (25) is proved in [1]. Using Lemma 5.3 we obtain the following one.

Lemma 5.4. *Almost surely, for any* $v^\varepsilon \in H^1(D_\varepsilon)$ *such that* $\|v^\varepsilon\|_{H^1(D_\varepsilon)} \leqslant C$ *and* $u \in C^\infty(\mathbb{R}^d)$, *as* $\varepsilon \to 0$, *the following limit relations hold*

$$\left|\int_{\Gamma_\varepsilon} \theta\big(x'\big) q\left(\frac{x'}{\varepsilon}, \omega\right) v^\varepsilon(x)\,\mathrm{d}\sigma_x - \int_{I_0} \theta\big(x'\big) Q\big(x'\big) v^\varepsilon\big(x', 0\big)\,\mathrm{d}x'\right| \to 0, \tag{26}$$

$$\left|\int_{\Gamma_\varepsilon} \theta\big(x'\big) p\left(\frac{x'}{\varepsilon}, \omega\right) v^\varepsilon(x) u(x)\,\mathrm{d}\sigma_x - \int_{I_0} \theta\big(x'\big) P\big(x'\big) v^\varepsilon\big(x', 0\big) u\big(x', 0\big)\,\mathrm{d}x'\right| \to 0 \tag{27}$$

with $P(x')$ *and* $Q(x')$ *defined in* (4).

We also need estimates for the trace of H^1 functions on Υ_ε.

Lemma 5.5. *For any functions* $v \in H^1(D_\varepsilon, \Gamma_1)$ *the following estimates are valid*

$$\varepsilon^{1-\alpha} \int_{\Upsilon_\varepsilon} v^2(x)\,\mathrm{d}\sigma_x \leqslant C_5 \varepsilon^\alpha \int_{G_\varepsilon} |\nabla v|^2\,\mathrm{d}x + C_6 \int_{I_0} v^2\big(x', 0\big)\,\mathrm{d}x' \tag{28}$$

and

$$\varepsilon^{1-\alpha} \int_{\Upsilon_\varepsilon} v^2(x)\,\mathrm{d}\sigma_x \leqslant C_7 \|v\|^2_{H^1(D_\varepsilon)}. \tag{29}$$

The proof of this lemma relies on trace and Friedrichs–Poincaré inequalities.

Similarly one can prove that for any $v \in H^1(D_\varepsilon, \Gamma_1)$ the estimate

$$\int_{I_0} \chi_\varepsilon\big(x'\big) v^2\big(x', 0\big)\,\mathrm{d}x' \leqslant C_8 \left(\varepsilon^\alpha \int_{G_\varepsilon} |\nabla v|^2\,\mathrm{d}x + \varepsilon^{1-\alpha} \int_{\Upsilon_\varepsilon} v^2(x)\,\mathrm{d}\sigma_x\right) \tag{30}$$

holds true, where $\chi_\varepsilon(x') = \chi(\frac{x'}{\varepsilon})$, and $\chi(\xi), \xi \in \mathbb{R}^{d-1}$, is 1-periodic in $\xi_1, \xi_2, \dots, \xi_{d-1}$ function defined as:

$$\chi(\xi) = \begin{cases} 1, & \xi \in B, \\ 0, & \xi \in \square \setminus \overline{B}, \end{cases} \qquad k = 1, \dots, K_1. \tag{31}$$

Obviously (see, for instance, [22,44]), the estimate

$$\big|\big(\chi_\varepsilon - |B|, v\big)_{L_2(\square)}\big| \leqslant C\sqrt{\varepsilon}\|v\|_{H^{1/2}(\square)}$$

holds for any $v \in H^{1/2}(\square)$.

Reasoning as in the proof of Lemma 1 from [21] yields the following lemma.

Lemma 5.6. *The inequality*

$$\|v\|_{L_2(G_\varepsilon)} \leqslant C\varepsilon^{\alpha/2}\|v\|_{H^1(D_\varepsilon)}$$

holds true for any function $v \in H^1(D_\varepsilon)$.

Proposition 5.1. *Let* u_ε *be a solution to problem* (1). *Then under assumptions* (h1)–(h4), *there exists* $C > 0$ *such that, almost surely for all sufficiently small* $\varepsilon > 0$, *the following estimates hold*

$$\|u_\varepsilon\|_{H^1(D_\varepsilon)} \leqslant C \tag{32}$$

and

$$\mathbb{E}\big(\|u_\varepsilon\|_{H^1(D_\varepsilon)}\big) \leqslant C. \tag{33}$$

Proof. The estimate (32) easily follows from the uniform coerciveness of the quadratic form of problem (1) in the space $H^1(D_\varepsilon, \varGamma_1)$. Taking the expectation of both sides in (32) yields (33). □

Suppose that $\phi \in H^1(D_\varepsilon^{(1)})$ with $D_\varepsilon^{(1)} = I_0 \times (-\varepsilon^\alpha l, 0)$. Then for almost all $t \in [-\varepsilon^\alpha l, 0]$ one can prove the estimate

$$\left\|\phi(\cdot, t) - \frac{1}{\varepsilon^\alpha l}\int_{-\varepsilon^\alpha l}^{0} \phi(\cdot, x_d)\,\mathrm{d}x_d\right\|_{L_2(I_0)} \leqslant C\varepsilon^{\alpha/2}\|\partial_{x_d}\phi\|_{L_2(D_\varepsilon^{(1)})}. \tag{34}$$

Indeed,

$$\begin{aligned}
&\int_{I_0}\left(\phi(x', t) - \frac{1}{\varepsilon^\alpha l}\int_{-\varepsilon^\alpha l}^{0} \phi(x', x_d)\,\mathrm{d}x_d\right)^2 \mathrm{d}x' \\
&\quad= \frac{1}{(\varepsilon^\alpha l)^2}\int_{I_0}\left(\int_{-\varepsilon^\alpha l}^{0}\big(\phi(x', t) - \phi(x', x_d)\big)\,\mathrm{d}x_d\right)^2 \mathrm{d}x' \\
&\quad\leqslant \frac{1}{\varepsilon^\alpha l}\int_{I_0}\int_{-\varepsilon^\alpha l}^{0}\big(\phi(x', t) - \phi(x', x_d)\big)^2\,\mathrm{d}x_d\,\mathrm{d}x' \\
&\quad= \frac{1}{\varepsilon^\alpha l}\int_{I_0}\int_{-\varepsilon^\alpha l}^{0}\left(\int_{x_d}^{t}\partial_{x_d}\phi(x', y)\,\mathrm{d}y\right)^2 \mathrm{d}x_d\,\mathrm{d}x' \\
&\quad\leqslant \int_{I_0}\int_{-\varepsilon^\alpha l}^{0}\int_{-\varepsilon^\alpha l}^{0}\big(\partial_{x_d}\phi(x', y)\big)^2\,\mathrm{d}y\,\mathrm{d}x_d\,\mathrm{d}x' \\
&\quad= \varepsilon^\alpha l\int_{D_\varepsilon^{(1)}}\big(\partial_{x_d}\phi(x', x_d)\big)^2\,\mathrm{d}x.
\end{aligned}$$

Lemma 5.7. *The following estimates hold for all sufficiently small* $\varepsilon > 0$:

$$\left|\varepsilon^{1-\alpha}\int_{\Upsilon_\varepsilon}\varphi(x)\,\mathrm{d}\sigma_x - \Xi\int_{I_0}\varphi\big(x'\big)\,\mathrm{d}x'\right| \leqslant C\varepsilon^{\alpha/2}\|\varphi\|_{H^1(D_\varepsilon)} \quad \forall\varphi\in H^1(D_\varepsilon),$$
$$\left|\varepsilon^{1-\alpha}\int_{\Upsilon_\varepsilon}\varphi(x)\psi(x)\,\mathrm{d}\sigma_x - \Xi\int_{I_0}\varphi\big(x'\big)\psi\big(x'\big)\,\mathrm{d}x'\right| \tag{35}$$
$$\leqslant C\varepsilon^{\alpha/2}\|\varphi\|_{H^1(D_\varepsilon)}\|\psi\|_{H^1(D_\varepsilon)} \quad \forall\varphi,\psi\in H^1(D_\varepsilon),$$

where $\Xi = l|\partial B|$.

The following statement relies on Lemma 7.1 in [1].

Lemma 5.8. *Assume that* $H(\xi')$, $\xi'\in\mathbb{R}^{d-1}$, *is* 1*-periodic in* ξ' *function with values in* $\mathbb{R}^k$. *Let* $h(\xi',\omega) = \widetilde{h}(T_{\xi'}\omega)$ *be a statistically homogeneous random field with values in* $\mathbb{R}^k$, *and suppose that condition* (12) *is fulfilled. Then, given a smooth function* $\mathcal{R}(x', z_1, z_2)$, $x'\in I_0$, $z_1\in\mathbb{R}^k$, $z_2\in\mathbb{R}^k$, *such that*

$$\big\|\mathcal{R}\big(x',\widetilde{h}(\cdot),z_2\big)\big\|_{L_2(\Omega)} \leqslant C,$$
$$\int_{[0,1]^{d-1}}\mathbb{E}\mathcal{R}\big(x',\widetilde{h}(\cdot),H\big(\xi'\big)\big)\,\mathrm{d}\xi' = 0, \quad \textit{for all } x'\in I_0,$$

we have

$$\mathbb{E}\left(\sup_{\|\varpi\|_{H^{1/2}(I_0)}=1}\left|\int_{I_0}\mathcal{R}\left(x',h\left(\frac{x}{\varepsilon},\omega\right),H\left(\frac{x}{\varepsilon}\right)\right)\varpi(x)\,\mathrm{d}x'\right|^2\right) \leqslant C\sqrt{\varepsilon}, \tag{36}$$

with a constant C *that does not depend on* ε.

6. Convergence and estimates

Proof of Theorem 4.1. The existence and uniqueness of u_ε follow from the coercivity of the quadratic form of problem (1) and the Lax–Milgram lemma (see [22] for further details).

• The case $\tau\geqslant 0$ and $\mu\geqslant 1-\alpha$.
If $\mu = 1-\alpha$ and $\tau = 0$, then after simple transformations we find

$$\int_{D_\varepsilon}\nabla(u_0-u_\varepsilon)\nabla v\,\mathrm{d}x + \int_{\Gamma_\varepsilon}\theta p(u_0-u_\varepsilon)v\,\mathrm{d}\sigma_x + \varepsilon^{1-\alpha}k_1\int_{\Upsilon_\varepsilon}(u_0-u_\varepsilon)v\,\mathrm{d}\sigma_x$$
$$= \int_{D_\varepsilon}\nabla u_0\nabla v\,\mathrm{d}x - \int_{D_\varepsilon}fv\,\mathrm{d}x - \int_{\Gamma_\varepsilon}\theta qv\,\mathrm{d}\sigma_x - \varepsilon^{1-\alpha}\int_{\Upsilon_\varepsilon}k_2v\,\mathrm{d}\sigma_x$$
$$+ \int_{\Gamma_\varepsilon}\theta pu_0v\,\mathrm{d}\sigma_x + \varepsilon^{1-\alpha}k_1\int_{\Upsilon_\varepsilon}u_0v\,\mathrm{d}\sigma_x$$

$$= \int_{D_\varepsilon\setminus D_0} \nabla u_0 \nabla v \,\mathrm{d}x - \int_{D_\varepsilon\setminus D_0} fv\,\mathrm{d}x + \int_{I_0} (\theta Q + \varXi k_2) v\,\mathrm{d}x' - \int_{I_0} (\theta P + \varXi k_1) u_0 v\,\mathrm{d}x'$$
$$- \int_{\Gamma_\varepsilon} \theta q v\,\mathrm{d}\sigma_x - \varepsilon^{1-\alpha} \int_{\Upsilon_\varepsilon} k_2 v\,\mathrm{d}\sigma_x + \int_{\Gamma_\varepsilon} \theta p u_0 v\,\mathrm{d}\sigma_x + \varepsilon^{1-\alpha} k_1 \int_{\Upsilon_\varepsilon} u_0 v\,\mathrm{d}\sigma_x. \tag{37}$$

Here u_0 is a solution of problem (3).

Let us estimate all the terms on the right-hand side of the last relation. By Lemmas 5.5 and 5.6 bearing in mind the smoothness of u_0, we have

$$\begin{aligned}
\left|\int_{D_\varepsilon\setminus D_0} \nabla u_0 \nabla v\,\mathrm{d}x\right| &\leqslant \left|\int_{\varPi_\varepsilon} \nabla u_0 \nabla v\,\mathrm{d}x\right| + \left|\int_{G_\varepsilon} \nabla u_0 \nabla v\,\mathrm{d}x\right| \\
&\leqslant \|\nabla u_0\|_{L_2(\varPi_\varepsilon)} \|v\|_{H^1(D_\varepsilon)} + \|\nabla u_0\|_{L_2(G_\varepsilon)} \|v\|_{H^1(D_\varepsilon)} \\
&\leqslant C\bigl(\sqrt{\varepsilon}\|u_0\|_{H^2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)} + \varepsilon^{\alpha/2}\|u_0\|_{H^2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)}\bigr) \\
&\leqslant C\varepsilon^{\alpha/2}\|u_0\|_{H^2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)}
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\left|\int_{D_\varepsilon\setminus D_0} fv\,\mathrm{d}x\right| &\leqslant \left|\int_{\varPi_\varepsilon} fv\,\mathrm{d}x\right| + \left|\int_{G_\varepsilon} fv\,\mathrm{d}x\right| \\
&\leqslant C\bigl(\sqrt{\varepsilon}\|f\|_{L_2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)} + \varepsilon^{\alpha/2}\|f\|_{L_2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)}\bigr) \\
&\leqslant C\varepsilon^{\alpha/2}\|f\|_{L_2(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)}.
\end{aligned} \tag{39}$$

Then, according to Lemma 5.4, as $\varepsilon \to 0$, almost surely we have

$$\left|\int_{\Gamma_\varepsilon} \theta q v\,\mathrm{d}\sigma_x - \int_{I_0} \theta Q v\,\mathrm{d}x'\right| \to 0 \tag{40}$$

and

$$\left|\int_{\Gamma_\varepsilon} \theta p u_0 v\,\mathrm{d}\sigma_x - \int_{I_0} \theta P u_0 v\,\mathrm{d}x'\right| \to 0$$

for any $v \in C^\infty(\mathbb{R}^d)$. By Lemma 5.7 the inequalities hold

$$\left|\varepsilon^{1-\alpha}\int_{\Upsilon_\varepsilon} k_2 v\,\mathrm{d}\sigma_x - \varXi\int_{I_0} k_2 v\,\mathrm{d}x'\right| \leqslant C\varepsilon^{\alpha/2}\|v\|_{H^1(D_\varepsilon)} \tag{41}$$

and

$$\left|\varepsilon^{1-\alpha}k_1\int_{\Upsilon_\varepsilon} u_0 v\,\mathrm{d}\sigma_x - \varXi k_1\int_{I_0} u_0 v\,\mathrm{d}x'\right| \leqslant C\varepsilon^{\alpha/2}\|u_0\|_{H^1(D_\varepsilon)}\|v\|_{H^1(D_\varepsilon)}.$$

It follows from Proposition 5.1 that, for a subsequence $\varepsilon_k \to 0$, we have $u_{\varepsilon_k} \to \widehat{u}$ weakly in $H^1(D_0)$, as $k \to \infty$. This implies that, for any $v \in C^\infty(\mathbb{R}^d)$,

$$\lim_{k\to\infty} \int_{D_{\varepsilon_k}} (\nabla u_0 - \nabla u_{\varepsilon_k})\nabla v \,\mathrm{d}x = \int_{D_0} \nabla(u_0 - \widehat{u})\nabla v \,\mathrm{d}x.$$

Passing to the limit, as $k \to \infty$, on both sides of (37) and exploiting Lemmata 5.4 and 5.7, we conclude that, for any $v \in C^\infty(\mathbb{R}^d) \cap H^1(D_0, \varGamma_1)$,

$$\int_{D_0} (\nabla u_0 - \nabla\widehat{u})\nabla v \,\mathrm{d}x + \int_{I_0} \theta P\big(x'\big)(u_0 - \widehat{u})v \,\mathrm{d}x' + \int_{I_0} \varXi k_1(u_0 - \widehat{u})v \,\mathrm{d}x' = 0.$$

By density arguments the last relation also holds true for any $v \in H^1(D_0, \varGamma_1)$. This implies that $u_0 = \widehat{u}$. Therefore, a.s. the whole family u_ε converges to u_0 weakly in $H^1(D_0)$, and (8) follows from the Rellich–Kondrashov theorem (see, for instance, [22]).

The convergence (9) follows from (8).

• The case $\mu > 1 - \alpha$ and $\tau > 0$. In this case, after simple transformations, we derive from (2) and (5) that

$$\begin{aligned}
&\int_{D_\varepsilon} \nabla(u_0 - u_\varepsilon)\nabla v \,\mathrm{d}x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p(u_0 - u_\varepsilon)v \,\mathrm{d}\sigma_x + \varepsilon^\mu k_1 \int_{\varUpsilon_\varepsilon} (u_0 - u_\varepsilon)v \,\mathrm{d}\sigma_x \\
&\quad = \int_{D_\varepsilon} \nabla u_0 \nabla v \,\mathrm{d}x - \int_{D_\varepsilon} f v \,\mathrm{d}x - \int_{\varGamma_\varepsilon} \theta q v \,\mathrm{d}\sigma_x \\
&\qquad - \varepsilon^{1-\alpha} \int_{\varUpsilon_\varepsilon} k_2 v \,\mathrm{d}\sigma_x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p u_0 v \,\mathrm{d}\sigma_x + \varepsilon^\mu k_1 \int_{\varUpsilon_\varepsilon} u_0 v \,\mathrm{d}\sigma_x \\
&\quad = \int_{D_\varepsilon \setminus D_0} \nabla u_0 \nabla v \,\mathrm{d}x - \int_{D_\varepsilon \setminus D_0} f v \,\mathrm{d}x + \int_{I_0} (\theta Q + \varXi k_2)v \,\mathrm{d}x' \\
&\qquad - \int_{\varGamma_\varepsilon} \theta q v \,\mathrm{d}\sigma_x - \varepsilon^{1-\alpha} \int_{\varUpsilon_\varepsilon} k_2 v \,\mathrm{d}\sigma_x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p u_0 v \,\mathrm{d}\sigma_x + \varepsilon^\mu k_1 \int_{\varUpsilon_\varepsilon} u_0 v \,\mathrm{d}\sigma_x.
\end{aligned} \tag{42}$$

Clearly, estimates (38)–(39), (41), as well as relation (40), remain valid. By Lemma 5.2 and Proposition 3.1, for any $v \in C^\infty(\mathbb{R}^d)$, we have a.s.

$$\varepsilon^\tau \left| \int_{\varGamma_\varepsilon} p(u_0 - u_\varepsilon)v \,\mathrm{d}\sigma_x \right| \leqslant C\varepsilon^\tau \int_{I_0} p|u_0 - u_\varepsilon||v| \,\mathrm{d}x' \leqslant C\varepsilon^\tau \|u_0 - u_\varepsilon\|_{H^1(D_\varepsilon)}$$

with a constant C that might depend on ω. Also, by Lemma 5.5 we have

$$\varepsilon^{\mu-1+\alpha} \left| \varepsilon^{1-\alpha} k_1 \int_{\varUpsilon_\varepsilon} (u_0 - u_\varepsilon)v \,\mathrm{d}\sigma_x \right| \leqslant C\varepsilon^{\mu-1+\alpha} \|u_0 - u_\varepsilon\|_{H^1(D_\varepsilon)}.$$

Combining the above relations we obtain that, for any $v \in C^\infty(\mathbb{R}^d) \cap H^1(D_0, \varGamma_1)$,

$$\left| \int_{D_\varepsilon} \nabla(u_0 - u_\varepsilon) \nabla v \, \mathrm{d}x \right| \to 0,$$

as $\varepsilon \to 0$. As in the previous case this yields (8) and (9).

The cases $\mu = 1 - \alpha$, $\tau > 0$ and $\mu > 1 - \alpha$, $\tau = 0$ can be considered in a similar way.

We now turn to the H^1 convergence (10). We choose $v = (u_0 - u_\varepsilon)$ as a test function in (37) (respectively (42)). The resulting relation reads

$$\begin{aligned}
&\int_{D_\varepsilon} \left|\nabla(u_0 - u_\varepsilon)\right|^2 \mathrm{d}x + \int_{\varGamma_\varepsilon} \theta p (u_0 - u_\varepsilon)^2 \, \mathrm{d}\sigma_x + \varepsilon^{1-\alpha} k_1 \int_{\varUpsilon_\varepsilon} (u_0 - u_\varepsilon)^2 \, \mathrm{d}\sigma_x \\
&\quad = \int_{D_\varepsilon \setminus D_0} \nabla u_0 \nabla (u_0 - u_\varepsilon) \, \mathrm{d}x - \int_{D_\varepsilon \setminus D_0} f (u_0 - u_\varepsilon) \, \mathrm{d}x + \int_{I_0} (\theta Q + \varXi k_2)(u_0 - u_\varepsilon) \, \mathrm{d}x' \\
&\qquad - \int_{I_0} (\theta P + \varXi k_1) u_0 (u_0 - u_\varepsilon) \, \mathrm{d}x' - \int_{\varGamma_\varepsilon} \theta q (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x \\
&\qquad - \varepsilon^{1-\alpha} \int_{\varUpsilon_\varepsilon} k_2 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x + \int_{\varGamma_\varepsilon} \theta p u_0 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x + \varepsilon^{1-\alpha} k_1 \int_{\varUpsilon_\varepsilon} u_0 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x
\end{aligned} \tag{43}$$

and, for (42), we have

$$\begin{aligned}
&\int_{D_\varepsilon} \left|\nabla(u_0 - u_\varepsilon)\right|^2 \mathrm{d}x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p (u_0 - u_\varepsilon)^2 \, \mathrm{d}\sigma_x + \varepsilon^{\mu} k_1 \int_{\varUpsilon_\varepsilon} (u_0 - u_\varepsilon)^2 \, \mathrm{d}\sigma_x \\
&\quad = \int_{D_\varepsilon \setminus D_0} \nabla u_0 \nabla (u_0 - u_\varepsilon) \, \mathrm{d}x - \int_{D_\varepsilon \setminus D_0} f (u_0 - u_\varepsilon) \, \mathrm{d}x + \int_{I_0} (\theta Q + \varXi k_2)(u_0 - u_\varepsilon) \, \mathrm{d}x' \\
&\qquad - \int_{\varGamma_\varepsilon} \theta q (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x - \varepsilon^{1-\alpha} \int_{\varUpsilon_\varepsilon} k_2 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p u_0 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x \\
&\qquad + \varepsilon^{\mu} k_1 \int_{\varUpsilon_\varepsilon} u_0 (u_0 - u_\varepsilon) \, \mathrm{d}\sigma_x.
\end{aligned} \tag{44}$$

We proceed with estimating the terms on the right-hand side of (43). First, since $u_0 \in H^2(D_\varepsilon)$, in view of Lemma 5.6, we have

$$\left| \int_{D_\varepsilon \setminus D_0} \nabla u_0 \nabla (u_0 - u_\varepsilon) \, \mathrm{d}x \right| \leqslant C \varepsilon^{\alpha/2} \tag{45}$$

for sufficiently small ε. Similarly,

$$\left| \int_{D_\varepsilon \setminus D_0} f (u_0 - u_\varepsilon) \, \mathrm{d}x \right| \leqslant C \varepsilon^{\alpha/2}. \tag{46}$$

By Lemma 5.7 and Proposition 5.1, the inequalities hold

$$\left|\varepsilon^{1-\alpha}\int_{\Upsilon_\varepsilon} k_2(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \Xi\int_{I_0} k_2(u_0-u_\varepsilon)\,\mathrm{d}x'\right| \leqslant C\varepsilon^{\alpha/2}$$

and

$$\left|\varepsilon^{1-\alpha}k_1\int_{\Upsilon_\varepsilon} u_0(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \Xi k_1\int_{I_0} u_0(u_0-u_\varepsilon)\,\mathrm{d}x'\right| \leqslant C\varepsilon^{\alpha/2}.$$

Denote

$$\mathcal{J}_\varepsilon = \int_{\Gamma_\varepsilon}\theta p u_0(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \int_{I_0}\theta P u_0(u_0-u_\varepsilon)\,\mathrm{d}x'.$$

Our next goal is to show that almost surely

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon = 0. \tag{47}$$

Remark 6.1. Similar convergence has been studied in [1].

For the sake of brevity we introduce the following notation:

$$\begin{aligned} &U_0(x') = u_0\left(x',\varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon},\omega\right)\right), \qquad U_\varepsilon(x') = u_\varepsilon\left(x',\varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon},\omega\right)\right),\\ &\mathcal{S}^\varepsilon(x') = \left(1+\Theta^2(x')\left|\partial_\omega\widetilde{F}(T_{x'/\varepsilon}\omega)\right|^2\right)^{1/2}. \end{aligned} \tag{48}$$

Notice that U_0 does depend on ε. We represent $\mathcal{J}_\varepsilon$ as the sum of four terms:

$$\begin{aligned} \mathcal{J}_\varepsilon = &\left(\int_{\Gamma_\varepsilon}\theta p u_0(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \int_{I_0\setminus B_\varepsilon^0}\theta p U_0(U_0-U_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x'\right)\\ &+\left(\int_{I_0\setminus B_\varepsilon^0}\theta p U_0(U_0-U_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x' - \int_{I_0\setminus B_\varepsilon^0}\theta p u_0(U_0-U_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x'\right)\\ &+\left(\int_{I_0\setminus B_\varepsilon^0}\theta p u_0(U_0-U_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x' - \int_{I_0\setminus B_\varepsilon^0}\theta p u_0(u_0-u_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x'\right)\\ &+\left(\int_{I_0\setminus B_\varepsilon^0}\theta p u_0(u_0-u_\varepsilon)\mathcal{S}^\varepsilon\,\mathrm{d}x' - \int_{I_0}\theta P u_0(u_0-u_\varepsilon)\,\mathrm{d}x'\right). \end{aligned}$$

Let us show that each of these terms vanishes as $\varepsilon\to 0$. By (19), (20), assumption (h2′) and the Hölder inequality, a.s. for sufficiently small ε we have

$$\begin{aligned} \int_{\Gamma_\varepsilon} p u_0^2\,\mathrm{d}\sigma_x &\leqslant 2\int_{I_0} p\left(1+\Theta^2(x')\left|\partial_\omega\widetilde{F}(T_{x'/\varepsilon}\omega)\right|^2\right)^{1/2} u_0^2\left(x',\varepsilon\Theta(x')F\left(\frac{x'}{\varepsilon},\omega\right)\right)\mathrm{d}x'\\ &\leqslant C\|\tilde{p}\|_{L_\infty(\Omega)}\left(1+\|\partial_\omega F\|_{L_{d/2\vee 2}(\Omega)}\right)\|u_0\|^2_{H^2(D^+)} \leqslant C_9\|u_0\|^2_{H^2(D^+)} \end{aligned} \tag{49}$$

with deterministic constants C and C_9; here we denote $\frac{d}{2} \vee 2 = \max(\frac{d}{2}, 2)$. From (2) and (32) it is easy to deduce that a.s.

$$\int_{\Gamma_\varepsilon} \theta p (u_\varepsilon)^2 \,\mathrm{d}\sigma_x \leqslant C.$$

Combining the last two inequalities we conclude that almost surely for sufficiently small ε the following estimate holds

$$\int_{\Gamma_\varepsilon} \theta p (u_0 - u_\varepsilon)^2 \,\mathrm{d}\sigma_x \leqslant C \tag{50}$$

with a deterministic constant C. From (49), (50) and (20) we get

$$\left| \int_{\Gamma_\varepsilon} \theta p u_0 (u_0 - u_\varepsilon) \,\mathrm{d}\sigma_x - \int_{I_0 \setminus B_\varepsilon^0} \theta p U_0 (U_0 - U_\varepsilon) \mathcal{S}^\varepsilon\big(x'\big) \,\mathrm{d}x' \right| \leqslant C\varepsilon.$$

Considering (49), (50) and (h2$'$), by Lemma 5.1 and the Hölder inequality we obtain

$$\begin{aligned}
&\left| \int_{I_0 \setminus B_\varepsilon^0} \theta p U_0 (U_0 - U_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' - \int_{I_0 \setminus B_\varepsilon^0} \theta p u_0 (U_0 - U_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' \right| \\
&\quad \leqslant \int_{I_0 \setminus B_\varepsilon^0} |U_0 - u_0| \big(\sqrt{\theta p \mathcal{S}^\varepsilon} |U_0 - U_\varepsilon| \big) \sqrt{\theta p \mathcal{S}^\varepsilon} \,\mathrm{d}x' \\
&\quad \leqslant C \|U_0 - u_0\|_{L_{(2d/(d-2))}(I_0)} \leqslant C\varepsilon^{(d+2)/(2d)} \leqslant C\sqrt{\varepsilon}.
\end{aligned}$$

Notice that Lemma 5.1 applies here since u_0 is extended in D^+. For $d = 2$ the desired inequality follows from the Hölder continuity of u_0.

In order to show that, almost surely,

$$\lim_{\varepsilon \to 0} \left| \int_{I_0 \setminus B_\varepsilon^0} \theta p u_0 (U_0 - U_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' - \int_{I_0 \setminus B_\varepsilon^0} \theta p u_0 (u_0 - u_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' \right| = 0 \tag{51}$$

we observe that, by the Sobolev embedding and trace theorems, $u_0 \in L_{2(d-1)/(d-4)}(I_0)$. Due to assumption (h2$'$) this implies that $\|\mathcal{S}^\varepsilon u_0\|_{L_2(I_0)} \leqslant C$ a.s. Therefore,

$$\left| \int_{I_0 \setminus B_\varepsilon^0} \theta p u_0 (U_0 - U_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' - \int_{I_0 \setminus B_\varepsilon^0} \theta p u_0 (u_0 - u_\varepsilon) \mathcal{S}^\varepsilon \,\mathrm{d}x' \right| \leqslant C\sqrt{\varepsilon};$$

here we have also used (16) and (32).

It is easy to see that

$$\int_{I_0 \setminus B_\varepsilon^0} \theta p \mathcal{S}^\varepsilon u_0 (u_0 - u_\varepsilon) \,\mathrm{d}x' = \int_{I_0} \theta p \mathcal{S}^\varepsilon u_0 (u_0 - u_\varepsilon)(1 - \chi_\varepsilon) \,\mathrm{d}x',$$

where $\chi_\varepsilon(x') = \chi(\frac{x'}{\varepsilon})$ and χ is defined in (31). Now, in order to prove (47) it remains to show that

$$\lim_{\varepsilon\to 0}\int_{I_0}\big(\theta p(1-\chi_\varepsilon)\mathcal{S}^\varepsilon - P\big)u_0(u_0-u_\varepsilon)\,\mathrm{d}x' = 0. \tag{52}$$

This convergence follows from the Birkhoff ergodic theorem. Indeed, by the definition of P we have

$$\mathbb{E}\big\{\theta\big(x'\big)\tilde{p}\big(1-|B|\big)\sqrt{1+\Theta^2\big(x'\big)|\partial_\omega\tilde{F}|^2} - P\big(x'\big)\big\} = 0,$$

for any $(x', 0) \in I_0$. Under assumption (h2$'$) this implies that the function $(\theta p(1-\chi_\varepsilon)\mathcal{S}^\varepsilon - P)$ converges almost surely to zero weakly in $L_{2\vee d/2}(I_0)$. Since $u_0 \in H^2(D^+)$ and $(u_0 - u_\varepsilon)$ is bounded in $H^{1/2}(I_0)$, the family $u_0(u_0-u_\varepsilon)$ is compact in $L_{2\wedge d/(d-2)}(I_0)$; here $2 \wedge \frac{d}{d-2} = \min(2, \frac{d}{d-2})$. This yields (52).

Combining now (45)–(47), we arrive at the conclusion that all the terms on the right-hand side of (43) almost surely tend to zero, as $\varepsilon \to 0$. This yields

$$\lim_{\varepsilon\to 0}\left|\int_{D_\varepsilon}\big|\nabla(u_0-u_\varepsilon)\big|^2\,\mathrm{d}x + \int_{\Gamma_\varepsilon} p(u_0-u_\varepsilon)^2\,\mathrm{d}\sigma_x + \varepsilon^{1-\alpha}k_1\int_{\Upsilon_\varepsilon}(u_0-u_\varepsilon)^2\,\mathrm{d}\sigma_x\right| = 0,$$

and by the uniform coercivity of the quadratic form of problem (1) we derive that a.s.

$$\lim_{\varepsilon\to 0}\|u_0-u_\varepsilon\|_{H^1(D^\varepsilon)} = 0.$$

The convergence (11) is a direct consequence of (10).

For $\tau > 0$ and $\mu > 1-\alpha$ the desired convergence can be justified by similar arguments. □

Proof of Theorem 4.2. Consider the case $\mu = 1-\alpha$ and $\tau = 0$. Since f vanishes in the vicinity of I_0, the solution u_0 is smooth in a sufficiently small neighborhood of I_0 and thus has a smooth extension in D^+; as above, we keep the same notation u_0 for the extended function.

Consider the relation (43). Recalling (48) one can derive from (16) and Lemma 5.2, that

$$\left|\int_{\Gamma_\varepsilon}\theta q(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \int_{I_0}\theta q(1-\chi_\varepsilon)\mathcal{S}^\varepsilon(u_0-u_\varepsilon)\,\mathrm{d}x'\right| \leqslant C\sqrt{\varepsilon}\|u_0-u_\varepsilon\|_{H^1(D_\varepsilon)}. \tag{53}$$

Taking into account the C^∞-smoothness of u_0 in the vicinity of I_0, one also has

$$\left|\int_{\Gamma_\varepsilon}\theta p u_0(u_0-u_\varepsilon)\,\mathrm{d}\sigma_x - \int_{I_0}\theta p(1-\chi_\varepsilon)\mathcal{S}^\varepsilon u_0(u_0-u_\varepsilon)\,\mathrm{d}x'\right| \leqslant C\sqrt{\varepsilon}\|u_0-u_\varepsilon\|_{H^1(D_\varepsilon)}. \tag{54}$$

Now, for $z_1 \in \mathbb{R}^{d-1}$ and $z_2 \in \mathbb{R}$, let us define

$$\mathcal{R}\big(x', z_1, z_2\big) = \theta\big(x'\big)z_2\sqrt{1+\big(\Theta\big(x'\big)\big)^2 z_1^2}\big(1-|B|\big) - \theta\big(x'\big)Q\big(x'\big).$$

It is easy to check that the function $\mathcal{R}(x', \partial_\omega \tilde{F}, \tilde{q})$ satisfies the assumptions of Lemma 5.8. Therefore, considering the boundedness of $(u_0 - u_\varepsilon)$ in $H^{1/2}(I_0)$, we have

$$\mathbb{E}\left\{\left|\int_{I_0} \theta\big(Q - q(1-\chi_\varepsilon)\mathcal{S}^\varepsilon\big)(u_0 - u_\varepsilon)\,\mathrm{d}x'\right|\right\} \leqslant C\sqrt{\varepsilon}. \tag{55}$$

Similarly, in view of the smoothness of u_0 in the neighborhood of I_0, we obtain

$$\mathbb{E}\left\{\left|\int_{I_0} \theta\big(p(1-\chi_\varepsilon)\mathcal{S}^\varepsilon - P\big)u_0(u_0 - u_\varepsilon)\,\mathrm{d}x'\right|\right\} \leqslant C\sqrt{\varepsilon}. \tag{56}$$

Combining (53)–(56) with the estimates obtained in the proof of Theorem 4.1, yields

$$\begin{aligned}
&\mathbb{E}\int_{D_\varepsilon} \big|\nabla u_\varepsilon(x) - \nabla u_0(x)\big|^2\,\mathrm{d}x + \mathbb{E}\int_{\varGamma_\varepsilon} \theta p\left(\frac{x'}{\varepsilon}, \omega\right)\big(u_\varepsilon(x) - u_0(x)\big)^2\,\mathrm{d}\sigma_x \\
&\quad + \varepsilon^{1-\alpha}k_1\mathbb{E}\int_{\varUpsilon_\varepsilon} \big(u_\varepsilon(x) - u_0(x)\big)^2\,\mathrm{d}\sigma_x \leqslant C\varepsilon^{\alpha/2}\mathbb{E}\|u_\varepsilon - u_0\|_{H^1(D_\varepsilon)} + C\sqrt{\varepsilon}.
\end{aligned}$$

Thanks to the coercivity of the quadratic form of problem (1) this implies the bound (13).

The case $\mu > 1 - \alpha$ and $\tau > 0$ can be treated in a similarly way. The theorem is proved. □

Proof of Theorem 4.3. Substituting u_ε for v in (2) and using Proposition 5.1 and Lemma 5.7, we get

$$\int_{D_\varepsilon} \big|\nabla u_\varepsilon(x)\big|^2\,\mathrm{d}x + \varepsilon^\tau \int_{\varGamma_\varepsilon} \theta p\left(\frac{x'}{\varepsilon}, \omega\right)u_\varepsilon^2(x)\,\mathrm{d}\sigma_x + \varepsilon^\mu k_1 \int_{\varUpsilon_\varepsilon} u_\varepsilon^2(x)\,\mathrm{d}\sigma_x \leqslant C. \tag{57}$$

If $\tau \leqslant \mu - 1 + \alpha$, then, dividing the last relation by ε^τ, we have

$$\int_{I_0\setminus B_\varepsilon^0} \theta p\left(\frac{x'}{\varepsilon}, \omega\right)u_\varepsilon^2\left(x', \varepsilon\varTheta F\left(\frac{x'}{\varepsilon}, \omega\right)\right)\mathrm{d}x' \leqslant \int_{\varGamma_\varepsilon} \theta p\left(\frac{x'}{\varepsilon}, \omega\right)u_\varepsilon^2(x)\,\mathrm{d}\sigma_x \leqslant C\varepsilon^{-\tau}. \tag{58}$$

Considering the upper bound $\|u_\varepsilon\|_{H^1(D_\varepsilon)} \leqslant C$, by condition (h3) we obtain

$$\lim_{\varepsilon\to 0}\left|\int_{I_0} \theta(\mathbb{E}p)\big(1 - |B|\big)u_\varepsilon^2(x)\,\mathrm{d}x' - \int_{I_0\setminus B_\varepsilon^0} \theta p\left(\frac{x'}{\varepsilon}, \omega\right)u_\varepsilon^2\left(x', \varepsilon\varTheta F\left(\frac{x'}{\varepsilon}, \omega\right)\right)\mathrm{d}x'\right| = 0 \quad \text{a.s.}$$

Since $\mathbb{E}p > 0$ and $\theta \geqslant \theta_- > 0$, this implies that a.s.

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L_2(I_0)} = 0. \tag{59}$$

The family $\{u_\varepsilon\}$ is uniformly bounded in $H^1(D_\varepsilon)$. Consider arbitrary convergent subsequence $u_{\varepsilon_k}, \varepsilon_k \to 0$. It is evident that the limit function $u'(x)$ satisfies the equation $-\Delta u' = f$ and the boundary conditions $u' = 0$ on $\varGamma_1$ and $\partial_\nu u' = 0$ on γ. In view of (59), it also holds $u'|_{I_0} = 0$. Hence, $u'(x) = u_0(x)$. Then, the whole family $\{u_\varepsilon\}$ converges to u_0, that is, $\|u_\varepsilon - u_0\|_{L^2(D_\varepsilon)} \to 0$, as $\varepsilon \to 0$.

If $\tilde{p} \geqslant p_- > 0$, then, as an immediate consequence of (58) and (16), we get

$$\|u_\varepsilon - u_0\|_{L_2(I_0\setminus B_\varepsilon^0)} = \|u_\varepsilon\|_{L_2(I_0\setminus B_\varepsilon^0)} \leqslant C\big(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\big). \tag{60}$$

Exploiting the upper bound $\|u_\varepsilon - u_0\|_{H^1(D_\varepsilon)} \leqslant C$ we conclude that

$$\|u_\varepsilon - u_0\|_{L_2(I_0)} \leqslant C\big(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\big). \tag{61}$$

Since $(u_\varepsilon - u_0)$ is a harmonic function in D_0 that satisfies the boundary conditions $(u_\varepsilon - u_0) = 0$ on Γ_1, and $\partial_\nu(u_\varepsilon - u_0) = 0$ on γ, then

$$\|u_\varepsilon - u_0\|_{L_2(D_0)} \leqslant C\big(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\big). \tag{62}$$

To justify this inequality it suffices to represent $(u_\varepsilon - u_0)$ in terms of the corresponding Green function and to use (61).

If $\tau > \mu - 1 + \alpha$, then, by the same arguments, we obtain

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{L_2(D_\varepsilon)} = 0 \quad \text{a.s.}$$

and, if $\tilde{p} \geqslant p_- > 0$,

$$\|u_\varepsilon - u_0\|_{L_2(D_0)} \leqslant C\big(\varepsilon^{(1-\alpha-\mu)/2} + \varepsilon^{\alpha/2}\big).$$

This completes the proof. $\square$

7. Estimates for eigenelements

In this section, we apply the results developed in [44] to the spectral problem associated with boundary value problem (1). Consider the following problems:

$$\begin{cases} -\Delta_x u_\varepsilon^k(x) = \lambda_\varepsilon^k u_\varepsilon^k(x), & x \in D_\varepsilon; \\ \partial_\nu u_\varepsilon^k(x) + \varepsilon^\tau p\left(\dfrac{x'}{\varepsilon}, \omega\right) u_\varepsilon^k(x) = 0, & x \in \Gamma_\varepsilon; \\ \partial_\nu u_\varepsilon^k(x) + \varepsilon^\mu k_1 u_\varepsilon^k(x) = 0, & x \in \Upsilon_\varepsilon; \\ u_\varepsilon^k(x) = 0, & x \in \Gamma_1; \\ \partial_\nu u_\varepsilon^k(x) = 0, & x \in \gamma_\varepsilon, \end{cases} \qquad k = 1, 2, \ldots, \tag{63}$$

$$\begin{cases} -\Delta_x u_0^k(x) = \lambda_0^k u_0^k(x), & x \in D_0, \\ \partial_\nu u_0^k(x) = 0, & x \in \partial D_0 \setminus (\Gamma_1 \cup I_0), \\ -\partial_{x_d} u_0^k\big(x',0\big) & \\ \quad + \big(\delta_{\tau,0} P\big(x'\big) + \delta_{\mu,1-\alpha} \Xi k_1\big) u_0^k\big(x',0\big) = 0, & \big(x',0\big) \in I_0 \\ u_0^k(x) = 0, & x \in \Gamma_1, \end{cases} \qquad k = 1, 2, \ldots, \tag{64}$$

and

$$\begin{cases} -\Delta_x u_0^k(x) = \lambda_0^k u_0^k(x), & x \in D_0, \\ \partial_\nu u_0^k(x) = 0, & x \in \partial D_0 \setminus (\Gamma_1 \cup I_0) \qquad k = 1, 2, \dots. \\ u_0^k(x) = 0, & x \in \Gamma_1 \cup I_0, \end{cases} \tag{65}$$

Here, $u_\varepsilon^k \in H^1(D_\varepsilon, \Gamma_1)$, $u_0^k \in H^1(D_0, \Gamma_1)$, $k = 1, 2, \dots$, are orthogonal basis in $L_2(D_\varepsilon)$ and $L_2(D_0)$, respectively. The sets $\{\lambda_\varepsilon^k\}$, $\{\lambda_0^k\}$, $k = 1, 2, \dots$, are the corresponding eigenvalues such that

$$0 < \lambda_\varepsilon^1 \leqslant \lambda_\varepsilon^2 \leqslant \dots \leqslant \lambda_\varepsilon^k \leqslant \dots, \qquad 0 \leqslant \lambda_0^1 \leqslant \lambda_0^2 \leqslant \dots \leqslant \lambda_0^k \leqslant \dots$$

and they repeat with respect to their multiplicities.

For the sake of completeness, we state here the results on spectral convergence for positive, selfadjoint and compact operators on Hilbert spaces (see Section III.1 in [44] for the proof).

Theorem 7.1. *Let H_ε and H_0 be two separable Hilbert spaces with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$, respectively. Let $A_\varepsilon \in \mathcal{L}(H_\varepsilon)$ and $A_0 \in \mathcal{L}(H_0)$. Let V be a linear subspace of H_0 such that $\{v\colon v = A_0 u, u \in H_0\} \subset V$. We assume that the following properties are satisfied*:

- C1. *There exists $R_\varepsilon \in \mathcal{L}(H_0, H_\varepsilon)$ such that $(R_\varepsilon F, R_\varepsilon F)_{H_\varepsilon} \overset{\varepsilon \to 0}{\to} \gamma_0 (F, F)_{H_0}$, for all $F \in V$ and certain positive constant γ_0.*
- C2. *The operators A_ε and A_0 are positive, compact and selfadjoint. Moreover, $\|A_\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ are bounded by a constant, independent of ε.*
- C3. $\|A_\varepsilon R_\varepsilon F - R_\varepsilon A_0 F\|_{H_\varepsilon} \overset{\varepsilon \to 0}{\to} 0$ *for all $F \in V$.*
- C4. *The family of operators A^ε is uniformly compact, i.e., for any sequence F^ε in H_ε such that $\sup_\varepsilon \|F^\varepsilon\|_{H_\varepsilon}$ is bounded by a constant independent of ε, we can extract a subsequence $F^{\varepsilon'}$, that verifies the following*:

  $$\left\| A_{\varepsilon'} F^{\varepsilon'} - R_{\varepsilon'} v^0 \right\|_{H_{\varepsilon'}} \to 0,$$

  as $\varepsilon' \to 0$, for certain $v^0 \in H_0$.

Let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ and $\{\mu_i^0\}_{i=1}^\infty$ be the sequences of the eigenvalues of A_ε and A_0, respectively, with the classical convention of repeated eigenvalues. Let $\{w_i^\varepsilon\}_{i=1}^\infty$ and ($\{w_i^0\}_{i=1}^\infty$, respectively) be the corresponding eigenfunctions in H_ε, which are assumed to be orthonormal (H_0, respectively).

Then, for each k, there exists a constant C_{10}^k, independent of ε, such that

$$\left|\mu_k^\varepsilon - \mu_k^0\right| \leqslant C_{10}^k \sup_{\substack{u \in \mathcal{N}(\mu_0^k, A_0), \\ \|u\|_{H_0} = 1}} \|A_\varepsilon R_\varepsilon u - R_\varepsilon A_0 u\|_{H_\varepsilon},$$

where $\mathcal{N}(\mu_0^k, A_0) = \{u \in H_0, A_0 u = \mu_0^k u\}$. Moreover, if μ_k^0 has multiplicity s ($\mu_k^0 = \mu_{k+1}^0 = \dots = \mu_{k+s-1}^0$), then for any w eigenfunction associated with μ_k^0, with $\|w\|_{H_0} = 1$, there exists a linear combination w^ε of eigenfunctions of A^ε, $\{w_j^\varepsilon\}_{j=k}^{j=k+s-1}$ associated with $\{\mu_j^\varepsilon\}_{j=k}^{j=k+s-1}$ such that

$$\left\| w^\varepsilon - R_\varepsilon w \right\|_{H_\varepsilon} \leqslant C_{11}^k \|A_\varepsilon R_\varepsilon w - R_\varepsilon A_0 w\|_{H_\varepsilon},$$

where the constant C_{11}^k is independent on ε.

We denote by H_ε the space $L_2(D_\varepsilon)$ with the scalar product

$$\left(f^\varepsilon, g^\varepsilon\right)_{H_\varepsilon} \equiv \int_{D_\varepsilon} f^\varepsilon(x) g^\varepsilon(x)\,\mathrm{d}x.$$

We denote by H_0 the space $L_2(D_0)$, where the scalar product is

$$\left(f^0, g^0\right)_{H_0} \equiv \int_{D_0} f^0 g^0\,\mathrm{d}x.$$

We define the operator

$$A_\varepsilon : L_2(D_\varepsilon) \to H^1(D_\varepsilon, \Gamma_1), \quad A_\varepsilon f = u_\varepsilon,$$

where u_ε is the solution to the problem (1) with $g \equiv 0$ and $k_2 = 0$.

In the case $\tau \geqslant 0$ and $\mu \geqslant 1 - \alpha$ we set:

$$A_0 : L_2(D_0) \to H^1(D_0, \Gamma_1), \quad A_0 f = u_0,$$

where u_0 is the solution to problem (3) with $Q \equiv 0$ and $k_2 = 0$.

In the case $\tau < 0$ and $\mu < 1 - \alpha$ we set:

$$A_0 : L_2(D_0) \to H^1(D_0, \Gamma_1 \cup I_0), \quad A_0 f = u_0,$$

where u_0 is the solution to problem (6).

In fact, A_ε and A_0 are operators associated with the eigenvalue problems (63), and (64) (if $\tau \geqslant 0$ and $\mu \geqslant 1 - \alpha$), and (65) (if $\tau < 0$ and $\mu < 1 - \alpha$), respectively.

Now, considering the operators $A_\varepsilon : H_\varepsilon \to H_\varepsilon$ and $A_0 : H_0 \to H_0$, it is easy to establish the positiveness, self-adjointness and compactness of the operators A_ε and A_0, respectively. In particular, the compactness of both operators follows from the compactness of the imbedding of $H^1(D_\varepsilon)$ into the space $L_2(D_\varepsilon)$ and $H^1(D_0)$ into the space $L_2(D_0)$, respectively.

Let V be $V = H^1(D_0, \Gamma_1)$ (if $\tau \geqslant 0$ and $\mu \geqslant 1 - \alpha$), and $V = H^1(D_0, \Gamma_1 \cup I_0)$ (if $\tau < 0$ and $\mu < 1 - \alpha$), which satisfies $\mathrm{Im}(A_0) \subseteq V \subset H_0$, and let R_ε be

$$R_\varepsilon : L_2(D_0) \to L_2(D_\varepsilon) \tag{66}$$

the extension operator (we extend functions by zero in $D_\varepsilon \setminus D_0$).

Let us verify the conditions C1–C4 of Theorem 7.1 (Theorem 1.4 from Section III.1 in [44]).

C1. The operator $R_\varepsilon : H_0 \to H_\varepsilon$ is defined in (66). Obviously,

$$(R_\varepsilon F, R_\varepsilon F)_{H_\varepsilon} = \int_{D_\varepsilon} F^2\,\mathrm{d}x \to \int_{D_0} F^2\,\mathrm{d}x = (F, F)_{H_0}$$

as $\varepsilon \to 0$. Hence, we conclude that this condition is fulfilled with $\gamma_0 = 1$.

C2. Let us prove that norms $\|A_\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ are uniformly bounded with respect to ε. Keeping in mind the equivalence of norms, we obtain that:

$$\begin{aligned}\|u_\varepsilon\|^2_{H^1(D_\varepsilon)} &\leqslant C_{12}\biggl(\int_{D_\varepsilon}|\nabla u_\varepsilon|^2\,\mathrm{d}x+\varepsilon^\tau\int_{\Gamma_\varepsilon}\theta p\biggl(\frac{x'}{\varepsilon},\omega\biggr)u_\varepsilon^2(x)\,\mathrm{d}\sigma_x+\varepsilon^\mu k_1\int_{\Upsilon_\varepsilon}u_\varepsilon^2(x)\,\mathrm{d}\sigma_x\biggr)\\ &\leqslant C_{13}\|f\|_{L_2(D_\varepsilon)}\|u_\varepsilon\|_{H^1(D_\varepsilon)}\end{aligned}$$

or

$$\|u_\varepsilon\|_{H^1(D_\varepsilon)}\leqslant C_{13}\|f\|_{L_2(D_\varepsilon)}.$$

Thus, $\|A_\varepsilon f\|_{H_\varepsilon}\leqslant C_{14}\|f\|_{H_\varepsilon}$, and condition C2 is fulfilled.

C3. By Theorems 4.2 and 4.3 condition C3 takes place. Indeed, by the definitions of the operators A_ε, A_0 for any $f\in V$, we obtain that:

$$\|A_\varepsilon R_\varepsilon f-R_\varepsilon A_0 f\|^2_{H_\varepsilon}=\|u_\varepsilon-u\|^2_{H_\varepsilon}\to 0\quad\text{as }\varepsilon\to 0.$$

Thus, condition C3 holds.

C4. If a sequence $\{f_\varepsilon\}$ is bounded in H_ε then, by Proposition 5.1, the solutions $\{u_\varepsilon=A_\varepsilon f_\varepsilon\}_\varepsilon$ to problem (1) are uniformly bounded in $H^1(D_\varepsilon,\Gamma_1)$. Therefore, there exists $w\in H^1(D_0,\Gamma_1)$ and a subsequence $\varepsilon'\to 0$ such that $u_{\varepsilon'}\to w$ in $L_2(D_0)$ and weakly in $H^1(D_0)$. Thus,

$$\|A_{\varepsilon'}f_{\varepsilon'}-R_{\varepsilon'}w\|^2_{H_{\varepsilon'}}=\int_{D_{\varepsilon'}}\bigl(u_{\varepsilon'}(x)-w(x)\bigr)^2\,\mathrm{d}x\to 0\quad\text{as }\varepsilon'\to 0,$$

and condition C4 is fulfilled.

Now, we consider the spectral problems:

$$\begin{aligned}&A_\varepsilon u_\varepsilon^k=\mu_\varepsilon^k u_\varepsilon^k,\quad u_\varepsilon^k\in H_\varepsilon,\\ &\mu_\varepsilon^1\geqslant\mu_\varepsilon^2\geqslant\cdots\geqslant\mu_\varepsilon^k\geqslant\cdots>0,\quad k=1,2,\dots,\\ &\bigl(u_\varepsilon^l,u_\varepsilon^k\bigr)_{H_\varepsilon}=\delta_{lk}\end{aligned}$$

and

$$\begin{aligned}&A_0 u_0^k=\mu_0^k u_0^k,\quad u_0^k\in H_0,\\ &\mu_0^1\geqslant\mu_0^2\geqslant\cdots\geqslant\mu_0^k\geqslant\cdots>0,\quad k=1,2,\dots,\\ &\bigl(u_0^l,u_0^k\bigr)_{H_0}=\delta_{lk}.\end{aligned}$$

According to our definitions $\mu_\varepsilon^k=\frac{1}{\lambda_\varepsilon^k}$, and $\mu_0^k=\frac{1}{\lambda_0^k}$, where λ_ε^k and λ_0^k are the eigenvalues of problems (63) and (64) (if $\tau\geqslant 0$ and $\mu\geqslant 1-\alpha$), and (65) (if $\tau<0$ and $\mu<1-\alpha$), respectively.

Finally, applying Theorem 7.1 (Theorems 1.4 and 1.7 in Section III.1 of [44]), we prove the following statements.

Theorem 7.2. *In the case* $\tau \geqslant 0$ *and* $\mu \geqslant 1-\alpha$ *for the eigenvalues* $\lambda_\varepsilon^k, \lambda_0^k$ *of problems* (63) *and* (64), *respectively, the estimate holds*

$$\left|\lambda_\varepsilon^k - \lambda_0^k\right| \leqslant C\left(\mathbf{1}_{\mu=1-\alpha}\varepsilon^{\mu-1+\alpha} + \varepsilon^{\alpha/2} + \varepsilon^{1/4} + \mathbf{1}_{\tau=0}\varepsilon^{\tau}\right).$$

In the case $\tau < 0$ *or* $\mu < 1-\alpha$ *for the eigenvalues* $\lambda_\varepsilon^k, \lambda_0^k$ *of problems* (63) *and* (65), *respectively, the estimates*

$$\left|\lambda_\varepsilon^k - \lambda_0^k\right| \leqslant C\left(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\right) \quad \textit{for } \tau \leqslant \mu - 1 + \alpha$$

and

$$\left|\lambda_\varepsilon^k - \lambda_0^k\right| \leqslant C\left(\varepsilon^{(1-\mu-\alpha)/2} + \varepsilon^{\alpha/2}\right) \quad \textit{for } \tau > \mu - 1 + \alpha$$

are valid.

Theorem 7.3. *Let us consider the same hypothesis as in Theorem* 7.2. *Suppose that* k, l *are integers,* $k \geqslant 0, l \geqslant 1$, *and* $\lambda_0^k < \lambda_0^{k+1} = \cdots = \lambda_0^{k+l} < \lambda_0^{k+l+1}$.

Then, in the case $\tau \geqslant 0$ *and* $\mu \geqslant 1-\alpha$ *for any* w, *eigenfunction of* (64), *associated with the eigenvalue* λ_0^{k+1}, *there exists a linear combination* $\overline{u}_\varepsilon$ *of eigenfunctions* $u_\varepsilon^{k+1}, \dots, u_\varepsilon^{k+l}$ *of problem* (63) *such that*:

$$\|\overline{u}_\varepsilon - R_\varepsilon w\|_{L_2(D_\varepsilon)} \leqslant C\left(\mathbf{1}_{\mu=1-\alpha}\varepsilon^{\mu-1+\alpha} + \varepsilon^{\alpha/2} + \varepsilon^{1/4} + \mathbf{1}_{\tau=0}\varepsilon^{\tau}\right).$$

Then, in the case $\tau < 0$ *and* $\mu < 1-\alpha$ *for any* w, *eigenfunction of* (65), *associated with the eigenvalue* λ_0^{k+1}, *there exists a linear combination* $\overline{u}_\varepsilon$ *of eigenfunctions* $u_\varepsilon^{k+1}, \dots, u_\varepsilon^{k+l}$ *of problem* (63) *such that*:

$$\|\overline{u}_\varepsilon - R_\varepsilon w\|_{L_2(D_\varepsilon)} \leqslant C\left(\varepsilon^{-\tau/2} + \varepsilon^{1/2}\right) \quad \textit{for } \tau \leqslant \mu - 1 + \alpha$$

and

$$\|\overline{u}_\varepsilon - R_\varepsilon w\|_{L_2(D_\varepsilon)} \leqslant C\left(\varepsilon^{(1-\mu-\alpha)/2} + \varepsilon^{\alpha/2}\right) \quad \textit{for } \tau > \mu - 1 + \alpha.$$

Acknowledgements

The first author was financially supported by the Government grant of the Russian Federation under the Resolution No. 220 "On measures designed to attract leading scientists to Russian institutions of higher education" according to the Agreement No. 11.G34.31.0054, signed by the Ministry of education and science of the Russian Federation, the leading scientist, and Lomonosov Moscow State University (on the basis of which the present research is organized) and also supported by the Russian Foundation for Basic Research (grant No. 12-01-00445).

This paper was partly written during the visit of Gregory Chechkin and Andrey Piatnitski at the Università degli Studi di Salerno in July 2010. They would like to express their gratitude for the hospitality.

References

[1] Y. Amirat, O. Bodart, G.A. Chechkin and A.L. Piatnitski, Boundary homogenization in domains with randomly oscillating boundary, *Stochastic Processes and Their Applications* **121** (2011), 1–23.

[2] Y. Amirat, O. Bodart, U. De Maio and A. Gaudiello, Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary, *SIAM J. Math. Anal*. **35**(6) (2004), 1598–1616.

[3] Y. Amirat, G.A. Chechkin and R.R. Gadyl'shin, Asymptotics for eigenelements of Laplacian in domain with oscillating boundary: Multiple eigenvalues, *Applicable Analysis* **86**(7) (2007), 873–897.

[4] Y. Amirat, G.A. Chechkin and R.R. Gadyl'shin, Asymptotic approximation of eigenelements of the Dirichlet problem for the Laplacian in a domain with shoots, *Mathematical Methods in the Applied Sciences (M2AS)* **33**(7) (2010), 811–830.

[5] N.S. Bakhvalov and G.P. Panasenko, *Homogenization: Averaging Processes in Periodic Media*, Math. Appl. (Soviet Ser.), Vol. 36, Kluwer Academic Publishers, 1989.

[6] A.G. Belyaev, Averaging the third boundary value problem for the Poisson equation in a domain with a rapidly oscillating boundary, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* (6) (1988), 63–66 (in Russian).

[7] A.G. Belyaev, A.G. Mikheev and A.S. Shamaev, Plane wave diffraction by a rapidly oscillating surface, *Comput. Math. Math. Phys.* **32**(8) (1992), 1121–1133. (Transl. from *Zh. Vychisl. Mat. i Mat. Fiz.* **32**(8) (1992), 1258–1272.)

[8] A.Yu. Belyaev and G.A. Chechkin, Averaging of operators with a fine-scaled structure of boundary conditions, *Math. Notes* **65**(3,4) (1999), 418–429.

[9] A. Bensoussan, J.L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[10] D. Blanchard, A. Gaudiello and G. Griso, Junction of a periodic family of elastic rods with 3d plate, *J. Math. Pures Appl.* **88**(9) (2007), 1–33 (Part I); **88**(9) (2007), 149–190 (Part II).

[11] D. Blanchard, A. Gaudiello and T.A. Mel'nyk, Boundary homogenization and reduction of dimension in a Kirchhoff–Love plate, *SIAM Journal on Mathematical Analysis* **39**(6) (2008), 1764–1787.

[12] D. Blanchard, A. Gaudiello and J. Mossino, Highly oscillating boundaries and reduction of dimension: the critical case, *Analysis and Application* **5** (2007), 137–163.

[13] G. Bouchitté, A. Lidouh and P. Suquet, Homogénéisation de frontière pour la modélisation du contact entre un corps déformable non linéaire et un corps rigide, *C. R. Acad. Sci. Paris* (Ser. I) **313** (1991), 967–972.

[14] A. Bourgeat, A. Mikelić and S. Wright, Stochastic two-scale convergence in the mean and applications, *J. Reine Angew. Math.* **456** (1994), 19–51.

[15] G. Buttazzo and R.V. Kohn, Reinforcement by a thin layer with oscillating thickness, *Appl. Math. Optim.* **16**(3) (1987), 247–261.

[16] G.A. Chechkin and T.P. Chechkina, On homogenization problems in domains of the "Infusorium" type, *Trudy Seminara Imeni I.G. Petrovskogo* **23** (2003), 379–400. (English transl.: *Journal of Mathematical Sciences* **120**(3) (2004), 1470–1482.)

[17] G.A. Chechkin, T.P. Chechkina, C. D'Apice and U. De Maio, Homogenization in domains randomly perforated along the boundary, *Discrete and Continuous Dynamical Systems. B* **12**(4) (2009), 713–730.

[18] G.A. Chechkin, T.P. Chechkina, C. D'Apice, U. De Maio and T.A. Mel'nyk, Homogenization of 3D thick cascade junction with the random transmission zone periodic in one direction, *Russian Journal of Mathematical Physics* **17**(1) (2010), 35–55.

[19] G.A. Chechkin, T.P. Chechkina, C. D'Apice, U. De Maio and T.A. Mel'nyk, Asymptotic analysis of a boundary-value problem in a cascade thick junction with a random transmission zone, *Applicable Analysis* **88**(10,11) (2009), 1543–1562.

[20] G.A. Chechkin, C. D'Apice and U. De Maio, On the rate of convergence of solutions in domain with periodic multilevel oscillating boundary, *Mathematical Methods in the Applied Sciences (M2AS)* **33**(17) (2010), 2019–2036.

[21] G.A. Chechkin, A. Friedman and A.L. Piatnitski, The boundary value problem in domains with very rapidly oscillating boundary, *Journal of Mathematical Analysis and Applications (JMAA)* **231**(1) (1999), 213–234.

[22] G.A. Chechkin, A.L. Piatnitski and A.S. Shamaev, *Homogenization: Methods and Applications*, Translations of Mathematical Monographs Serie, Vol. 234, AMS Press, Providence, RI, 2007. (Transl. from *Homogenization: Methods and Applications*, Tamara Rozhkovskaya Press, Novosibirsk, 2007.)

[23] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford Lecture Ser. Math. Appl., Vol. 17, Oxford University Press, 1999.

[24] E.N. Dancer and D. Daners, Domain perturbation for elliptic equations subject to Robin boundary conditions, *Journal of Differential Equations* **138**(1) (1997), 86–132.

[25] U. De Maio, T. Durante and T.A. Mel'nyk, Asymptotic approximation for the solution to the Robin problem in a thick multi-level junction, *Mathematical Models and Methods in Applied Sciences* **15**(12) (2005), 1897–1921.

[26] A. Friedman, B. Hu and Y. Liu, A boundary value problem for the Poisson equation with multi-scale oscillating boundary, *J. Differential Equations* **137**(1) (1997), 54–93.

[27] A. Gaudiello, Asymptotic behavior of non-homogeneous Neumann problems in domains with oscillating boundary, *Ricerche di Math.* **43** (1994), 239–292.
[28] W. Jäger and A. Mikelić, On the roughness-induced boundary conditions for an incompressible viscous flow, *J. Differential Equations* **170**(1) (2001), 96–122.
[29] V.V. Jikov, S.M. Kozlov and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.
[30] Yu.A. Kazmerchuk and T.A. Mel'nyk, Homogenization of the Signorini boundary-value problem in a thick plane junction, *Nonlinear Oscillations* **12**(1) (2009), 44–58.
[31] R. Kohn and M. Vogelius, A new model for thin plates with rapidly varying thickness, *J. Solid Structures* **20**(4) (1984), 333–350.
[32] S.M. Kozlov, Averaging of random operators, *Math. USSR Sb.* **37** (1980), 167–180.
[33] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Applied Mathematical Sciences, Vol. 49, Springer-Verlag, Berlin, 1985.
[34] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, Heidelberg, 1972. (Transl. by P. Kenneth from *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris, 1968.)
[35] V.A. Marchenko and E.Ya. Khruslov, *Homogenized Models of Microinhomogeneous Media*, Naukova Dumka, Kiev, 2005. (English transl.: *Homogenization of Partial Differential Equations*, Progress in Mathematical Physics, Vol. 46, Birkhäuser, Boston, MA, 2006.)
[36] E. Marušić-Paloka, Average of the Navier's law on the rapidly oscillating boundary, *J. Math. Anal. Appl.* **259**(2) (2001), 685–701.
[37] V.G. Maz'ya, Classes of spaces, measures and capacities in the theory of spaces of differentiable functions, in: *Modern Problems of Mathematics. Fundamental Investigations (Itogi Nauki i Tekhniki VINITI AN SSSR)* **26** (1987), 159–228 (in Russian).
[38] T.A. Mel'nyk, Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3:2:1, *Mathematical Models and Methods in Applied Sciences* **31**(9) (2008), 1005–1027.
[39] T.A. Mel'nyk and G.A. Chechkin, Homogenization of a boundary-value problem in a thick 3-dimensional multilevel junction, *Russian Academy of Sciences Sbornik Mathematics* **200**(3) (2009), 357–383. (Transl. from *Matematicheskii Sbornik* **200**(3) (2009), 49–74.)
[40] T.A. Mel'nyk, G.A. Chechkin and T.P. Chechkina, Convergence theorems for solutions and energy functional of boundary value problems in thick multilevel junctions of a new type with perturbed Neumann conditions on the boundary of thin rectangles, *Journal of Mathematical Sciences* **159**(1) (2009), 113–132. (Transl. from *Problemy Mat. Analiza* **40** (2009), 113–131.)
[41] T.A. Mel'nyk and P.S. Vaschuk, Homogenization of the Neumann–Fourier problem in a thick two-level junction of type 3:2:1, *J. Math. Physics, Analysis and Geometry* **2** (2006), 318–337.
[42] V.P. Mikhailov, *Partial Differential Equations*, Mir, Moscow, 1978.
[43] S.A. Nazarov, *Asymptotic Analysis of Thin Plates and Rods, Vol. 1, Dimension Reduction and Integral Estimates,* Nauchnaja kniga, Novosibirsk, 2002 (in Russian).
[44] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.
[45] G.C. Papanicolaou and S.R.S. Varadhan, Boundary value problems with rapidly oscillating random coefficients. Random fields, Vols. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai, Vol. 27, North-Holland, Amsterdam, 1981, pp. 835–873.
[46] E. Sánchez-Palencia, *Homogenization Techniques for Composite Media*, Springer-Verlag, Berlin, 1987.
[47] E. Sánchez-Palencia and P. Suquet, Friction and homogenization of a boundary, in: *Free Boundary Problems: Theory and Applications,* A. Fasano and M. Primicerio, eds, Pitman, London, 1983, pp. 561–571.
[48] S.L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, 3rd edn, Translations of Mathematical Monographs Series, Vol. 90, AMS Press, Providence, RI, 1991.
[49] V.V. Yurinskii, Averaging an elliptic boundary-value problem with random coefficients, *Siberian Math. J.* **21**(3) (1980), 470–482.
[50] V.V. Zhikov and A.L. Pyatnitskii, Homogenization of random singular structures and random measures, *Izv. Math.* **70**(1) (2006), 19–67.