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Homogenization of Ferromagnetic Energies on Poisson Random Sets in the Plane

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Abstract

We prove that by scaling nearest-neighbour ferromagnetic energies defined on Poisson random sets in the plane one obtains an isotropic perimeter energy with a surface tension characterised by an asymptotic formula. The result relies on proving that cells with 'very long' or 'very short' edges of the corresponding Voronoi tessellation can be neglected. In this way we may apply Geometry Measure Theory tools to define a compact convergence, and a characterisation of metric properties of clusters of Voronoi cells using limit theorems for subadditive processes.

1. Introduction

In this paper we study a prototypical model of pair-interaction energies on Poisson random sets in the plane, and some interesting extension. These energies are a random version of nearest-neighbour 'ferromagnetic' systems defined on Bravais lattices, whose overall behaviour is that of an interfacial energy [1,20]. Aside from their theoretical interest, the analysis of ferromagnetic energies is relevant for numerical approximations and modeling issues in view of the possibility of lattice approximations for arbitrary interfacial energies makes (we refer to [15] for optimal constructions on regular lattices, available even with constraints on the interaction potentials). Surface energies are, in turn, an important building block in the study of general functionals defined on more complex spaces of functions of bounded variation, passing through the generalization to functions with a discrete number of values [5] and using the latter to approximate arbitrary functions by coarea-type arguments (see e.g. [4]). Furthermore, the study of energies involving bulk and surface part can often be decoupled in the analysis of each part, which justifies the analysis of surface energies separately also in that context (see [14,21] and the recent advances in the analysis and derivation of complex energies from discrete systems in [6,7]). The present contribution can be then viewed as a step towards the extension of the analysis of discrete systems producing bulk and surface integrals to general random distribution of points. The simplest case of parameters taking only two values (equivalently, characteristic functions) will allow us to concentrate on the basic geometric features of the underlying discrete environment.

Discrete energies with randomness producing surface effects have been previously considered under various hypotheses. Results on regular lattices with random interactions comprise: random weak membrane models in [17], random ferromagnetic energies with positive coefficients in [19] and ferromagnetic energies with a random distribution of degenerate coefficients in [18]. Stochastic lattices have been considered under the hypothesis that sites be distributed in such a way that no 'great holes' or 'concentration of sites' may occur, so that we obtain uniform upper and lower estimates for the size of the Voronoi cells of the underlying tessellation. This implies that those lattices can be treated in average as a regular periodic lattice (see [2,3,8,13]). Our focus is precisely in avoiding such an hypothesis considering points distributed according to a Poisson point process in the plane (*Poisson random set*). We denote by $\mathcal{N}$ such a set of points and by $\mathcal{E}$ the set of the edges of the underlying Delaunay triangulation, which are identified with pairs of points (i, j) in $\mathcal{N} \times \mathcal{N}$ (the *nearest neighbours* in $\mathcal{N}$). The energy we consider can be viewed as defined on subsets $\mathcal{I}$ of $\mathcal{N}$ by

$$E(\mathcal{I}) = \#\{(i, j) \in \mathcal{E} : i \in \mathcal{I}, j \notin \mathcal{I}\}. \tag{1}$$

Note that the same energy can be interpreted as the number of edges of the boundary of the set

$$A_{\mathcal{I}} = \bigcup_{i \in \mathcal{I}} C_i, \tag{2}$$

where C_i is the cell of the Voronoi tessellation containing the point $i \in \mathcal{N}$. Another way to write the same energy is by identifying each set $\mathcal{I}$ with a (scalar) spin function parameterized by indices in $\mathcal{N}$ and defined by $u_i^{\mathcal{I}} = 1$ if $i \in \mathcal{I}$ and $u_i^{\mathcal{I}} = -1$ if $i \notin \mathcal{I}$, so that we may rewrite $E(\mathcal{I})$ as depending on $u^{\mathcal{I}}$, setting

$$E(u^{\mathcal{I}}) = \frac{1}{8} \sum_{i,j \in \mathcal{N}} (u_i - u_j)^2 = \frac{1}{4} \sum_{i,j \in \mathcal{N}} |u_i - u_j|, \tag{3}$$

the factors coming from double counting and the fact that $|u_i - u_j| \in \{0, 2\}$. Conversely, we may take this as the definition of the energy, and correspondingly pass to subsets of $\mathcal{N}$ by noting that $E(\mathcal{I}_u) = E(u)$, where $\mathcal{I}_u = \{I \in \mathcal{N} : u_i = 1\}$.

In order to describe the overall properties of E we perform a discrete-to-continuum analysis through a scaling procedure. We intoduce a small parameter $\varepsilon > 0$ and consider the scaled energy E_ε defined on subsets of $\varepsilon\mathcal{E}$ by

$$E_\varepsilon(\mathcal{I}) = \varepsilon \, \#\{(i, j) \in \varepsilon\mathcal{E} : i \in \varepsilon\mathcal{I}, j \notin \varepsilon\mathcal{I}\}, \tag{4}$$

which again can be interpreted as ε times the number of edges of the boundary of the scaled set

$$A^\varepsilon_{\mathcal{I}} = \varepsilon \bigcup_{i \in \mathcal{I}} C_{i/\varepsilon}. \tag{5}$$

Note that if we had a uniform upper and lower bound of the size of each of these edges, then $E_\varepsilon(\mathcal{I})$ would be comparable with the perimeter of $A^\varepsilon_{\mathcal{I}}$. In that case, given a family $\mathcal{I}_\varepsilon$ with equibounded $E_\varepsilon(\mathcal{I})$, the sets $A_\varepsilon = A^\varepsilon_{\mathcal{I}_\varepsilon}$ would be (locally) precompact in the sense of sets of finite perimeter; i.e., there would exist a set of finite perimeter A such that, up to subsequences, $|(A_\varepsilon \triangle A) \cap Q| \to 0$ for any cube Q.

For Poisson random sets, the edges of Voronoi cells do not satisfy a uniform estimate. Nevertheless, very long or very short edges are in a sense negligible. Indeed, a result by Pimentel [35] implies that a path in which a large proportion of such sets is present must be 'short', and hence, by an isoperimetric argument encircle a 'small' set. Using this result, we can show that if $E_\varepsilon(\mathcal{I})$ is equibounded and A_ε are defined above, then there exists families of sets B'_ε and B''_ε such that $|B'_\varepsilon \cup B''_\varepsilon| \to 0$ and the perimeter of the sets

$$(A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon \tag{6}$$

is equibounded. We deduce then that, up to subsequences, A_ε still (locally) converge to a set of finite perimeter A.

We can then characterize the behaviour of the energies E_ε by computing their Γ-limit with respect to this convergence. Note that, by the isotropy of Poisson random sets, if the limit is of perimeter type, it must be of the form

$$F(A) = \tau_0\, \mathcal{H}^1(\partial A); \tag{7}$$

i.e., a constant τ_0 (the *surface tension*) times the perimeter of A (in this notation ∂A denotes the *reduced boundary* of A). The main issue is then to characterize such τ_0 so as to adapt the discrete-to-continuum technique of [18,19] to this case. A central observation is that the union of the boundaries of all Voronoi cells C_i for which we have suitable outer and inner bounds determine a set which possesses a unique infinite connected component. We then introduce a parameter $\alpha > 0$ that quantifies these bounds so that they become less and less stringent when $\alpha \to 0$. We denote by $\mathcal{V}_\alpha$ the union of boundaries of such 'α-regular' Voronoi cells. The properties of $\mathcal{V}_\alpha$ are derived from percolation argument as in [17–19], and can be used to prove that a first-passage percolation formula holds for paths in $\mathcal{V}$ and at the same time permit to use the blow-up technique [16,24] for proving a lower bound. An upper bound is finally shown by using the subadditive properties of the problems defining τ_0.

The techniques used to prove the homogenization theorem for nearest-neighbour interactions can be used to prove an analogous result when we take into account interactions corresponding to pairs of nodes in $\mathcal{N}$ with distance not exceeding a constant $\mathcal{R}$. Namely, in place of energies (4), we consider

$$E^{\mathcal{R}}_\varepsilon(\mathcal{I}) = \varepsilon\, \#\{i \in \varepsilon\mathcal{I},\, j \notin \varepsilon\mathcal{I} : \|i - j\| \le \mathcal{R}\varepsilon\}. \tag{8}$$

Note that this energy cannot be directly compared with E_ε in (4) since some nearest neighbours in $\mathcal{N}$ may be at a distance larger than $\mathcal{R}$. Nevertheless, using the properties of α-regular Voronoi cells we are able to show that for $\mathcal{R}$ large enough energies $E_\varepsilon^{\mathcal{R}}$ are equicoercive and still converge to an isotropic perimeter

$$F^{\mathcal{R}}(A) = \tau_0^{\mathcal{R}} \mathcal{H}^1(\partial A) \tag{9}$$

almost surely as $\varepsilon \to 0$. This result is interesting in view of applications to inhomogeneous interactions depending on the distance between the nodes (see [2,25]), which may be of use in Data Science.

A further result is an 'approximate homogenization theorem', in which for each $\alpha > 0$ we consider energies E_ε^α defined as the restriction of energy (4) to sets whose boundary is in $\varepsilon\mathcal{V}_\alpha$. By the properties of α-regular Voronoi cells the length of the boundary of sets is automatically estimated by the energy and compactness arguments are immediate. We prove that the Γ-limit of E_ε^α is still an isotropic perimeter $F_\alpha(A) = \tau_\alpha \mathcal{H}^1(\partial A)$, with τ_α decreasing to τ_0.

It is worth noting that some of the results extend to arbitrary dimension (mainly, the compactness lemma for sets with equibounded energy), but the properties of regular Voronoi cells as stated and the characterization of τ_0 with a first-passage percolation formula are particular of the planar case. The treatment of the asymptotic analysis of the energies in higher dimension will require different tools and homogenization formulas, which justify a separate treatment.

2. Notation and Statement of the Results

$\mathcal{L}^2(A)$ or $|A|$ denotes the 2-dimensional Lebesgue measure of a set A, $\mathbf{1}_A$ the characteristic function of the set A, $Q = [-1/2, 1/2]^2$ the unit cube in $\mathbb{R}^2$.

2.1. Poisson random sets

$\mathcal{N}$ denotes a *Poisson random set* with *intensity* $\lambda > 0$ in $\mathbb{R}^2$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We recall that a Poisson random set or stationary Poisson point process with intensity λ in $\mathbb{R}^2$ is a map from Ω to the set of locally finite subsets of $\mathbb{R}^2$ such that for any bounded Borel set $B \in \mathbb{R}^2$ the function $\#\{B \cap \mathcal{N}\}$ is a random variable, and the following two conditions are fulfilled:

• for any bounded Borel set $B \subset \mathbb{R}^2$ the number of points in $B \cap \mathcal{N}$ has a Poisson law with parameter $\lambda|B|$

$$\mathbf{P}\{\#(B \cap \mathcal{N}) = n\} = e^{-\lambda|B|}\frac{(\lambda|B|)^n}{n!};$$

• for any collection of bounded disjoint Borel subsets in $\mathbb{R}^2$ the random variables defined as the number of points of $\mathcal{N}$ in these subsets are independent.

We refer for instance to [23] for equivalent definitions of a Poisson random set and its main properties.

We also assume that the probability space is equipped with a dynamical system $T_x : \Omega \mapsto \Omega$, $x \in \mathbb{R}^2$, and that for any bounded Borel set B and any $x \in \mathbb{R}^2$ we have $\#\big((B + x) \cap \mathcal{N}\big)(\omega) = \#\big(B \cap \mathcal{N}\big)(T_x\omega)$. We recall that T_x is a group of measurable measure-preserving transformations in Ω, also measurable as a function $T_\cdot : \Omega \times \mathbb{R}^2 \mapsto \Omega$. We suppose that T_x is ergodic. Further details can be found for instance in [29, Chapter 7].

In what follows, we only consider a Poisson random set with intensity 1, since the results for a Poisson random set with intensity λ may be obtained by considering the case with intensity 1 and then applying a scaling transformation $\mathcal{N} \longrightarrow \sqrt{\lambda}\mathcal{N}$.

The cells of the *Voronoi tessellation* of $\mathcal{N}$ are denoted by

$$C_i := \{x \in \mathbb{R}^2 : |x - i| \leq |x - j| \text{ for all } j \in \mathcal{N}\},$$

with $i \in \mathcal{N}$. Each C_i thus defined is a polyhedral set; the set of edges of the Voronoi cells is denoted by $\mathcal{V}$. The set of the vertices of C_i (or endpoints of elements of $\mathcal{V}$) is denoted by $\mathcal{N}^*$

Note that we may assume that each point in $\mathbb{R}^2$ belongs to at most three Voronoi cells or three elements of $\mathcal{E}$, since this is an event of probability 1.

The set of edges of the *Delaunay triangulation* of $\mathcal{N}$ is denoted by $\mathcal{E}$ and is identified with the set of pairs (i, j) in $\mathcal{N} \times \mathcal{N}$ such that C_i and C_j share a common edge.

We define a *path* of Voronoi cells as a collection $\{C_{i_j} : 1 \leq j \leq K\}$ such that C_{i_j} and $C_{i_{j+1}}$ have an edge in common, or, equivalently, such that $(i_j, i_{j+1}) \in \mathcal{E}$ for all $j \in \{1, \ldots, K-1\}$. From the latter standpoint, we also talk of a path in $\mathcal{E}$. We say that such a path connects two sets X and Y if $X \cap C_1 \neq \emptyset$ and $Y \cap C_K \neq \emptyset$. If $X = \{x\}$ and $Y = \{y\}$ then we simply say that the path connects x and y.

2.2. Asymptotic behaviour of ferromagnetic energies on poisson random sets

For future reference and comparison with the existing literature, we state our results in terms of energies on (scalar) spin functions, keeping in mind the possible alternative formulations as energies on sets or on set of points. The (scaled) ferromagnetic energy of the Poisson random set is defined on spin functions $u : \varepsilon\mathcal{N} \to \{-1, 1\}$ by

$$\begin{aligned} E_\varepsilon(u) &= \frac{1}{8} \sum_{(i,j)\in\varepsilon\mathcal{E}} \varepsilon(u_i - u_j)^2 \\ &= \frac{1}{2}\varepsilon\,\#\{(i, j) \in \varepsilon\mathcal{E} : u_i \neq u_j\} \\ &= \varepsilon\,\#\{(i, j) \in \varepsilon\mathcal{E} : u_i = 1, u_j = -1\}, \end{aligned} \tag{10}$$

where the scaling factor $\frac{1}{8}$ is due to double counting and to the fact that $(u_j - u_j)^2 \in \{0, 4\}$.

To each $u : \varepsilon\mathcal{N} \to \{-1, 1\}$ we associate the (scaled) *Voronoi set of u* defined by

$$V_\varepsilon(u) = \bigcup_{\{i : u_i = 1\}} \varepsilon\, C_{i/\varepsilon}, \tag{11}$$

and the *piecewise-constant interpolation* (with underlying set $\varepsilon\mathcal{N}$), still denoted $u : \mathbb{R}^2 \to \{-1, 1\}$, defined by

$$u(x) = \begin{cases} 1 & \text{if } x \in V_\varepsilon(u) \\ -1 & \text{if } x \in \mathbb{R}^2 \setminus V_\varepsilon(u). \end{cases} \tag{12}$$

Definition 2.1. We say that a family $u^\varepsilon : \varepsilon\mathcal{N} \to \{-1, 1\}$ *converges to a set* A if the piecewise-constant interpolations u^ε converge to the function $\mathbf{1}_A - \mathbf{1}_{\mathbb{R}^2\setminus A}$ locally in $L^1(\mathbb{R}^2)$, or, equivalently, if $\mathbf{1}_{V_\varepsilon(u^\varepsilon)}$ converge to $\mathbf{1}_A$ locally in $L^1(\mathbb{R}^2)$.

The following compactness lemma justifies the use of the convergence in Definition 2.1 in the computation of the Γ-limit of E_ε [11,12]. Note that the result cannot be directly deduced from the compactness property of sets of equibounded perimeter, since we cannot deduce the equiboundedness of the perimeters of the sets $V_\varepsilon(u^\varepsilon)$ from the equiboundedness of $E_\varepsilon(u_\varepsilon)$:

Lemma 2.2. (compactness) *Let u^ε be such that* $\sup_\varepsilon E_\varepsilon(u^\varepsilon) < +\infty$. *Then, up to subsequences u^ε converges to some set A in the sense of Definition* 2.1. *Moreover, the limit set is a set of finite perimeter.*

The compactness lemma above ensures that the domain of the Γ-limit of E_ε be the family of sets of finite perimeter in $\mathbb{R}^2$. The asymptotic behaviour of E_ε will be described by an asymptotic formula similar to those encountered in first-passage percolation, involving minimal paths on $\mathcal{E}$ between points of $\mathbb{R}^2$. To that end we define for all $x \in \mathbb{R}^2$

$$\pi_0(x) = \text{ closest point of } \mathcal{N}^* \text{ to } x,$$

For almost all x this point is uniquely defined. For the remaining points we choose one of the closest points of $\mathcal{N}^*$ to x. For $x,\ y \in \mathbb{R}^2$ we define

$$m_0(x, y) = \min\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{E} \text{connecting } \pi_0(x) \text{ and } \pi_0(y)\}, \tag{13}$$

where a path of segments (in our case edges in $\mathcal{V}$) connecting two points $\overline{x}$ and $\overline{y}$ is a collection of segments $[x_{k-1}, x_k]$ with $1 \leq k \leq K$ for some $K \in \mathbb{N}$ such that $x_0 = \overline{x}$ and $x_K = \overline{y}$, and such that the related piecewise-linear curve is not self-intersecting.

Theorem 2.3. (homogenization theorem) *Let $\mathcal{E}$ be a Poisson random set with intensity* 1. *Then there exists a deterministic constant $\tau_0 \in (0, +\infty)$ (the* surface tension*) such that almost surely the energies E_ε defined in* (10) Γ*-converge to the energy $F(A) = \tau_0\, \mathcal{H}^1(\partial A)$, defined on sets of finite perimeter, with respect to the convergence in Remark* 3.4. *Furthermore, the constant τ_0 satisfies*

$$\tau_0 = \lim_{t\to\infty} \frac{m_0\big((0, 0), (t, 0)\big)}{t}$$

almost surely, where m_0 is given by (13).

The proof of this result will be the content of Section 4.

Remark 2.4. By the scaling argument $\mathcal{N} \to \sqrt{\lambda}\mathcal{N}$, we deduce that if $\mathcal{E}$ is a Poisson random set with intensity λ then the Γ-limit of the corresponding E_ε is $\sqrt{\lambda}\, \tau_0\, \mathcal{H}^1(\partial A)$.

3. Compactness

This section is devoted to the proof of the Compactness Lemma 2.2. Even though we will use it in the planar case, we note that that result can be proved in any space dimension d up to minor changes (see Remark 3.3 below).

Π denotes the set of finite connected unions of Voronoi cells (here connected means that the corresponding set of edges of the Delaunay triangulation is connected). If $P \in \Pi$ we set

$$\mathbf{A}(P) = \{z \in \mathbb{Z}^2 : (z + Q) \cap P \neq \emptyset\}.$$

In what follows, if it does not lead to an ambiguity, we identify $\mathbf{A}(P)$ with the union of unit squares centered at the points of $\mathbf{A}(P)$.

Connected sets of Voronoi cells might have rather irregular geometric structure. It is more comfortable to deal with their covering by elements of a regular grid of squares. The lemma below states that for sufficiently large $P \in \Pi$ the numbers of elements in P and in $\mathbf{A}(P)$ are comparable.

Lemma 3.1. (Pimentel's polyomino lemma [35]) *Let $R > 0$ and $\gamma > 0$. Then there exists a deterministic constant C such that, for almost all ω, there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that if $P \in \Pi$ and $\varepsilon < \varepsilon_0$ satisfy*

$$P \cap \frac{R}{\varepsilon} Q \neq \emptyset, \qquad \max\{\#\{i : C_i \subset P\}, \#\mathbf{A}(P)\} \geq \varepsilon^{-\gamma}, \tag{14}$$

then we have

$$\frac{1}{C}\#\{i : C_i \subset P\} \leq \#\mathbf{A}(P) \leq C\,\#\{i : C_i \subset P\}. \tag{15}$$

Proof. Denote $\Pi^z_{\leqslant r} = \{P \in \Pi : z + Q \cap P \neq \emptyset,\ \#\{i : C_i \subset P\} \leqslant r\}$ and $\Pi^z_{\geqslant r} = \{P \in \Pi : z + Q \cap P \neq \emptyset,\ \#\{i : C_i \subset P\} \geqslant r\}$. According to [35, Theorem 1] and comments to this theorem there exist constants $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 > 0$ such that for any $z \in \mathbb{Z}^2$

$$\mathbf{P}\left\{\min_{P \in \Pi^z_{\geqslant r}} \#\mathbf{A}(P) \leqslant s\right\} \leqslant e^{-r/2}, \quad \text{if } r \geqslant \kappa_1 s, \tag{16}$$

and

$$\mathbf{P}\left\{\max_{P \in \Pi^z_{\leqslant r}} \#\mathbf{A}(P) \geqslant s\right\} \leqslant e^{-\kappa_3 s}, \quad \text{if } s \geqslant \kappa_2 r, \tag{17}$$

Letting $r = \kappa_1 s$ in (16) and summing up over $z \in R s^{\frac{1}{\gamma}} Q \cap \mathbb{Z}^2$ we obtain

$$\mathbf{P}\left\{\min_{z \in R s^{\frac{1}{\gamma}} Q \cap \mathbb{Z}^2} \ \min_{P \in \Pi^z_{\geqslant \kappa_1 s}} \#\mathbf{A}(P) \leqslant s\right\} \leqslant R^2 s^{2/\gamma} e^{-\frac{\kappa_1}{2} s}.$$

By the Borel-Cantelli lemma a.s. for sufficiently large s we have

$$\min_{z\in Rs^{\frac{1}{\gamma}}Q\cap\mathbb{Z}^2}\ \min_{P\in\Pi^z_{\geqslant\kappa_1 s}} \#\mathbf{A}(P)\geqslant s.$$

Letting $s=\varepsilon^{-\gamma}$ we obtain the first estimate in (15). The second one can be derived from (17) in the same way. □

Note in particular that in the hypotheses of the lemma, we also have

$$\min\{\#\{i : C_i\subset P\}, \#\mathbf{A}(P)\}\geq \frac{1}{C}\varepsilon^{-\gamma}. \tag{18}$$

Further geometric properties of such Voronoi tessellations can be found in [22].

Lemma 2.2 will be a consequence of the following result:

Lemma 3.2. (compactness of Voronoi sets) *Let u^ε be such that* $\sup_\varepsilon E_\varepsilon(u^\varepsilon)<+\infty$. *Then we can write*

$$V_\varepsilon(u^\varepsilon)=(A_\varepsilon\cup B'_\varepsilon)\setminus B''_\varepsilon,$$

where $|B'_\varepsilon|+|B''_\varepsilon|\to 0$, $\{A_\varepsilon\}$ is a family of sets with equibounded perimeter, the family $\mathbf{1}_{A_\varepsilon}$ is precompact in $L^1_{\rm loc}(\mathbb{R}^2)$ and each its limit point is the characteristic function of a set of finite perimeter A, so that the same holds for $\mathbf{1}_{V_\varepsilon(u^\varepsilon)}$.

Proof. Since we reason locally, in order to ease the notation we assume that e.g. all u^ε are identically -1 outside a fixed cube (or equivalently that $V_\varepsilon(u^\varepsilon)$ are contained in a fixed cube).

We fix $\gamma>0$ small enough. We subdivide $\partial V_\varepsilon(u^\varepsilon)$ into its connected components. We denote by $\mathcal{C}^{\gamma,+}_\varepsilon$ the family of such connected components S with

$$\#\{i\in\mathcal{N} : u_{\varepsilon i}=1, \varepsilon C_i\cap S\neq\emptyset\}\geqq\varepsilon^{-\gamma}. \tag{19}$$

Note that each such connected component can be identified with the set

$$P=P(S)=\bigcup\{C_i : u_{\varepsilon i}=1, \varepsilon C_i\cap S\neq\emptyset\}, \tag{20}$$

which belongs to the set Π. We denote by $\mathcal{C}^{\gamma,-}_\varepsilon$ the family of the remaining connected components.

The first step will be to identify the small sets B'_ε and B''_ε as the 'interior' of contours in $\mathcal{C}^{\gamma,-}_\varepsilon$ where the inner trace of $\mathbf{1}_{V_\varepsilon(u^\varepsilon)}$ is 0 and 1, respectively. In this way the remaining set will have a boundary only composed of 'large' components from $\mathcal{C}^{\gamma,+}_\varepsilon$. This argument needs a little more formalization since we may have contours contained in other contours.

By the finiteness of the energy we have

$$\#\mathcal{C}^{\gamma,-}_\varepsilon\leq\frac{C}{\varepsilon}$$

Note that

$$\#\left(\mathbf{A}\left(\frac{1}{\varepsilon}S\right)\right)\leq C\varepsilon^{-\gamma}\ \text{for every } S\in\mathcal{C}^{\gamma,-}_\varepsilon.$$

Indeed, otherwise $\#(\mathbf{A}(\frac{1}{\varepsilon}S)) > C\varepsilon^{-\gamma} > \varepsilon^{-\gamma}$, so that the hypotheses of Lemma 3.1 are satisfied and (15) implies that (19) holds, which gives a contradiction. Hence each $S \in \mathcal{C}_\varepsilon^{\gamma,-}$ is contained in a set with boundary at most of length $C\varepsilon^{1-\gamma}$. By an isoperimetric estimate, the measure of the bounded set sorrounded by each $S \in \mathcal{C}_\varepsilon^{\gamma,-}$ is $O(\varepsilon^{2-2\gamma})$. Hence, the total measure of such sets is $O(\varepsilon^{1-2\gamma})$.

Consider now each maximal $S \in \mathcal{C}_\varepsilon^{\gamma,-}$; i.e., which is not contained in any other bounded set whose boundary is another element in $\mathcal{C}_\varepsilon^{\gamma,-}$. For each such S, let P be defined from S by (20). We have two cases, whether εP is interior to S or not. We denote by $\mathcal{C}_{1,\varepsilon}^{\gamma,-}$ the first family, by $\mathcal{C}_{2,\varepsilon}^{\gamma,-}$ the second one, and define B'_ε as the union of the $\varepsilon C_{i/\varepsilon}$ in the interior of S for some $S \in \mathcal{C}_{1,\varepsilon}^{\gamma,-}$ and such that $u_i^\varepsilon = 1$, and B''_ε as the union of the $\varepsilon C_{i/\varepsilon}$ in the interior of S for some $S \in \mathcal{C}_{2,\varepsilon}^{\gamma,-}$ and such that $u_i^\varepsilon = -1$. If we set

$$V_\varepsilon = (V_\varepsilon(u^\varepsilon) \setminus B'_\varepsilon) \cup B''_\varepsilon$$

then ∂V_ε consists only of components in $\mathcal{C}_\varepsilon^{\gamma,+}$, and

$$|B'_\varepsilon \cup B''_\varepsilon| \le C\varepsilon^{1-2\gamma}.$$

We now write $V_\varepsilon = A_\varepsilon \cup A'_\varepsilon$, where

$$\begin{aligned} A_\varepsilon &= \bigcup \{(\varepsilon z + \varepsilon Q) : \varepsilon z + \varepsilon Q \subset V_\varepsilon\} \\ A'_\varepsilon &= V_\varepsilon \setminus A_\varepsilon \end{aligned}$$

Note that

$$\partial A_\varepsilon \subset \varepsilon \bigcup_{S \in \mathcal{C}_\varepsilon^{\gamma,+}} \partial \mathbf{A}(P(S))$$

with $P(S)$ defined in (20). By Lemma 3.1 we have

$$\mathcal{H}^1(\partial \mathbf{A}(P(S))) \le C\#\{i \in \mathcal{N} : u_{\varepsilon i} = 1, \varepsilon C_i \cap S \ne \emptyset\}$$

Summing up over all $S \in \mathcal{C}_\varepsilon^{\gamma,+}$, we obtain

$$\mathcal{H}^1(\partial A_\varepsilon) \le C\, E_\varepsilon(u^\varepsilon).$$

Hence, $\{A_\varepsilon\}$ is a family of sets with equibounded perimeter, and the functions $\mathbf{1}_{A_\varepsilon}$ are locally precompact in $L^1(\mathbb{R}^2)$ by the precompactness of sets of equibounded perimeter [10,34].

Again by Lemma 3.1 we have

$$|A'_\varepsilon| \le C\varepsilon^2 \sum_{S \in \mathcal{C}_\varepsilon^{\gamma,+}} \#\mathbf{A}(P(S)) \le C\varepsilon E_\varepsilon(u^\varepsilon).$$

This shows that $|A'_\varepsilon| \to 0$, and proves the claim, upon adding A'_ε to B'_ε defined above. $\square$

Remark 3.3. The previous compactness result holds in any dimension d with minor changes in the proof, upon noting that Pimentel's lemma holds with

$$\mathbf{A}(P) = \{z \in \mathbb{Z}^d : (z + Q) \cap P \neq \emptyset\},$$

and Q the coordinate unit cube in $\mathbb{R}^d$ [35].

Remark 3.4. (convergence in terms of the empirical measures) To each $u^\varepsilon : \varepsilon\mathcal{N} \to \{-1, 1\}$ we can associate the so-called *empirical measure*

$$\mu(u^\varepsilon) = \sum_{\{i\in\varepsilon\mathcal{N}:u_i^\varepsilon=1\}} \varepsilon^2\delta_i.$$

If u^ε are such that $\sup_\varepsilon E_\varepsilon(u^\varepsilon) < +\infty$ and u^ε converge to A as in Definition 2.1, then the measures $\mu(u^\varepsilon)$ locally converge to the measure $\mathbf{1}_A\mathcal{L}^2$ with respect to the weak* convergence of measures. Thanks to Lemma 3.2, then these two convergences are equivalent.

To check the convergence of $\mu(u^\varepsilon)$, we first note that we may suppose that $\mu(u^\varepsilon) \rightharpoonup f\mathcal{L}^2$ for some $f : \mathbb{R}^2 \to [0, 1]$. It suffices to show that $f = 0$ at almost every point of density 0 for A (a symmetric argument then shows that $f = 1$ at almost every point of density 1 for A).

For almost all such x_0 we have that

$$\limsup_{\varepsilon\to0} |V_\varepsilon(u^\varepsilon) \cap (x_0 + \rho Q)| = o(\rho^2)$$

and $\limsup\limits_{\varepsilon\to0} E_\varepsilon(u^\varepsilon, Q_\rho) = o(\rho)$, where we have set

$$E_\varepsilon(u^\varepsilon, Q_\rho) = \frac{1}{2}\varepsilon\,\#\{(i, j) \in \varepsilon\mathcal{E} : u_i^\varepsilon \neq u_j^\varepsilon, i \text{ or } j \in \rho Q\}.$$

We may subdivide $V_\varepsilon(u^\varepsilon) \cap (x_0 + \rho Q)$ into disjoint connected components:

$$V_\varepsilon(u^\varepsilon) \cap (x_0 + \rho Q) = \bigcup_{\#(P_j\cap\varepsilon\mathcal{N})\leq\varepsilon^{-\gamma}} P_j \cup \bigcup_{\#(L_k\cap\varepsilon\mathcal{N})>\varepsilon^{-\gamma}} L_k,$$

We may apply Lemma 3.1 to each L_k to obtain

$$\sum_k \varepsilon^2\#(L_k \cap \varepsilon\mathcal{N}) \leq C\varepsilon^2 \sum_k \#\mathbf{A}\left(\frac{1}{\varepsilon}L_k\right) \leq C|V_\varepsilon(u^\varepsilon)| = o(\rho^2).$$

As for P_j we have

$$\#(\{P_j\}) \leq \frac{1}{\varepsilon}E_\varepsilon(u^\varepsilon, Q_\rho) = \frac{1}{\varepsilon}o(\rho), \qquad \sum_j \#(P_j \cap \varepsilon\mathcal{N}) \leq \frac{1}{\varepsilon^{1+\gamma}}o(\rho).$$

In conclusion,

$$\mu(u^\varepsilon)(x_0 + \rho Q) = \varepsilon^2\#\{u_i^\varepsilon = 1,\ i \in x_0 + \rho Q\} \leq o(\rho^2) + \varepsilon^{1-\gamma}o(\rho).$$

Letting first $\varepsilon \to 0$ and then $\rho \to 0$ we prove the claim.

4. Proof of the Homogenization Theorem

In this section we prove Theorem 2.3, first characterizing the surface tension and then computing the Γ-limit. Preliminarily, we introduce regular Voronoi cells and study their geometry.

4.1. Geometry of clusters of regular voronoi cells

The surface tension characterizing the Γ-limit will be expressed by an asymptotic average length of minimal paths analogous to first-passage percolation formulas. A difficulty in our case is that in principle one of the end-points of such paths could be located in an 'exceptional region' where very small Voronoi cells accumulate. In order to treat this case, we first introduce regular Voronoi cells and study some percolation characteristics of the grid of such cells.

For $\alpha > 0$ we set

$$\mathcal{N}_\alpha^0 = \left\{ i \in \mathcal{N} : C_i \text{ contains a ball of radius } \alpha, \text{ diam } C_i \le \frac{1}{\alpha}, \#\text{edges of } C_i \le \frac{1}{\alpha} \right\} \tag{21}$$

the family of *regular Voronoi cells with parameter* α. The following lemma describes some geometrical features of regular Voronoi tessellations.

Lemma 4.1. (a channel property of $\mathcal{N}_\alpha^0$) *Let* $\delta > 0$. *For every* $T \in \mathbb{R}$, $\nu \in S^1$ *and* $x \in \mathbb{R}^2$ *we define*

$$R^\nu_{T,\delta}(x) = \left\{ x : |\langle x - x_i, \nu_i \rangle| \le \delta T, \, |\langle x - x_i, \nu_i^\perp \rangle| \le \frac{1}{2} T \right\}.$$

Then there exist $\alpha_0, C_\delta > 0$ *such that a.s. there exists* $T_0(\omega) > 0$ *such that for all* $T > T_0(\omega)$ *the rectangle* $R^\nu_{T,\delta}(x)$ *contains at least* $C_\delta T$ *disjoint paths of Voronoi cells* C_i *with* $i \in \mathcal{N}_\alpha^0$ *connecting the two opposite sides of* $R^\nu_{T,\delta}(x)$ *parallel to* ν. *This property is uniform as* x/T *vary on a bounded set of* $\mathbb{R}^2$.

Proof. Our arguments rely on the result known as channel property in the Bernoulli site percolation model in $\mathbb{Z}^2$. The idea of the proof is to consider a regular grid of squares in $\mathbb{R}^2$ and to choose the squares where the Poisson random set possesses a number of geometric properties (properties (**c**1)– (**c**3) below). These properties are designed in such a way that
i. the events that they hold in disjoint squares are independent, and the probability of such event does not depend on the position of a square.
ii. the probability that these properties hold in a unit square is close enough to 1,
iii. there exists $\alpha > 0$ such that any channel of squares where the mentioned properties are fulfilled contains a path of Voronoi cells C_i with $i \in \mathcal{N}_\alpha^0$ (α-channel).

We proceed to the detailed construction. Denote $Q_{5L} := [-5L, 5L]^2$, and for $L, K, \alpha \in \mathbb{R}^+$ and $j \in \mathbb{Z}^2$ denote by $\mathcal{E}(L, K, \alpha, j)$ the event that the following conditions are fulfilled:

($\mathbf{c}_1$) any square $[0, L]^2 + Li$ with $i \in \mathbb{Z}^2 \cap [-4.5, 5.5]^2$ contains at least one point of $\mathcal{N} - 10Lj$,
($\mathbf{c}_2$) the total number of points $\#((\mathcal{N} - 10Lj) \cap Q_{5L})$ does not exceed K,
($\mathbf{c}_3$) the distance between any two points of $(\mathcal{N} - 10Lj) \cap Q_{5L}$ as well as the distance from any point of $(\mathcal{N} - 10Lj) \cap Q_{5L}$ to ∂Q_{5L} is greater than 2α.

Letting ξ_j be the characteristic function of $\mathcal{E}(L, K, \alpha, j)$ and considering the properties of the Poisson random set we conclude that ξ_j, $j \in \mathbb{Z}^2$, are i.i.d. random variables. For any $\gamma > 0$ one can choose sufficiently large L and K and sufficiently small $\alpha > 0$ so that

$$\mathbf{P}(\mathcal{E}(L, K, \alpha, j)) > 1 - \gamma. \tag{22}$$

Indeed, the probability that any cube of size L in Q_{5L} contains at least one point of the Poisson random set tends to 1 as $L \to \infty$. Then, given $L > 0$, the probability that the number of points in Q_{5L} does not exceed K tends to 1 as $K \to \infty$. The probability that in the cube Q_{5L} the smallest distance between two points is less than α goes to zero as $\alpha \to 0$. Finally, the probability that α-neighbourhood of ∂Q_{5L} contains at least one point also goes to zero. Combining this relations we obtain the desired property.

For any two points j', $j'' \in \mathbb{Z}^2$ such that $|j' - j''| = 1$ denote by $I_{j',j''}$ the segment $[10Lj', 10Lj'']$ in $\mathbb{R}^2$. If $\xi_{j'} = \xi_{j''} = 1$ then

($\mathbf{s}1$) any Voronoi cell C_i that has a non-trivial intersection with $I_{j',j''}$ belongs to $(Q_{5L} + 10Lj') \cup (Q_{5L} + 10Lj'')$,
($\mathbf{s}2$) any such a cell C_i contains a ball of radius α,
($\mathbf{s}3$) the number of edges of each such C_i is not greater than K.

In particular, due to ($\mathbf{s}1$) and ($\mathbf{c}_2$), the total number of the cells C_i having a non-empty intersection with $I_{j',j''}$ does not exceed $2K$.

Statement ($\mathbf{s}1$) can be justified as follows: Let x' be an arbitrary point of $I_{j',j''}$. Denote by C_i the Voronoi cell that contains x' and by x_i the corresponding point of the Poisson random set. Due to ($\mathbf{c}_1$) we have $|x' - x_i| \leqq \sqrt{2}L$. Then any point $y \in \partial\big((Q_{5L} + 10Lj') \cup (Q_{5L} + 10Lj'')\big)$ satisfies the inequality $|y - x_i| \geqq (5 - \sqrt{2})L$. On the other hand, by ($\mathbf{c}_1$) the distance of y from $\mathcal{N}$ is not greater than $\sqrt{2}L$. This implies that $y \notin C_i$. Therefore, $C_i \subset (Q_{5L} + 10Lj') \cup (Q_{5L} + 10Lj'')$, and ($\mathbf{s}1$) follows.

In a similar way one can show that for any C_i that has a nontrivial intersection with $I_{j',j''}$ and any $x_j \in \mathcal{N}$ such that C_i and C_j have an edge in common we have $x_j \in (Q_{5L} + 10Lj') \cup (Q_{5L} + 10Lj'')$. In view of ($\mathbf{c}_2$) this yields ($\mathbf{s}3$).

Statement ($\mathbf{s}2$) is an immediate consequence of ($\mathbf{c}_3$).

Now the desired channel property follows from the well-known channel property in the Bernoulli site percolation model. For the reader convenience we formulate it here. Let η_j, $j \in \mathbb{Z}^2$, be a collection of i.i.d. random variables taking on the value 1 with probability p and the value 0 with probability $(1 - p)$. We say that $\{j_i\}_{i=1}^M$ is a 1-path if j_i and j_{i+1}, $i = 1, 2, \ldots, M - 1$, are neighbouring points of $\mathbb{Z}^2$ and $\eta_{j_i} = 1$ for all i. Then there exists $p_{\rm cr} \in (0, 1)$ such that for all $p > p_{\rm cr}$ the following statement holds: for any $\delta > 0$ there exists $\mathcal{K}_\delta > 0$ such that for almost each $\omega \in \Omega$ there exists $T_0 = T_0(\omega) > 0$ such that any rectangle $R^{\nu}_{T,\delta}(x)$ with $T \geqq T_0$ and $x \in [-T, T]^2$ contains at least $\mathcal{K}_\delta > 0$ disjoint 1-paths connecting the two opposite sides of $R^{\nu}_{T,\delta}(x)$ parallel to ν. We refer to [30] for further details.

It remains to choose γ in (22) in such a way that $1 - \gamma > p_{\rm cr}$. Labeling the squares $Q_{5L} + 10j$ with the corresponding points $j \in \mathbb{Z}^2$ and recalling the just formulated channel property of the Bernoulli site percolation model with $\eta_j = \xi_j$ we obtain the desired statement. □

From the proof of the previous lemma, in particular we obtain the following proposition:

Proposition 4.2. *There exists α_0 such that if $\alpha < \alpha_0$ there exists a unique infinite connected component of $\mathcal{N}^0_\alpha$, and its complement is composed of bounded connected sets.*

With this proposition in mind, we may define *clusters* of regular Voronoi cells.

Definition 4.3. (α-*clusters*) Let $\alpha < \alpha_0$ be as in Proposition 4.2. We denote by $\mathcal{N}_\alpha$ the infinite connected component of $\mathcal{N}^0_\alpha$ defined therein. Moreover, we denote by $\mathcal{N}^*_\alpha$ the set of vertices of edges of C_i with $i \in \mathcal{N}_\alpha$, by $\mathcal{V}_\alpha$ the set of the edges of such C_i, and by $\mathcal{E}_\alpha$ the set of edges of the Delaunay triangulation defined by set of pairs (i, j) in $\mathcal{N}^2_\alpha$ such that C_i and C_j share a common edge.

Remark 4.4. (a channel property of $\mathcal{N}_\alpha$)With the notation of Definition 4.3, note that the paths of cells C_i in Lemma 4.1 can be taken with $i \in \mathcal{N}_\alpha$.

4.2. Geometric properties of voronoi tessellation of poisson set. Surface tension

In this section we consider the geometric properties of the Poisson-Voronoi tessellation and introduce the surface tension in terms of an asymptotic distance between two points of the grid. In order to apply the subadditive theorem we should show that the grid distance between two arbitrary points has a finite expectation. The symbol $\mathbf{E}$ stands for the expectation in Ω.

Proposition 4.5. *For all $t > 0$ we have*

$$\mathbf{E}(m_0((0,0),(t,0))) < +\infty. \tag{23}$$

Furthermore, the limit

$$\tau_0 = \lim_{t\to+\infty} \frac{m_0((0,0),(t,0))}{t}$$

exists almost surely and is deterministic.

Proof. We say that a set $S \subset \mathbb{Z}^2$ is l^∞-connected if for any two points i and j in S there is a path $i = i_0, i_1, \dots, i_m = j$ in S such that $|i_k - i_{k-1}|_\infty = 1, k = 1, \dots, m$.

Consider all l^∞-connected sets in $\mathbb{Z}^2$ of size n that contain the origin. According to [28, Proof of Theorem 4.20] for any $n \geqq 0$ the number of such sets is not greater than C_2^n for some constant $C_2 > 0$.

Next we choose L, K and $\alpha > 0$ in such a way that $C_2^2 \gamma < \frac{1}{4}$, where γ is defined in (22).

We say that a site $j \in \mathbb{Z}^2$ is open if conditions ($\mathbf{c}_1$)-($\mathbf{c}_3$) in the proof of Lemma 4.1 are satisfied; otherwise j is closed. The probability that a l^∞-connected set in $\mathbb{Z}^2$ consists of closed points, has size n and is a maximum l^∞-connected component of closed points does not exceed γ^n. We denote such a set by $S(n)$.

Consider the sets

$$\mathcal{S}_0(n) = \bigcup_{j \in S(n)} \big(Q_{5L} + 10Lj\big),$$

$$\mathcal{S}_1(n) = \mathcal{S}_0(n) \bigcup \Big\{ x \in \mathbb{R}^2 \,:\, \mathrm{dist}_\infty(x, 10LS(n)) \leqq 10L \Big\}.$$

If $\mathcal{S}_0(n)$ contains k points of $\mathcal{N}$, then the length of the shortest path from $(0, 0)$ to $(1, 0)$ does not exceed $(k + 8nK)^2$. The probability that $\mathcal{S}_0(n)$ contains exactly k points of $\mathcal{N}$ is equal to

$$\frac{(100L^2 n)^k}{k!} \exp(-100L^2 n).$$

Denote $L_0 = 100L^2$.

The probability that $S(n)$ is a maximum connected component of closed sites and that $\mathcal{S}_0(n)$ contains exactly k points of $\mathcal{N}$ is not greater than

$$p_{kn} = \Big(\gamma^n\Big)^{\frac{1}{2}} \Big(\frac{(L_0 n)^k}{k!} \exp(-L_0 n) \Big)^{\frac{1}{2}}.$$

Summing up over all connected sets in $\mathbb{Z}^2$ that contain the origin and over all k from 0 to $+\infty$, we obtain that the expectation of the shortest path from $(0, 0)$ to $(1, 0)$ admits the following upper bound:

$$\mathbf{E}(m_0((0,0),(1,0))) \leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_2^n p_{kn} (k + 8nK)^2$$

$$\leqq \sum_{n,k=0}^{\infty} \exp\left((\log(C_2) + \frac{1}{2}\log(\gamma))n \right) \left(\frac{(L_0 n)^k}{k!} \exp(-L_0 n) \right)^{\frac{1}{2}} (k + 8nK)^2.$$

Since $\frac{(L_0 n)^k}{k!} \exp(-L_0 n) < 1$, using the Stirling formula and considering our choice of γ, one concludes that the series converges. This yields the relation in (23) for $t \leqq 1$. For larger t we use the subadditive property of $m_0((0, 0), (t, 0))$. Namely, for any $s_1,\ t_1$ and $s_2,\ t_2$ we have

$$m_0((0, 0), (s_2, t_2)) \leqq m_0((0, 0), (s_1, t_1)) + m_0((s_1, t_1), (s_2, t_2)).$$

This ensures the relation $\mathbf{E}(m_0((0, 0), (t, 0))) < +\infty$ for any $t > 0$.

In the same way one can show that

$$\mathbf{E}\left(\sup_{0 \leqq t \leqq 1} m_0((0, 0), (t, 0)) \right) < +\infty.$$

Then the second statement of Proposition follows from the Kingman subadditive ergodic theorem, see [32, Theorem 5.6] for the continuous-time version of this theorem that applies in the case under consideration. □

Proposition 4.6. (isotropy and uniformity of the surface tension) *We have*

$$\tau_0 = \lim_{t\to+\infty} \frac{m_0(x, x+tv)}{t} \tag{24}$$

for all $v \in S^1$, *and the limit is uniform for* $x = x(t)$ *if* $|x| \leq Ct$ *and* $v \in S^1$.

Proof. Our first goal is to show that there exists a constant C_0 such that a.s. for any $\varkappa > 0$ and $c_1 > 0$ and for all $t \geqq t_0(\omega, c_1)$ we have

$$m_0(x, y) \leqq C_0|x-y| + \varkappa t + \sqrt{t} \tag{25}$$

for all x and y from the cube $\{x \in \mathbb{R}^2 : |x|_\infty \leqq c_1 t\}$. To this end we use again the definition of a cube Q_{5L} given in the proof of Lemma 4.1 and recall that a site $j \in \mathbb{Z}^2$ is open if conditions $(\mathbf{c}_1)$–$(\mathbf{c}_3)$ are fulfilled. We then choose the parameter γ in (22) sufficiently small so that the open sites form a.s. an infinite open cluster that we call $\mathcal{C}$. Then a.s. for sufficiently large t the diameter of any l^∞-connected component of sites in in the complement to the infinite open cluster in $\{x \in \mathbb{R}^2 : |x|_\infty \leqq (10L)^{-1}c_1 t\}$ does not exceed $c_2 \log t$ with $c_2 > 0$, see [28]. Computing the probability to have in a cube of size $c_2 \log t$ more than $\sqrt{t}$ points of $\mathcal{N}$, considering the fact that the number of such cubes centred at $j \in \mathbb{Z}^2$ and belonging to $\{x \in \mathbb{R}^2 : |x|_\infty \leqq (10L)^{-1}c_1 t\}$ grows polynomially in t and using the Borel-Cantelli lemma we conclude that a.s. for sufficiently large t we have

$$m_0(x, \tilde{\pi}_\alpha(x)) \leqq \sqrt{t}, \qquad m_0(y, \tilde{\pi}_\alpha(y)) \leqq \sqrt{t}, \tag{26}$$

where $\tilde{\pi}_\alpha(x)$ is the nearest to x vertex of the union of the Voronoi cells that contain points of the scaled infinite open cluster $10L\mathcal{C}$.

From the results in [26] it follows that a.s. for sufficiently large t, for any two points j^1 and j^2 of the open infinite cluster such that $j^1, j^2 \in \{x \in \mathbb{R}^2 : |x|_\infty \leqq (10L)^{-1}c_1 t\}$, and for any $\varkappa > 0$ the cluster distance between j^1 and j^2 is not greater than $C_2|j^1 - j^2| + \varkappa t$; here C_2 is a positive constant that does not depend on $\varkappa$. Combining this estimate with (26) we obtain (25).

Next, we are going to show that for any $x \in \mathbb{R}^2$ with $|x| \leqq C$ and any $v \in S^1$ the limit relation

$$\tau_0 = \lim_{t\to+\infty} \frac{m_0(tx, tx+tv)}{t} \tag{27}$$

holds a.s. In view of (25) it suffices to prove this relation for integer t that tends to ∞. In the remaining part of the proof we call this parameter n instead of t.

We fix a small positive $\theta > 0$ and denote by $\mathcal{A}_N$ the event

$$\mathcal{A}_N = \left\{\omega \in \Omega : \left|\frac{m_0(0, kv)}{k} - \tau_0\right| \leqq \theta \text{ for all } k \geqq N\right\}.$$

Since $\mathbf{P}(\mathcal{A}_N)$ tends to 1 as $N \to \infty$, for any $\delta > 0$ there exists $N_0 = N_0(\delta)$ such that

$$\mathbf{P}(\mathcal{A}_{N_0}) \geqq 1-\delta.$$

By the Birkhoff ergodic theorem a.s. for any $\nu > 0$ and $\varkappa > 0$ there exists $k_0 = k_0(\omega, \nu, \varkappa)$ such that

$$\left| \frac{1}{k} \sum_{j=1}^{k} \mathbf{1}_{\mathcal{A}_{N_0}}(T_{jx}\omega) - \mathbf{P}(\mathcal{A}_{N_0}) \right| \leqq \nu$$

for all $k \geqq \frac{1}{2}k_0$ and moreover inequality (25) holds for all such k. We assume that ν and δ are small enough so that $3(\nu+\delta) \leqq \frac{1}{2}$.

For $k \geqq k_0$ denote by ℓ the maximum of integers j such that $j > k+1$ and for all $i \in (k, j)$ we have $T_{ix}\omega \notin \mathcal{A}_{N_0}$.

Let M be the number of unities in the sequence $\left\{\mathbf{1}_{\mathcal{A}_{N_0}}(T_{ix}\omega)\right\}_{i=1}^{k}$. By the definition of ℓ, the number of unities in $\left\{\mathbf{1}_{\mathcal{A}_{N_0}}(T_{ix}\omega)\right\}_{i=1}^{k+\ell}$ is equal to M as well.

Since $k+\ell > k_0$, we have

$$\nu > \left| \frac{M}{k+\ell} - \mathbf{P}(\mathcal{A}_{N_0}) \right| = \left| 1 - \mathbf{P}(\mathcal{A}_{N_0}) - \frac{\ell + (k-M)}{k+\ell} \right|.$$

This yields

$$\frac{\ell + (k-M)}{k+\ell} < \nu + 1 - \mathbf{P}(\mathcal{A}_{N_0}) \leqq \nu + \delta.$$

Since $k - M \geqq 0$, recalling that $\nu + \delta \leqq \frac{1}{6}$ we obtain $\ell \leqq 2(\nu+\delta)k$.

For an arbitrary $k > \max\big(k_0, N_0\big)$ and $L = 3(\nu+\delta)k$ there exists $n \in [k, k+L]$ such that $T_{nx}\omega \in \mathcal{A}_{N_0}$. Then we have

$$\left| \frac{1}{k} m_0^{\omega}(nx, nx+kv) - \tau_0 \right| = \left| \frac{1}{k} m_0^{T_{nx}\omega}(0, kv) - \tau_0 \right| \leqq \theta. \tag{28}$$

Since $n - k \leqq 3(\nu+\delta)k$ and $k > k_0$, then by (25)

$$|m_0(nx, nx+kv) - m_0(kx, kx+kv)| \leqq [3C_0C(\nu+\delta) + \varkappa]k + \sqrt{k}.$$

Dividing by k and considering (28) we obtain

$$\left| \frac{1}{k} m_0^{\omega}(kx, kx+kv) - \tau_0 \right| \leqq \theta + [3C_0C(\nu+\delta) + \varkappa] + \frac{1}{\sqrt{k}}.$$

It remains to take into account the fact that θ, ν, δ and $\varkappa$ are arbitrary positive number, and (27) follows.

In view of estimate (25) the pointwise convergence in (27) implies the uniform convergence in (24) for $|x| \leqq Ct$. This completes the proof. □

Proposition 4.7. (coerciveness of the surface tension) *We have* $\tau_0 > 0$.

Proof. Given $t > 0$ take a minimal path $\{e_k\}$ for $m_0((0,0),(0,t))$. We can apply Lemma 3.1 with $\varepsilon = 1/t$, $R = 1$, $\gamma = 1/2$, and $P \in \Pi$ with $e_k \subset P$ for all k. We then have

$$t(1+o(1)) \le \#\mathbf{A}(P) \le C\#\{C_i : C_i \subset P\} \le C\#\{e_k\},$$

which shows the claim, since the constant in this estimate are independent of t. $\square$

Proposition 4.8. *There exists a constant C_0 such that if t is large enough then if $\{e_k\}$ is a test path for $m_0(x, x+tv)$ with x as in Proposition 4.6 with $\#(\{e_k\}) \le tM$, then each point of $\{e_k\}$ is at most at distance C_0Mt from x.*

Proof. It suffices to apply Lemma 3.1 to the set of all Voronoi cells with non empty intersection with $\bigcup_k e_k$ and $\varepsilon = 1/t$. We then cover $\bigcup_k e_k$ with the union of at most $2C_0Mt$ cubes, from which the claim follows. $\square$

4.3. Computation of the Γ*-limit*

Lower bound. We use an argument typical of the blow-up technique [16,24].

Let $u^\varepsilon \to A$. Since A is of finite perimeter, with fixed $\sigma > 0$ and $\delta > 0$ we consider a disjoint finite family of rectangles

$$R_i = \left\{ x : |\langle x - x_i, \nu_i\rangle| \le \delta\rho_i, |\langle x - x_i, \nu_i^\perp\rangle| \le \frac{1}{2}\rho_i \right\}$$

such that

$$\mathcal{H}^1\left(\partial A \setminus \bigcup_i R_i\right) \le \sigma \qquad \text{and} \qquad \left|\sum_i \rho_i - \mathcal{H}^1(\partial A)\right| \le \sigma.$$

Since $A_\varepsilon \to A$ we may assume that

$$\mathcal{L}^2(A_\varepsilon \cap R_i^+) = o(1), \qquad \mathcal{L}^2((A \setminus A_\varepsilon) \cap R_i^+) = o(1)$$

as $\varepsilon \to 0$, where

$$R_i^\pm = R_i \pm 2\delta\rho_i\nu_i.$$

We now fix an index i. We use the channel property in Lemma 4.1 to find a path $\{\varepsilon C_j^+\}$ joining the two sides of R_i^+ parallel to ν_i, with j endpoints of segments of a path in $\mathcal{E}_\alpha$, and such that

$$\mathcal{L}^2\left(A_\varepsilon \cap R_i^+ \cap \varepsilon \bigcup_j C_j^+\right) \le \frac{\varepsilon}{C_\delta \rho_i}\mathcal{L}^2(A_\varepsilon \cap R_i^+),$$

which follows from the existence of a number of disjoints paths proportional to ρ_i.

Note that, since $|\varepsilon C_j^+| \ge \pi\varepsilon^2\alpha^2$, we have

$$\#\{j : \varepsilon C_j^+ \subset A_\varepsilon\} \le \frac{1}{\pi\varepsilon^2\alpha^2}\mathcal{L}^2\left(A_\varepsilon \cap R_i^+ \cap \varepsilon \bigcup_j C_j^+\right)$$
$$\le \frac{1}{\varepsilon\pi C_\delta\rho_i\alpha^2}\mathcal{L}^2(A_\varepsilon \cap R_i^+).$$

Similarly, we define $\{C_j^-\}$ joining the two sides of R_i^- parallel to ν_i, and such that

$$\mathcal{L}^2\left((A\backslash A_\varepsilon)\cap R_i^-\cap\varepsilon\bigcup_j C_j^-\right)\le\frac{\varepsilon}{C_\delta\rho_i}\mathcal{L}^2((A\backslash A_\varepsilon)\cap R_i^-),$$

so that

$$\#\{j:\varepsilon C_j^-\subset(A\setminus A_\varepsilon)\}\le\frac{1}{\varepsilon\pi C_\delta\rho_i\alpha^2}\mathcal{L}^2(((A\setminus A_\varepsilon)\cap R_i^-).$$

We define U_ε^+ as the connected component of $R_i^+\setminus\varepsilon\bigcup_j C_j^+$ containing the upper side $S_i^+=\{x\in R_i^+:\langle x-x_i,\nu_i\rangle=3\delta\rho_i\}$ and U_ε^- as the connected component of $R_i^-\setminus\varepsilon\bigcup_j C_j^-$ containing the lower side $S_i^-=\{x\in R_i^-:\langle x-x_i,\nu_i\rangle=-3\delta\rho_i\}$, and define

$$\widetilde{A}_\varepsilon=(A_\varepsilon\setminus U_\varepsilon^+)\cup U_\varepsilon^-.$$

We now consider the connected component of the set $(R_i\cup R_i^+\cup R_i^-)\setminus\widetilde{A}_\varepsilon$ containing the upper side S_i^+. Note that this connected component does not contain S_i^-, so that it contains a path of edges $\{e_k^\varepsilon\}$ in $\mathcal{V}$ connecting the two sides of $R_i\cup R_i^+\cup R_i^-$ parallel to ν_i. We denote by $x_\varepsilon^\pm$ the extreme points of this path.

Using Proposition 4.6, we can now estimate

$$\begin{aligned}\#\{\text{edges of }\partial V_\varepsilon(u^\varepsilon)\text{ inside }R_i\}&\ge\#\{\text{edges of }\partial A_\varepsilon\text{ inside }R_i\}\\&\ge\#\{e_k^\varepsilon\}-\frac{1}{\varepsilon\pi C_\delta\rho_i\alpha^2}o(1)\\&\ge m_0(x_\varepsilon^-,x_\varepsilon^+)-\frac{1}{\varepsilon\pi C_\delta\rho_i\alpha^2}o(1)\\&\ge(\tau_0+o(1))\frac{\rho_i}{\varepsilon}-\frac{1}{\varepsilon\pi C_\delta\rho_i\alpha^2}o(1).\end{aligned}$$

Summing up in i we then get

$$\liminf_{\varepsilon\to0}E_\varepsilon(u^\varepsilon)\ge\sum_i\rho_i\tau_0\ge\tau_0(\mathcal{H}^1(\partial A)-\sigma)$$

and prove the claim by the arbitrariness of σ.

Upper bound. By an approximation argument [10,11] it is sufficient to prove the upper bound for polyhedral sets. Moreover, we can just deal with a single connected bounded polyhedron A with a connected boundary since all other cases can be reduced to that by considering union or complements of such sets.

We write the boundary of A as the union of segments $[x_{j-1},x_j]$ with endpoints $x_0,\dots,x_N\in\mathbb{R}^2$ with $x_N=x_0$. With fixed $m\in\mathbb{N}$ and $\delta>0$, for all $j\in\{1,\dots,N\}$ and $l\in\{1,\dots,m\}$ we consider a non-intersecting path $\{e_k^{j,l}\}$ in $\mathcal{V}$ between $\pi_0(x_{j,m-1}^\varepsilon)$ and $\pi_0(x_{j,m}^\varepsilon)$, where

$$x_{j,m}^\varepsilon=\frac{1}{\varepsilon}\Big(x_{j-1}+\frac{l}{m}(x_j-x_{j-1})\Big),$$

such that

$$\#\{e_k^{j,l}\} \leq \frac{1}{m\varepsilon}|x_j - x_{j-1}|(\tau_0 + \delta). \tag{29}$$

Denoting the union of the rescaled paths

$$B_\varepsilon^{\delta,m} = \varepsilon \bigcup_{j,l,k} e_k^{j,l}$$

let $A_\varepsilon^{\delta,m}$ be the complement of the infinite connected component of $\mathbb{R}^2 \setminus B_\varepsilon^{\delta,m}$ (note that the paths $\{e_k^{j,l}\}$ may intersect, so that there may be more than one bounded connected component of the complement of their union). If u^ε is defined as

$$u_i^\varepsilon = \begin{cases} 1 & \text{if } i \in A_\varepsilon^{\delta,m} \\ -1 & \text{if } i \notin A_\varepsilon^{\delta,m}, \end{cases} \tag{30}$$

then we have

$$\begin{aligned} E_\varepsilon(u^\varepsilon) &\leq \varepsilon \sum_{j,l,k} \#\{e_k^{j,l}\} \leq \sum_{j,l} \frac{1}{m}|x_j - x_{j-1}|(\tau_0 + \delta) \\ &= \mathcal{H}^1(\partial A)(\tau_0 + \delta), \end{aligned} \tag{31}$$

since the boundary of $A_\varepsilon^{\delta,m}$ is contained in $B_\varepsilon^{\delta,m}$.

By Lemma 2.2, thanks to (31) these sets converge as $\varepsilon \to 0$ to a set of finite perimeter $A^{\delta,m}$, and

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon(A^{\delta,m}) \leq \mathcal{H}^1(\partial A)(\tau_0 + \delta). \tag{32}$$

Thanks to Proposition 4.8 each point of $\varepsilon\{e_k^{j,l}\}$ is at most at a distance C/m from the segment $[\varepsilon\pi_0(x_{j,m-1}^\varepsilon), \varepsilon\pi_0(x_{j,m}^\varepsilon)]$, and hence, since

$$\lim_{\varepsilon\to 0} \varepsilon\pi_0(x_{j,m}^\varepsilon) = x_{j-1} + \frac{l}{m}(x_j - x_{j-1}),$$

the boundary of $A^{\delta,m}$ is contained in a C/m-neighbourhood of ∂A. This implies that $A^{\delta,m}$ converge to A as $m \to +\infty$ independently of δ. By the lower semicontinuity of the Γ-limsup [11] we then deduce that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon(A) \leq \lim_{m\to+\infty} \Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon(A^{\delta,m}) \leq \mathcal{H}^1(\partial A)(\tau_0 + \delta),$$

and the claim is proved.

5. Finite Range of Interactions

In this section we consider a Poisson model with finite range of interactions. Given $\mathcal{R} > 0$ denote by $\mathcal{E}^{\mathcal{R}}$ the subset of $\mathcal{N} \times \mathcal{N}$ defined by

$$\mathcal{E}^{\mathcal{R}} = \{i, j \in \mathcal{N}, : |i - j| \leqq \mathcal{R}\}.$$

The corresponding (scaled) ferromagnetic energy on spin functions $u : \varepsilon\mathcal{N} \to \{-1, 1\}$ takes the form

$$E_\varepsilon^{\mathcal{R}}(u) = \frac{1}{8} \sum_{(i,j)\in\varepsilon\mathcal{E}^{\mathcal{R}}} \varepsilon(u_i - u_j)^2 \; = \; \frac{1}{2}\varepsilon\,\#\{(i,j)\in\varepsilon\mathcal{E}^{\mathcal{R}} : u_i \neq u_j\}$$
$$= \varepsilon\,\#\{(i,j)\in\varepsilon\mathcal{E}^{\mathcal{R}} : u_i = 1, u_j = -1\}. \qquad (33)$$

We assume that $\mathcal{R}$ is sufficiently large. In order to clarify this, we fix $\gamma \in (0, \frac{1}{2})$ and denote by $\mathcal{E}_L(j)$, $j \in \mathbb{Z}^2$, the event that the following condition is fulfilled:
• any square $[0, L]^2 + Li$ with $i \in \mathbb{Z}^2 \cap [-9.5, 10.5]^2$ contains at least one point of $\mathcal{N} - 10Lj$.
The squares $10Lj + Q_{20L}$ for which this condition holds are called L-good. As explained in the proof of Lemma 4.1 the probability of $\mathcal{E}_L(j)$ tends to 1 as $L \to \infty$. Letting $Q_{20L}(j) = 10Lj + Q_{20L}$ with $j \in \mathbb{Z}^2$ we say that a collection of squares $\{Q_{20L}(j_k)\}_{k=1}^N$ is admissible if the interiors of these squares do not intersect. We say that $Q_{20L}(j)$ and $Q_{20L}(m)$ are neighbouring if $|m - j| = 2$ or $|m - j| = \sqrt{5}$. The notion of connectedness is introduced accordingly.

Lemma 5.1. *There exist $L_0 > 0$ and $\beta > 0$ such that a.s. for any $R > 0$, there exists $\varepsilon_0(\omega, R)$ such that for any $L \geqq L_0$ and for $\varepsilon \leqq \varepsilon_0$, any admissible connected subset $\mathcal{S}$ of $\{Q_{20L}(j)\}_{j\in\mathbb{Z}^2}$ with $\#(\mathcal{S}) \geqq \varepsilon^{-\gamma}$ and $\mathcal{S} \cap [-\varepsilon^{-1}R, \varepsilon^{-1}R]^2 \neq \emptyset$ contains at least $\beta\#(\mathcal{S})$ L-good squares.*

Proof. The proof relies on the standard counting arguments. Each $(20L)$-square has 12 neighbouring squares. Therefore, the total number of connected admissible sets of squares that contain $Q_{20L}(0)$ and have cardinality N does not exceed $e^{\bar{c}}N$ with a constant $\bar{c} > 0$, see [28]. For any finite collection of admissible $(20L)$-squares the events $\mathcal{E}_L(j)$ are independent. Thus, for any admissible connected set of such squares that has exactly N squares the probability that the proportion of good L-squares is less than β is less than $\exp\big[\big(\bar{c} + \log 2 + \beta(1 - \mathcal{E}_L(0))\big)N\big]$. Since, the probability of $\mathcal{E}_L(0)$ tends to 1 as $L \to \infty$, this yields the desired statement by the Borel-Cantelli lemma. □

From now on we assume that $\mathcal{R} \geqq 5L_0$. For the energies defined in (33) we obtain a Γ-limit (homogenization) result. Since the techniques used here are quite similar to those used in the previous sections, for the majority of statements we provide just a sketch of the proofs.

Lemma 5.2. *Let $\mathcal{R}$ be sufficiently large, and assume that a family $\{u^\varepsilon\}$ is such that $\sup_\varepsilon E_\varepsilon^{\mathcal{R}}(u^\varepsilon) < +\infty$. Then $V_\varepsilon(u^\varepsilon)$ admits the following representation:*

$$V_\varepsilon(u^\varepsilon) = (A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon,$$

where $|B'_\varepsilon| + |B''_\varepsilon| \to 0$ as $\varepsilon \to 0$, the family $\mathbf{1}_{A_\varepsilon}$ is precompact in $L^1_{\rm loc}(\mathbb{R}^2)$ and each its limit point is the characteristic function of a set of finite perimeter A, so that the same holds for $\mathbf{1}_{V_\varepsilon(u^\varepsilon)}$.

Proof. The proof follows the line of the proof of Lemma 3.2 after replacing the unit square Q with Q_{10L_0}. In particular, the sets B'_ε, B''_ε, A'_ε and A''_ε are introduced in the same way as in Lemma 3.2. The inequality

$$\mathcal{H}^1(\partial A'_\varepsilon) \leq C\, E_\varepsilon^{\mathcal{R}}(u^\varepsilon)$$

follows from Lemma 5.1. □

Denote $J^{\mathcal{R}}$ the set of segments $[i, j] \subset \mathbb{R}^2$ with $(i, j) \in \mathcal{E}^{\mathcal{R}}$. Then for any smooth non-selfintersecting curve $\varphi : [0, 1] \to R^2$ such that φ has no points in common with $\mathcal{N}$ we set

$$m^{\mathcal{R}}_{\varphi} = \#\left(\{\varphi(t) : t \in [0, 1]\} \cap J^{\mathcal{R}}\right)$$

and

$$m^{\mathcal{R}}(x, y) = \min_{\varphi,\, \varphi(0)=x, \varphi(1)=y} m^{\mathcal{R}}_{\varphi}.$$

The proof of the next statement is exactly the same as that of Proposition 4.5. It relies on the properties of α-squares introduced above and the subadditive theorem.

Proposition 5.3. *For all* $t > 0$ *we have*

$$\mathbf{E}(m^{\mathcal{R}}((0, 0), (t, 0))) < +\infty \tag{34}$$

Furthermore, the limit

$$\tau^{\mathcal{R}} = \lim_{t \to +\infty} \frac{m^{\mathcal{R}}((0, 0), (t, 0))}{t}$$

exists almost surely and is deterministic.

The convergence stated in the previous proposition is uniform if the starting point satisfies the estimate $x \leqq Ct$. This is granted by the following proposition:

Proposition 5.4. *We have*

$$\tau^{\mathcal{R}} = \lim_{t \to +\infty} \frac{m^{\mathcal{R}}(x, x + tv)}{t} \tag{35}$$

for all $v \in S^1$*, and the limit is uniform for* $x = x(t)$ *if* $|x| \leq Ct$ *and* $v \in S^1$.

The proof of this statement is the same as that of Proposition 4.6.

We proceed with the main result of this section.

Theorem 5.5. *(homogenization theorem) Let* $\mathcal{N}$ *be a Poisson point process with intensity* 1*. Then for sufficiently large* $\mathcal{R}$

the energies $E^{\mathcal{R}}_{\varepsilon}$ *defined in* (33) Γ*-converge to the energy* $F^{\mathcal{R}}(A) = \tau^{\mathcal{R}} \mathcal{H}^1(\partial A)$*, defined on sets of finite perimeter, with respect to the convergence introduced in Definition* 2.1.

Proof. The statement of this Theorem can be derived from Lemmas 5.1–5.2 and Propositions 5.3–5.4, it follows the line of the proof of Theorem 2.3. □

6. Approximate Surface Tensions

In this final section we consider the restriction of the energies E_{ε} to (spin functions with corresponding) sets whose boundary is composed of edges of α-regular Voronoi cells. We denote by E^{α}_{ε} such energies. Note that in this case $E^{\alpha}_{\varepsilon}(u^{\varepsilon})$

immediately gives the equiboundedness of the perimeter of the sets $V_\varepsilon(u^\varepsilon)$ and hence their precompactness. We briefly describe the limit of E_ε^α at fixed α.

With given $\alpha < \alpha_0$ as in Proposition 4.2 we define, for all $x \in \mathbb{R}^2$,that

$$\pi_\alpha(x) = \text{ closest point of } \mathcal{N}_\alpha^* \text{ to } x.$$

For almost all x this point is uniquely defined. For the remaining points we choose one of the closest points of $\mathcal{N}_\alpha^*$ to x. For all $x, y \in \mathbb{R}^2$ we set

$$m_\alpha(x, y) = \min\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{V}_\alpha \text{ connecting } \pi_\alpha(x) \text{ and } \pi_\alpha(y)\}.$$

Proposition 6.1. *For all $\alpha < \alpha_0$ a.s. the limit*

$$\tau_\alpha = \lim_{t\to+\infty} \frac{m_\alpha(x, x + tv)}{t}$$

exists for all $v \in S^1$, and the limit is uniform for $x = x(t)$ if $|x| \le Ct$ and $v \in S^1$. Furthermore $\tau_\alpha \in (0, +\infty)$.

Proof. The proof follows that for τ_0, and is actually simpler since bounds for $m_\alpha(x, x + tv)$ are easier. □

Theorem 6.2. *(homogenization on the α-cluster) For $\alpha < \alpha_0$ almost surely there exists the Γ-limit of E_ε^α and it equals $\tau_\alpha \mathcal{H}^1(\partial A)$.*

Proof. The proof is the same as for the homogenization theorem in the previous section, taking care of using the same α as the one labeling the energies in the proof of the lower bound. Note that it is not necessary to use Proposition 4.8 for the proof of the upper inequality. □

Proposition 6.3. *We have* $\inf\limits_{\alpha<\alpha_0} \tau_\alpha = \lim\limits_{\alpha\to 0} \tau_\alpha$.

Proof. Choose $\alpha_0 > 0$ in such a way that for some L and K we have

$$\mathbf{P}(\mathcal{E}(L, K, \alpha_0, j)) > p_{\rm cr}.$$

It suffices to show that $\tau_{\alpha_1} \leqq \tau_{\alpha_2}$, if $0 < \alpha_1 < \alpha_2 \leqq \alpha_0$. Since $\mathcal{N}_{\alpha_2}^* \subset \mathcal{N}_{\alpha_1}^*$, then

$$\begin{gathered}\min\big\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{V}_{\alpha_1} \text{ connecting } \pi_{\alpha_2}(x_0) \text{ and } \pi_{\alpha_2}(x_t)\big\} \\ \leqq m_{\alpha_2}(x_0, x_t),\end{gathered} \tag{36}$$

where $x_0 = 0$ and $x_t = (t, 0)$. We should estimate

$$\min\big\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{V}_{\alpha_1} \text{ connecting } \pi_{\alpha_1}(x_t) \text{ and } \pi_{\alpha_2}(x_t)\big\}.$$

To this end we consider the cubes $Q_{5L} + 10Lj$, $j \in \mathbb{Z}^2$, that were introduced in the proof of Lemma 4.1 and take those of them that satisfy conditions $(\mathbf{c}_1)$–$(\mathbf{c}_3)$ for $\alpha = \alpha_0$. Under our choice of α_0 a.s. these exists a unique infinite cluster of such cubes. The complement to the infinite cluster consists of connected bounded sets. Moreover, according to [28], for sufficiently large t the maximal size of the connected components in the complement to the infinite cluster that have a non-trivial intersection with $[-2t, 2t]^2]$ does not exceed $c \log(t)$. This implies that the size of the maximal connected component of $[-2t, 2t]^2] \setminus \mathcal{V}_{\alpha_0}$ does not exceed $c \log(t)$. Since $\mathcal{N}^*_{\alpha_0} \subset \mathcal{N}^*_{\alpha_2} \subset \mathcal{N}^*_{\alpha_1}$, then $\pi_{\alpha_1}(x_t)$ and $\pi_{\alpha_2}(x_t)$ belong to the closure of the same connected component of $[-2t, 2t]^2] \setminus \mathcal{V}_{\alpha_0}$. Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \min \left\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{V}_{\alpha_1} \text{ connecting } \pi_{\alpha_1}(x_t) \text{ and } \pi_{\alpha_2}(x_t)\right\} = 0.$$

Similarly,

$$\lim_{t\to\infty} \frac{1}{t} \min \left\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{V}_{\alpha_1} \text{ connecting } \pi_{\alpha_1}(x_0) \text{ and } \pi_{\alpha_2}(x_0)\right\} = 0.$$

Combining these two relations with (36) we obtain the desired inequality $\tau_{\alpha_1} \leqq \tau_{\alpha_2}$. □

It turns out that for vanishing α the approximate surface tension τ_α converges to τ_0; this is the subject of the following statement:

Proposition 6.4. *The following relation holds:*

$$\lim_{\alpha\to 0} \tau_\alpha = \tau_0. \tag{37}$$

Proof. Due to Proposition 4.6 for large n we have

$$m_\alpha((0,0),(0,n)) \geqq m_0((0,0),(0,n))(1+o(1)),$$

where a.s. $o(1)$ tends to zero as $n \to \infty$. Therefore, $\tau_\alpha \geqq \tau_0$. Our goal is to prove the opposite inequality. To this end consider a cube $Q_n = [-n, n]^2$, and denote by C_i^n the cells of Voronoi tessellation corresponding to the set $\mathcal{N}^n = \mathcal{N} \cap Q_n$, the dual set is denoted by $\mathcal{N}^{*,n}$, and $\mathcal{E}^n$ is defined accordingly. Letting

$$\pi_0^n(x) = \text{ closest point of } \mathcal{N}^{*,n} \text{ to } x$$

for all $x \in Q_n$, we introduce

$$m_0^n(x, y) = \min\{\#\{e_k\} : \{e_k\} \text{ is a path in } \mathcal{E}^n \text{ connecting } \pi_0^n(x) \text{ and } \pi_0^n(y)\} \tag{38}$$

for $x,\ y \in Q_n$. By the same arguments as those used in the proof of Proposition 4.6 one can show that a.s.

$$\lim_{n\to\infty} \frac{m_0^n((0,0),(0,n))}{n} = \tau_0. \tag{39}$$

Then for any $\varkappa > 0$ and $\delta > 0$ there exists $n_0 > 0$ such that

$$\mathbf{P}\left\{m_0^{n_0}((0,0),(0,n_0)) \geqq \tau_0 n_0(1+\delta)\right\} \leqq \varkappa. \tag{40}$$

Next we choose $\alpha_1 > 0$, K and L such that, for any $j \in \mathbb{Z}^2$,

$$\mathbf{P}(\mathcal{E}(L, K, \alpha_1, j)) > 1 - \varkappa. \tag{41}$$

The choice of α_1, K and L does not depend on n_0. Without loss of generality we may assume that

$$K \leqq \delta\tau_0 n_0 \tag{42}$$

and that $n_0 = (5L + 10l_1 L)$ for some $l_1 \in \mathbb{Z}^+$, that is the point $(n_0, 0)$ belongs to the right side of the cube $Q_{5L} + 10Lj_1$ with $j_1 = (l_1, 0)$.

As was shown in the proof of Lemma 4.1 the inequality

$$\mathbf{P}\left\{\mathcal{E}\left(\tfrac{n_0}{5}, K_2, \alpha_2, j\right)\right\} > 1 - \varkappa \tag{43}$$

holds for sufficiently large K_2 and sufficiently small α_2; here we assume that n_0 is large enough.

Denote $e_i, i = 1,\ 2$, the standard basis in $\mathbb{R}^2$. We say that a cube Q_{n_0} is α_2-good, if

- $m_0^{n_0}(0, \pm n_0 e_i) \leqq \tau_0 n_0(1 + \delta), \quad i = 1, 2;$
- $\omega \in \mathcal{E}(L, K, \alpha_1, \pm l_1 e_i), i = 1, 2;$
- $\omega \in \mathcal{E}\big(\frac{n_0}{5}, K_2, \alpha_2, 0\big)$.

A cube $Q_{n_0} + 2n_0 j$, $j \in \mathbb{Z}^2$, is said to be α_2-good, if the cube Q_{n_0} is α_2-good with respect to the point process $\mathcal{N} - 2n_0 j$.

Define a random variable θ_j which is equal to 1, if the cube $Q_{n_0} + 2n_0 j$ is α_2-good, and 0 otherwise. The random variables $\{\theta_j\}_{j\in\mathbb{Z}^2}$ are i.i.d. From (40)–(43) it follows that $\mathbf{P}\{\theta_j = 1\} \geqq 1 - 9\varkappa$. Furthermore, for any two neighbouring α_2-good cubes $Q_{n_0} + 2n_0 j_1$ and $Q_{n_0} + 2n_0 j_2$ with $|j_1 - j_2|_\infty = 1$ we have

$$m_{\alpha_2}(2n_0 j_1, 2n_0 j_2) \leqq 2\tau_0 n_0(1 + 2\delta). \tag{44}$$

For small enough $\varkappa$ the α_2-good cubes $\{Q_{n_0} + 2n_0 j\}$ form a unique infinite cluster which is identified with the corresponding cluster for the variables θ_j. For $t = 2n_0 k$ with $k \in \mathbb{Z}^+$ denote by $\rho_\varkappa(k)$ the cluster distance between the α_2-good cubes which are closest to 0 and to t, respectively. According to [28] the limit $\widetilde{\rho}(\varkappa) = \lim\limits_{k\to\infty} \big(k^{-1}\rho_\varkappa(k)\big)$ exists a.s. and is deterministic. Moreover, $\widetilde{\rho}(\varkappa) \to 1$ as $\varkappa \to 0$. Then for large k we obtain

$$m_{\alpha_2}\big((0,0), (0,t)\big) \leqq 2\tau_0 n_0(1 + 2\delta)\big(\rho_\varkappa(k) + o(k)\big),$$

where $o(k)$ tends to zero as $k \to \infty$. Dividing the last relation by t and passing to the limit $t \to \infty$ yields

$$\tau_{\alpha_2} \leqq \tau_0(1 + 2\delta)(\widetilde{\rho}(\varkappa)).$$

Since δ and $\varkappa$ are arbitrary positive numbers and $\widetilde{\rho}(\varkappa)$ tends to 1 as $\varkappa \to 0$, we obtain the desired convergence (37). □

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