Asymptotic Analysis 39 (2004) 281–308
IOS Press

A variational approach to double-porosity problems

Andrea Braides [a,*], Valeria Chiadò Piat [b] and Andrey Piatnitski [c,d]
[a] *Dipartimento di Matematica, Università di Roma 'Tor Vergata' via della Ricerca Scientifica, 00133 Rome, Italy*
E-mail: braides@mat.uniroma2.it
[b] *Dipartimento di Matematica, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy*
[c] *Department of Mathematics, Narvik Institute of Technology, HiN, Postbox 385, 8505 Narvik, Norway*
[d] *P.N. Lebedev Physical Institute, RAS, 53 Leninski prospect, Moscow 117924, Russia*

1. Introduction

In this paper we outline an approach by Γ-convergence to some problems related to 'double-porosity' homogenization. Various such models have been discussed in the mathematical literature, the first rigorous result for a linear double-porosity model having been obtained by Arbogast, Douglas and Hornung in [7]. Multicomponent high contrast homogenization problems were studied by Panasenko in [23,22]. The two-scale convergence approach to double-porosity problems was developed by Allaire in [4]. A random model has been studied by Bourgeat, Mikelic and Piatnitski in [10]. Other double-porosity type problems for linear and some nonlinear operators have been considered in [9,18,24–26,16, 29,27] and [8].

In our framework, the homogenization process involves the analysis of energies defined on some (mutually disconnected) highly oscillating connected sets (*hard components*), in whose complement (*soft component*) an energy density satisfying weaker coerciveness conditions is considered. To be more precise, we fix $N \geqslant 1$ and 1-periodic Lipschitz open connected sets $E_1, \dots, E_N \subset \mathbb{R}^n$ with $\mathrm{dist}(E_i, E_j) > 0$ if $i \neq j$. If $n = 2$ the connectedness condition can be satisfied only if $N = 1$; note that even this case will give nontrivial results. We also set

$$E_0 = \mathbb{R}^n \setminus (E_1 \cup \dots \cup E_N);$$

note that we do not make any connectedness assumption on E_0, which may be composed only of isolated bounded components if $N = 1$. For each $j = 0, \dots, N$ we consider energy densities $f_j : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to \mathbb{R}$ and 'low order terms' $g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$. We suppose that g_j, f_j are Borel functions and 1-periodic in the first variable. For the sake of simplicity of presentation we suppose that there exists $p > 1$ such that all f_j satisfy a p-growth condition, each f_j is quasiconvex and f_0 is positively homogeneous of

[*] Corresponding author.

0921-7134/04/$17.00

degree p. In this way, given an open set $\Omega \subset \mathbb{R}^n$, we consider the energy

$$F_\varepsilon(u) = \sum_{j=1}^{N} \int_{\Omega \cap \varepsilon E_j} \left(f_j\left(\frac{x}{\varepsilon}, Du\right) + g_j\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x + \int_{\Omega \cap \varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du\right) + g_0\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x \tag{1.1}$$

defined on $W^{1,p}(\Omega; \mathbb{R}^m)$. Let us note here that, due to the presence of a microstructure in the geometry of the sets E_j, the oscillating behaviour of f_j and g_j is of secondary importance. Indeed, even the following simple type of energies

$$F_\varepsilon(u) = \sum_{j=1}^{N} \int_{\Omega \cap \varepsilon E_j} |Du|^2 \,\mathrm{d}x + \int_{\Omega \cap \varepsilon E_0} \varepsilon^2 |Du|^2 \,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x \tag{1.2}$$

exhibits a double-porosity phenomenon. We will show that the energies (1.1) Γ-converge as $\varepsilon \to 0$, and have as their Γ-limit a multi-phase system of the form

$$F_0(u_1, \ldots, u_N) = \sum_{j=1}^{N} \int_\Omega \left(f^j_{\mathrm{hom}}(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \int_\Omega \varphi(u_1, \ldots, u_N) \,\mathrm{d}x \tag{1.3}$$

defined on $(W^{1,p}(\Omega; \mathbb{R}^m))^N$. We emphasize that the Γ-limit is computed with respect to a particular choice of convergence. Namely, we say that $u_\varepsilon \to (u_1, \ldots, u_N)$ if

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{N} \int_{\Omega \cap \varepsilon E_j} |u_\varepsilon - u_j|^p \,\mathrm{d}x = 0.$$

It should be noted that even in the case $N = 1$ this Γ-limit is not equivalent to that performed with respect to the strong convergence in $L^p(\Omega)$; hence, in particular, the lower-order term on εE_0, not being a continuous perturbation with respect to the introduced topology, gives a nontrivial contribution to φ. The fact that the Γ-limit depends crucially on the choice of the reference convergence is an interesting feature of the functionals under considerations. An alternative choice of the convergence with respect to which the Γ-limit is computed, could be the weak L^p-convergence. However, in that case the Γ-limit is a nonlocal functional whose form seems to bear less information about the limit process. The computation of the Γ-limit with respect to strong and weak L^p-convergences as well as multivariate L^p convergence of all $N + 1$ phases including the soft one, and their comparison with the result above is contained in Section 7 of the paper. In the same section we also compute the Γ-limit of energies of the form

$$F^h_\varepsilon(u) = F_\varepsilon(u) - \int_\Omega h \cdot u \,\mathrm{d}x,$$

where $h \in L^{p'}(\Omega; \mathbb{R}^m)$ and p' is the conjugate exponent of p $(1/p + 1/p' = 1)$.

The asymptotic result above needs some words of explanation. It is convenient to reason in terms of minimum problems. Consider fixed boundary data $\phi_1, \dots, \phi_N \in W^{1,p}(\Omega; \mathbb{R}^m)$ and for every $\varepsilon > 0$ let u_ε be a solution of

$$\min\{F_\varepsilon(u)\colon\ u = \phi_j \text{ on } \varepsilon E_j \cap \partial\Omega,\ j = 1, \dots, N\}. \tag{1.4}$$

Then for $j = 1, \dots, N$ the suitably defined $W^{1,p}$-extension of the restriction of u_ε to $\varepsilon E_j \cap \Omega$ converges (upon subsequences, locally weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$) to a function u_j (see Acerbi et al. [1]). The result above implies that $(u_1, \dots, u_N)$ is a minimum point of

$$\min\{F_0(u_1, \dots, u_N)\colon\ u_j = \phi_j \text{ on } \partial\Omega,\ j = 1, \dots, N\}. \tag{1.5}$$

The form of the limit F_0 highlights a decoupling process. The energy densities f^j_{hom} and $\tilde{g}_j$ are given by the independent process of homogenization (see e.g., Braides and Chiadò Piat [13] and Braides and Garroni [15]) on each εE_j, respectively. The contribution of u_ε on εE_0 is 'integrated out' and appears in the limit through the form of φ (note that no L^p compactness of this part can be deduced from the energy estimates; in fact, in general, u_ε is not compact in $L^p(\Omega)$). In the simplest case when all functions are convex, the latter function is defined by

$$\varphi(z_1, \dots, z_N) = \min\biggl\{\int_{E_0\cap(0,1)^n} \bigl(f_0(y, Dv) + g_0(y, v)\bigr)\,\mathrm{d}y\colon\ v \in W^{1,p}\bigl((0,1)^n; \mathbb{R}^m\bigr),$$
$$v = z_j \text{ on } E_j \cap (0,1)^n,\ j = 1, \dots, N,\ v \text{ is 1-periodic}\biggr\}. \tag{1.6}$$

Note that φ may contain different types of contributions. One type is given by the interaction between the different phases E_j. Consider a simple case when $m = 1,\ N = 2,\ f_0(y, \xi) = |\xi|^2$ and $g_0(y, u) = 0$. In this situation, a translation and positive-homogeneity argument easily shows that

$$\varphi(z_1, z_2) = |z_1 - z_2|^2 \min\biggl\{\int_{E_0\cap(0,1)^n} |Dv|^2\,\mathrm{d}y\colon\ v \in W^{1,2}\bigl((0,1)^n\bigr),$$
$$v = 0 \text{ on } E_1 \cap (0,1)^n,\ v = 1 \text{ on } E_2 \cap (0,1)^n,\ v \text{ is 1-periodic}\biggr\}. \tag{1.7}$$

Another type of contribution is due to the presence of the zero-order term g_0, as illustrated by the following example: we consider $m = N = 1,\ f_0(y, \xi) = |\xi|^2$ and $g_0(y, u) = \lambda|u|^2$. In this case, we have only one limit phase, so that no interaction between phases is possible, but the function φ is nontrivial:

$$\varphi(z) = |z|^2 \min\biggl\{\int_{E_0\cap(0,1)^n} \bigl(|Dv|^2 + \lambda|v|^2\bigr)\,\mathrm{d}y\colon\ v \in W^{1,2}\bigl((0,1)^n\bigr),$$
$$v = 1 \text{ on } E_1 \cap (0,1)^n,\ v \text{ is 1-periodic}\biggr\}. \tag{1.8}$$

The introduction of the parameter λ allows us to highlight that in (1.8) φ can be considered as a transition-layer effect. Suppose for simplicity that $E_0 \cap (0,1)^n = E \Subset (0,1)^n$. In this case it is well known (from [21,20,11,3] for example) that

$$\lim_{\lambda\to+\infty}\min\left\{\int_E\left(\frac{|Dv|^2}{\sqrt{\lambda}}+\sqrt{\lambda}|v|^2\right)\mathrm{d}y\colon\ v\in W^{1,2}\big((0,1)^n\big),\ v=1\text{ on }\partial E\right\}=\mathcal{H}^{n-1}(\partial E),\tag{1.9}$$

hence $\varphi(z)$ behaves as $\sqrt{\lambda}\mathcal{H}^{n-1}(\partial E)|z|^2$ for large values of λ. In the case of the evolution version of double-porosity model related to problem (1.2), the presence of the above term $\varphi(z)$ of sublinear growth in λ is responsible for the appearance of a nonlocal integral operator in the homogenized evolution equation.

In the last section of the paper we consider energies with other scalings. Namely, we assume that the scaling factor of the soft phase does not match the growth conditions. In this case the energy functional takes the form

$$F_\varepsilon^q(u)=\sum_{j=1}^N\int_{\Omega\cap\varepsilon E_j}\left(f_j\left(\frac{x}{\varepsilon},Du\right)+g_j\left(\frac{x}{\varepsilon},u\right)\right)\mathrm{d}x+\int_{\Omega\cap\varepsilon E_0}\left(\varepsilon^q f_0\left(\frac{x}{\varepsilon},Du\right)+g_0\left(\frac{x}{\varepsilon},u\right)\right)\mathrm{d}x$$

with $q\neq p$. We show that for $0<q<p$ the Γ-limit of F_ε^q with respect to the convergence $u_\varepsilon\to(u_1,\dots,u_N)$ if finite only if $u_1=\dots=u_N$, and it coincides with that computed with respect to the strong L^p convergence, while for $q>p$ the phases are asymptotically decoupled, the contribution of the soft phase being reduced to a constant.

2. Notation and preliminaries

We use standard notation for Lebesgue and Sobolev spaces. By $\mathbb{M}^{m\times n}$ we denote the space of $m\times n$ matrices. The Lebesgue measure of a set E is denoted by $|E|$. The Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^{n-1}$. By $[t]$ we denote the integer part of $t\in\mathbb{R}$. The average of a function f on a nonempty set A is denoted by $⨍_A f\,\mathrm{d}x=|A|^{-1}\int_A f\,\mathrm{d}x$. If $T>0$, a function f defined on $\mathbb{R}^n$ is said to be T-periodic if $f(x+Te_i)=f(x)$ for all x and i, $\{e_i\}$ being the standard orthonormal basis of $\mathbb{R}^n$. The space $W^{1,p}_\#((0,T)^n;\mathbb{R}^m)$ is the space of T-periodic $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$-functions. The symbol C denotes a generic strictly positive constant.

We recall the definition of *Γ-convergence* of a sequence of functionals G_k defined on $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$, with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$-convergence. We say that (G_k) Γ-converges to G_0 on $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ as $k\to+\infty$, if for all $u\in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$

(i) (*Γ-liminf inequality*) for all sequences (u_k) of functions in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ that converge to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$, we have

$$G_0(u)\leqslant\liminf_k G_k(u_k);$$

(ii) (*Γ-limsup inequality*) there exists a sequence (u_k) in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ such that

$$G_0(u)\geqslant\limsup_k G_k(u_k).$$

We will say that a family (G_ε) Γ-converges to G_0 if for all sequences (ε_k) of positive numbers converging to 0 (i) and (ii) above are satisfied with G_{ε_k} in place of G_k. For a simplified introduction to Γ-convergence we refer to [12] (for a comprehensive study see [17]), while a detailed analysis of some of its applications to homogenization theory can be found in [14]. We will use the following Γ-convergence result proved in [13] (see also [1,15,14]).

Theorem 2.1. *Let E be a 1-periodic connected set in $\mathbb{R}^n$ with Lipschitz boundary, and let $f:\mathbb{R}^n\times\mathbb{M}^{M\times n}\to\mathbb{R}$ be a Borel function, 1-periodic in the first variable and satisfying a growth condition of order $p>1$. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, and let $H_\varepsilon: W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^M)\to\overline{\mathbb{R}}$ be defined by*

$$H_\varepsilon(u)=\int_{\Omega\cap\varepsilon E} f\left(\frac{x}{\varepsilon},Du\right)\mathrm{d}x.$$

Then H_ε Γ-converge on $W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^M)$, with respect to the $L^1_{\rm loc}(\Omega;\mathbb{R}^M)$-convergence to the functional $H_{\rm hom}$ given by

$$H_{\rm hom}(u)=\int_\Omega f_{\rm hom}(Du)\,\mathrm{d}x,$$

where $f_{\rm hom}$ satisfies the formula

$$f_{\rm hom}(\xi)=\lim_{T\to+\infty}\inf\left\{\frac{1}{T^n}\int_{(0,T)^n\cap E} f(y,Dv+\xi)\,\mathrm{d}y\colon\ v\in W^{1,p}_0\big((0,T)^n;\mathbb{R}^M\big)\right\},\tag{2.1}$$

and a growth condition of order p, so that the domain of $H_{\rm hom}$ is $W^{1,p}(\Omega;\mathbb{R}^M)$. Moreover, recovery sequences for $u\in W^{1,p}(\Omega;\mathbb{R}^M)$ can be chosen converging weakly in $W^{1,p}(\Omega;\mathbb{R}^M)$.

We will widely use the following extension result (see [1], [14, Theorem B2]).

Theorem 2.2. *Let $p>1$; let E be a periodic connected open subset of $\mathbb{R}^n$ with Lipschitz boundary. Given $\varepsilon>0$ there exists a linear and continuous extension operator $T_\varepsilon: W^{1,p}(\Omega\cap\varepsilon E;\mathbb{R}^m)\to W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^m)$ and two constants $k_0,k_1>0$ such that, letting*

$$\Omega(r)=\big\{x\in\Omega\colon\ \mathrm{dist}(x,\partial\Omega)>r\big\},$$

we have

$$\begin{aligned}&T_\varepsilon u=u\quad\text{in }\Omega\cap\varepsilon E,\\&\int_{\Omega(\varepsilon k_0)}|T_\varepsilon u|^p\,\mathrm{d}x\leqslant k_1\int_{\Omega\cap\varepsilon E}|u|^p\,\mathrm{d}x,\\&\int_{\Omega(\varepsilon k_0)}\big|D(T_\varepsilon u)\big|^p\,\mathrm{d}x\leqslant k_1\int_{\Omega\cap\varepsilon E}|Du|^p\,\mathrm{d}x\end{aligned}\tag{2.2}$$

for all $u\in W^{1,p}(\Omega\cap\varepsilon E;\mathbb{R}^m)$. The constants k_i are independent of ε and Ω.

3. Statement of the main result

Let $N \geqslant 1$. We fix N periodic connected open subsets of $\mathbb{R}^n$ with Lipschitz boundary, that we denote by $E_1, \ldots, E_N$. We assume that $\overline{E}_i \cap \overline{E}_j = \emptyset$ for $i \neq j$ and set

$$E_0 = \mathbb{R}^n \setminus \bigcup_{j=1}^{N} E_j.$$

Then we consider Borel functions $f_k : E_k \times \mathbb{M}^{m\times n} \to \mathbb{R}$ and $g_k : E_k \times \mathbb{R}^m \to \mathbb{R}$, $k = 0, \ldots, N$, and suppose that they possess the following properties:

(i) (*periodicity*) $f_k(\cdot, \xi)$ and $g_k(\cdot, z)$ are 1-periodic;
(ii) (*p-growth condition*) there exist $p > 1$ and constants $c_0, c_1, c_2 > 0$ such that

$$c_0\big(|\xi|^p - 1\big) \leqslant f_k(y, \xi) \leqslant c_1\big(|\xi|^p + 1\big), \tag{3.1}$$

$$-c_2 \leqslant g_k(y, z) \leqslant c_1\big(|z|^p + 1\big); \tag{3.2}$$

(iii) (*Lipschitz continuity*)

$$\big|g_k(y, z) - g_k(y, z')\big| \leqslant c_1\big(1 + |z|^{p-1} + |z'|^{p-1}\big)|z - z'|. \tag{3.3}$$

We define the energies

$$F_\varepsilon(u) = \sum_{j=1}^{N} \int_{\Omega\cap\varepsilon E_j} \left(f_j\left(\frac{x}{\varepsilon}, Du\right) + g_j\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x + \int_{\Omega\cap\varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du\right) + g_0\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x \tag{3.4}$$

for $u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

Let the extension operators T_ε^j corresponding to $\Omega \cap E_j$ be defined as in Theorem 2.2. We define the extension operator $\mathbf{T}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \to (W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m))^N$ by

$$(\mathbf{T}_\varepsilon u)_j = T_\varepsilon^j(u|_{\Omega\cap\varepsilon E_j}).$$

We consider the convergence on $W^{1,p}(\Omega; \mathbb{R}^m)$ defined as the $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ convergence of these extensions. Namely, we will write that

$$u_\varepsilon \to (u_1, \ldots, u_N) \quad \text{if } \mathbf{T}_\varepsilon u_\varepsilon \to (u_1, \ldots, u_N) \text{ in } \big(L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)\big)^N. \tag{3.5}$$

Note that by Theorem 2.2 F_ε are equicoercive on bounded sets of L^1 with respect to this convergence. Note also that the limit function in (3.5) and the results below do not depend on the particular choice of the extension operators, as it happens in the case of homogenization of Neumann boundary-value problems in perforated domains (see, for instance, [1]). An alternative equivalent definition of the topology is given in Remark 3.4.

We study the Γ-convergence of the energies F_ε with respect to the convergence in (3.5). Before stating our main result, we introduce some notation. Let $f^j_{\mathrm{hom}}, \tilde g_j, \varphi : \mathbb{R}^{mN} \to \mathbb{R}$ ($j = 1,\dots,N$) be the functions defined by

$$f^j_{\mathrm{hom}}(\xi) = \lim_{T\to+\infty} \inf\left\{ \frac{1}{T^n} \int_{(0,T)^n\cap E_j} f_j(y, Dv+\xi)\,\mathrm{d}y:\ v \in W^{1,p}_0\big((0,T)^n;\mathbb{R}^m\big)\right\} \tag{3.6}$$

(this is a good definition by (2.1) in Theorem 2.1),

$$\tilde g_j(z) = \int_{(0,1)^n\cap E_j} g_j(y,z)\,\mathrm{d}y, \tag{3.7}$$

and

$$\varphi(z_1,\dots,z_N) = \lim_{T\to+\infty} \inf\left\{ \frac{1}{T^n} \int_{E_0\cap(0,T)^n} \big(f_0(y,Dv) + g_0(y,v)\big)\,\mathrm{d}y:\ v \in W^{1,p}\big((0,T)^n;\mathbb{R}^m\big),\ v = z_j \text{ on } E_j,\ j = 1,\dots,N\right\}. \tag{3.8}$$

The existence of the limit in the definition of the function φ will be proved in Proposition 4.1.

Note that by Theorem 2.1, if we set

$$F^j_\varepsilon(u) = \int_{\Omega\cap\varepsilon E_j} f_j\left(\frac{x}{\varepsilon}, Du\right)\mathrm{d}x \tag{3.9}$$

on $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, then F^j_ε Γ-converge on $W^{1,p}(\Omega;\mathbb{R}^m)$ (with respect to the L^1_{loc} convergence) to

$$F^j_{\mathrm{hom}}(u) = \int_\Omega f^j_{\mathrm{hom}}(Du)\,\mathrm{d}x. \tag{3.10}$$

The main result of this work is the following theorem.

Theorem 3.1. *Suppose in addition to hypotheses* (i)–(iii) *above, that $f_0(z,\cdot)$ is positively homogeneous of degree p and that $|\partial\Omega| = 0$. Then the functionals F_ε defined by* (3.4) *Γ-converge with respect to the convergence* (3.5) *to the functional F_0 with domain $(W^{1,p}(\Omega;\mathbb{R}^m))^N$ defined by*

$$F_0(u_1,\dots,u_N) = \sum_{j=1}^N \int_\Omega \big(f^j_{\mathrm{hom}}(Du_j) + \tilde g_j(u_j)\big)\,\mathrm{d}x + \int_\Omega \varphi(u_1,\dots,u_N)\,\mathrm{d}x,$$

with f^j_{hom}, $\tilde g_j$ and φ given by (3.6)–(3.8). *Namely,*

(i) (coerciveness) *from any sequence (u_ε) that is bounded in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and satisfies $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, one can extract a subsequence* (*not relabelled*) *such that $\mathbf{T}_\varepsilon u_\varepsilon$ converges in $(L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m))^N$ to some $(u_1,\dots,u_N) \in (W^{1,p}(\Omega;\mathbb{R}^m))^N$;*

(ii) (liminf inequality) *for all* $(u_1, \dots, u_N) \in (W^{1,p}(\Omega;\mathbb{R}^m))^N$ *and* $u_\varepsilon \to (u_1, \dots, u_N)$ *we have*

$$F_0(u_1, \dots, u_N) \leqslant \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon);$$

(iii) (limsup inequality) *for all* $(u_1, \dots, u_N) \in (W^{1,p}(\Omega;\mathbb{R}^m))^N$ *there exists* $u_\varepsilon \to (u_1, \dots, u_N)$ *such that we have*

$$F_0(u_1, \dots, u_N) \geqslant \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

Corollary 3.2 (convergence of minimum problems). *Let* Ω *have a Lipschitz boundary, and let* $\phi_j \in W^{1,p}(\Omega;\mathbb{R}^m)$ *be given, for* $j = 1, \dots, N$. *We can consider the minimum problems*

$$m_\varepsilon = \min\bigl\{F_\varepsilon(u)\colon\ u = \phi_j \text{ on } \varepsilon E_j \cap \partial\Omega,\ j = 1, \dots, N\bigr\}, \tag{3.11}$$

where the boundary condition means that the function

$$v_j = \begin{cases} u & \text{on } \Omega \cap \varepsilon E_j, \\ \phi_j & \text{on } (\mathbb{R}^n \setminus \Omega) \cap \varepsilon E_j \end{cases}$$

belongs to $W^{1,p}_{\mathrm{loc}}(\varepsilon E_j;\mathbb{R}^m)$. *Then the values* m_ε *converge to*

$$m = \min\bigl\{F_0(u_1, \dots, u_N)\colon\ u_j = \phi_j \text{ on } \partial\Omega,\ j = 1, \dots, N\bigr\}$$

and from every sequence of minimizers $\{u_\varepsilon\}$ *for* m_ε *we can extract a subsequence* (*not relabelled*) *such that* $u_\varepsilon \to (u_1, \dots, u_N)$, *where* $(u_1, \dots, u_N)$ *is a minimizer for* m.

Proof. The proof of the corollary follows immediately from Theorem 3.1 and the equicoerciveness of F_ε thanks to the properties of Γ-convergence, once we notice two facts. First, the boundary condition is ‘closed’ under our convergence provided that $F_\varepsilon(u_\varepsilon) \leqslant C$, i.e., if $u_\varepsilon \to (u_1, \dots, u_N)$ and $u_\varepsilon = \phi_j$ on $\varepsilon E_j \cap \partial\Omega$, for every $j = 1, \dots, N$, and if $F_\varepsilon(u_\varepsilon) \leqslant C$, then $u_j = \phi_j$ on $\partial\Omega$, for every $j = 1, \dots, N$ (see [13, Section 4]). Second, for each function $(u_1, \dots, u_N)$ with $u_j = \phi_j$ on $\partial\Omega$ it is possible to find a recovery sequence $u_\varepsilon \to (u_1, \dots, u_N)$ satisfying the boundary conditions in (3.11) by modifying a recovery sequence for the limsup inequality (see, e.g., [14, Section 11.3]). □

Remark 3.3 (*simplified formulae*). (1) If E_0 is composed of bounded disconnected components, i.e., if $E_0 = \mathbb{Z}^n + E$ where E is bounded and such that $(i + E) \cap (j + E) = \emptyset$ for $i \neq j$, then necessarily $N = 1$ and we have

$$\varphi(z) = \inf\biggl\{\int_E \bigl(f_0(y, Dv) + g_0(y, v)\bigr)\,\mathrm{d}y\colon\ v \in W^{1,p}\bigl((0,1)^n;\mathbb{R}^m\bigr),\ v = z \text{ on } \partial E\biggr\}. \tag{3.12}$$

(2) In the convex case formulae (3.6) and (3.8) simplify to periodic cell problems:

$$f^j_{\mathrm{hom}}(\xi) = \inf\biggl\{\int_{E_j \cap (0,1)^n} f_j(y, Dv + \xi)\,\mathrm{d}y\colon\ v \in W^{1,p}_{\#}\bigl((0,1)^n;\mathbb{R}^m\bigr)\biggr\} \tag{3.13}$$

and

$$\varphi(z_1,\dots,z_N)=\inf\biggl\{\int_{E_0\cap(0,1)^n}\bigl(f_0(y,Dv)+g_0(y,v)\bigr)\,\mathrm{d}y:\\ v\in W^{1,p}_{\#}\bigl((0,1)^n;\mathbb{R}^m\bigr),\ v=z_j \text{ on } E_j,\ j=1,\dots,N\biggr\},\tag{3.14}$$

(see Remark 4.3 below).

Remark 3.4 (*alternative definition of convergence*). The convergence $u_\varepsilon\to(u_1,\dots,u_N)$ can be equivalently defined by

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{N}\int_{\Omega\cap\varepsilon E_j}|u_\varepsilon-u_j|^p\,\mathrm{d}x=0.$$

Note that in this way we do not have to suppose that E_j is Lipschitz, but only connected (see, for instance, [28]).

3.1. Some generalizations

We briefly describe a number of generalizations of our result, with references to the technical points that can be easily reworked.

The nonpositively homogeneous case. The hypothesis that f_0 is positively homogeneous of degree p can be removed at the expense of heavier notation and the necessity of passing to a subsequence. Consider for example the homogeneous case of $f_0(y,\xi)=f_0(\xi)$; we then introduce the functions

$$h_\varepsilon(\xi)=\varepsilon^p Qf_0\biggl(\frac{\xi}{\varepsilon}\biggr),$$

where Q denotes the operation of *quasiconvexification* (see, e.g., [14, Section 6.2]). Upon extracting a subsequence we may define

$$h_0(\xi)=\lim_j \varepsilon_j^p Qf_0\biggl(\frac{\xi}{\varepsilon_j}\biggr).$$

In this case the function φ is characterized by

$$\varphi(z_1,\dots,z_N)=\lim_{T\to+\infty}\inf\biggl\{\frac{1}{T^n}\int_{E_0\cap(0,T)^n}\bigl(h_0(Dv)+g_0(y,v)\bigr)\,\mathrm{d}y:\\ v\in W^{1,p}\bigl((0,T)^n;\mathbb{R}^m\bigr),\ v=z_j \text{ on } E_j\biggr\}.\tag{3.15}$$

For details on this procedure we refer to the analogue definition in the framework of perforated domains in the nonlinear vector case (see [5]).

The locally periodic case. We can take into account a locally periodic dependence on x in our energy densities by considering F_ε of the form

$$F_\varepsilon(u)=\sum_{j=1}^{N}\int_{\Omega\cap\varepsilon E_j}\Big(f_j\Big(x,\frac{x}{\varepsilon},Du\Big)+g_j\Big(x,\frac{x}{\varepsilon},u\Big)\Big)\,\mathrm{d}x + \int_{\Omega\cap\varepsilon E_0}\Big(\varepsilon^p f_0\Big(x,\frac{x}{\varepsilon},Du\Big)+g_0\Big(x,\frac{x}{\varepsilon},u\Big)\Big)\,\mathrm{d}x. \tag{3.16}$$

If locally uniform continuity conditions are satisfied, of the form

$$\begin{aligned}&\big|f_j(x,y,\xi)-f_j(x',y,\xi)\big|\leqslant\omega\big(|x-x'|\big)\big(1+|\xi|^p\big),\\ &\big|g_j(x,y,z)-g_j(x',y,z)\big|\leqslant\omega\big(|x-x'|\big)\big(1+|z|^p\big),\end{aligned} \tag{3.17}$$

where ω is a continuous function with $\omega(0)=0$, and if $f_j(x,\cdot,\cdot)$, $g_j(x,\cdot,\cdot)$ satisfy hypotheses (i)–(iii) for all x, then Theorem 3.1 still holds with F_0 of the form

$$F_0(u_1,\dots,u_N)=\sum_{j=1}^{N}\int_\Omega\big(f^j_{\mathrm{hom}}(x,Du_j)+\tilde g_j(x,u_j)\big)\,\mathrm{d}x+\int_\Omega\varphi(x,u_1,\dots,u_N)\,\mathrm{d}x.$$

The definitions of the energy densities f^j_{hom}, $\tilde g_j$, φ are the same as before, the variable x acting as a parameter.

The proof of this generalization is easily obtained by locally 'freezing' the variable x (see, e.g., [14, Exercise 14.6] for details). Note that the continuity in the variable x in (3.17) is not easily removed. We can nevertheless consider the particular case $g(x,y,z)=g(y,z)-h(x)\cdot z$, where we add the term $h(x)\cdot z$ being measurable in x. This situation is considered in detail in Section 7.4.

4. Definition of the limit energy densities

We now turn to the characterization of φ. For all $T>0$ let

$$\varphi_T(z_1,\dots,z_N)=\inf\Big\{\frac{1}{T^n}\int_{(0,T)^n\cap E_0}\big(f_0(y,Dv)+g_0(y,v)\big)\,\mathrm{d}y:\ v\in W^{1,p}\big((0,T)^n;\mathbb{R}^m\big),\ v=z_j \text{ on } E_j,\ j=1,\dots,N\Big\}. \tag{4.1}$$

Proposition 4.1. *For all $z_1,\dots,z_N$ there exists the limit*

$$\varphi(z_1,\dots,z_N)=\lim_{T\to+\infty}\varphi_T(z_1,\dots,z_N).$$

Moreover, for any $u \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)$ *such that* $u = z_j$ *on* E_j *for all* $j = 1, \dots, N$*, we have*

$$\varphi(z_1,\dots,z_N) = \lim_{T\to+\infty} \inf\biggl\{ \frac{1}{T^n} \int_{(0,T)^n\cap E_0} \bigl(f_0(y,Dv) + g_0(y,v)\bigr)\,\mathrm{d}y\colon \\ v \in W^{1,p}\bigl((0,T)^n;\mathbb{R}^m\bigr),\ v = u \text{ on } \bigcup_{j=1}^N E_j \cup \partial(0,T)^n \biggr\}. \tag{4.2}$$

Proof. Let $0 < T < S$. For all $v \in W^{1,p}((0,S)^n;\mathbb{R}^m)$ with $v = z_j$ on E_j we clearly have

$$\begin{aligned} &\frac{1}{S^n}\int_{(0,S)^n\cap E_0} \bigl(f_0(y,Dv) + g_0(y,v)\bigr)\,\mathrm{d}y \\ &\quad\geqslant \frac{1}{S^n} \sum_{i\in\mathbb{Z}^n\colon\, 0\leqslant[T+1](i_j+1)\leqslant S} \int_{(i[T+1]+(0,T)^n)\cap E_0} \bigl(f_0(y,Dv) + g_0(y,v)\bigr)\,\mathrm{d}y \\ &\quad\geqslant \frac{1}{S^n}\biggl[\frac{S}{T+1}\biggr]^n T^n \varphi_T(z_1,\dots,z_N); \end{aligned}$$

hence, by the arbitrariness of v

$$\varphi_S(z_1,\dots,z_N) \geqslant \frac{T^n}{S^n}\biggl[\frac{S}{T+1}\biggr]^n \varphi_T(z_1,\dots,z_N),$$

from which we deduce, by letting $S \to +\infty$, that

$$\liminf_{S\to+\infty} \varphi_S(z_1,\dots,z_N) \geqslant \biggl(\frac{T}{T+1}\biggr)^n \varphi_T(z_1,\dots,z_N),$$

and finally, by letting $T \to +\infty$, that

$$\liminf_{S\to+\infty} \varphi_S(z_1,\dots,z_N) \geqslant \limsup_{T\to+\infty} \varphi_T(z_1,\dots,z_N);$$

that is the first characterization of φ.

Let $\{u^T\}$ be a sequence of test functions such that

$$\int_{(0,T)^n\cap E_0} \bigl(f_0\bigl(y,Du^T\bigr) + g_0\bigl(y,u^T\bigr)\bigr)\,\mathrm{d}y \leqslant T^n\bigl(\varphi_T(z_1,\dots,z_N) + \mathrm{o}(1)\bigr),$$

where o(1) tends to zero as $T \to \infty$. We then have

$$\lim_{T\to+\infty} \frac{1}{T^n} \int_{((0,T)^n\setminus(1,T-1)^n)\cap E_0} \bigl(f_0\bigl(y,Du^T\bigr) + g_0\bigl(y,u^T\bigr)\bigr)\,\mathrm{d}y = 0 \tag{4.3}$$

(otherwise we obtain a contradiction by considering $v(x) = u^T(x + (1,\dots,1))$ as a test function in $\varphi_{T-2}(z_1,\dots,z_N)$). Now, let $u \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)$ be such that $u = z_j$ on E_j; let ϕ_T be a smooth cut-off function such that $\phi^T = 1$ on $(1, T-1)^n$, $\phi^T = 0$ on $\partial(0,T)^n$ and $|D\phi^T| \leqslant 2$. We then define $v^T = \phi^T u^T + (1-\phi^T)u$ and get

$$\begin{aligned}
\varphi_T^0(z_1,\dots,z_N) &\stackrel{\text{def}}{=} \inf\left\{\frac{1}{T^n}\int_{(0,T)^n\cap E_0} \big(f_0(y,Dv)+g_0(y,v)\big)\,\mathrm{d}y\colon\ v=u \ \text{ on } \bigcup_{j=1}^N E_j\cup\partial(0,T)^n\right\}\\
&\leqslant \frac{1}{T^n}\int_{(0,T)^n\cap E_0}\big(f_0\big(y,Dv^T\big)+g_0\big(y,v^T\big)\big)\,\mathrm{d}y\\
&\leqslant \frac{1}{T^n}\int_{((0,T)^n\setminus(1,T-1)^n)\cap E_0}\big(f_0\big(y,Dv^T\big)+g_0\big(y,v^T\big)\big)\,\mathrm{d}y+\varphi_{T-2}(z_1,\dots,z_N)+\mathrm{o}(1)\\
&\leqslant \frac{1}{T^n}\int_{((0,T)^n\setminus(1,T-1)^n)\cap E_0} c\big(1+\big|Du^T\big|^p+|Du|^p+\big|u^T\big|^p+|u|^p\big)\,\mathrm{d}y\\
&\quad+\varphi_{T-2}(z_1,\dots,z_N)+\mathrm{o}(1),
\end{aligned}$$

where o(1) vanishes as $T\to+\infty$. Letting $T\to+\infty$ we deduce by (4.3), (3.1) and by the Poincaré inequality, that

$$\limsup_{T\to+\infty}\varphi_T^0(z_1,\dots,z_N)\leqslant\varphi(z_1,\dots,z_N).$$

Since we trivially have $\varphi_T^0(z_1,\dots,z_N)\geqslant\varphi_T(z_1,\dots,z_N)$ the equality in (4.2) is proved. □

Note that in the statement and proof above we can replace the 1-periodicity of u by an equi-integrability condition for $|u|^p$ and $|Du|^p$. Namely, the following statement holds, with the same proof as that of the previous proposition.

Proposition 4.2. *Let $\{v_T\}$ be a sequence of $W^{1,p}_{\mathrm{loc}}$ functions such that $v_T = z_j$ on E_j and $T^{-n}\|v_T\|^p_{W^{1,p}((0,T)^n\setminus(1,T-1)^n)}\to 0$ as $T\to\infty$, and assume that $\{u_T\}$ is a sequence of test functions that satisfies the bounds*

$$\int_{(0,T)^n\cap E_0}\big(f_0\big(y,Du^T\big)+g_0\big(y,u^T\big)\big)\,\mathrm{d}y\leqslant T^n\varphi_T(z_1,\dots,z_N)+1$$

and the boundary conditions $u^T=z_j$ on E_j, and $u^T=v_T$ on $\partial(0,T)^n$. We then have

$$\lim_{T\to+\infty}\frac{1}{T^n}\int_{((0,T)^n\setminus(1,T-1)^n)\cap E_0}\big(f_0\big(y,Du^T\big)+g_0\big(y,u^T\big)\big)\,\mathrm{d}y=0.$$

Remark 4.3 (*the convex case*). If $f_0(y,\cdot)$ and $g_0(y\cdot)$ are convex then we can reduce to the *cell-problem formula*

$$\begin{aligned}
&\varphi(z_1,\dots,z_N)\\
&\quad=\inf\left\{\int_{(0,1)^n\cap E_0}\big(f_0(y,Dv)+g_0(y,v)\big)\,\mathrm{d}y\colon\ v\in W^{1,p}_{\#}\big((0,1)^n;\mathbb{R}^m\big),\ v=z_j\ \text{on } E_j\right\}.
\end{aligned}\tag{4.4}$$

In fact, regardless to the convexity assumption, from Proposition 4.1 we immediately obtain that φ can be equivalently expressed by

$$\varphi(z_1,\dots,z_N)=\lim_{K\to+\infty}\inf\biggl\{\frac{1}{K^n}\int_{(0,K)^n\cap E_0}\bigl(f_0(y,Dv)+g_0(y,v)\bigr)\,\mathrm{d}y\colon\ v\in W^{1,p}_{\#}\bigl((0,K)^n;\mathbb{R}^m\bigr),\ v=z_j \text{ on } E_j\biggr\} \tag{4.5}$$

($K\in\mathbb{N}$). The proof of formula (4.4) may be then obtained from (4.5) following standard convexity arguments (see, e.g., [14, Section 14.3]).

Example 4.4. Let $f_0(\xi)=|\xi|^p$ and $N=m=1$. Let us suppose that $E_0=E+\mathbb{Z}^n$ with $(E+i)\cap(E+j)=\emptyset$ for $i\neq j$ and $g_0(z)=\lambda|z|^p$. Then for large values of λ and fixed z the function $\varphi(z)$ behaves as $c\lambda^{1/p'}$ where $p'=p/(p-1)$. In fact, by Remark 3.3(1) we can write

$$\varphi(z)=\varphi_\lambda(z)=|z|^p\inf\biggl\{\int_E\bigl(|Dv|^p+\lambda|v|^p\bigr)\,\mathrm{d}y\colon\ v\in W^{1,p}(E),\ v=1 \text{ on } \partial E\biggr\}. \tag{4.6}$$

It is well-known that the functionals defined on $W^{1,p}(E)$ by

$$\Lambda_\eta(v)=\begin{cases}\displaystyle\int_E\biggl(\eta^{p-1}|Dv|^p+\frac{1}{\eta}|v|^q\biggr)\,\mathrm{d}y & \text{if } v=1 \text{ on } \partial E,\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converge as $\eta\to 0$ in $L^1(E)$ to the trivial functional

$$\Lambda(v)=\begin{cases}C\mathcal{H}^{n-1}(\partial E) & \text{if } v=0 \text{ a.e. in } E,\\ +\infty & \text{otherwise,}\end{cases}$$

where $C=(q+p')^{-1}$ (see, e.g., [11] for details) and in particular

$$\lim_{\eta\to 0}\min\Lambda_\eta=\min\Lambda=C\mathcal{H}^{n-1}(\partial E).$$

By taking $\eta=\lambda^{-1/p}$ and $q=p$ we easily see that

$$\lim_{\lambda\to+\infty}\frac{\varphi_\lambda(z)}{|z|^p}\lambda^{-1/p'}=\lim_{\eta\to 0+}\min\Lambda_\eta=\frac{p-1}{p^2}\mathcal{H}^{n-1}(\partial E).$$

5. Proof of the lower bound

Upon a relaxation argument (see [2]) it is not restrictive to suppose that f_0 is quasiconvex and in particular that it satisfies the local Lipschitz condition

$$\bigl|f_0(y,\xi)-f_0(y,\xi')\bigr|\leqslant c\bigl(1+|\xi|^{p-1}+|\xi'|^{p-1}\bigr)|\xi-\xi'| \tag{5.1}$$

(see [14, Section 4.3])

We now choose a sequence $u_\varepsilon \to (u_1, \dots, u_N)$ with $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Note that for all $j = 1, \dots, N$ we have

$$\lim_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_j} g_j\left(\frac{x}{\varepsilon}, v_\varepsilon\right) \mathrm{d}x = \int_\Omega \tilde{g}_j(v)\,\mathrm{d}x$$

whenever $v_\varepsilon \to v$ in $L^p(\Omega;\mathbb{R}^m)$. Hence, since $T_\varepsilon^j u_\varepsilon$ converges to u_j locally weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, thanks to the convergence in (3.9) and (3.10) we obtain

$$\liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_j} \left(f_j\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_j\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \geqslant \int_\Omega \left(f^j_{\mathrm{hom}}(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x. \tag{5.2}$$

It remains to estimate the contribution on $\Omega \cap \varepsilon E_0$. To this end we choose $K \in \mathbb{N}$; let k_0 be defined by Theorem 2.2 and set

$$I_\varepsilon^K = \left\{i \in \mathbb{Z}^n\colon\ \varepsilon K i + \left(-\varepsilon k_0, \varepsilon(K + k_0)\right)^n \subset \Omega\right\}.$$

For all $i \in I_\varepsilon^K$ and $j = 1, \dots, N$ we denote $Q_K^i = \varepsilon K i + (0, \varepsilon K)^n$,

$$u^i_{j,\varepsilon} = \frac{1}{K^n\varepsilon^n|E_j \cap (0,1)^n|} \int_{Q_K^i\cap\varepsilon E_j} u_\varepsilon\,\mathrm{d}x, \tag{5.3}$$

and then define $w^i_{j,\varepsilon}$ on $\varepsilon K i + (-\varepsilon k_0, \varepsilon(K + k_0))^n \cap E_j$ by setting

$$w^i_{j,\varepsilon}(x) = u_\varepsilon(x) - u^i_{j,\varepsilon}.$$

Applying Theorem 2.2 with $E = E_j$ and $\varepsilon K i + (-\varepsilon k_0, \varepsilon(K + k_0))^n$ in place of Ω one can extend $w^i_{j,\varepsilon}$ to $\varepsilon K i + (-\varepsilon k_0, \varepsilon(K + k_0))^n$ in such a way that

$$\int_{Q_K^i} |w^i_{j,\varepsilon}|^p\,\mathrm{d}x \leqslant c(K) \int_{\varepsilon K i+(-\varepsilon k_0,\varepsilon(K+k_0))^n\cap\varepsilon E_j} |u_\varepsilon - u^i_{j,\varepsilon}|^p\,\mathrm{d}x$$

and

$$\int_{Q_K^i} |Dw^i_{j,\varepsilon}|^p\,\mathrm{d}x \leqslant c(K) \int_{\varepsilon K i+(-\varepsilon k_0,\varepsilon(K+k_0))^n\cap\varepsilon E_j} |Du_\varepsilon|^p\,\mathrm{d}x. \tag{5.4}$$

Upon enlarging $\varepsilon K i + (-\varepsilon k_0, \varepsilon(K + k_0))^n \cap \varepsilon E_j$ to a connected set (whose shape does not depend on i) we can use Poincaré's inequality and (5.4) to get

$$\int_{Q_K^i} |w^i_{j,\varepsilon}|^p\,\mathrm{d}x \leqslant \varepsilon^p c(K) \int_{\varepsilon K i+(-\varepsilon k_0,\varepsilon(K+k_0))^n\cap\varepsilon E_j} |Du_\varepsilon|^p\,\mathrm{d}x. \tag{5.5}$$

Let now ϕ_j be smooth functions with $\phi_j = 1$ on E_j and 0 on E_i if $i \neq j$, and set

$$w^i_\varepsilon(x) = \sum_{j=1}^N \phi_j\left(\frac{x}{\varepsilon}\right) w^i_{j,\varepsilon}.$$

Then, by means of (5.4) and (5.5), we have

$$\int_{Q_K^i} |w_\varepsilon^i|^p \,\mathrm{d}x \leqslant \varepsilon^p c(K) \sum_{j=1}^N \int_{\varepsilon K i + (-\varepsilon k_0, \varepsilon(K+k_0))^n \cap \varepsilon E_j} |Du_\varepsilon|^p \,\mathrm{d}x$$

and

$$\int_{Q_K^i} |Dw_\varepsilon^i|^p \,\mathrm{d}x \leqslant c(K) \sum_{j=1}^N \int_{\varepsilon K i + (-\varepsilon k_0, \varepsilon(K+k_0))^n \cap \varepsilon E_j} |Du_\varepsilon|^p \,\mathrm{d}x.$$

By (5.1), (3.3) and Hölder's inequality we then obtain

$$\begin{aligned}
&\left| \sum_{i \in I_\varepsilon^K} \int_{Q_K^i \cap \varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon - Dw_\varepsilon^i\right) - \varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_0\left(\frac{x}{\varepsilon}, u_\varepsilon - w_\varepsilon^i\right) - g_0\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \right| \\
&\quad \leqslant c(K)\left(\varepsilon^p |\Omega| + \varepsilon^p \int_\Omega |Du_\varepsilon|^p + \varepsilon^p \sum_i \int_{Q_K^i} |Dw_\varepsilon^i|^p \right)^{(p-1)/p} \left(\varepsilon^p \int_{\Omega \cap \varepsilon \bigcup_{j=1}^N E_j} |Dw_\varepsilon^i|^p \right)^{1/p} \\
&\quad \leqslant c(K)\varepsilon.
\end{aligned} \tag{5.6}$$

As a consequence of (5.6) we obtain

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \sum_{i \in I_\varepsilon^K} \int_{Q_K^i \cap \varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_0\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \\
&\quad = \liminf_{\varepsilon \to 0} \sum_{i \in I_\varepsilon^K} \int_{Q_K^i \cap \varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon - Dw_\varepsilon^i\right) + g_0\left(\frac{x}{\varepsilon}, u_\varepsilon - w_\varepsilon^i\right)\right) \mathrm{d}x \\
&\quad \geqslant \liminf_{\varepsilon \to 0} \sum_{i \in I_\varepsilon^K} \varepsilon^n K^n \varphi_K(u_{1,\varepsilon}^i, \dots, u_{N,\varepsilon}^i),
\end{aligned} \tag{5.7}$$

where in the last inequality we have used the definition of φ_K (after a suitable change of variables) and the fact that $u_\varepsilon - w_{j,\varepsilon}^i = u_{j,\varepsilon}^i$ on $Q_K^i \cap \varepsilon E_j$.

We now remark that

$$\lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon^K} \varepsilon^n K^n \varphi_K(u_{1,\varepsilon}^i, \dots, u_{N,\varepsilon}^i) = \int_\Omega \varphi_K(u_1, \dots, u_N) \,\mathrm{d}x. \tag{5.8}$$

Indeed, by the growth conditions on f_0 and g_0, $v \mapsto \varphi_K(v)$ is a continuous operator in L^p, so that it suffices to show that the piecewise constant functions u_ε^K defined by

$$u_{j,\varepsilon}^K(x) = u_{j,\varepsilon}^i \quad \text{on } Q_K^i \tag{5.9}$$

converge locally in $L^p(\Omega)$ to u_j. This is easily seen by applying Poincaré's inequality and using the L^p convergence of $T_\varepsilon^j u_\varepsilon$ to u_j.

Summing up the inequalities in (5.2), (5.7) and (5.8) we obtain

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geqslant \sum_{j=1}^{N} \int_\Omega \big(f^j_{\mathrm{hom}}(Du_j) + \tilde g_j(u_j)\big)\,\mathrm{d}x + \int_\Omega \varphi_K(u_1,\dots,u_N)\,\mathrm{d}x.$$

The liminf inequality is now obtained by taking the limit as $K \to +\infty$ and using Proposition 4.1 together with Fatou's lemma. □

6. Proof of the upper bound

We first remark that it suffices to restrict to the case when the target function $U = (u_1,\dots,u_N)$ is linear. The case of a piecewise linear U can be obtained by a localization argument (see the proof of Proposition 4.3 in [6] for a direct construction) and the general case is obtained by density (see [12, Remark 1.29]).

We fix $\xi_1,\dots,\xi_N$. For all $\eta > 0$ we will construct a recovery sequence $u_\varepsilon \to (\xi_1 x,\dots,\xi_N x)$ such that

$$F_0(\xi_1 x,\dots,\xi_N x) \geqslant \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - \eta,$$

which implies the limsup inequality.

For every $j = 1,\dots,N$ and $\delta > 0$, let $K \in \mathbb{N}$ and $u^K_j \in W^{1,p}_0((0,K)^n;\mathbb{R}^m)$ be such that

$$\int_{(0,K)^n\cap E_j} f_j\big(y, Du^K_j + \xi_j\big)\,\mathrm{d}y \leqslant K^n\big(f^j_{\mathrm{hom}}(\xi_j) + \delta\big), \tag{6.1}$$

by (3.10). Upon extending u^K_j outside $(0,K)^n$ we can suppose that the support of u^K_j is contained in $(1,K-1)^n$ and its intersection with E_j is connected. For all $i \in \mathbb{Z}^n$ let x^ε_i be the center of the cube

$$Q^\varepsilon_i = iK\varepsilon + (0,\varepsilon K)^n$$

and let v^ε_i be such that $v^\varepsilon_i = \xi_j x^\varepsilon_i$ in $E_j,\ j = 1,\dots,N$, and

$$\int_{(0,K)^n\cap E_0} \big(f_0(y, Dv^\varepsilon_i) + g_0(y, v^\varepsilon_i)\big)\,\mathrm{d}y \leqslant K^n \varphi_K\big(\xi_1 x^\varepsilon_i,\dots,\xi_N x^\varepsilon_i\big) + 1,$$

by (4.1).

For each $j = 1,\dots,N$, denote by ϕ_j a smooth 1-periodic cut-off functions with $\phi_j = 1$ on E_j and 0 on E_i if $i \neq j$, and by ϕ^K a K-periodic function with $\phi^K = 1$ on $(1,K-1)^n$ and 0 on $\partial(0,K)^n$. The function u_ε is defined piecewisely on each Q^ε_i by

$$\begin{aligned} u_\varepsilon(x) = {}& \phi^K\Big(\frac{x}{\varepsilon}\Big)\left(v^\varepsilon_i\Big(\frac{x-\varepsilon iK}{\varepsilon}\Big) + \sum_{j=1}^{N} \phi_j\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon u^K_j\Big(\frac{x}{\varepsilon}\Big) + \xi_j(x - x^\varepsilon_i)\Big)\right) \\ & + \Big(1 - \phi^K\Big(\frac{x}{\varepsilon}\Big)\Big)\sum_{j=1}^{N} \phi_j\Big(\frac{x}{\varepsilon}\Big)\xi_j x. \end{aligned} \tag{6.2}$$

Note that

$$u_\varepsilon(x) = \sum_{j=1}^{N} \phi_j\left(\frac{x}{\varepsilon}\right)\xi_j x, \quad \text{for } x \in \partial Q_i^\varepsilon,$$

so that $u_\varepsilon \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$. By construction $u_\varepsilon \to (\xi_1 x, \dots, \xi_N x)$ in the sense of (3.5).

We have by (6.2) and Proposition 4.2

$$\begin{aligned}
&\int_{Q_i^\varepsilon} \left|u_\varepsilon - v_i^\varepsilon\left(\frac{x-\varepsilon iK}{\varepsilon}\right)\right|^p \mathrm{d}x \\
&\quad \leqslant c\left(\varepsilon^p \sum_{j=1}^{N} \int_{Q_i^\varepsilon} \left|u_j^K\left(\frac{x}{\varepsilon}\right)\right|^p \mathrm{d}x + (\varepsilon K)^{p+n} + \varepsilon^n K^{n-1} + \mathrm{o}(1)\varepsilon^n K^n\right),
\end{aligned} \tag{6.3}$$

where $\mathrm{o}(1) \to 0$ as $K \to 0$, and also

$$\begin{aligned}
&\int_{Q_i^\varepsilon} \left|Du_\varepsilon - Dv_i^\varepsilon\left(\frac{x-\varepsilon iK}{\varepsilon}\right)\right|^p \mathrm{d}x \leqslant c\left(\sum_{j=1}^{N} \int_{Q_i^\varepsilon} \left|u_j^K\left(\frac{x}{\varepsilon}\right)\right|^p \mathrm{d}x + \varepsilon^n K^n + \varepsilon^{n-p} K^{n-1}\right) \\
&\quad + \varepsilon^{n-p}\mathrm{o}\big(K^n\big) + \sum_j \int_{Q_i^\varepsilon} \left|u_j^K\left(\frac{x}{\varepsilon}\right) + \xi_j(x - x_i^\varepsilon)\right|^p \mathrm{d}x + \sum_j \int_{Q_i^\varepsilon} \left|Du_j^K\left(\frac{x}{\varepsilon}\right) + \xi_j\right|^p \mathrm{d}x.
\end{aligned} \tag{6.4}$$

We now denote

$$J_\varepsilon^K = \big\{i \in \mathbb{Z}^n\colon\ \varepsilon iK + (0, \varepsilon K)^n \cap \Omega \neq \emptyset\big\}.$$

By proceeding similarly as in (5.6), we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_0\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \\
&\quad \leqslant \sum_{i\in J_\varepsilon^K} \limsup_{\varepsilon\to 0} \int_{Q_i^\varepsilon\cap\varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_0\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \\
&\quad = \sum_{i\in J_\varepsilon^K} \limsup_{\varepsilon\to 0} \int_{Q_i^\varepsilon\cap\varepsilon E_0} \left(\varepsilon^p f_0\left(\frac{x}{\varepsilon}, Dv_i^\varepsilon\left(\frac{x-\varepsilon iK}{\varepsilon}\right)\right) + g_0\left(\frac{x}{\varepsilon}, v_i^\varepsilon\left(\frac{x-\varepsilon iK}{\varepsilon}\right)\right)\right) \mathrm{d}x \\
&\quad \leqslant \limsup_{\varepsilon\to 0} \sum_{i\in J_\varepsilon^K} \varepsilon^n K^n \varphi_K\big(\xi_1 x_i^\varepsilon, \dots, \xi_N x_i^\varepsilon\big) + \mathrm{o}(1) = \int_\Omega \varphi_K(\xi_1 x, \dots, \xi_N x)\,\mathrm{d}x + \mathrm{o}(1)
\end{aligned} \tag{6.5}$$

where $\mathrm{o}(1)$ tends to zero, as $K \to +\infty$. Finally, since $u_\varepsilon = \varepsilon u_j^K(x/\varepsilon) + \xi_j$ on εE_j, by (6.1) we have

$$\limsup_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_j} \left(f_j\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) + g_j\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right) \mathrm{d}x \leqslant |\Omega| f^j_{\mathrm{hom}}(\xi_j) + \int_\Omega \tilde{g}_j(\xi_j x)\,\mathrm{d}x. \tag{6.6}$$

Summing up the inequalities in (6.5) and (6.6) we finally obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant F_0(\xi_1 x, \dots, \xi_N x)$$

as desired. □

7. Comparison with other types of convergence

In this section we compute the Γ-limits of the functionals F_ε with respect to other convergences. First, we describe such a Γ-limit with respect to the strong convergence in $L^p(\Omega;\mathbb{R}^m)$. It should be noted that this result cannot be applied to the study of convergence of energies in minimum problems since no compactness argument applies; but the form of the Γ-limit highlights the difference with the approach of the previous sections and provides an upper estimate for the Γ-limit in Theorem 3.1 when $N = 1$.

Subsequently, we treat the Γ-limit with respect to the weak convergence of all $N+1$ phases of u_ε so that the soft phase is also taken into account; this may be an alternate way of dealing with the asymptotic behaviour of energies in minimum problems since coerciveness properties with respect to the weak topology of $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ are easily available. To state the convergence result we first generalize the approach of the preceding sections by also considering the behaviour of the soft phase. In this way we obtain an integral Γ-limit defined on $N+1$ phases. It must be remarked that the derivatives of the 'limit soft phase' u_0 do not appear in the Γ-limit so that this variable can be easily minimized out to re-obtain the previous result. Note however that this $(N+1)$-phase formulation can be useful; for example, if integral constraints of the form $\int_\Omega u_\varepsilon\,\mathrm{d}x = C$ are added. Finally, we obtain the Γ-convergence result with respect to the weak $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$-convergence by averaging on the $N+1$ phases, thus obtaining a functional in a nonlocal form.

7.1. *Strong convergence in $L^p(\Omega;\mathbb{R}^m)$*

We now compute the Γ-limit of F_ε with respect to the strong L^p-convergence; i.e., when we consider $u_\varepsilon \to u$ strongly in $L^p(\Omega;\mathbb{R}^m)$ in the definition of the Γ-liminf and Γ-limsup inequalities. In the notation introduced in (3.5), in this case we also have $u_\varepsilon \to (u,\dots,u)$; i.e., $u_j = u$ for all $j = 1,\dots,N$. We introduce the function g_0 as

$$\tilde g_0(z) = \int_{(0,1)^n\cap E_0} g_0(y,z)\,\mathrm{d}y, \tag{7.1}$$

for every $z\in\mathbb{R}^m$. As for $g_j,\ j = 1,\dots,N$, if $u_\varepsilon \to u$ strongly in $L^p(\Omega;\mathbb{R}^m)$, then

$$\lim_{\varepsilon\to 0}\int_{\Omega\cap\varepsilon E_0} g_0\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\mathrm{d}x = \int_\Omega \tilde g_0(u)\,\mathrm{d}x. \tag{7.2}$$

With this observation in mind, the following proposition can be easily proven.

Proposition 7.1 (strong L^p-convergence). *Under the hypotheses in Theorem* 3.1*, the Γ-limit of* (F_ε) *with respect to the strong convergence in* $L^p(\Omega;\mathbb{R}^m)$ *is given by*

$$F_0^s(u) = \int_\Omega f^s_{\mathrm{hom}}(Du)\,\mathrm{d}x + \int_\Omega g^s(u)\,\mathrm{d}x, \quad u\in W^{1,p}(\Omega;\mathbb{R}^m), \tag{7.3}$$

where

$$f^s_{\mathrm{hom}}(\xi) = \sum_{j=1}^N f^j_{\mathrm{hom}}(\xi), \qquad g^s(z) = \sum_{j=0}^N \tilde g_j(z). \tag{7.4}$$

Proof. Since $u_\varepsilon \to u$ implies $u_\varepsilon \to (u, \dots, u)$, then the Γ-lim inf inequality follows as in the proof of Theorem 3.1, remarking in addition that (7.2) holds.

As for the Γ-lim sup inequality, we construct an optimal sequence $u_\varepsilon \to u$ as follows. Let $\Phi_j,\ j = 0, \dots, N$, be 1-periodic C^∞-functions such that $0 \leqslant \Phi_j \leqslant 1$,

$$\sum_{j=0}^N \Phi_j(y) = 1 \quad \text{and} \quad \begin{cases} \Phi_j = 1 & \text{in } E_j, \\ \Phi_j = 0 & \text{in } E_i, \text{for } i \neq j,\ i \in \{1, \dots, N\}, \end{cases}$$

for $j = 1, \dots, N$. For all $j = 1, \dots, N$ let (u^j_ε) be a recovery sequence for $\int_\Omega f^j_{\mathrm{hom}}(Du)\,\mathrm{d}x$, converging to u weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, and let $u^0_\varepsilon = u$. Then we set

$$u_\varepsilon(x) = \sum_{j=0}^N \Phi_j\left(\frac{x}{\varepsilon}\right) u^j_\varepsilon(x).$$

The Γ-lim sup inequality follows, upon remarking that, taking into account that $\sum_{j=0}^N D\Phi_j u^j_\varepsilon = \sum_{j=1}^N D\Phi_j(u^j_\varepsilon - u^0_\varepsilon)$ since $\sum_{j=0}^N D\Phi_j = 0$, we have

$$\varepsilon^p \int_{\Omega\cap\varepsilon E_0} f_0\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) \mathrm{d}x \leqslant C \sum_{j=1}^N \left(\varepsilon^p \int_\Omega |Du^j_\varepsilon|^p \,\mathrm{d}x + \int_\Omega |u^j_\varepsilon - u|^p \,\mathrm{d}x + \varepsilon^p |\Omega|\right) + C\varepsilon^p \int_\Omega \left(1 + |Du|^p\right) \mathrm{d}x,$$

so that this term is negligible in the limit. □

7.2. Weak convergence of $N+1$ phases

We can introduce an additional variable to describe the limit behaviour of sequences u_ε on the soft phase εE_0. Since we do not have strong-L^p coerciveness properties on this phase, we have to consider weak L^p limits.

Definition 7.2 (*weak convergence of phases*). Let (u_ε) be a family in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$; we say that $u_\varepsilon \rightharpoonup (u_0, u_1, \dots, u_N)$ (as $\varepsilon \to 0$) if

$$u_\varepsilon \chi_{\varepsilon E_j} \rightharpoonup u_j \left|E_j \cap (0,1)^n\right|$$

weakly in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ for all $j = 0, 1, \dots, N$.

Remark 7.3. If $u_\varepsilon \rightharpoonup (u_0, u_1, \dots, u_N)$ and $u_\varepsilon \to (v_1, \dots, v_N)$ in the sense of (3.5), then $u_j = v_j$ for $j = 1, \dots, N$. In fact,

$$u_\varepsilon \chi_{\varepsilon E_j} = (T^j_\varepsilon u_\varepsilon)\chi_{\varepsilon E_j} \rightharpoonup v_j \big|E_j \cap (0,1)^n\big|$$

by weak-strong convergence, so that $u_j = v_j$. Note that $\{u_\varepsilon\}$ is compact with respect to both topologies, upon requiring a boundedness assumption on $F_\varepsilon(u_\varepsilon)$.

In order to describe the effect of the additional variable u_0 on the shape of the Γ-limit (computed with respect to this new convergence) we introduce the energy density

$$\Phi(z_0, z_1 \dots, z_N) = \lim_{T\to+\infty} \Phi_T(z_0, z_1 \dots, z_N), \tag{7.5}$$

where

$$\Phi_T(z_0, z_1, \dots, z_N) = \inf\bigg\{ \frac{1}{T^n} \int_{E_0\cap(0,T)^n} \big(f_0(y, Dv) + g_0(y, v)\big)\, \mathrm{d}y: \\ v = z_j \text{ on } E_j \cap (0,T)^n,\ j = 1, \dots, N,\ \fint_{E_0\cap(0,T)^n} v \,\mathrm{d}x = z_0 \bigg\}. \tag{7.6}$$

The existence of the limit in (7.5) can be proved as in Proposition 4.1.

Remark 7.4. Following the reasonings of Section 4, we can prove some properties of Φ. Namely, for $(z_0, z_1, \dots, z_N) \in \mathbb{R}^{N+1}$ and a function $u \in W^{1,p}_\#((0,1)^n; \mathbb{R}^m)$ such that $u = z_j$ on E_j for $j = 1, \dots, N$, we define

$$\Phi^0_T(z_0, z_1 \dots, z_N) = \inf\bigg\{ \frac{1}{T^n} \int_{E_0\cap(0,T)^n} \big(f_0(y, Dv) + g_0(y, v)\big)\, \mathrm{d}y: \\ v = z_j \quad \text{on } E_j \cap (0,T)^n,\ j = 1, 2 \dots, N; \tag{7.7}$$

$$v = u \text{ on } \bigcup_{j=1}^N E_j \cup \partial(0,T)^n,\ \fint_{E_0\cap(0,T)^n} v \,\mathrm{d}x = z_0 \bigg\}. \tag{7.8}$$

We then have

(i) $\Phi = \Phi^0$, where $\Phi^0 = \lim_{T\to+\infty} \Phi^0_T$;
(ii) by the fact that we take u in Proposition 4.1 independent of z_0, it can be easily seen that the mapping $z_0 \mapsto \Phi^0(z_0, z_1, \dots, z_N)$ is convex for each $z_1, \dots, z_N$; moreover, $(z_1, \dots, z_N) \mapsto \Phi^0(z_0, z_1, \dots, z_N)$ is locally Lipschitz-continuous. In particular, by the growth condition on Φ^0 it follows that the functional

$$I(u_0, u_1, \dots, u_N) = \int_\Omega \Phi^0(u_0, u_1, \dots, u_N)\, \mathrm{d}x$$

is lower semicontinuous with respect to the weak convergence for u_0 and the strong convergence for $u_1, \dots, u_N$ in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ (see [19]);

(iii) if g_0 satisfies a growth condition from below of the type $g_0(z) \geqslant \psi(z)$, with ψ convex and

$$\liminf_{|z|\to+\infty} \frac{\psi(z)}{|z|} = +\infty, \tag{7.9}$$

then we have

$$\Phi_T(z_0, z_1, \dots, z_N) \geqslant \left|E_0 \cap (0,1)^n\right| \psi(z_0). \tag{7.10}$$

Since Φ_T are equilocally Lipschitz-continuous, they converge locally uniformly to Φ. By the growth condition and the convexity of Φ, we also have

$$\lim_{T\to+\infty} \Phi_T^{**}(z_0, z_1, \dots, z_N) = \Phi(z_0, z_1, \dots, z_N), \tag{7.11}$$

where Φ_T^{**} indicates the convex envelope of Φ_T with respect to the variable z_0, at fixed $z_1, \dots, z_N$.

Theorem 7.5 (Γ-limit on $N+1$ phases). *Let F_ε satisfy the hypotheses of Theorem* 3.1, *and suppose in addition that g_0 satisfies the growth condition of Remark* 7.4(iii). *Then the functionals F_ε Γ-converge with respect to the convergence introduced in Definition* 7.2 (*in the sense explained in Theorem* 3.1) *to the functional $\overline{F}_0$ defined on $L^p(\Omega;\mathbb{R}^m) \times (W^{1,p}(\Omega;\mathbb{R}^m))^N$ by*

$$\overline{F}_0(u_0, u_1, \dots, u_N) = \sum_{j=1}^N \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \int_\Omega \Phi(u_0, u_1, \dots, u_N)\, \mathrm{d}x. \tag{7.12}$$

Proof. The proof of the Γ-liminf inequality follows as in Section 5, noting that in the last inequality of (5.7) we obtain a term of the form

$$\sum_{i\in I_\varepsilon^K} \varepsilon^n K^n \Phi_K\left(u_{0,\varepsilon}^i, u_{1,\varepsilon}^i, \dots, u_{N,\varepsilon}^i\right),$$

where $u_{j,\varepsilon}^i$ are defined as in (5.3). After defining $u_{j,\varepsilon}^K$ as in (5.9), we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) &\geqslant \sum_{j=1}^N \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \liminf_{\varepsilon\to0} \int_\Omega \Phi_K\left(u_{0,\varepsilon}^K, u_{1,\varepsilon}^K, \dots, u_{N,\varepsilon}^K\right) \mathrm{d}x \\
&\geqslant \sum_{j=1}^N \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \liminf_{\varepsilon\to0} \int_\Omega \Phi_K^{**}\left(u_{0,\varepsilon}^K, u_{1,\varepsilon}^K, \dots, u_{N,\varepsilon}^K\right) \mathrm{d}x \\
&\geqslant \sum_{j=1}^N \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \int_\Omega \Phi_K^{**}(u_0, u_1, \dots, u_N)\, \mathrm{d}x,
\end{aligned} \tag{7.13}$$

since $\int_\Omega \Phi_K^{**}(u_0, u_1, \dots, u_N)\, \mathrm{d}x$ is lower semicontinuous with respect to the L^p-weak (in u_0)$\times(L^p)^N$-strong (in $u_1, \dots, u_N$) convergence (see [19]) and $u_{j,\varepsilon}^K \rightharpoonup u_j$, for $j = 0, \dots, N$. We can then let $K \to +\infty$ and conclude the proof by (7.11).

As for the proof of the Γ-limsup inequality, it follows exactly that of Section 6, upon remarking that it suffices to deal with the case where the target function $U = (u_0, u_1, \dots, u_N)$ is constant in the first component and linear in the others. If $U = (z_0, \xi_1 \cdot x, \dots, \xi_N \cdot x)$, then the construction can be repeated word for word, taking care of choosing v_i^ε satisfying $⨍_{E_0 \cap (0,K)^n} v_i^\varepsilon \, dx = z_0$. □

7.3. Weak convergence in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$

At this point, we can easily describe the Γ-limit of F_ε with respect to the weak L^p-convergence. In this case, the limit is a nonlocal functional.

We suppose that

$$g_0(y, z) \geqslant c\big(|z|^p - 1\big) \tag{7.14}$$

for all $y,\ z$, so that the definition of Γ-limit in the weak topology of L^p can be expressed through the Γ-liminf and Γ-limsup inequality in the same way as for the strong topology (see [17]).

Theorem 7.6 (Γ-limit with respect to the L^p-weak convergence). *Under the hypotheses of Theorem* 3.1, *and the additional assumption* (7.14) *the functionals F_ε Γ-converge with respect to the weak convergence in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ to the functional $F_0^w(u)$ given by*

$$F_0^w(u) = \inf\left\{ \overline{F}_0(u_0, u_1, \dots, u_N) \colon \sum_{j=0}^N |E_j \cap (0,1)^n| u_j = u \right\}.$$

Proof. To prove the Γ-liminf inequality, let $u_\varepsilon \rightharpoonup u$. We can suppose (up to extracting a subsequence) that there exists the limit $\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)$, and that $u_\varepsilon \rightharpoonup (u_0, u_1, \dots, u_N)$ with $\sum_{j=0}^N |E_j \cap (0,1)^n| u_j = u$; in fact, for every $v \in C_0^\infty(\Omega; \mathbb{R}^m)$

$$\int_\Omega u \cdot v \, dx = \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \cdot v \, dx = \lim_{\varepsilon \to 0} \sum_{j=0}^N \int_\Omega u_\varepsilon \chi_{\varepsilon E_j} \cdot v \, dx = \sum_{j=0}^N \int_\Omega u_j |E_j \cap (0,1)^n| \cdot v \, dx.$$

By Theorem 7.5 we then obtain

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geqslant \overline{F}_0(u_0, u_1, \dots, u_N) \geqslant F_0^w(u).$$

To prove the Γ-limsup inequality, given $u \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $\eta > 0$, we can choose $u_0, u_1, \dots, u_N$ such that $\sum_{j=0}^N |E_j \cap (0,1)^n| u_j = u$ and

$$\overline{F}_0(u_0, u_1, \dots, u_N) \leqslant F_0^w(u) + \eta.$$

By Theorem 7.5, there exists $u_\varepsilon \rightharpoonup (u_0, u_1, \dots, u_N)$ (and hence $u_\varepsilon \rightharpoonup u$) such that

$$F_\varepsilon(u_\varepsilon) \to \overline{F}_0(u_0, u_1, \dots, u_N),$$

from which the conclusion easily follows. □

7.4. Energies with additional forcing terms

The Γ-limit computed with respect to weak convergences is helpful when studying the functionals with additional inhomogeneous terms that are continuous with respect to the L^p-norm. Namely, we can describe the Γ-limit of energies of the form

$$F_\varepsilon^h(u) = F_\varepsilon(u) - \int_\Omega h \cdot u \,\mathrm{d}x,$$

where $h \in L^{p'}(\Omega;\mathbb{R}^m)$. In fact, the last term is continuous with respect to the convergence $u_\varepsilon \rightharpoonup (u_0,\dots,u_N)$ in the sense that

$$\lim_{\varepsilon\to 0}\int_\Omega h\cdot u_\varepsilon \,\mathrm{d}x = \sum_{j=0}^N \left|E_j\cap(0,1)^n\right|\int_\Omega h\cdot u_j\,\mathrm{d}x,$$

so that the Γ-limit is given by

$$\begin{aligned} F_0^h(u_0,\dots,u_N) = {} & \sum_{j=1}^N \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde g_j(u_j) - h\cdot u_j\left|E_j\cap(0,1)^n\right|\right)\mathrm{d}x \\ & + \int_\Omega \left(\Phi(u_0,u_1,\dots,u_N) - h\cdot u_0\left|E_0\cap(0,1)^n\right|\right)\mathrm{d}x. \end{aligned} \tag{7.15}$$

It is interesting to note that we can also recover the Γ-limit with respect to the convergence $u_\varepsilon \to (u_1,\dots,u_N)$ by minimizing out the dependence on u_0 in the last integral. For simplicity we deal with the convex case only. With fixed $(u_1,\dots,u_N)$ we have

$$\begin{aligned} & \min_{u_0}\int_\Omega \left(\Phi(u_0,u_1,\dots,u_N) - h\cdot u_0\left|E_0\cap(0,1)^n\right|\right)\mathrm{d}x \\ & \quad = \int_\Omega \min_{z\in\mathbb{R}^m}\left(\Phi(z,u_1,\dots,u_N) - h(x)\cdot z\left|E_0\cap(0,1)^n\right|\right)\mathrm{d}x = \int_\Omega \Psi(h,u_1,\dots,u_N)\,\mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} \Psi(t,z_1,\dots,z_N) = {} & \min_{z\in\mathbb{R}^m}\biggl\{\min\biggl\{\int_{E_0\cap(0,1)^n}\bigl(f_0(y,Dv)+g_0(y,v)\bigr)\,\mathrm{d}y: \\ & \qquad v\in W_\#^{1,p}\bigl((0,1)^n;\mathbb{R}^m\bigr),\ v = z_j \text{ on } E_j,\ ⨍_{E_0\cap(0,1)^n} v\,\mathrm{d}x = z\biggr\} - t\cdot z\biggr\} \\ = {} & \min_{z\in\mathbb{R}^m}\min\biggl\{\int_{E_0\cap(0,1)^n}\bigl(f_0(y,Dv)+g_0(y,v) - t\cdot v\bigr)\,\mathrm{d}y: \\ & \qquad v\in W_\#^{1,p}\bigl((0,1)^n;\mathbb{R}^m\bigr),\ v = z_j \text{ on } E_j,\ ⨍_{E_0\cap(0,1)^n} v\,\mathrm{d}x = z\biggr\} \end{aligned}$$

$$= \min\biggl\{\int_{E_0\cap(0,1)^n} \bigl(f_0(y,Dv)+g_0(y,v)-t\cdot v\bigr)\,\mathrm{d}y\colon$$
$$v\in W^{1,p}_{\#}\bigl((0,1)^n;\mathbb{R}^m\bigr),\ v=z_j \text{ on } E_j\biggr\}.$$

Note that the definition of $\Psi(t,z_1,\dots,z_N)$ coincides with that of φ in Theorem 3.1 (or more precisely with that in Remark 3.3(2)) when $g_0(y,v)$ is replaced by $g_0(y,v)-t\cdot v$ ($t\in\mathbb{R}^m$ fixed). This observation shows that the functional given by

$$\sum_{j=1}^{N}\int_\Omega \bigl(f^j_{\text{hom}}(Du_j)+\tilde g_j(u_j)-h\cdot u_j\bigl|E_j\cap(0,1)^n\bigr|\bigr)\,\mathrm{d}x+\int_\Omega \Psi(h,u_1,\dots,u_N)\,\mathrm{d}x$$

is indeed the Γ-limit of (F^h_ε) with respect to the convergence $u_\varepsilon\to(u_1,\dots,u_N)$. In fact, a lower bound is proven above by pointwise minimization, while an upper bound can be proven by approximation: if $h(x)=t$ is constant then we can directly apply Theorem 3.1; if it is piecewise constant then we can construct recovery sequences by reasoning locally, while in the general case we proceed by approximation of h with piecewise constant functions.

8. Limits with other scalings

In this section we study the Γ-limit of F_ε in the case when the scaling factor of the soft phase does not match the growth conditions. Namely, fixed $q>0$ we consider the energies

$$F^q_\varepsilon(u)=\sum_{j=1}^{N}\int_{\Omega\cap\varepsilon E_j}\biggl(f_j\Bigl(\frac{x}{\varepsilon},Du\Bigr)+g_j\Bigl(\frac{x}{\varepsilon},u\Bigr)\biggr)\,\mathrm{d}x$$
$$+\int_{\Omega\cap\varepsilon E_0}\biggl(\varepsilon^q f_0\Bigl(\frac{x}{\varepsilon},Du\Bigr)+g_0\Bigl(\frac{x}{\varepsilon},u\Bigr)\biggr)\,\mathrm{d}x, \tag{8.1}$$

with q being different from p. The computation of the Γ-limit F^q_0 of those energies with respect to the convergence $u_\varepsilon\to(u_1,\dots,u_N)$ can be reduced to the case $p=q$ by some comparison arguments.

We note the two cases:

(a) $0<q<p$. In this case we obtain

$$F^q_0(u_1,\dots,u_N)=\begin{cases}F^s(u) & \text{if } u_1=\dots=u_N(=u),\\ +\infty & \text{otherwise;}\end{cases}$$

i.e., F^q_0 is equivalent to the limit in the strong L^p-convergence. Note that from the equiboundedness of the energies $F_\varepsilon(u_\varepsilon)$ we cannot directly deduce that (u_ε) is strongly-L^p compact.

If u is affine then the limsup inequality is immediately obtained by constructing a recovery sequence as for $F^s(u)$: if $u = \xi \cdot x$, for a fixed $\eta > 0$ consider $K \in \mathbb{N}$ and $v^1, \dots, v^N \in W^{1,p}_{\#}((0,K)^n; \mathbb{R}^m)$ such that $\int_{(0,K)^n \cap E_j} f_j(y, Dv^j)\, \mathrm{d}y \leqslant K^n(f^j_{\mathrm{hom}}(\xi) + \eta)$. We set

$$u_\varepsilon(x) = \sum_{j=1}^N \varepsilon \Phi_j\left(\frac{x}{\varepsilon}\right) u^j\left(\frac{x}{\varepsilon}\right) + \xi \cdot x,$$

with Φ_j as in the proof of Theorem 7.1. Then $u_\varepsilon \to \xi \cdot x$ and $\limsup_{\varepsilon \to 0^+} F^q_\varepsilon(u_\varepsilon) \leqslant F^s(\xi \cdot x) + N\eta$, so that the Γ-limsup inequality is proved by the arbitrariness of η. As usually, the passage from affine to piecewise-affine, and then to arbitrary u, is standard.

As for the liminf inequality, for a fixed $\lambda > 0$ we can use the inequality $F^q_\varepsilon(u) \geqslant G^\lambda_\varepsilon(u)$, valid for small enough ε, where

$$\begin{aligned} G^\lambda_\varepsilon(u) = & \sum_{j=1}^N \int_{\Omega \cap \varepsilon E_j} \left(f_j\left(\frac{x}{\varepsilon}, Du\right) + g_j\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x \\ & + \int_{\Omega \cap \varepsilon E_0} \left(\lambda \varepsilon^p f_0\left(\frac{x}{\varepsilon}, Du\right) + g_0\left(\frac{x}{\varepsilon}, u\right)\right) \mathrm{d}x. \end{aligned} \tag{8.2}$$

We then obtain a bound from below given by the functional

$$G^\lambda_0(u_1, \dots, u_N) = \sum_{j=1}^N \int_\Omega \left(f^j_{\mathrm{hom}}(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \int_\Omega \psi^\lambda(u_1, \dots, u_N)\, \mathrm{d}x$$

with

$$\begin{aligned} \psi^\lambda(z_1, \dots, z_N) = \sup_{K \in \mathbb{N} \setminus \{0\}} \inf\bigg\{ & \frac{1}{K^n} \int_{E_0 \cap (0,K)^n} \big(\lambda f_0(y, Dv) + g_0(y, v)\big)\, \mathrm{d}y: \\ & v \in W^{1,p}_{\#}\big((0,K)^n; \mathbb{R}^m\big),\ v = z_j \text{ on } E_j,\ j = 1, \dots, N \bigg\} \end{aligned} \tag{8.3}$$

(note that instead of passage to the limit in (8.3) we have equivalently written a supremum). We can then take the limit (that is a supremum as well) as $\lambda \to +\infty$ and (by the Monotone Convergence Theorem) obtain the lower bound

$$\sup_\lambda G^\lambda_0(u_1, \dots, u_N) = \sum_{j=1}^N \int_\Omega \left(f^j_{\mathrm{hom}}(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \int_\Omega \psi^\infty(u_1, \dots, u_N)\, \mathrm{d}x,$$

where

$$\begin{aligned} \psi^\infty(z_1, \dots, z_N) = \sup_\lambda \sup_K \inf\bigg\{ & \frac{1}{K^n} \int_{E_0 \cap (0,K)^n} \big(\lambda f_0(y, Dv) + g_0(y, v)\big)\, \mathrm{d}y: \\ & v \in W^{1,p}_{\#}\big((0,K)^n; \mathbb{R}^m\big),\ v = z_j \text{ on } E_j,\ j = 1, \dots, N \bigg\} \end{aligned}$$

$$= \sup_K \sup_\lambda \inf\biggl\{ \frac{1}{K^n} \int_{E_0\cap(0,K)^n} \bigl(\lambda f_0(y, Dv) + g_0(y, v)\bigr)\,\mathrm{d}y\colon \\ v \in W^{1,p}_\#\bigl((0,K)^n;\mathbb{R}^m\bigr),\ v = z_j \text{ on } E_j,\ j = 1,\dots,N \biggr\}$$

$$= \sup_K \inf\biggl\{ \frac{1}{K^n} \int_{E_0\cap(0,K)^n} g_0(y,v)\,\mathrm{d}y\colon\ Dv = 0,\ v \in W^{1,p}_\#\bigl((0,K)^n;\mathbb{R}^m\bigr), \\ v = z_j \text{ on } E_j,\ j = 1,\dots,N \biggr\}.$$

This last minimum problem is trivial, since either there is no possible test function or (in the case $z_1 = \cdots = z_N (= z)$) the only test function is the constant $v = z$. Hence,

$$\psi^\infty(z_1,\dots,z_N) = \begin{cases} \tilde g_0(z) & \text{if } z_1 = \cdots = z_N (= z), \\ +\infty & \text{otherwise,} \end{cases}$$

and we recover the desired inequality;

(b) $q > p$ (*decoupled phases*). In this case the contribution of the soft phase reduces to a constant, and the Γ-limit is given by

$$F_0^q(u_1,\dots,u_N) = \sum_{j=1}^N \int_\Omega \bigl(f^j_{\mathrm{hom}}(Du_j) + \tilde g_j(u_j)\bigr)\,\mathrm{d}x + C_0|\Omega|,$$

where

$$C_0 = \int_{E_0\cap(0,1)^n} \min_s g_0(y,s)\,\mathrm{d}y.$$

In this case the lower bound is trivial, since

$$\varepsilon^q f_0(y,z) + g_0(y,u) \geqslant \min_s g_0(y,s)$$

for all $z,\ u$. To construct a recovery sequence for $(u_1,\dots,u_N)$, with fixed $\eta > 0$ we can choose a smooth 1-periodic function u_0 with

$$\int_{E_0\cap(0,1)^n} g_0\bigl(y, u_0(y)\bigr)\,\mathrm{d}y \leqslant C_0 + \eta,$$

set $u^0_\varepsilon = u_0(x/\varepsilon)$ and let (u^j_ε) be a recovery sequence for $\int_\Omega f^j_{\mathrm{hom}}(Du_j)\,\mathrm{d}x$ converging to u_j weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$. Choose Φ_j as in the proof of Proposition 7.1 with the additional property that

$$\limsup_{\varepsilon\to 0^+} \sum_{j=1}^N \int_{\Omega\cap\varepsilon(E_0\cap\{\Phi_j>0\})} \bigl(1 + |u^j_\varepsilon|^p + |u^0_\varepsilon|^p\bigr)\,\mathrm{d}x \leqslant \eta. \tag{8.4}$$

We then define

$$u_\varepsilon(x) = \sum_{j=0}^{N} \Phi_j\left(\frac{x}{\varepsilon}\right) u_\varepsilon^j(x),$$

and estimate

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant \sum_{j=1}^{N} \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + \limsup_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_0} \left(\varepsilon^q f_0\left(\frac{x}{\varepsilon}, \sum_{j=0}^{N}\left(\frac{1}{\varepsilon} D\Phi_j\left(\frac{x}{\varepsilon}\right) u_\varepsilon^j + \Phi_j\left(\frac{x}{\varepsilon}\right) Du_\varepsilon^j\right)\right) + g_0\left(\frac{x}{\varepsilon}, \sum_{j=0}^{N} \Phi_j\left(\frac{x}{\varepsilon}\right) u_\varepsilon^j\right)\right) \mathrm{d}x.$$

The first term in the limsup on the right-hand side is estimated by

$$\sum_{j=0}^{N} \int_{\Omega\cap\varepsilon E_0} \varepsilon^q c\left(1 + \frac{1}{\varepsilon^p}|u_\varepsilon^j|^p + |Du_\varepsilon^j|^p\right) \mathrm{d}x,$$

and hence vanishes as $\varepsilon \to 0$. As for the second term, we can write

$$\int_{\Omega\cap\varepsilon E_0} g_0\left(\frac{x}{\varepsilon}, \sum_{j=0}^{N} \Phi_j\left(\frac{x}{\varepsilon}\right) u_\varepsilon^j\right) \mathrm{d}x = \int_{\Omega\cap\varepsilon E_0} g_0\left(\frac{x}{\varepsilon}, u_0\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x + \sum_{j=1}^{N} \int_{\Omega\cap\varepsilon(E_0\cap\{\Phi_j>0\})} \left(g_0\left(\frac{x}{\varepsilon}, \sum_{j=0}^{N} \Phi_j\left(\frac{x}{\varepsilon}\right) u_\varepsilon^j\right) - g_0\left(\frac{x}{\varepsilon}, u_\varepsilon^0\right)\right) \mathrm{d}x.$$

The last sum can be estimated by $c\eta$ thanks to (8.4), thus we obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant \sum_{j=1}^{N} \int_\Omega \left(f_{\mathrm{hom}}^j(Du_j) + \tilde{g}_j(u_j)\right) \mathrm{d}x + |\Omega| C_0 + c\eta,$$

and the desired inequality is proved.

Acknowledgements

We gratefully acknowledge the support of GNAMPA through the project *Homogenization techniques and asymptotic methods for problems with multiple scales*. The work of A. Piatnitski has also been supported in part by RFBR, grants 00-01-22000 and 02-01-00868.

References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso and D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, *Nonlinear Anal.* **18** (1992), 481–496.

[2] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rational Mech. Anal.* **86** (1984), 125–145.

[3] G. Alberti, Variational models for phase transitions, an approach via Γ-convergence, in: *Calculus of Variations and Partial Differential Equations*, L. Ambrosio and N. Dancer, eds, Springer-Verlag, Berlin, 2000, pp. 95–114.

[4] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482–1518.

[5] N. Ansini and A. Braides, Asymptotic analysis of periodically-perforated nonlinear media, *J. Math. Pures Appl.* **81** (2002), 439–451.

[6] N. Ansini, A. Braides and V. Chiadò Piat, Homogenization of periodic multi-dimensional structures, *Boll. Un. Mat. Ital. B* **2** (1999), 735–758.

[7] T. Arbogast, J. Douglas Jr. and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.* **21** (1990), 823–836.

[8] A. Boughammoura and M. Mabrouk, Homogeneisation d'un milieu elastique fortement heterogene, *C. R. Mecanique* **330** (2002), 543–548.

[9] A. Bourgeat, S. Luckhaus and A. Mikelic, Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow, *SIAM J. Math. Anal.* **27** (1996), 1520–1543.

[10] A. Bourgeat, A. Mikelic and A. Piatnitski, Modèle de double porosité aléatoire, *C. R. Acad. Sci. Paris, Ser. I* **327** (1998), 99–104.

[11] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1998.

[12] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, Oxford, 2002.

[13] A. Braides and V. Chiadò Piat, Remarks on the homogenization of connected media, *Nonlinear Anal.* **22** (1994), 391–407.

[14] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[15] A. Braides and A. Garroni, Homogenization of nonlinear media with soft and stiff inclusions, *Math. Models Methods Appl. Sci.* **5** (1995), 543–564.

[16] M. Briane, Homogenization in some weakly connected domains, *Ricerche Mat.* **47** (1998), 51–94.

[17] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[18] A. Fasano, A. Mikelić and M. Primicerio, Homogenization of flows throgh porous media with permeable grains, *Adv. Math. Sci. Appl.* **8** (1998), 1–31.

[19] A.D. Ioffe, On lower semicontinuity of integral functionals. I, *SIAM J. Control Optim.* **15** (1977), 521–537.

[20] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. Anal.* **98** (1987), 123–142.

[21] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital. B* **14** (1977), 285–299.

[22] G. Panasenko, Homogenization of processes in strongly non-homogeneous media, *Soviet Phys. Dokl.* **33** (1988), 76–79.

[23] G. Panasenko, Multicomponent homogenization of processes in strongly non-homogeneous structures, *Math. USSR-Sb.* **69** (1991), 143–153.

[24] L. Pankratov and A. Piatnitski, Nonlinear "double porosity" type model, *C. R. Acad. Sci. Paris, Ser. I* **334** (2002), 435–440.

[25] G.V. Sandrakov, Homogenization of elasticity equations with contrasting coefficients, *Mat. Sb.* **190** (1999), 1749–1806.

[26] G.V. Sandrakov, Homogenization of parabolic equations with contrasting coefficients, *Izv. Ross. Akad. Nauk Ser. Mat.* **63** (1999), 1015–1061.

[27] S.B. Schul'ga, Homogenization of nonlinear variational problems by means of two-scale convergence, *Proc. Steklov Inst. Math.* **236** (2002), 357–364.

[28] V.V. Zhikov, Connectedness and homogenization. Examples of fractal conductivity, *Mat. Sb.* **187** (1996), 1109–1147.

[29] V.V. Zhikov, On an extension and application of the two-scale convergence method, *Mat. Sb.* **191** (2000), 973–1014.

[30] V.V. Zhikov, S.K. Kozlov and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.