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Homogenization of Periodic Systems with Large Potentials

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Abstract

We consider the homogenization of a system of second-order equations with a large potential in a periodic medium. Denoting by ε the period, the potential is scaled as ε^{-2}. Under a generic assumption on the spectral properties of the associated cell problem, we prove that the solution can be approximately factorized as the product of a fast oscillating cell eigenfunction and of a slowly varying solution of a scalar second-order equation. This result applies to various types of equations such as parabolic, hyperbolic or eigenvalue problems, as well as fourth-order plate equation. We also prove that, for well-prepared initial data concentrating at the bottom of a Bloch band, the resulting homogenized tensor depends on the chosen Bloch band. Our method is based on a combination of classical homogenization techniques (two-scale convergence and suitable oscillating test functions) and of Bloch waves decomposition.

1. Introduction

We study the homogenization of evolution problems for a singularly perturbed second-order elliptic system with periodically oscillating coefficients. To fix ideas, let us consider the following parabolic problem

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \left(\varepsilon^{-2}c\left(\frac{x}{\varepsilon}\right) + d\left(x, \frac{x}{\varepsilon}\right)\right) u_\varepsilon &= 0 \quad \text{in } \Omega \times (0, T), \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(t = 0, x) &= u_\varepsilon^0(x) \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set and $T > 0$ a final time. The unknown $u_\varepsilon(t, x)$ is a vector-valued function from $\Omega \times (0, T)$ into $\mathbb{R}^K$. The coefficients $A(y)$, $c(y)$ and $d(x, y)$ are real and bounded functions defined for $x \in \Omega$ and $y \in \mathbb{T}^N$ (the unit torus). Furthermore, the tensor $A(y)$ is symmetric, uniformly positive definite, while $c(y)$ and $d(x, y)$ are symmetric with no positivity assumption. The parabolic equation (1) is just an example: other evolution problems of interest covered by this paper are the wave equation, parabolic fourth-order equations, or spectral problems. A generalization to the Schrödinger equation is the topic of another work [10]. The scalar case of (1) (i.e., $K = 1$ and u_ε is a real-valued function) is well understood (see, e.g., [5–9, 13, 20, 28]) and the goal of this paper is to solve the case of systems of several coupled equations. However, the method, as well as some results, are very different in the system case. In order to convince the reader, we first describe the main results and ideas of proof in the scalar case.

For $K = 1$, introduce the first eigencouple of the spectral cell problem

$$-\mathrm{div}_y \left(A(y)\nabla_y \psi_1\right) + c(y)\psi_1 = \lambda_1 \psi_1 \quad \text{in } \mathbb{T}^N, \tag{2}$$

which, by the Krein-Rutman theorem, is simple and satisfies $\psi_1(y) > 0$ in $\mathbb{T}^N$. The first eigenvalue λ_1 can be interpreted physically as a measure of the balance between the diffusion and potential terms. Since ψ_1 does not vanish, the unknown can be changed by writing a so-called *factorization principle*

$$v_\varepsilon(t, x) = e^{\frac{\lambda_1 t}{\varepsilon^2}} \frac{u_\varepsilon(t, x)}{\psi_1\left(\frac{x}{\varepsilon}\right)}, \tag{3}$$

and, after some algebra, it can easily be shown that the new unknown v_ε is a solution of a simpler equation

$$\begin{aligned}
\psi_1^2\left(\frac{x}{\varepsilon}\right)\frac{\partial v_\varepsilon}{\partial t} - \mathrm{div}\left((\psi_1^2 A)\left(\frac{x}{\varepsilon}\right)\nabla v_\varepsilon\right) + (\psi_1^2 d)\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon &= 0 && \text{in } \Omega\times(0, T),\\
v_\varepsilon &= 0 && \text{on } \partial\Omega\times(0, T),\\
v_\varepsilon(t=0, x) &= \frac{u_\varepsilon^0(x)}{\psi_1\left(\frac{x}{\varepsilon}\right)} && \text{in } \Omega.
\end{aligned} \tag{4}$$

The new parabolic equation (4) is simple to homogenize since it does not contain any singularly perturbed term, and we thus obtain the following result.

Theorem 1. *Assume that* (1) *is a scalar problem* ($K = 1$). *If* $u_\varepsilon^0(x) = v^0(x)\psi_1\left(\frac{x}{\varepsilon}\right)$, *then* v_ε, *defined by* (3), *converges weakly in* $L^2\left((0, T); H_0^1(\Omega)\right)$ *to the solution* v *of the following homogenized problem:*

$$\begin{aligned}
\frac{\partial v}{\partial t} - \mathrm{div}\left(A^*\nabla v\right) + d^*(x)\, v &= 0 && \text{in } \Omega \times (0, T),\\
v &= 0 && \text{on } \partial\Omega \times (0, T),\\
v(t = 0, x) &= v^0(x) && \text{in } \Omega,
\end{aligned} \tag{5}$$

where A^* *is a constant homogenized tensor and* $d^*(x)$ *a homogenized coefficient.*

It is clear from the above brief summary of the scalar case that the main idea, namely the factorization principle (3), does not usually work in the case of systems, i.e., $K > 1$. Indeed, in general there is no maximum principle, and therefore no Krein-Rutman theorem, for systems. Thus, ψ_1 may change sign and the change of unknowns (3) is meaningless because v_ε blows up at some points (see however

[5] for a special system for which the maximum principle holds true). Even if we perform a formal computation by assuming that (3) is valid, the system satisfied by v_ε has not a simple structure and it is not clear that it admits a homogenized limit, and even so, there is no reason why the homogenized tensor should be coercive.

In order to homogenize (1) in the system case, our main new idea is to use Bloch wave theory. Under a generic simplicity assumption for the first eigenvalue and a non-degenerate quadratic behavior near its minimum (see (9)) we obtain a result similar to Theorem 1 (see Theorem 2 for details). The two main features are that the homogenized equation is always scalar and that the cell problem must sometimes be shifted, namely the usual periodicity condition in (2) has to be replaced by a Bloch periodicity condition. Technically, the Bloch wave theory allows us to prove a new compactness result (Lemma 3) which shows that sequences satisfying some weak *a priori* estimates can be written approximately as the product of a periodically oscillating sequence and another compact sequence. Our analysis applies not only to the parabolic problem (1) but also to the corresponding spectral problem and hyperbolic system. In the latter case, different limit regimes are obtained according to the sign of the minimal cell eigenvalue λ_1. Section 2 contains our notation, a brief review of Bloch wave theory and our main assumption. Our main results are stated in Section 3 while the proofs are distributed in Sections 4, 5 and 6.

In Section 7 we also obtain new homogenization results for some specific well-prepared initial data (assuming that $\Omega = \mathbb{R}^N$). More precisely, recall that Bloch wave theory introduces the notion of Bloch bands, corresponding to the range of cell eigenvalues or, in physical terms, to energy levels of Fermi surfaces. Theorem 1 is concerned with the first Bloch band (or ground state). If we assume that the initial data u_ε^0 is concentrating at the bottom of a higher level Bloch band (see Section 7 for a precise statement), we obtain a convergence result similar to Theorem 1 but with a different homogenized tensor (depending on the level of the chosen Bloch band). Even in the scalar case this result is new. In the context of the Schrödinger equation it is known as an effective mass theorem (see, the e.g., [21, 23, 24]). The fact that the homogenized tensor depends on the initial data is very striking in homogenization theory since usually effective properties are proved to be intrinsic in the sense that they do not depend on the domain, the applied forces or source terms, and the initial data.

In Section 8 we show that under a new assumption on the first Bloch eigenvalue, a different homogenized limit can be obtained for (1). Indeed, the homogenized problem is a parabolic fourth-order equation.

Finally, Section 9 is devoted to an extension of our previous results to a different model, namely we consider a fourth-order equation. We first obtain homogenized limits similar to those of Section 3 but with a fourth-order operator instead of a second-order one. Then, under a different assumption on the first Bloch eigenvalue, we prove that a second-order homogenized limit can also be obtained (a situation which is symmetric to that in Section 8). Our method could be generalized to other models. In particular, its application to the Schrödinger equation is of paramount interest. However, since much more can be deduced in the Schrödinger case, we address this problem in a separate work [10].

There are several motivations for studying the homogenization of the singularly perturbed system (1). First, (1) is a model of reaction-diffusion equations in periodic media (like a porous medium or a crystal in solid state physics) and the large potential is classical when studying long-time asymptotics. Second, the spectral problem for (1) is the usual model in nuclear reactor physics, the so-called simplified transport equation. This is a set of diffusion equations for the even moments of the neutron flux (moments with respect to the angular velocity variable). One of the main features of this simplified transport system is that it does not satisfy a maximum principle. So our work is the first rigorous study of homogenization for this problem, which is of paramount interest for fast numerical computations in the nuclear industry (see [27] for more details and numerical applications). Third, as a limit case of large potentials we recover perforated domains with periodic holes supporting Dirichlet boundary conditions (take $c = +\infty$ in the holes and $c = 0$ elsewhere). In such a case the term of order ε^{-2} disappears from the equation (1) although there is still a singular perturbation due to the presence of Dirichlet holes. The scalar setting, $K = 1$, was studied in [28] and we extend this result to the vector-valued case. One possible application is the study of a composite material with fixed inclusions in the context of linear elasticity. Fourth, even in the case when $c \equiv 0$ (i.e., without singular perturbation) our homogenization result for initial data concentrating at the bottom of high-level Bloch bands is new and can be seen as a type of corrector result for capturing an initial layer in time in the context of classical homogenization [11, 12, 18] (see Remark 16).

2. Notation and Bloch decomposition

We first give our precise notation and assumptions on the real coefficients $A(y)$, $c(y)$ and $d(x, y)$ involved in equation (1). Our tensorial notation is the following. Recall that N is the space dimension, and K is the system dimension, i.e., all unknown functions are defined with values in $\mathbb{R}^K$. We adopt the convention that Latin indices i, j belong to $\{1, .., N\}$, i.e., refer to spatial coordinates, while Greek indices α, β vary in $\{1, .., K\}$. The $K \times K$ matrices c and d are symmetric, with entries $c_{\alpha\beta}$, $d_{\alpha\beta}$ respectively, and have no specific positivity properties. The tensor A acts on $K \times N$ matrices. Denoting by $(u_\alpha)_{1 \leqq \alpha \leqq K}$ the components of a vector-valued function u, its gradient is the $K \times N$ matrix ∇u defined by its entries

$$\nabla u = \left(\frac{\partial u_\alpha}{\partial x_i} \right)_{1 \leqq \alpha \leqq K,\ 1 \leqq i \leqq N}, \tag{6}$$

and the product $A \nabla u$ is also a $K \times N$ matrix defined with the Einstein summation convention by

$$A \nabla u = \left(A_{\alpha\beta i j} \frac{\partial u_\alpha}{\partial x_i} \right)_{1 \leqq \beta \leqq K,\ 1 \leqq j \leqq N}. \tag{7}$$

The tensor A is symmetric in the sense that

$$A\xi \cdot \xi' = A\xi' \cdot \xi \text{ for any } \xi, \xi' \in \mathbb{R}^{K \times N},$$

and it is uniformly coercive, i.e., there exists $\nu > 0$ such that for a.e. $y \in \mathbb{T}^N$

$$A(y)\xi \cdot \xi \geqq \nu |\xi|^2 \text{ for any } \xi \in \mathbb{R}^{K\times N}.$$

We assume that $A(y)$ and $c(y)$ are real, measurable, bounded, periodic functions, i.e. their entries belong to $L^\infty(\mathbb{T}^N)$, while $d(x, y)$ is real, measurable and bounded with respect to x, and periodic continuous with respect to y, i.e., its entries belong to $L^\infty\left(\Omega; C(\mathbb{T}^N)\right)$ (other assumptions are possible).

A formal two-scale asymptotic expansion (in the spirit of [11]) shows that the leading term in the ansatz of u_ε is the solution of an equation in the unit cell $\mathbb{T}^N$. Therefore, we need to study a microscopic version of (1). It turns out that the key cell problem is the following Bloch (or shifted) spectral cell equation

$$-(\mathrm{div}_y + 2i\pi\theta)\Big(A(y)(\nabla_y + 2i\pi\theta)\psi_n\Big) + c(y)\psi_n = \lambda_n(\theta)\psi_n \quad \text{in } \mathbb{T}^N, \tag{8}$$

which, as a compact self-adjoint complex-valued operator on $L^2(\mathbb{T}^N)^K$, admits a countable sequence of real increasing eigenvalues $(\lambda_n)_{n\geqq 1}$ and normalized eigenfunctions $(\psi_n)_{n\geqq 1}$ with $\|\psi_n\|_{L^2(\mathbb{T}^N)^K} = 1$. The dual parameter θ is called the Bloch frequency and it runs in the dual cell of $\mathbb{T}^N$, i.e., by periodicity it is enough to consider $\theta \in \mathbb{T}^N$. We refer to [11, 15, 25] for more details about the Bloch spectral problem (8).

Our main assumption is that there exists a Bloch parameter $\theta_0 \in \mathbb{T}^N$ such that

$$\begin{array}{ll} \text{(i)} & \theta_0 \text{ is the unique minimizer of } \lambda_1(\theta) \text{ in } \mathbb{T}^N, \\ \text{(ii)} & \lambda_1(\theta_0) \text{ is a simple eigenvalue,} \\ \text{(iii)} & \text{the Hessian matrix } \nabla_\theta\nabla_\theta\lambda_1(\theta_0) \text{ is positive definite.} \end{array} \tag{9}$$

Remark 1. In the scalar case, $K = 1$, assumption (9) is satisfied with $\theta_0 = 0$. Indeed, by using the maximum principle, it is easily seen that the minimum of $\lambda_1(\theta)$ is uniquely attained at 0, and then that the Hessian matrix $\nabla_\theta\nabla_\theta\lambda_1(0)$, being equal to the usual homogenized matrix (see, e.g., [16]), is positive definite. On the other hand, for any $K > 1$ and in the absence of a zero-order term, i.e., $c \equiv 0$, it is easy to check that $\theta_0 = 0$ is the unique minimizer of $\lambda_1(\theta)$ (however, $\lambda_1(0)$ is not simple and, if it exists, the Hessian matrix may be not positive definite). In full generality, there always exists a minimizer of $\lambda_1(\theta)$ but it may be non-unique and $\lambda_1(\theta_0)$ has no reason to be simple (although, by extending the results of [2], it is possible to show that $\lambda_1(\theta_0)$ is generically simple).

Remark 2. The range of possible values of θ_0 is limited. The coefficients A and c being real, it is clear that taking the complex conjugate of (8) amounts to changing θ to $-\theta$. In other words the function $\lambda_1(\theta) = \lambda_1(-\theta)$ is even. Since by periodicity it is enough to minimize $\lambda_1(\theta)$ on $[-1/2, +1/2]^N$, the assumed uniqueness of the minimizer θ_0 implies that necessarily all the components of θ_0 are either 0 or 1/2.

We do not know if it is possible to obtain a non-zero value of θ_0. We performed numerical experiments in 2-d to compute θ_0 for the simplified transport equations (the SPN model) which is a system of two coupled equations [27]. Even for numerical values of the coefficients out of their range of physical validity, we always obtain $\theta_0 = 0$. Nevertheless, in a slightly different context, namely for a system of linear elasticity which is not uniformly elliptic but simply satisfies the Hadamard ellipticity condition (in other words the associated energy is rank-one convex but not convex), there is numerical and physical evidence that the minimal value θ_0 in (9) is not zero [17]. Similarly, numerical computations in [1] show that, for a different model of fluid-structure interaction, in 2-d there are two minimal values θ_0: $(0, 1/2)$ and $(1/2, 0)$.

Remark 3. Assumption (9) can be slightly weakened, see Remarks 11, 12 and 13. However, if we remove the simplicity assumption for $\lambda_1(\theta_0)$ the homogenized limit is not any longer a scalar equation but rather a system (see Remark 13 for details). For example, when $c \equiv 0$, the minimal eigenvalue $\lambda_1(0) = 0$ is of multiplicity K (with constant eigenvectors), and it is well known that, in such a case, (1) admits an homogenized limit which is again a system of K equations.

Under assumption (9) it is a classical matter to prove that the first eigencouple of (8) is smooth at θ_0 (see, e.g., [19]). Introducing the operator $\mathbb{A}(\theta)$ defined on $L^2(\mathbb{T}^N)^K$ by

$$\mathbb{A}(\theta)\psi = -(\mathrm{div}_y + 2i\pi\theta)\Big(A(y)(\nabla_y + 2i\pi\theta)\psi\Big) + c(y)\psi - \lambda_1(\theta)\psi, \tag{10}$$

it is easy to compute the derivatives of (8) for $n = 1$. Denoting by $(e_k)_{1\leqq k\leqq N}$ the canonical basis of $\mathbb{R}^N$, the first derivative satisfies

$$\begin{aligned}\mathbb{A}(\theta)\frac{\partial\psi_1}{\partial\theta_k} &= 2i\pi e_k A(y)(\nabla_y + 2i\pi\theta)\psi_1 + (\mathrm{div}_y + 2i\pi\theta)\left(A(y)2i\pi e_k\psi_1\right)\\ &\quad + \frac{\partial\lambda_1}{\partial\theta_k}(\theta)\psi_1,\end{aligned} \tag{11}$$

and the second derivative is

$$\begin{aligned}\mathbb{A}(\theta)\frac{\partial^2\psi_1}{\partial\theta_k\partial\theta_l} &= 2i\pi e_k A(y)(\nabla_y + 2i\pi\theta)\frac{\partial\psi_1}{\partial\theta_l} + (\mathrm{div}_y + 2i\pi\theta)\left(A(y)2i\pi e_k\frac{\partial\psi_1}{\partial\theta_l}\right)\\ &\quad + 2i\pi e_l A(y)(\nabla_y + 2i\pi\theta)\frac{\partial\psi_1}{\partial\theta_k} + (\mathrm{div}_y + 2i\pi\theta)\left(A(y)2i\pi e_l\frac{\partial\psi_1}{\partial\theta_k}\right)\\ &\quad + \frac{\partial\lambda_1}{\partial\theta_k}(\theta)\frac{\partial\psi_1}{\partial\theta_l} + \frac{\partial\lambda_1}{\partial\theta_l}(\theta)\frac{\partial\psi_1}{\partial\theta_k}\\ &\quad - 4\pi^2 e_k A(y)e_l\psi_1 - 4\pi^2 e_l A(y)e_k\psi_1 + \frac{\partial^2\lambda_1}{\partial\theta_l\partial\theta_k}(\theta)\psi_1.\end{aligned} \tag{12}$$

For $\theta = \theta_0$ we have $\nabla_\theta\lambda_1(\theta_0) = 0$, thus equations (11) and (12) simplify and we find

$$\frac{\partial\psi_1}{\partial\theta_k} = 2i\pi\zeta_k, \qquad \frac{\partial^2\psi_1}{\partial\theta_k\partial\theta_l} = -4\pi^2\chi_{kl}, \tag{13}$$

where ζ_k is the solution of

$$\mathbb{A}(\theta_0)\zeta_k = e_k A(y)(\nabla_y + 2i\pi\theta_0)\psi_1 + (\mathrm{div}_y + 2i\pi\theta_0)\left(A(y)e_k\psi_1\right) \quad \text{in } \mathbb{T}^N, \tag{14}$$

and χ_{kl} is the solution of

$$\begin{aligned}\mathbb{A}(\theta_0)\chi_{kl} = & e_k A(y)(\nabla_y + 2i\pi\theta_0)\zeta_l + (\mathrm{div}_y + 2i\pi\theta_0)\,(A(y)e_k\zeta_l) \\ & + e_l A(y)(\nabla_y + 2i\pi\theta_0)\zeta_k + (\mathrm{div}_y + 2i\pi\theta_0)\,(A(y)e_l\zeta_k) \\ & + e_k A(y)e_l\psi_1 + e_l A(y)e_k\psi_1 - \frac{1}{4\pi^2}\frac{\partial^2\lambda_1}{\partial\theta_l\partial\theta_k}(\theta_0)\psi_1 \quad \text{in } \mathbb{T}^N. \end{aligned} \tag{15}$$

There exists a unique solution of (14), up to the addition of a multiple of ψ_1. Indeed, the right-hand side of (14) satisfies the required compatibility condition (i.e., it is orthogonal to ψ_1) because ζ_k is just a multiple of the partial derivative of ψ_1 with respect to θ_k which necessarily exists, see (11). By the same token, there exists a unique solution of (15), up to the addition of a multiple of ψ_1. The compatibility condition of (15) yields a formula for the Hessian matrix $\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$.

We now recall some results on the Bloch decomposition associated with the spectral problem (8) (see, e.g., [11, 15]).

Lemma 1. *Let $u(y) \in L^2(\mathbb{R}^N)^K$. Define $\alpha_k(\theta) = \int_{\mathbb{R}^N} u(y)\cdot\overline{\psi}_k(y,\theta)e^{-2i\pi\theta\cdot y}dy$. Then,*

$$u(y) = \sum_{k\geqq 1}\int_{\mathbb{T}^N}\alpha_k(\theta)\psi_k(y,\theta)e^{2i\pi\theta\cdot y}d\theta.$$

Furthermore, if $v(y) = \sum_{k\geqq 1}\int_{\mathbb{T}^N}\beta_k(\theta)\psi_k(y,\theta)e^{2i\pi\theta\cdot y}d\theta$ in $L^2(\mathbb{R}^N)^K$, then

$$\int_{\mathbb{R}^N} u(y)\cdot\overline{v}(y)\,dy = \sum_{k\geqq 1}\int_{\mathbb{T}^N}\alpha_k(\theta)\overline{\beta}_k(\theta)\,d\theta.$$

In what follows we shall need a rescaled version of Lemma 1 that we now describe. Upon the change of variable $y = \frac{x}{\varepsilon}$, we define $u^\varepsilon(x) = \varepsilon^{-N/2}u(y)$ which satisfies $\|u^\varepsilon\|_{L^2(\mathbb{R}^N)^K} = \|u\|_{L^2(\mathbb{R}^N)^K}$. Applying Lemma 1 we deduce the following rescaled Bloch transform:

$$u_\varepsilon(x) = \sum_{k\geqq 1}\int_{\varepsilon^{-1}\mathbb{T}^N}\alpha_k^\varepsilon(\eta)\psi_k\left(\frac{x}{\varepsilon},\theta_0+\varepsilon\eta\right)e^{2i\pi\eta\cdot x}e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}d\eta, \tag{16}$$

with $\eta = \frac{\theta-\theta_0}{\varepsilon}$ and $\alpha_k^\varepsilon(\eta) = \varepsilon^{N/2}\alpha_k(\theta)$. The same orthogonality property holds true:

$$\int_{\mathbb{R}^N} u^\varepsilon(x)\cdot\overline{v}^\varepsilon(x)\,dx = \sum_{k\geqq 1}\int_{\varepsilon^{-1}\mathbb{T}^N}\alpha_k^\varepsilon(\eta)\overline{\beta}_k^\varepsilon(\eta)\,d\eta.$$

3. Main results

Let $\Omega \subset \mathbb{R}^N$ be an open set (bounded or not). Let $0 < T < +\infty$ be a final time. We first consider the parabolic problem

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \left(\frac{c\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2} + d\left(x, \frac{x}{\varepsilon}\right)\right) u_\varepsilon &= 0 && \text{in } \Omega \times (0, T), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(t = 0, x) &= u_\varepsilon^0(x) && \text{in } \Omega. \end{aligned} \tag{17}$$

The unknown $u_\varepsilon(t, x)$ is vector-valued, i.e., it is a function from $(0, T) \times \Omega$ into $\mathbb{C}^K$ with $K \geqq 1$. Since Bloch waves are involved in our results, we always consider complex-valued unknown functions. Assuming that the initial data u_ε^0 belongs to $L^2(\Omega)^K$, it is a classical result that there exists a unique solution of (17) in $C\left([0, T]; L^2(\Omega)^K\right) \cap L^2\left((0, T); H_0^1(\Omega)^K\right)$.

Since the matrix c does not satisfy any positivity property, we cannot obtain any *a priori* estimate directly from (17). On the other hand, the cell spectral problem and assumption (9) indicate that $\lambda_1(\theta_0)$ governs the time decay (or growth, according to its sign) of the solution u_ε. Therefore, we first perform a time renormalization in the spirit of the factorization principle (3) and we introduce a new unknown,

$$\tilde{u}_\varepsilon(t, x) = e^{\frac{\lambda_1(\theta_0)t}{\varepsilon^2}} u_\varepsilon(t, x), \tag{18}$$

which satisfies

$$\begin{aligned} \frac{\partial \tilde{u}_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_\varepsilon\right) + \frac{c\left(\frac{x}{\varepsilon}\right) - \lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon + d\left(x, \frac{x}{\varepsilon}\right)\tilde{u}_\varepsilon &= 0 && \text{in } \Omega \times (0, T), \\ \tilde{u}_\varepsilon &= 0 && \text{on } \partial\Omega \times (0, T), \\ \tilde{u}_\varepsilon(t = 0, x) &= u_\varepsilon^0(x) && \text{in } \Omega. \end{aligned} \tag{19}$$

Then, we can obtain the following *a priori* estimate.

Lemma 2. *There exists a constant $C > 0$ which does not depend on ε(but may depend on T) such that the solution of* (19) *satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon \|\nabla \tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^{N\times K}} \leqq C \|u_\varepsilon^0\|_{L^2(\Omega)^K}. \tag{20}$$

Theorem 2. *Assume* (9) *and that the initial data $u_\varepsilon^0 \in L^2(\Omega)^K$ is of the form*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} v^0(x), \tag{21}$$

with $v^0 \in W^{1,\infty}(\Omega)$. The solution of (17) *can be written as*

$$u_\varepsilon(t, x) = e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^2}} \left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} v_\varepsilon(t, x) + r_\varepsilon(t, x)\right), \tag{22}$$

where r_ε is a vector-valued remainder term, defined on $(0,T)\times\mathbb{R}^N$, such that

$$\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K}=0 \text{ for any compact set } \omega\subset\mathbb{R}^N, \tag{23}$$

and v_ε is a scalar sequence which converges weakly in $L^2\left((0,T);H^1(\Omega)\right)$, and strongly in $L^2\left((0,T);L^2_{\text{loc}}(\Omega)\right)$, to the solution v of the scalar homogenized problem

$$\begin{aligned}\frac{\partial v}{\partial t}-\operatorname{div}\left(A^*\nabla v\right)+d^*(x)\,v&=0 && \text{in } \Omega\times(0,T),\\ v&=0 && \text{on } \partial\Omega\times(0,T),\\ v(t=0,x)&=v^0(x) && \text{in } \Omega,\end{aligned} \tag{24}$$

with $A^*=\frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$ and $d^*(x)=\int_{\mathbb{T}^N}d(x,y)\psi_1(y)\cdot\overline{\psi_1}(y)\,dy$.

Remark 4. Of course, if Ω is bounded, we can take $\omega=\Omega$ in (23) and replace $L^2_{\text{loc}}(\Omega)$ by $L^2(\Omega)$ in the above theorem.

Remark 5. It is only for simplicity that we make assumption (21) on the "well-prepared" character of the initial data. Indeed, we use it only for proving the strong convergence of v_ε to v in $L^2\left((0,T);L^2_{\text{loc}}(\Omega)\right)$. The rest of Theorem 2 holds true with the weaker assumption that $u^0_\varepsilon(x)e^{-2i\pi\frac{\theta_0\cdot x}{\varepsilon}}$ two-scale converges to $\psi_1(y,\theta_0)v^0(x)$ with $v^0\in L^2(\Omega)$ (see [3, 22] and Proposition 1 for the notion of two-scale convergence). What is more, for any kind of initial data we can still obtain a similar result, but the homogenized initial condition v^0 is just defined as some type of weak two-scale limit (which may well be zero). In other words, there is no need to have well-prepared initial data in Theorem 2.

Remark 6. Theorem 2 still holds true if we add to equation (17) a non-linear term of order ε^0. Typically, we can add a non-linear term of the type $g(x,\frac{x}{\varepsilon},u_\varepsilon)$ where $g(x,y,\xi)$ is an homogeneous of degree one, Lipschitz function with respect to ξ such that

$$|g(x,y,\xi)-g(x,y,\xi')|\leqq C|\xi-\xi'|,\quad g(x,y,t\xi)=tg(x,y,\xi)\quad \forall t>0.$$

In such a case, the homogenized problem (24) has an additional zero-order term which is $g^*(x,v)$ with $g^*(x,v)=\int_{\mathbb{T}^N}g(x,y,\psi_1(y,\theta_0)v)\cdot\overline{\psi_1}(y,\theta_0)\,dy$. Similarly, it is possible to add to (17) a source term of the type

$$f_\varepsilon(t,x)=e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^2}}e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}f\left(t,x,\frac{x}{\varepsilon}\right).$$

It yields a source term $f^*(t,x)=\int_{\mathbb{T}^N}f(t,x,y)\cdot\overline{\psi_1}(y)\,dy$ in the homogenized equation (24).

We now consider the eigenvalue problem in a bounded domain Ω:

$$\begin{aligned}-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)+\left(\frac{c\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2}+d\left(x,\frac{x}{\varepsilon}\right)\right)u_\varepsilon&=\lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega,\\ u_\varepsilon&=0 \qquad \text{on } \partial\Omega.\end{aligned} \tag{25}$$

Since Ω is assumed to be bounded, problem (25) has a real discrete spectrum

$$\lambda_1^\varepsilon \leqq \lambda_2^\varepsilon \leqq \cdots \leqq \lambda_n^\varepsilon \cdots \to +\infty,$$

with real eigenfunctions denoted by u_k^ε, normalized by $\|u_k^\varepsilon\|_{L^2(\Omega)^K} = 1$.

Theorem 3. *Under assumption* (9), *for each $k \geqq 1$ there is*

$$\lambda_k^\varepsilon = \frac{\lambda_1(\theta_0)}{\varepsilon^2} + \mu_k + o(1) \quad \text{with } \lim_{\varepsilon\to 0} o(1) = 0,$$

and the corresponding eigenvector $u_k^\varepsilon(x)$ admits the representation

$$u_k^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_k^\varepsilon(x) + r_k^\varepsilon(x) \tag{26}$$

where $v_k^\varepsilon \in H_0^1(\Omega)$ and $r_k^\varepsilon \in L^2(\Omega)^K$ satisfy

$$\lim_{\varepsilon\to 0}\|r_k^\varepsilon\|_{L^2(\Omega)^K} = 0, \quad \|v_k^\varepsilon\|_{H_0^1(\Omega)} \leqq C, \quad \lim_{\varepsilon\to 0}\|v_k^\varepsilon\|_{L^2(\Omega)} = 1,$$

and any limit point v_k, as $\varepsilon \to 0$, of the scalar sequence v_k^ε is a normalized eigenfunction associated with the k^{th} eigenvalue μ_k of the scalar homogenized spectral problem

$$\begin{aligned} -\operatorname{div}\left(A^*\nabla v\right) + d^*(x)v &= \mu v \quad \text{in } \Omega,\\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{27}$$

with $A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$ and $d^*(x) = \int_{\mathbb{T}^N} d(x,y)\psi_1(y)\cdot\overline{\psi}_1(y)\,dy$.

Furthermore, if μ_k is a simple eigenvalue of (27), *the entire sequence v_k^ε converges to the homogenized eigenfunction v_k.*

Finally we address the hyperbolic problem

$$\begin{aligned} \frac{\partial^2 u_\varepsilon}{\partial t^2} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \frac{c\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2}u_\varepsilon &= 0 \quad \text{in } \Omega\times(0,T),\\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega\times(0,T),\\ u_\varepsilon(t=0,x) &= u_\varepsilon^0(x) \quad \text{in } \Omega,\\ \frac{\partial u_\varepsilon}{\partial t}(t=0,x) &= u_\varepsilon^1(x) \quad \text{in } \Omega, \end{aligned} \tag{28}$$

where $u_\varepsilon(t,x)$ takes its values in $\mathbb{C}^K$ with $K \geqq 1$. Assuming that the initial data are $u_\varepsilon^0 \in H_0^1(\Omega)^K$ and $u_\varepsilon^1 \in L^2(\Omega)^K$, (28) admits a unique solution $u_\varepsilon \in C\left([0,T]; H_0^1(\Omega)^K\right) \cap C^1\left([0,T]; L^2(\Omega)^K\right)$. The scalar case $K = 1$ was addressed in [4]. Depending on the sign of the minimal eigenvalue $\lambda_1(\theta_0)$ of the cell problem (8), we obtain different asymptotic behavior for (28). We begin with the case $\lambda_1(\theta_0) = 0$ which does not require any time renormalization.

Theorem 4. *Assume* (9), $\lambda_1(\theta_0) = 0$ *and that the initial data are of the form*

$$\begin{aligned} u_\varepsilon^0(x) &= \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^0(x) \in H_0^1(\Omega)^K, \\ u_\varepsilon^1(x) &= \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^1(x) \in L^2(\Omega)^K, \end{aligned} \tag{29}$$

with $v^0 \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ *and* $v^1 \in L^2(\Omega)$*. The solution of* (28) *can be written as*

$$u_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x), \tag{30}$$

where r_ε *is a vector-valued remainder term such that*

$$\lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \text{ for any compact set } \omega \subset \mathbb{R}^N, \tag{31}$$

and v_ε *is a scalar sequence which converges weakly in* $L^2\left((0,T); H^1(\Omega)\right)$ *to the solution* v *of the scalar homogenized problem*

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \operatorname{div}\left(A^*\nabla v\right) = 0 & \text{in } \Omega\times(0,T), \\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ v(t=0,x) = v^0(x) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial t}(t=0,x) = v^1(x) & \text{in } \Omega, \end{cases} \tag{32}$$

with $A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$.

When $\lambda_1(\theta_0) \neq 0$, we cannot homogenize directly (28). As in the scalar case [4] we must rather perform a time rescaling and consider large times of order ε^{-1}. In other words, instead of (28) we now consider

$$\begin{cases} \varepsilon^2\dfrac{\partial^2 u_\varepsilon}{\partial t^2} - \operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \dfrac{c\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2} u_\varepsilon = 0 & \text{in } \Omega\times(0,T), \\ u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ u_\varepsilon(t=0,x) = u_\varepsilon^0(x) & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial t}(t=0,x) = u_\varepsilon^1(x) & \text{in } \Omega. \end{cases} \tag{33}$$

Let us first assume that $\lambda_1(\theta_0) < 0$. We perform a time renormalization analogous to (18) and we introduce a new unknown,

$$\tilde{u}_\varepsilon(t,x) = e^{-\frac{\sqrt{-\lambda_1(\theta_0)}t}{\varepsilon^2}} u_\varepsilon(t,x), \tag{34}$$

which satisfies

$$\begin{cases} \varepsilon^2\dfrac{\partial^2 \tilde{u}_\varepsilon}{\partial t^2} + 2\sqrt{-\lambda_1(\theta_0)}\dfrac{\partial \tilde{u}_\varepsilon}{\partial t} \\ \quad -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon\right) \\ \quad + \frac{c\left(\frac{x}{\varepsilon}\right)-\lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon = 0 & \text{in } \Omega\times(0,T), \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ \tilde{u}_\varepsilon(t=0,x) = u_\varepsilon^0(x) & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}_\varepsilon}{\partial t}(t=0,x) = u_\varepsilon^1(x) - \dfrac{\sqrt{-\lambda_1(\theta_0)}}{\varepsilon^2} u_\varepsilon^0(x) & \text{in } \Omega. \end{cases} \tag{35}$$

In this case we obtain a parabolic homogenized equation.

Theorem 5. *Assume* (9), $\lambda_1(\theta_0) < 0$ *and that the initial data is*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^0(x) \in H_0^1(\Omega)^K, \tag{36}$$

with $v^0 \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$, *and that* $\varepsilon^2 u_\varepsilon^1(x)$ *is bounded in* $L^2(\Omega)^K$ *while* $\varepsilon^2\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) \cdot u_\varepsilon^1(x)$ *converges weakly to* 0 *in* $L^2(\Omega)$. *The solution of* (33) *can be written as*

$$u_\varepsilon(t,x) = e^{\frac{\sqrt{-\lambda_1(\theta_0)}t}{\varepsilon^2}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x)\right), \tag{37}$$

where r_ε *is a vector-valued remainder term such that*

$$\lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \textit{ for any compact set } \omega \subset \mathbb{R}^N, \tag{38}$$

and v_ε *converges weakly in* $L^2\left((0,T); H^1(\Omega)\right)$ *to the solution* v *of the scalar homogenized problem*

$$\begin{aligned} 2\sqrt{-\lambda_1(\theta_0)}\frac{\partial v}{\partial t} - \operatorname{div}\left(A^*\nabla v\right) &= 0 && \textit{in } \Omega \times (0,T), \\ v &= 0 && \textit{on } \partial\Omega \times (0,T), \\ v(t=0,x) &= \tfrac{1}{2}v^0(x) && \textit{in } \Omega, \end{aligned} \tag{39}$$

with $A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$.

Remark 7. The $\frac{1}{2}$ factor in front of the initial data in the homogenized problem (39) is quite surprising. It arises because the initial velocity in (35) contains some contribution of u_ε^0. As already explained in the scalar case [4], there is an initial layer in time in (35) which is not taken into account by Theorem 5.

Let us now assume that $\lambda_1(\theta_0) > 0$. We perform another time renormalization and we introduce a new unknown,

$$\tilde{u}_\varepsilon(t,x) = e^{-i\frac{\sqrt{\lambda_1(\theta_0)}t}{\varepsilon^2}} u_\varepsilon(t,x), \tag{40}$$

which satisfies

$$\begin{aligned} \varepsilon^2\frac{\partial^2\tilde{u}_\varepsilon}{\partial t^2} + 2i\sqrt{\lambda_1(\theta_0)}\frac{\partial\tilde{u}_\varepsilon}{\partial t} & \\ -\operatorname{div}\left(A\left(\tfrac{x}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon\right) & \\ +\frac{c\left(\frac{x}{\varepsilon}\right)-\lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon &= 0 && \text{in } \Omega\times(0,T), \\ \tilde{u}_\varepsilon &= 0 && \text{on } \partial\Omega\times(0,T), \\ \tilde{u}_\varepsilon(t=0,x) &= u_\varepsilon^0(x) && \text{in } \Omega, \\ \frac{\partial\tilde{u}_\varepsilon}{\partial t}(t=0,x) &= u_\varepsilon^1(x) - i\frac{\sqrt{\lambda_1(\theta_0)}}{\varepsilon^2}u_\varepsilon^0(x) && \text{in } \Omega. \end{aligned} \tag{41}$$

In this case we obtain a Schrödinger-type homogenized equation. Remark that, although there is no remainder term in (43), the convergence of v_ε is much weaker than in the previous cases (see also Remark 14).

Theorem 6. *Assume* (9), $\lambda_1(\theta_0) > 0$ *and that the initial data is*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^0(x) \in H_0^1(\Omega)^K, \tag{42}$$

with $v^0 \in W^{1,\infty}(\Omega)$*, and that* $\varepsilon^2 u_\varepsilon^1(x)$ *is bounded in* $L^2(\Omega)^K$ *while* $\varepsilon^2\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\cdot u_\varepsilon^1(x)$ *converges weakly to* 0 *in* $L^2(\Omega)$. *The solution of* (33) *can be written as*

$$u_\varepsilon(t,x) = e^{i\frac{\sqrt{\lambda_1(\theta_0)}t}{\varepsilon^2}} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x), \tag{43}$$

where v_ε *two-scale converges to* $\psi_1(y,\theta_0)v(t,x)$ *and* $v \in L^2\left((0,T); H_0^1(\Omega)\right)$ *is the solution of the scalar homogenized problem*

$$\begin{array}{lr} 2i\sqrt{\lambda_1(\theta_0)}\dfrac{\partial v}{\partial t} - \operatorname{div}\left(A^*\nabla v\right) = 0 & \text{in } \Omega\times(0,T), \\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ v(t=0,x) = \frac{1}{2}v^0(x) & \text{in } \Omega, \end{array} \tag{44}$$

with $A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$.

Remark 8. All the results in the hyperbolic case (Theorems 4, 5, and 6) hold true when we add a zero-order term of the type $d\left(x, \frac{x}{\varepsilon}\right)u_\varepsilon$, where $d(x,y)$ is a real symmetric non-negative matrix with entries in $L^\infty\left(\Omega; C(\mathbb{T}^N)\right)$. This yields a zero-order term in the homogenized problem which is precisely $d^*(x) = \int_{\mathbb{T}^N} d(x,y)\psi_1(y)\cdot\overline{\psi}_1(y)\,dy$.

4. Proofs in the parabolic case

Notation. for any function $\phi(x,y)$ defined on $\mathbb{R}^N\times\mathbb{T}^N$, we denote by ϕ^ε the function $\phi(x, \frac{x}{\varepsilon})$.

Proof of Lemma 2. We multiply equation (19) by $\overline{\tilde{u}_\varepsilon}$ and we integrate by parts to obtain

$$\frac{1}{2}\int_\Omega |\tilde{u}_\varepsilon(t,x)|^2 dx - \frac{1}{2}\int_\Omega |u_\varepsilon^0(x)|^2 dx + \int_0^t\int_\Omega d\left(x,\frac{x}{\varepsilon}\right)\tilde{u}_\varepsilon\cdot\overline{\tilde{u}_\varepsilon}\,ds\,dx$$
$$+\int_0^t\int_\Omega\left(A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon\cdot\nabla\overline{\tilde{u}_\varepsilon} + \frac{c\left(\frac{x}{\varepsilon}\right)-\lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon\cdot\overline{\tilde{u}_\varepsilon}\right)ds\,dx = 0. \tag{45}$$

If we can check that the last integral in (45) is non-negative, the lemma is proved by a standard Gronwall inequality. Extending $\tilde{u}_\varepsilon$ by zero outside Ω and changing the variable to $y = \frac{x}{\varepsilon}$, a sufficient condition is to prove that, for any $u \in H^1(\mathbb{R}^N)^K$,

$$\int_{\mathbb{R}^N}\left(A(y)\nabla u\cdot\nabla\overline{u} + (c(y)-\lambda_1(\theta_0))\,u\cdot\overline{u}\right)dy \geqq 0.$$

Applying the Bloch decomposition of Lemma 1 to u yields

$$\int_{\mathbb{R}^N}\left(A(y)\nabla u\cdot\nabla\overline{u} + (c(y)-\lambda_1(\theta_0))\,u\cdot\overline{u}\right)dy$$

$$= \sum_{k \geqq 1} \int_{\mathbb{T}^N} |\alpha_k(\theta)|^2 \, (\lambda_k(\theta) - \lambda_1(\theta_0))\, d\theta$$

which is non-negative by assumption (9). □

We now briefly recall the notion of two-scale convergence (see [3, 22]).

Proposition 1. *Let w_ε be a bounded sequence in $L^2(\Omega)$. There exist a subsequence, still denoted by ε, and a limit $w(x, y) \in L^2(\Omega \times \mathbb{T}^N)$ such that w_ε* two-scale con-verges *to w in the sense that*

$$\lim_{\varepsilon \to 0} \int_\Omega w_\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_{\mathbb{T}^N} w(x, y) \phi(x, y)\, dx\, dy$$

for all functions $\phi(x, y) \in L^2\left(\Omega; C(\mathbb{T}^N)\right)$. The two-scale convergence is denoted by $w_\varepsilon \overset{2s}{\rightharpoonup} w$.

Furthermore, if $\varepsilon \nabla w_\varepsilon$ is also bounded in $L^2(\Omega)^N$, then, up to another subsequence, $\varepsilon \nabla w_\varepsilon \overset{2s}{\rightharpoonup} \nabla_y w$ and w belongs to $L^2\left(\Omega; H^1(\mathbb{T}^N)\right)$.

Proof of Theorem 2. To simplify the exposition we forget the notation $\tilde{\cdot}$ for the solution $\tilde{u}_\varepsilon$ of (19). Equivalently, we could have subtracted from $c(y)$ an adequate constant, so that $\lambda_1(\theta_0) = 0$ and $u_\varepsilon = \tilde{u}_\varepsilon$. Define a sequence w_ε by

$$w_\varepsilon(t, x) = u_\varepsilon(t, x) e^{-2i\pi \frac{\theta_0 \cdot x}{\varepsilon}}.$$

By the *a priori* estimate of Lemma 2 we have

$$\|w_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon \|\nabla w_\varepsilon\|_{L^2((0,T)\times\Omega)^K} \leqq C,$$

and applying Proposition 1, up to a subsequence, we find that there exists a limit $w(t, x, y) \in L^2\left((0, T) \times \Omega; H^1(\mathbb{T}^N)^K\right)$ such that

$$w_\varepsilon \overset{2s}{\rightharpoonup} w \quad \text{and} \quad \varepsilon \nabla w_\varepsilon \overset{2s}{\rightharpoonup} \nabla_y w$$

in the sense of two-scale convergence.

First step. We multiply (19) by the complex conjugate of $\varepsilon^2 \phi(t, x, \frac{x}{\varepsilon}) e^{2i\pi \frac{\theta_0 \cdot x}{\varepsilon}}$, where $\phi(t, x, y)$ is a smooth test function defined on $[0, T) \times \Omega \times \mathbb{T}^N$, with compact support in $[0, T) \times \Omega$, and with values in $\mathbb{C}^K$. Integrating by parts yields

$$\begin{aligned} &\varepsilon^2 \int_\Omega u_\varepsilon^0 \cdot \overline{\phi}^\varepsilon e^{-2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} dx - \varepsilon^2 \int_0^T \int_\Omega w_\varepsilon \cdot \frac{\partial \overline{\phi}^\varepsilon}{\partial t} dt\, dx \\ &+ \int_0^T \int_\Omega A^\varepsilon (\varepsilon \nabla + 2i\pi\theta_0) w_\varepsilon \cdot (\varepsilon \nabla - 2i\pi\theta_0) \overline{\phi}^\varepsilon \, dt\, dx \\ &+ \int_0^T \int_\Omega (c^\varepsilon - \lambda_1(\theta_0) + \varepsilon^2 d^\varepsilon) w_\varepsilon \cdot \overline{\phi}^\varepsilon \, dt\, dx = 0. \end{aligned}$$

Passing to the two-scale limit yields the variational formulation of

$$-(\mathrm{div}_y + 2i\pi\theta)\Big(A(y)(\nabla_y + 2i\pi\theta)w\Big) + c(y)w = \lambda_1(\theta_0)w \quad \text{in } \mathbb{T}^N.$$

By the simplicity of $\lambda_1(\theta_0)$, this implies that there exists a scalar function $v(t, x) \in L^2((0, T) \times \Omega)$ (possibly complex-valued) such that

$$w(t, x, y) = v(t, x)\psi_1(y, \theta_0). \tag{46}$$

Second step. We multiply (19) by the complex conjugate of

$$\Psi_\varepsilon = e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\phi(t, x) + \varepsilon\sum_{k=1}^{N}\frac{\partial\phi}{\partial x_k}(t, x)\zeta_k\left(\frac{x}{\varepsilon}\right)\right), \tag{47}$$

where $\phi(t, x)$ is a smooth, compactly supported, test function defined from $[0, T) \times \Omega$ into $\mathbb{C}$, and $\zeta_k(y)$ is the solution of (14). After some algebra we find that

$$\begin{aligned}
\int_\Omega A^\varepsilon\nabla u_\varepsilon\cdot\nabla\overline{\Psi}_\varepsilon dx = &\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)(\overline{\phi}w_\varepsilon)\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\psi}_1^\varepsilon \\
&+\varepsilon\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)\left(\frac{\partial\overline{\phi}}{\partial x_k}w_\varepsilon\right)\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\zeta}_k^\varepsilon \\
&-\int_\Omega A^\varepsilon e_k\frac{\partial\overline{\phi}}{\partial x_k}w_\varepsilon\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\psi}_1^\varepsilon \\
&+\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)\left(\frac{\partial\overline{\phi}}{\partial x_k}w_\varepsilon\right)\cdot e_k\overline{\psi}_1^\varepsilon \\
&-\int_\Omega A^\varepsilon w_\varepsilon\nabla\frac{\partial\overline{\phi}}{\partial x_k}\cdot e_k\overline{\psi}_1^\varepsilon \\
&-\int_\Omega A^\varepsilon w_\varepsilon\nabla\frac{\partial\overline{\phi}}{\partial x_k}\cdot(\varepsilon\nabla - 2i\pi\theta_0)\overline{\zeta}_k^\varepsilon \\
&+\int_\Omega A^\varepsilon\overline{\zeta}_k^\varepsilon(\varepsilon\nabla + 2i\pi\theta_0)w_\varepsilon\cdot\nabla\frac{\partial\overline{\phi}}{\partial x_k}.
\end{aligned} \tag{48}$$

Now, for any smooth compactly supported test function Φ from Ω into $\mathbb{C}^K$, we deduce from the definition of ψ_1 that

$$\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)\psi_1^\varepsilon\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\Phi} + \frac{1}{\varepsilon^2}\int_\Omega(c^\varepsilon - \lambda_1(\theta_0))\psi_1^\varepsilon\cdot\overline{\Phi} = 0, \tag{49}$$

and from the definition of ζ_k,

$$\begin{aligned}
&\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)\zeta_k^\varepsilon\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\Phi} + \frac{1}{\varepsilon^2}\int_\Omega(c^\varepsilon - \lambda_1(\theta_0))\zeta_k^\varepsilon\cdot\overline{\Phi} \\
&\quad = \varepsilon^{-1}\int_\Omega A^\varepsilon\left(\nabla + 2i\pi\frac{\theta_0}{\varepsilon}\right)\psi_1^\varepsilon\cdot e_k\overline{\Phi} - \varepsilon^{-1}\int_\Omega A^\varepsilon e_k\psi_1^\varepsilon\cdot\left(\nabla - 2i\pi\frac{\theta_0}{\varepsilon}\right)\overline{\Phi}.
\end{aligned} \tag{50}$$

Combining (48) with the potential term, we easily check that the first line of its right-hand side cancels out because of (49) with $\Phi = \overline{\phi} w_\varepsilon$, and the next three lines cancel out because of (50) with $\Phi = \frac{\partial \overline{\phi}}{\partial x_k} w_\varepsilon$. On the other hand, we can pass to the limit in the three last terms of we find that (48). Finally, using the above information, we find that (19) multiplied by $\overline{\Psi}_\varepsilon$ yields after simplification,

$$\begin{aligned}
&\int_\Omega u_\varepsilon^0 \cdot \overline{\Psi}_\varepsilon(t=0)dx - \int_0^T \int_\Omega w_\varepsilon \cdot \left(\overline{\psi}_1^\varepsilon \frac{\partial \overline{\phi}}{\partial t} + \varepsilon \frac{\partial^2 \overline{\phi}}{\partial x_k \partial t} \overline{\zeta}_k^\varepsilon \right) dt\, dx \\
&\quad - \int_0^T \int_\Omega A^\varepsilon w_\varepsilon \nabla \frac{\partial \overline{\phi}}{\partial x_k} \cdot e_k \overline{\psi}_1^\varepsilon dt\, dx \\
&\quad - \int_0^T \int_\Omega A^\varepsilon w_\varepsilon \nabla \frac{\partial \overline{\phi}}{\partial x_k} \cdot (\varepsilon \nabla - 2i\pi\theta_0) \overline{\zeta}_k^\varepsilon dt\, dx \\
&\quad + \int_0^T \int_\Omega A^\varepsilon \overline{\zeta}_k^\varepsilon (\varepsilon\nabla + 2i\pi\theta_0) w_\varepsilon \cdot \nabla \frac{\partial \overline{\phi}}{\partial x_k} dt\, dx \\
&\quad + \int_0^T \int_\Omega d^\varepsilon w_\varepsilon \cdot \overline{\Psi}_\varepsilon \, dt\, dx = 0.
\end{aligned} \tag{51}$$

Passing to the two-scale limit in each term of (51) gives

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{T}^N} \psi_1 v^0 \cdot \overline{\psi}_1 \overline{\phi}(t=0) dx\, dy - \int_0^T \int_\Omega \int_{\mathbb{T}^N} \psi_1 v \cdot \overline{\psi}_1 \frac{\partial \overline{\phi}}{\partial t} dt\, dx\, dy \\
&\quad - \int_0^T \int_\Omega \int_{\mathbb{T}^N} A\psi_1 v \nabla \frac{\partial \overline{\phi}}{\partial x_k} \cdot e_k \overline{\psi}_1 dt\, dx\, dy \\
&\quad - \int_0^T \int_\Omega \int_{\mathbb{T}^N} A\psi_1 v \nabla \frac{\partial \overline{\phi}}{\partial x_k} \cdot (\nabla_y - 2i\pi\theta_0) \overline{\zeta}_k dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega \int_{\mathbb{T}^N} A \overline{\zeta}_k (\nabla_y + 2i\pi\theta_0) \psi_1 v \cdot \nabla \frac{\partial \overline{\phi}}{\partial x_k} dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega \int_{\mathbb{T}^N} d\psi_1 v \cdot \overline{\psi}_1 \overline{\phi}\, dt\, dx\, dy = 0.
\end{aligned} \tag{52}$$

Recalling the normalization $\int_{\mathbb{T}^N} |\psi_1|^2 dy = 1$, and introducing

$$\begin{aligned}
2A_{jk}^* = \int_{\mathbb{T}^N} \Big(& A\psi_1 e_j \cdot e_k \overline{\psi}_1 + A\psi_1 e_k \cdot e_j \overline{\psi}_1 \\
& + A\psi_1 e_j \cdot (\nabla_y - 2i\pi\theta_0) \overline{\zeta}_k + A\psi_1 e_k \cdot (\nabla_y - 2i\pi\theta_0) \overline{\zeta}_j \\
& - A\overline{\zeta}_k (\nabla_y + 2i\pi\theta_0) \psi_1 \cdot e_j - A\overline{\zeta}_j (\nabla_y + 2i\pi\theta_0) \psi_1 \cdot e_k \Big) dy,
\end{aligned} \tag{53}$$

and $d^*(x) = \int_{\mathbb{T}^N} d(x, y) \psi_1(y) \cdot \overline{\psi}_1(y)\, dy$, (52) is equivalent to

$$\int_\Omega v^0 \overline{\phi}(0) dx - \int_0^T \int_\Omega \left(v \frac{\partial \overline{\phi}}{\partial t} + A^* v \cdot \nabla\nabla \overline{\phi} - d^*(x) v \overline{\phi} \right) dt\, dx = 0,$$

which is a very weak form of the homogenized equation (24). Note, however, that we cannot recover the Dirichlet boundary condition from (52). To this end we shall use the compactness Lemma 3 below (which was not required so far) or, more precisely, its Corollary 1 which implies the existence of a bounded scalar sequence v_ε in $L^2\left((0,T); H^1(\mathbb{R}^N)\right)$ such that

$$u_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x), \tag{54}$$

and $\lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0$ for any compact set $\omega \subset \mathbb{R}^N$. Up to a subsequence, v_ε converges weakly to a limit v in $L^2\left((0,T); H^1(\mathbb{R}^N)\right)$, which necessarily coincides with the two-scale limit obtained in (46). If the compact set ω lies outside Ω, i.e. $\omega \subset \left(\mathbb{R}^N \setminus \Omega\right)$, we deduce from (54) that

$$\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) = -r_\varepsilon(t,x) \quad \text{in } \omega\times(0,T),$$

and since ψ_1 is normalized, we obtain

$$\begin{aligned}\|r_\varepsilon\|^2_{L^2((0,T)\times\omega)^K} &= \int_0^T\int_\omega |\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)|^2 |v_\varepsilon(t,x)|^2 dt\,dx \\ &\to \int_0^T\int_\omega |v(t,x)|^2 dt\,dx = 0.\end{aligned}$$

Therefore, we deduce that $v = 0$ in any compact set ω outside Ω. This implies that v belongs to $L^2\left((0,T); H^1_0(\Omega)\right)$.

The compatibility condition of (15) for the second derivative of ψ_1 shows that the matrix A^*, defined by (53), is indeed equal to $\frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$, and thus is real, symmetric, positive definite by assumption (9). Therefore, the homogenized problem (24) is well posed. By uniqueness of the solution of the homogenized problem (24), we deduce that the entire sequence v_ε converges to v (which is a real-valued function if the initial data v^0 is so). □

Remark 9. As usual in periodic homogenization, the choice of the test function Ψ_ε, defined by (47), is dictated by the formal two-scale asymptotic expansion that can be obtained for the solution u_ε of (17). Indeed, if we admit that the ansatz of u_ε starts with the following two exponential terms (which is not obvious *a priori*!), then a simple and formal computation shows that

$$u_\varepsilon(t,x) \approx e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^2}} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} \left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) v(t,x) + \varepsilon \sum_{k=1}^N \frac{\partial v}{\partial x_k}(t,x)\zeta_k\left(\frac{x}{\varepsilon}\right)\right),$$

where v is the homogenized solution of (24).

Lemma 3. *Let u_ε be a bounded sequence in $L^2(\mathbb{R}^N)^K$. Assume that there exists a finite constant C such that*

$$\int_{\mathbb{R}^N} \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\nabla\overline{u_\varepsilon} + \frac{c\left(\frac{x}{\varepsilon}\right) - \lambda_1(\theta_0)}{\varepsilon^2} u_\varepsilon\cdot\overline{u_\varepsilon}\right)dx \leqq C. \tag{55}$$

Then, under assumption (9)*,*

$$u_\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(x) + r_\varepsilon(x), \tag{56}$$

where v_ε is a bounded scalar sequence in $H^1(\mathbb{R}^N)$ and $\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2(\omega)^K} = 0$ *for any compact set $\omega \subset \mathbb{R}^N$.*

Remark 10. If the sequence u_ε further vanishes outside an open set Ω, then we can obtain the representation (56) with v_ε uniformly bounded in $H_0^1(\Omega)$. Indeed, it is enough to project the function $v_\varepsilon \in H^1(\mathbb{R}^N)$, given by Lemma 3, on $H_0^1(\Omega)$.

Corollary 1. *Let $\tilde{u}_\varepsilon$ be the solution of the parabolic system* (19)*. Then, under assumptions* (9) *and* (21)*,*

$$\tilde{u}_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon},\theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x),$$

where v_ε is a bounded scalar sequence in $L^2((0,T);H^1(\mathbb{R}^N))$, such that $\frac{\partial v_\varepsilon}{\partial t}$ is bounded in $L^2((0,T)\times\mathbb{R}^N)$, and $\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0$ *for any compact set $\omega \subset \mathbb{R}^N$. In particular, v_ε is relatively compact in $L^2\big((0,T);L^2_{\rm loc}(\mathbb{R}^N)\big)$.*

Proof of Lemma 3. Our proof is in the spirit of the previous works [14, 16, 26]. Applying the rescaled Bloch decomposition (16) to $u_\varepsilon(x)$ with $\eta = \frac{\theta-\theta_0}{\varepsilon}$, we have

$$u_\varepsilon(x) = \sum_{k\geqq 1}\int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_k^\varepsilon(\eta)\psi_k\left(\frac{x}{\varepsilon},\theta_0+\varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\,d\eta, \tag{57}$$

and

$$\int_{\mathbb{R}^N}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\nabla\overline{u_\varepsilon} + \frac{c\left(\frac{x}{\varepsilon}\right)-\lambda_1(\theta_0)}{\varepsilon^2}u_\varepsilon\cdot\overline{u_\varepsilon}\right)dx$$
$$= \varepsilon^{-2}\sum_{k\geqq 1}\int_{\varepsilon^{-1}\mathbb{T}^N}|\alpha_k^\varepsilon(\eta)|^2\Big(\lambda_k(\theta_0+\varepsilon\eta)-\lambda_1(\theta_0)\Big)d\eta.$$

Since $\lambda_k(\theta)-\lambda_1(\theta_0) \geqq 0$ and, for $k \geqq 2$, $\lambda_k(\theta)-\lambda_1(\theta_0) \geqq C > 0$, we deduce from the bound (55) that

$$\sum_{k\geqq 2}\int_{\varepsilon^{-1}\mathbb{T}^N}|\alpha_k^\varepsilon(\eta)|^2 d\eta \leqq C\varepsilon^2.$$

For $k = 1$, by assumption (9) there exists $C > 0$ such that

$$\lambda_1(\theta)-\lambda_1(\theta_0) \geqq C|\theta-\theta_0|^2 \quad \forall\theta\in\mathbb{T}^N,$$

and thus (55) implies

$$\int_{\varepsilon^{-1}\mathbb{T}^N}|\eta|^2|\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq C.$$

Extending $\alpha_1^\varepsilon(\eta)$ by zero outside $\varepsilon^{-1}\mathbb{T}^N$, and using the inverse Fourier transform, we deduce that the scalar sequence $\tilde{v}_\varepsilon$, defined by

$$\tilde{v}_\varepsilon(x) = \int_{\mathbb{R}^N} \alpha_1^\varepsilon(\eta) e^{2i\pi\eta\cdot x} d\eta,$$

is bounded in $H^1(\mathbb{R}^N)$.

Introducing a parameter $q \in (0, 1)$ (to be chosen later) we define a cut-off of $\tilde{v}_\varepsilon$ by

$$v_\varepsilon = \int_{|\eta|<\varepsilon^{-q}} \alpha_1^\varepsilon(\eta) e^{2i\pi\eta\cdot x} d\eta. \tag{58}$$

The difference between v_ε and $\tilde{v}_\varepsilon$ is small since

$$\|\tilde{v}_\varepsilon - v_\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \int_{|\eta|>\varepsilon^{-q}} |\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq \varepsilon^{2q} \int_{\mathbb{R}^N} |\eta|^2 |\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq C\varepsilon^{2q}.$$

Similarly we have

$$\begin{aligned} &\int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_1^\varepsilon(\eta)\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} d\eta \\ &\quad = \int_{|\eta|<\varepsilon^{-q}} \alpha_1^\varepsilon(\eta)\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} d\eta + t_\varepsilon(x), \end{aligned}$$

where t_ε is small, i.e.,

$$\|t_\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \int_{\eta\in\varepsilon^{-1}\mathbb{T}^N,\ |\eta|>\varepsilon^{-q}} |\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq \varepsilon^{2q} \int_{\varepsilon^{-1}\mathbb{T}^N} |\eta|^2 |\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq C\varepsilon^{2q}.$$

Since the first eigencouple of (8) is differentiable with respect to θ at θ_0, there exists a constant $C > 0$ such that

$$\|\psi_1(\cdot, \theta) - \psi_1(\cdot, \theta_0)\|_{L^2(\mathbb{T}^N)^K} \leqq C|\theta - \theta_0| \quad \forall\theta \in \mathbb{T}^N.$$

Therefore, we have

$$\begin{aligned} &\int_{|\eta|<\varepsilon^{-q}} \alpha_1^\varepsilon(\eta)\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} d\eta \\ &\quad = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(x) + s_\varepsilon(x) \end{aligned}$$

where s_ε is small, i.e.,

$$\begin{aligned} &\|s_\varepsilon\|^2_{L^2(\omega)^K} \\ &\quad = \int_\omega \left| \int_{|\eta|<\varepsilon^{-q}} \alpha_1^\varepsilon(\eta) \left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) - \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} d\eta \right|^2 dx \\ &\quad \leqq \int_\omega \left(\int_{|\eta|<\varepsilon^{-q}} d\eta\right) \left(\int_{|\eta|<\varepsilon^{-q}} |\alpha_1^\varepsilon(\eta)|^2 \left|\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) - \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\right|^2 d\eta\right) dx \\ &\quad \leqq C\varepsilon^{-Nq} \int_{|\eta|<\varepsilon^{-q}} |\alpha_1^\varepsilon(\eta)|^2 \left(\int_\omega \left|\psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) - \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\right|^2 dx\right) d\eta \\ &\quad \leqq C|\omega|\varepsilon^{-Nq} \int_{|\eta|<\varepsilon^{-q}} \varepsilon^2 |\eta|^2 |\alpha_1^\varepsilon(\eta)|^2 d\eta \\ &\quad \leqq C|\omega|\varepsilon^{2-Nq} \end{aligned}$$

for any compact set $\omega \subset \mathbb{R}^N$ (we cannot obtain a uniform estimate on $\mathbb{R}^N$ since s_ε is not defined as a Bloch decomposition). Collecting all the intermediate steps we deduce

$$u_\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} v_\varepsilon(x) + r_\varepsilon(x)$$

and $\|r_\varepsilon\|^2_{L^2(\omega)^K} \leqq C|\omega|\varepsilon^{\frac{2}{N+2}}$ with the optimal value of q equal to $2/(N+2)$. □

Proof of Corollary 1. The parabolic energy estimate for (19) yields

$$\frac{1}{2}\int_\Omega |\tilde{u}_\varepsilon(T)|^2 dx + \int_0^T \int_\Omega \left(A^\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2} \tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon} \right) dx\, dt \leqq C.$$

This implies assumption (55) (integrated in time) and thus, mimicking the proof of Lemma 3, we obtain the same result with v_ε bounded in $L^2((0,T); H^1(\mathbb{R}^N))$ and r_ε converging strongly to 0 in $L^2((0,T) \times \omega)^K$.

To obtain the bound on $\frac{\partial v_\varepsilon}{\partial t}$ we now multiply (19) by $\frac{\partial \overline{\tilde{u}_\varepsilon}}{\partial t}$ to obtain

$$\int_0^T \int_\Omega \left| \frac{\partial \overline{\tilde{u}_\varepsilon}}{\partial t} \right|^2 dx + \frac{1}{2}\int_\Omega \left(A^\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2} \tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon} \right)(T) dx$$
$$= \frac{1}{2}\int_\Omega \left(A^\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2} \tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon} \right)(0) dx. \tag{59}$$

When we use assumption (21) of well-prepared initial data, and take into account the equation satisfied by ψ^1, a simple computation shows that the right-hand side of (59) is equal to

$$\frac{1}{2}\int_\Omega A^\varepsilon (\psi^1_\varepsilon \otimes \nabla v^0) \cdot \left(\overline{\psi}^1_\varepsilon \otimes \nabla \overline{v}^0 \right) dx,$$

which is bounded since $v^0 \in W^{1,\infty}(\Omega)$. Thus, it implies that $\frac{\partial \tilde{u}_\varepsilon}{\partial t}$ is bounded in $L^2((0,T) \times \Omega)^K$. Recalling the Bloch wave decomposition (57) of $\tilde{u}_\varepsilon$, we have

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t}(t,x) = \sum_{k \geqq 1} \int_{\varepsilon^{-1}\mathbb{T}^N} \frac{\partial \alpha_k^\varepsilon}{\partial t}(t,\eta) \psi_k\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} d\eta,$$

and

$$\left\| \frac{\partial v_\varepsilon}{\partial t} \right\|^2_{L^2((0,T)\times\mathbb{R}^N)} = \int_0^T \int_{\varepsilon^{-1}\mathbb{T}^N} \left| \frac{\partial \alpha_k^\varepsilon}{\partial t} \right|^2 d\eta\, dt \leqq \left\| \frac{\partial \tilde{u}_\varepsilon}{\partial t} \right\|^2_{L^2((0,T)\times\Omega)^K},$$

which proves that v_ε is bounded in $H^1((0,T) \times \mathbb{R}^N)$ and thus locally relatively compact in $L^2((0,T) \times \mathbb{R}^N)$.

If $\Omega = \mathbb{R}^N$, we can obtain the same compactness of v_ε without using assumption (21). Indeed, it suffices to multiply (19) by a test function

$$\phi_\varepsilon(t,x) = \int_{\varepsilon^{-1}\mathbb{T}^N} \beta(t,\eta) \psi_1\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} d\eta,$$

where $\beta(t, \eta)$ is the Fourier transform of a function $\phi(t, x) \in L^2((0, T); H^1(\mathbb{R}^N))$. Then, using the Bloch decomposition, we can prove that $\frac{\partial v_\varepsilon}{\partial t}$ is bounded in $L^2((0, T); H^{-1}(\mathbb{R}^N))$ which, by a standard embedding theorem, yields the result. This trick does not work for $\Omega \neq \mathbb{R}^N$ because ϕ_ε does not satisfy the Dirichlet boundary condition. □

Remark 11. If we remove from assumption (9) the positive definite character of the Hessian matrix $\nabla_\theta \nabla_\theta \lambda_1(\theta_0)$, we can still obtain an homogenization result, weaker than Theorem 2. Indeed, the same proof shows that w_ε two-scale converges, up to a subsequence, to $\psi_1(y, \theta_0)v(t, x)$ where v is a solution of the homogenized equation (24) with a possibly degenerate matrix A^* (which is nevertheless always non-negative because θ_0 is a minimum point). However, Lemma 3 holds true only if $\nabla_\theta \nabla_\theta \lambda_1(\theta_0)$ is positive definite. Thus, we cannot recover the Dirichlet boundary condition, neither can we obtain the uniqueness of the homogenized solution and the convergence of the entire sequence w_ε.

Remark 12. If we remove from assumption (9) the condition that the minimum point θ_0 of $\lambda_1(\theta)$ is unique, then we can also prove a weaker version of Theorem 2. For each minimum and associated Hessian matrix $\nabla_\theta \nabla_\theta \lambda_1$, we can extract a subsequence such that w_ε two-scale converges $\psi_1(y, \theta_0)v(t, x)$ where v is a solution of the homogenized equation (24). However, since Lemma 3 does not hold true in this case, we cannot recover the Dirichlet boundary condition. Nevertheless, if $\Omega = \mathbb{R}^N$ and $\nabla_\theta \nabla_\theta \lambda_1$ is positive definite, we do not need any boundary condition to obtain the unique resolvability of the homogenized equation. Thus, in such a case, the entire sequence w_ε is converging. Recall that $w_\varepsilon = e^{\frac{\lambda_1(\theta_0)t}{\varepsilon^2}} e^{-2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} u_\varepsilon$, so that for different minima we have different values of θ_0, thus different sequences w_ε, and eventually different homogenized problems. If the initial condition is a superposition of well-prepared initial data for each minimum point θ_0, then, by linearity, we can decompose the solution in a superposition of elementary solutions, each of them converging to its own homogenized limit depending on θ_0.

Remark 13. If we replace, in assumption (9), the simplicity of $\lambda_1(\theta_0)$ by the condition that its multiplicity is $k \geqq 1$, and if we make suitable assumptions on the smoothness of the k first branches of eigenvalues $\lambda_n(\theta)$ (and corresponding eigenvectors) in the vicinity of θ_0, then we can generalize Theorem 2. The main difference is that, in such a case, the homogenized problem is now a system of k diffusion equations which are coupled only by zero-order terms. The diffusion tensor of each equation is the Hessian of the corresponding branch of eigenvalues at θ_0. This is clearly seen in the first step of the proof of Theorem 2 where the conclusion is now that the two-scale limit $w(t, x, y)$ is a combination of k independent eigenvectors associated with $\lambda_1(\theta_0)$. In the second step of the proof, we now choose a test function which is a similar combination of k test functions associated with each smooth branch of eigenvectors (the functions ζ_i are the corresponding derivatives with respect to θ_i of these eigenvectors and may thus change from one branch to another). Passing to the limit is as before and there is no coupling of the second-order terms because of the orthogonality property of the chosen family of eigenvectors.

5. Proofs for the spectral problem

This section is devoted to the proof of Theorem 3.

Lemma 4. *There exists a finite constant C, which does not depend on ε, such that*

$$\frac{\lambda_1(\theta_0)}{\varepsilon^2} + C \leqq \lambda_1^\varepsilon \leqq \frac{\lambda_1(\theta_0)}{\varepsilon^2} + \mu_1 + o(1), \tag{60}$$

where $o(1)$ vanishes as $\varepsilon \to 0$.

Proof. Let (μ_1, v_1) be the first eigencouple of the homogenized problem (27). For each small $\delta > 0$, we introduce a smooth and compactly supported in Ω function w^δ, such that $\|w^\delta\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega \left(A^* \nabla w^\delta \cdot \nabla \bar{w}^\delta + d^*(x)|w^\delta|^2 \right) dx < \mu_1 + \delta.$$

In other words, w^δ is an approximation of v_1. In the variational formulation

$$\lambda_1^\varepsilon = \min_{\|u\|_{L^2(\Omega)^K}=1} \int_\Omega \left(A^\varepsilon \nabla u \cdot \nabla \overline{u} + \left(\varepsilon^{-2} c^\varepsilon + d^\varepsilon \right) u \cdot \overline{u} \right) dx \tag{61}$$

we substitute a test function of the form

$$U^\varepsilon = \gamma_\varepsilon e^{2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} \left(\psi_1 \left(\frac{x}{\varepsilon}, \theta_0 \right) w^\delta(x) + \varepsilon \sum_{k=1}^N \frac{\partial w^\delta}{\partial x_k}(x) \zeta_k \left(\frac{x}{\varepsilon} \right) \right), \tag{62}$$

where ζ_k is the solution of (14) and γ_ε is a normalization constant chosen in such a way that $\|U^\varepsilon\|_{L^2(\Omega)^K} = 1$. Since ψ_1 and ζ_k are periodic functions, and since w^δ is normalized, we have $\lim_{\varepsilon \to 0} \gamma_\varepsilon = 1$. In view of (10) and (14), after simple rearrangements we obtain

$$\begin{aligned}
\lambda_1^\varepsilon \leqq {} & \frac{\lambda_1(\theta_0)}{\varepsilon^2} + o(1) + \gamma_\varepsilon^2 \int_\Omega A^\varepsilon_{\alpha\beta,kl} \psi^\varepsilon_{1,\alpha} \bar{\psi}^\varepsilon_{1,\beta} \frac{\partial w^\delta}{\partial x_k} \frac{\partial w^\delta}{\partial x_l} dx \\
& + \gamma_\varepsilon^2 \int_\Omega \left\{ \bar{\psi}^\varepsilon_{1,\alpha} A^\varepsilon_{\alpha\beta,ml} \left(\frac{\partial}{\partial y_m} + 2i\pi\theta_{0,m} \right) \zeta^\varepsilon_{k,\beta} \frac{\partial w^\delta}{\partial x_l} \frac{\partial w^\delta}{\partial x_k} \right. \\
& \left. + \psi^\varepsilon_{1,\alpha} \left(\frac{\partial}{\partial y_k} - 2i\pi\theta_{0,k} \right) \left(A^\varepsilon_{\alpha\beta,km} \bar{\zeta}^\varepsilon_{l,\beta} \right) \frac{\partial w^\delta}{\partial x_m} \frac{\partial w^\delta}{\partial x_l} \right\} dx \\
& + \gamma_\varepsilon^2 \int_\Omega \bar{\psi}^\varepsilon_{1,\alpha} \psi^\varepsilon_{1,\beta} d^\varepsilon_{\alpha\beta} |w^\delta|^2 dx + \varepsilon^2 \gamma_\varepsilon^2 \int_\Omega d^\varepsilon (\zeta^\varepsilon \nabla w^\delta) \cdot (\bar{\zeta}^\varepsilon \nabla w^\delta) dx \\
& + 2\varepsilon \gamma_\varepsilon^2 \mathcal{R} \left(\int_\Omega \left(\bar{\psi}_1^\varepsilon \zeta^\varepsilon A^\varepsilon \nabla\nabla w^\delta w^\delta + d^\varepsilon (w^\delta \bar{\psi}_1^\varepsilon) \cdot (\zeta^\varepsilon \nabla w^\delta) \right) dx \right).
\end{aligned}$$

From the definitions of A^* and d^*, we deduce

$$\lambda_1^\varepsilon \leqq \frac{\lambda_1(\theta_0)}{\varepsilon^2} + \mu_1 + \delta + o(1), \tag{63}$$

where $o(1)$ vanishes as $\varepsilon \to 0$. Since δ is an arbitrary positive number, this yields the required upper bound in (60). On the other hand, by using Lemma 1 we have

$$\min_{\|u\|_{L^2(\Omega)^K}=1} \int_\Omega \left(A^\varepsilon \nabla u \cdot \nabla \overline{u} + \left(\varepsilon^{-2} c^\varepsilon + d^\varepsilon\right) u \cdot \overline{u}\right) dx \geqq \frac{\lambda_1(\theta_0)}{\varepsilon^2} + \inf_{x\in\Omega, y\in\mathbb{T}^N, |\eta|=1} d(x,y)\eta\cdot\eta \tag{64}$$

which yields the desired lower bound. □

Lemma 5. *There exists a scalar sequence v_ε, which is uniformly bounded in $H_0^1(\Omega)$, such that $\|v_\varepsilon\|_{L^2(\mathbb{R}^N)} = 1 + o(1)$ and*

$$u_1^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(x) + r_\varepsilon(x) \quad \textit{with} \quad \lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2(\Omega)^K} = 0. \tag{65}$$

Proof of Lemma 5. From the upper bound of Lemma 4 and from the Bloch decomposition applied to u_1^ε, we deduce

$$\frac{\lambda_1(\theta_0)}{\varepsilon^2} + \int_\Omega d^\varepsilon u_1^\varepsilon \cdot \overline{u}_1^\varepsilon\, dx \leqq \lambda_1^\varepsilon \leqq \frac{\lambda_1(\theta_0)}{\varepsilon^2} + \mu_1 + o(1),$$

which, together with the normalization $\|u_1^\varepsilon\|_{L^2(\Omega)^K} = 1$, implies that

$$-\infty < \inf_{x\in\Omega, y\in\mathbb{T}^N, |\eta|=1} d(x,y)\eta\cdot\eta \leqq \int_\Omega d^\varepsilon u_1^\varepsilon \cdot \overline{u}_1^\varepsilon\, dx \leqq C. \tag{66}$$

Then, the existence of $\tilde{v}_\varepsilon$, bounded in $H^1(\mathbb{R}^N)$, and such that

$$u_1^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} \tilde{v}_\varepsilon(x) + r_\varepsilon(x),$$

is a consequence of Lemma 3 since

$$\int_\Omega \left(A^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \overline{u}_1^\varepsilon + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2} u_1^\varepsilon \cdot \overline{u}_1^\varepsilon\right) dx = \lambda_1^\varepsilon - \frac{\lambda_1(\theta_0)}{\varepsilon^2} - \int_\Omega d^\varepsilon u_1^\varepsilon \cdot \overline{u}_1^\varepsilon\, dx \leqq C.$$

As explained in Remark 10, we can replace $\tilde{v}_\varepsilon \in H^1(\mathbb{R}^N)$ by $v_\varepsilon \in H_0^1(\Omega)$ defined as the solution of

$$-\Delta v_\varepsilon = -\Delta \tilde{v}_\varepsilon \quad \text{in } \Omega, \qquad v_\varepsilon \in H_0^1(\Omega).$$

Since u_1^ε vanishes outside Ω and r_ε converges locally strongly to 0, it is easy to show that (65) is satisfied with such a sequence v_ε. □

Proof of Theorem 3. We focus on the first eigenfunction, $k = 1$. For $k > 1$ a similar proof holds true.

By Lemma 5 the family v_ε is relatively compact in $L^2(\Omega)$, and any limit point v^0 of a converging subsequence satisfies the relation $\|v^0\|_{L^2(\Omega)} = 1$. By Lemma 4 we can also extract a subsequence such that $\lambda_1^\varepsilon - \frac{\lambda_1(\theta_0)}{\varepsilon^2}$ converges to a limit μ. According to (60),

$$C \leqq \mu \leqq \mu_1. \tag{67}$$

The proof is now very similar to that of Theorem 2 (see Section 4). Up to another subsequence, $e^{-2\pi i x\cdot\theta_0/\varepsilon}u_1^\varepsilon(x)$ two-scale converges to a limit $u_1^0(x, y)$ and $\varepsilon\nabla\left(e^{-2\pi i x\cdot\theta_0/\varepsilon}u_1^\varepsilon\right)$ two-scale converges to $\nabla_y u_1^0(x, y)$. As in the first step of the proof of Theorem 2, it can easily be shown that

$$u_1^0(x, y) = v^0(x)\psi_1(y, \theta_0),$$

where v^0 is a limit point of v_ε. To find the equation satisfied by v^0, we proceed as in the second step of the proof of Theorem 2. We multiply (25) by the test function

$$\Psi_\varepsilon(x) = e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right)\phi(x) + \varepsilon\sum_{k=1}^{N}\frac{\partial\phi}{\partial x_k}(x)\zeta_k\left(\frac{x}{\varepsilon}\right)\right),$$

where ϕ is smooth with compact support. This yields

$$\begin{aligned}\int_\Omega A^\varepsilon(x)\nabla u_1^\varepsilon(x)\nabla\overline{\Psi}_\varepsilon(x)dx + \int_\Omega \frac{c^\varepsilon(x) - \lambda_1(\theta_0)}{\varepsilon^2}u_1^\varepsilon(x)\cdot\overline{\Psi}_\varepsilon(x)dx \\ + \int_\Omega d^\varepsilon(x)u_1^\varepsilon(x)\cdot\overline{\Psi}_\varepsilon(x) = \frac{\lambda_1^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2}\int_\Omega u_1^\varepsilon(x)\cdot\overline{\Psi}_\varepsilon(x).\end{aligned}$$

As before, using (8) and (14), we can pass to the two-scale limit to obtain

$$\int_\Omega\left(A^* v^0\cdot\nabla\nabla\overline{\phi} + d^*(x)v^0\overline{\phi}\right)dx = \mu\int_\Omega v^0\overline{\phi}\,dx$$

which is a weak variational formulation of

$$-A^*\cdot\nabla\nabla v^0 + d^* v^0 = \mu v^0 \quad \text{in} \quad \Omega. \tag{68}$$

The Dirichlet boundary condition for the limit v^0 is recovered as in the parabolic case. Since $v^0 \neq 0$ and $\mu \leqq \mu_1$, we necessarily have

$$\mu = \mu_1,$$

and v^0 is an eigenfunction of (27) associated with μ_1. If μ_1 is simple, up to a convenient renormalization, the entire sequence u_1^ε is converging (and not merely a subsequence). □

6. Proofs in the hyperbolic case

We begin with a proof of Theorem 4 when $\lambda_1(\theta_0) = 0$. Actually, as soon as uniform *a priori* estimates are obtained for the solution of equation (28), the proof of convergence is very similar to that of Theorem 2 in the parabolic case. Therefore, for the sake of brevity, we content ourselves with establishing those *a priori* estimates.

Lemma 6. *Under the assumptions of Theorem 4, the solution u_ε of* (28) *satisfies*

$$\|u_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon\,\|\nabla u_\varepsilon\|_{L^2((0,T)\times\Omega)^{N\times K}} + \|\frac{\partial u_\varepsilon}{\partial t}\|_{L^\infty((0,T);L^2(\Omega)^K)} \leqq C, \tag{69}$$

where $C > 0$ is a constant which does not depend on ε. Furthermore, there exists a scalar sequence v_ε, uniformly bounded in $L^2\left((0,T);\,H^1(\Omega)\right)$, such that

$$u_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon},\theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x), \tag{70}$$

where r_ε is a remainder term such that

$$\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \textit{ for any compact set } \omega\subset\mathbb{R}^N. \tag{71}$$

Proof. We multiply (28) by $\frac{\partial\overline{u}_\varepsilon}{\partial t}$ to obtain the usual energy conservation

$$E_\varepsilon(t) = E_\varepsilon(0) \text{ with } E_\varepsilon(t) = \frac{1}{2}\int_\Omega\left(\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 + A^\varepsilon\nabla u_\varepsilon\cdot\nabla\overline{u}_\varepsilon + \frac{c^\varepsilon}{\varepsilon^2}u_\varepsilon\cdot\overline{u}_\varepsilon\right)dx. \tag{72}$$

Since $\lambda_1(\theta_0) = 0$, by using the well-prepared character of the initial data (29) and equation (8), a classical computation shows that

$$E_\varepsilon(0) = \frac{1}{2}\int_\Omega\left(\left|u_\varepsilon^1\right|^2 + A^\varepsilon(\psi_\varepsilon^1\otimes\nabla v^0)\cdot(\overline{\psi}_\varepsilon^1\otimes\nabla\overline{v}^0)\right)dx,$$

which is uniformly bounded by assumption. Then, the Bloch wave analysis of Lemma 2 yields

$$\int_\Omega\left(A^\varepsilon\nabla u_\varepsilon\cdot\nabla\overline{u}_\varepsilon + \frac{c^\varepsilon}{\varepsilon^2}u_\varepsilon\cdot\overline{u}_\varepsilon\right)dx \geqq 0.$$

Therefore, we deduce (69) from (72). To obtain (70) and (71) we use Lemma 3 since (72) implies that assumption (55) is satisfied. □

We now turn to the proof of Theorem 5 when $\lambda_1(\theta_0) < 0$. Once again the proof of convergence is very similar to that of Theorem 2 as soon as uniform *a priori* estimates are established (see [4] in the scalar case if necessary). Therefore, we restrict ourselves to obtaining *a priori* estimates for the rescaled hyperbolic system (35).

Lemma 7. *Under the assumptions of Theorem 5 the solution $\tilde{u}_\varepsilon$ of* (35) *satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon\|\nabla\tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^{N\times K}} + \varepsilon\left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\Omega)^K} \leqq C, \tag{73}$$

where $C > 0$ is a constant which does not depend on ε. Furthermore, there exists a scalar sequence v_ε, uniformly bounded in $L^2\left((0,T); H^1(\Omega)\right)$, such that

$$\tilde{u}_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x), \tag{74}$$

where r_ε is a remainder term such that

$$\lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \textit{ for any compact set } \omega \subset \mathbb{R}^N. \tag{75}$$

Proof. In a first step we multiply (35) by $\frac{\partial\overline{\tilde{u}_\varepsilon}}{\partial t}$ to obtain the usual energy conservation

$$E_\varepsilon(T) + 2\sqrt{-\lambda_1(\theta_0)} \int_0^T \int_\Omega \left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right|^2 dx\, dt = E_\varepsilon(0) \tag{76}$$

with

$$E_\varepsilon(t) = \frac{1}{2}\int_\Omega \left(\varepsilon^2 \left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right|^2 + A^\varepsilon \nabla\tilde{u}_\varepsilon \cdot \nabla\overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon}\right) dx.$$

As in the proof of Lemma 6, using (8) yields

$$\int_\Omega \left(A^\varepsilon \nabla u_\varepsilon^0 \cdot \nabla\overline{u_\varepsilon^0} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2} u_\varepsilon^0 \cdot \overline{u_\varepsilon^0}\right) dx = \int_\Omega A^\varepsilon (\psi_\varepsilon^1 \otimes \nabla v^0)\cdot(\overline{\psi}_\varepsilon^1 \otimes \nabla \overline{v}^0)\, dx,$$

which is however not sufficient to show that $E_\varepsilon(0)$ is uniformly bounded. Indeed we have

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t}(0) = u_\varepsilon^1 - \frac{\sqrt{-\lambda_1(\theta_0)}}{\varepsilon^2} u_\varepsilon^0$$

which merely implies

$$E_\varepsilon(0) \leqq C\varepsilon^{-2}.$$

Nevertheless, from the Bloch wave analysis of Lemma 2 we deduce

$$\int_\Omega \left(A^\varepsilon \nabla\tilde{u}_\varepsilon \cdot \nabla\overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon}\right) dx \geqq 0,$$

which, combined with (76), yields

$$\varepsilon^2 \left\|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon\sqrt{-\lambda_1(\theta_0)} \left\|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right\|_{L^2((0,T)\times\Omega)^K} \leqq C. \tag{77}$$

In a second step we multiply (35) by $\overline{\tilde{u}_\varepsilon}$ to obtain a better energy estimate:

$$\begin{aligned}
&\sqrt{-\lambda_1(\theta_0)} \int_\Omega |\tilde{u}_\varepsilon(T)|^2\, dx + \int_0^T \int_\Omega \left(A^\varepsilon \nabla\tilde{u}_\varepsilon \cdot \nabla\overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon - \lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon}\right) dx\, dt \\
&\quad = \sqrt{-\lambda_1(\theta_0)} \int_\Omega |\tilde{u}_\varepsilon(0)|^2\, dx + \varepsilon^2 \int_0^T \int_\Omega \left|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right|^2 dx\, dt \\
&\qquad + \varepsilon^2 \int_\Omega \overline{\tilde{u}_\varepsilon}(0) \frac{\partial \tilde{u}_\varepsilon}{\partial t}(0)\, dx - \varepsilon^2 \int_\Omega \overline{\tilde{u}_\varepsilon}(T) \frac{\partial \tilde{u}_\varepsilon}{\partial t}(T)\, dx.
\end{aligned} \tag{78}$$

Using (77) we deduce, from (78),

$$\sqrt{-\lambda_1(\theta_0)}\|\tilde{u}_\varepsilon(T)\|^2_{L^2(\Omega)^K} \leqq C\left(1+\|\tilde{u}_\varepsilon(T)\|_{L^2(\Omega)^K}\right),$$

which implies that $\tilde{u}_\varepsilon$ is bounded in $L^\infty\left((0,T);L^2(\Omega)^K\right)$. Using this information in (78) shows that assumption (55) is satisfied: thus, Lemma 3 can be applied to obtain (74) and (75). □

Finally we arrive at the proof of Theorem 6 when $\lambda_1(\theta_0) > 0$ and again we simply address the question of uniform *a priori* estimates for (41) (the proof of convergence is an adaptation of Theorem 2 and of the arguments of [4] in the scalar case).

Lemma 8. *Under the assumptions of Theorem 6 the solution* $\tilde{u}_\varepsilon$ *of* (41) *satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon\|\nabla\tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^{N\times K}} + \varepsilon^2\left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^\infty((0,T);L^2(\Omega)^K)} \leqq C, \tag{79}$$

where $C > 0$ *is a constant which does not depend on* ε.

Remark 14. The result of Lemma 8 is weaker than those of Lemmas 6 and 7 since it does not give any strong compactness of $\tilde{u}_\varepsilon$. In particular, it implies that we can not straightforwardly recover the homogenized Dirichlet boundary condition. As in the scalar case [4], in order to obtain the homogenized boundary condition the trick is to study the homogenization of a time integral of (41) which has less oscillating initial data. Indeed, defining $w_\varepsilon(t,x) = \int_0^t \tilde{u}_\varepsilon(s,x)\,ds + \chi_\varepsilon(x)$ with a suitable choice of χ_ε (so that w_ε satisfies the same partial differential equation as (41) without a source term), we can obtain better *a priori* estimates for w_ε than for $\tilde{u}_\varepsilon$. We thus obtain an homogenized equation with a Dirichlet boundary condition for a limit of w_ε, and upon differentiating in time we deduce the desired Dirichlet boundary condition for the limit of $\tilde{u}_\varepsilon$ (see [4] for details).

Proof. In a first step we multiply (41) by $\frac{\partial\overline{\tilde{u}_\varepsilon}}{\partial t}$ and we take the real part to obtain the usual energy conservation

$$E_\varepsilon(t) = E_\varepsilon(0) \tag{80}$$

with

$$E_\varepsilon(t) = \frac{1}{2}\int_\Omega\left(\varepsilon^2\left|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right|^2 + A^\varepsilon\nabla\tilde{u}_\varepsilon\cdot\nabla\overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon-\lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon\cdot\overline{\tilde{u}_\varepsilon}\right)dx.$$

As in the proof of Lemma 7, the assumptions merely imply

$$E_\varepsilon(0) \leqq C\varepsilon^{-2}.$$

Nevertheless, from the Bloch wave analysis of Lemma 2 we deduce

$$\int_\Omega\left(A^\varepsilon\nabla\tilde{u}_\varepsilon\cdot\nabla\overline{\tilde{u}_\varepsilon} + \frac{c^\varepsilon-\lambda_1(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon\cdot\overline{\tilde{u}_\varepsilon}\right)dx \geqq 0,$$

which, combined with (80), yields

$$\varepsilon^2 \| \frac{\partial \tilde{u}_\varepsilon}{\partial t} \|_{L^\infty((0,T);L^2(\Omega)^K)} \leqq C. \tag{81}$$

In a second step we multiply (41) by $\overline{\tilde{u}_\varepsilon}$ and we take the imaginary part

$$\begin{aligned} &\sqrt{\lambda_1(\theta_0)} \int_\Omega |\tilde{u}_\varepsilon(T)|^2 \, dx - \sqrt{\lambda_1(\theta_0)} \int_\Omega |\tilde{u}_\varepsilon(0)|^2 \, dx \\ &+\varepsilon^2 \mathcal{I} \left(\int_\Omega \overline{\tilde{u}_\varepsilon}(T) \frac{\partial \tilde{u}_\varepsilon}{\partial t}(T) \, dx - \int_\Omega \overline{\tilde{u}_\varepsilon}(0) \frac{\partial \tilde{u}_\varepsilon}{\partial t}(0) \, dx \right) = 0. \end{aligned} \tag{82}$$

Using (81) we deduce, from (82),

$$\sqrt{\lambda_1(\theta_0)} \| \tilde{u}_\varepsilon(T) \|^2_{L^2(\Omega)^K} \leqq C \left(1 + \| \tilde{u}_\varepsilon(T) \|_{L^2(\Omega)^K} \right),$$

which implies that $\tilde{u}_\varepsilon$ is bounded in $L^\infty \left((0, T); L^2(\Omega)^K \right)$. Remark that (82), unlike (78), does not include any gradient term, so we cannot apply Lemma 3 to obtain a better estimate. □

7. Generalization to higher-level bands

We generalize the homogenization of a parabolic system established in Section 3 for initial data concentrating at the bottom of the first Bloch band to another type of initial data concentrating at the bottom of an higher level band. Such a generalization holds true only in the case of the whole space $\Omega = \mathbb{R}^N$ because otherwise we lack an adequate generalization of the compactness Lemma 3. From now on in this section we replace assumption (9) by the following one: for an energy level $n \geqq 1$, there exists a Bloch parameter $\theta_0 \in \mathbb{T}^N$ such that

(i) θ_0 is the unique minimizer of $\lambda_n(\theta)$ in $\mathbb{T}^N$,
(ii) $\lambda_n(\theta_0)$ is a simple eigenvalue, (83)
(iii) the Hessian matrix $\nabla_\theta \nabla_\theta \lambda_n(\theta_0)$ is positive definite.

Under assumption (83) the n^{th} eigencouple of (8) is smooth at θ_0. It is easily seen that the first derivative $\frac{\partial \psi_n}{\partial \theta_k}$ and the second derivative $\frac{\partial^2 \psi_n}{\partial \theta_k \partial \theta_l}$ satisfy equations similar to (11) and (12) respectively, up to changing the index 1 to n. In particular, for $\theta = \theta_0$ we still use the notation

$$\frac{\partial \psi_n}{\partial \theta_k} = 2i\pi \zeta_k, \quad \frac{\partial^2 \psi_n}{\partial \theta_k \partial \theta_l} = -4\pi^2 \chi_{kl}, \tag{84}$$

where ζ_k and χ_{kl} are solutions of (14) and (15) respectively, up to changing the label 1 to n.

We study a parabolic system with purely periodic coefficients:

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \frac{c\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2}u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^N \times (0,T),\\ &\qquad\qquad\qquad\qquad u_\varepsilon(t=0,x) = u_\varepsilon^0(x) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{85}$$

We also need an assumption on the initial data which must be "well prepared", namely concentrating at the bottom on the n^{th} Bloch band. Recall from Lemma 1 that any function $u_\varepsilon^0 \in L^2(\mathbb{R}^N)$ can be decomposed as

$$u_\varepsilon^0(x) = \sum_{k\geqq 1} \int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_k^\varepsilon(\eta)\psi_k\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{-2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\, d\eta,$$

with $\eta = \frac{\theta-\theta_0}{\varepsilon}$. We denote by Π_ε^n the projection operator on the Bloch bands above the n^{th} level

$$\Pi_\varepsilon^n u_\varepsilon^0(x) = \sum_{k\geqq n} \int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_k^\varepsilon(\eta)\psi_k\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{-2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\, d\eta. \tag{86}$$

Our assumption on the initial data is that

$$u_\varepsilon^0 = \Pi_\varepsilon^n u_\varepsilon^0. \tag{87}$$

Typically, we are interested in initial data of the type

$$u_\varepsilon^{0,1}(x) = \Pi_\varepsilon^n\left(v^0(x)\psi_n\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\right), \tag{88}$$

with $v^0 \in L^2(\mathbb{R}^N)$. However, since the projection operator Π_ε^n is not very explicit, we also consider another type of initial data which satisfies assumption (87), namely

$$u_\varepsilon^{0,2}(x) = \int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_n(\eta)\psi_n\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\, d\eta, \tag{89}$$

with $\alpha_n \in L^2(\mathbb{R}^N)$ being the Fourier transform of $v^0(x)$. Actually, it is easy to check that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon^{0,1} - u_\varepsilon^{0,2}\|_{L^2(\mathbb{R}^N)^K} = 0.$$

For such well-prepared initial data, we perform a time renormalization similar to (18),

$$\tilde{u}_\varepsilon(t,x) = e^{\frac{\lambda_n(\theta_0)t}{\varepsilon^2}} u_\varepsilon(t,x), \tag{90}$$

such that $\tilde{u}_\varepsilon$ satisfies

$$\begin{aligned} &\frac{\partial \tilde{u}_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_\varepsilon\right) + \frac{c\left(\frac{x}{\varepsilon}\right) - \lambda_n(\theta_0)}{\varepsilon^2}\tilde{u}_\varepsilon = 0 \qquad \text{in } \mathbb{R}^N \times (0,T),\\ &\qquad\qquad\qquad\qquad \tilde{u}_\varepsilon(t=0,x) = u_\varepsilon^0(x) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{91}$$

Lemma 9. *Under assumption* (87)*, the solution of* (91) *satisfies*

$$\|\tilde{u}_\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N)^K)} + \varepsilon\|\nabla\tilde{u}_\varepsilon\|_{L^2((0,T)\times\mathbb{R}^N)^{N\times K}} \leqq C\|u_\varepsilon^0\|_{L^2(\mathbb{R}^N)^K}, \qquad (92)$$

and there exists a bounded scalar sequence v_ε *in* $L^2\left((0,T); H^1(\mathbb{R}^N)\right)$ *such that*

$$\tilde{u}_\varepsilon(t,x) = \psi_n\left(\frac{x}{\varepsilon},\theta_0\right)e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}v_\varepsilon(t,x) + r_\varepsilon(t,x), \qquad (93)$$

where $\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0$ *for any compact set* $\omega \subset \mathbb{R}^N$.

Theorem 7. *Assume that the initial data* $u_\varepsilon^0 \in L^2(\mathbb{R}^N)^K$ *is of the form* (88) *or* (89)*. The solution of* (85) *can be written as*

$$u_\varepsilon(t,x) = e^{-\frac{\lambda_n(\theta_0)t}{\varepsilon^2}}\left(\psi_n\left(\frac{x}{\varepsilon},\theta_0\right)e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}v_\varepsilon(t,x) + r_\varepsilon(t,x)\right), \qquad (94)$$

where r_ε *is a remainder term such that*

$$\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \text{ for any compact set } \omega \subset \mathbb{R}^N, \qquad (95)$$

and v_ε *converges weakly in* $L^2\left((0,T); H^1(\mathbb{R}^N)\right)$ *to the solution* v *of the scalar homogenized problem*

$$\frac{\partial v}{\partial t} - \operatorname{div}\left(A_n^*\nabla v\right) = 0 \text{ in } \mathbb{R}^N \times (0,T), \qquad (96)$$
$$v(t=0,x) = v^0(x) \text{ in } \mathbb{R}^N,$$

with $A_n^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_n(\theta_0)$.

Remark 15. In the context of the Schrödinger equation Theorem 7 is called an effective mass theorem [21, 23, 24]. Even in the case of a scalar equation, Theorem 7 is new since the factorization principle does not work for an energy level $n > 1$, namely one cannot divide the unknown u_ε by $\psi_n\left(\frac{x}{\varepsilon},\theta_0\right)$, which necessarily vanishes at some points in $\mathbb{T}^N$.

Remark 16. Initial data of the type (88) or (89) would yield a zero limit if homogenized in the setting of Theorem 2. The solution u_ε, given by (94), decays much faster than that given by (22) because $\lambda_n(\theta_0) > \lambda_1(\theta_0)$. Therefore, we can interpret Theorem 7 as describing initial layers in time, compared to Theorem 2 which captures the average behavior. This is consistent with the classical homogenization of parabolic equations, when $c \equiv 0$, where initial layers in time are known to exist [12] but cannot be characterized by the classical homogenization theory.

Proof of Lemma 9. We apply the rescaled Bloch decomposition (16) to equation (91):

$$\tilde{u}_\varepsilon(t,x) = \sum_{k\geqq 1}\int_{\varepsilon^{-1}\mathbb{T}^N}\alpha_k^\varepsilon(t,\eta)\psi_k\left(\frac{x}{\varepsilon},\theta_0+\varepsilon\eta\right)e^{2i\pi\eta\cdot x}e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}d\eta,$$

with

$$\alpha_k^\varepsilon(t,\eta) = \alpha_k^\varepsilon(0,\eta) e^{\frac{(\lambda_n - \lambda_k)(\theta_0 + \varepsilon\eta)}{\varepsilon^2} t}.$$

From assumption (87) we deduce that $\alpha_k^\varepsilon(t,\eta) = 0$ for any $k < n$. Therefore, for any time t, we have $\Pi_\varepsilon^n \tilde{u}_\varepsilon(t,x) = \tilde{u}_\varepsilon(t,x)$. Thus,

$$\int_{\mathbb{R}^N} \left(A\left(\frac{x}{\varepsilon}\right) \nabla \tilde{u}_\varepsilon \cdot \nabla \overline{\tilde{u}_\varepsilon} + \frac{c\left(\frac{x}{\varepsilon}\right) - \lambda_n(\theta_0)}{\varepsilon^2} \tilde{u}_\varepsilon \cdot \overline{\tilde{u}_\varepsilon} \right) dx \geqq 0,$$

which easily yields the *a priori* estimate (92). We now mimic the arguments of the proof of Lemma 3 (replacing the label 1 by n) to obtain the compactness result (93). □

Proof of Theorem 7. The proof is very similar to that of Theorem 2 so we simply sketch the main points. We introduce, as before, a sequence w_ε defined by

$$w_\varepsilon(t,x) = \tilde{u}_\varepsilon(t,x) e^{-2i\pi \frac{\theta_0 \cdot x}{\varepsilon}}.$$

By the a priori estimates of Lemma 9, there exist a subsequence and a limit $w(t,x,y) \in L^2\left((0,T)\times\mathbb{R}^N; H^1(\mathbb{T}^N)^K\right)$ such that w_ε and $\varepsilon\nabla w_\varepsilon$ two-scale converges to w and $\nabla_y w$ respectively (see [3, 22]). Similarly, by its very definition, $w_\varepsilon(0,x)$ two-scale converges to $\psi_n(y,\theta_0)\, v^0(x)$. In a first step we multiply (91) by the complex conjugate of $\varepsilon^2 \phi(t,x,\frac{x}{\varepsilon}) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}$, where $\phi(t,x,y)$ is a smooth test function defined in $[0,T)\times\mathbb{R}^N\times\mathbb{T}^N$ with values in $\mathbb{C}^K$. Passing to the two-scale limit yields the existence of a scalar function $v(t,x) \in L^2\left((0,T)\times\mathbb{R}^N\right)$ such that $w(t,x,y) = v(t,x)\psi_n(y,\theta_0)$. In a second step we multiply (91) by the complex conjugate of

$$\Psi_\varepsilon = e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} \left(\psi_n\left(\frac{x}{\varepsilon},\theta_0\right)\phi(t,x) + \varepsilon \sum_{k=1}^N \frac{\partial\phi}{\partial x_k}(t,x)\zeta_k\left(\frac{x}{\varepsilon}\right) \right),$$

where $\phi(t,x)$ is a smooth, compactly supported, test function defined from $[0,T)\times\mathbb{R}^N$ into $\mathbb{C}$, and $\zeta_k(y)$ is the solution of (14) where the label 1 is replaced by n. Passing to the two-scale limit yields a very weak form of the homogenized equation (96). It is routine to show that its solution $v(t,x)$ is indeed a classical weak solution. Then, by uniqueness of the solution, we deduce that the entire sequence w_ε two-scale converges to $\psi_n(y,\theta_0)\, v(t,x)$. □

Remark 17. All the results of this section are specific to the case of the whole space, i.e., $\Omega = \mathbb{R}^N$, and cannot be extended to the case of an additional zero-order term $d(x,\frac{x}{\varepsilon})$ because we crucially use the Bloch diagonalization to get *a priori* estimates.

8. Fourth-order homogenized problem

By changing the main assumption on the Bloch spectrum it is possible to obtain a fourth-order homogenized equation from a second-order parabolic problem. Specifically we consider

$$\begin{aligned} &\varepsilon^2 \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) \\ &+ \left(\varepsilon^{-2} c\left(\tfrac{x}{\varepsilon}\right) + \varepsilon^2 d\left(x, \tfrac{x}{\varepsilon}\right)\right) u_\varepsilon = 0 \quad \text{in } \Omega \times (0, T), \\ &u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T), \\ &u_\varepsilon(t = 0, x) = u_\varepsilon^0(x) \quad \text{in } \Omega. \end{aligned} \tag{97}$$

Remark that the time scaling in (97) is not the same as that in (17): this means that we are looking for an asymptotic for longer time of order ε^{-2} in (97), compared to (17). Instead of (9), we now make the following assumption

$$\begin{aligned} &\text{(i)}\ \ \theta_0 \text{ is the unique minimizer of } \lambda_1(\theta) \text{ in } \mathbb{T}^N, \\ &\text{(ii)}\ \ \lambda_1(\theta_0) \text{ is a simple eigenvalue}, \\ &\text{(iii)}\ \ \nabla_\theta \nabla_\theta \lambda_1(\theta_0) = 0, \\ &\text{(iv)}\ \ \text{the fourth-order tensor } \nabla_\theta \nabla_\theta \nabla_\theta \nabla_\theta \lambda_1(\theta_0) \text{ is positive definite.} \end{aligned} \tag{98}$$

Remark 18. We do not know if assumption (98) is satisfied in any practical example.

Since $\lambda_1(\theta_0)$ is a minimum, we also have $\nabla_\theta \lambda_1(\theta_0) = 0$ and $\nabla_\theta \nabla_\theta \nabla_\theta \lambda_1(\theta_0) = 0$. Under assumption (98) the first eigencouple of (8) is smooth at θ_0. Recall that, for $\theta = \theta_0$, the two first derivatives of ψ_1 are given by

$$\frac{\partial \psi_1}{\partial \theta_k} = 2i\pi \zeta_k, \qquad \frac{\partial^2 \psi_1}{\partial \theta_k \partial \theta_l} = -4\pi^2 \chi_{kl}, \tag{99}$$

where ζ_k is the solution of (14) and χ_{kl} is the solution of (15) (remark that this last equation simplifies since $\nabla_\theta \nabla_\theta \lambda_1(\theta_0) = 0$). Similarly, the third derivative is

$$\frac{\partial^3 \psi_1}{\partial \theta_j \partial \theta_k \partial \theta_l} = -8i\pi^3 \xi_{jkl}, \tag{100}$$

where

$$\begin{aligned} \mathbb{A}(\theta_0)\xi_{jkl} = {} & e_j A(y)(\nabla_y + 2i\pi\theta_0)\chi_{kl} + (\operatorname{div}_y + 2i\pi\theta_0)\left(A(y) e_j \chi_{kl}\right) \\ & + e_k A(y)(\nabla_y + 2i\pi\theta_0)\chi_{jl} + (\operatorname{div}_y + 2i\pi\theta_0)\left(A(y) e_k \chi_{jl}\right) \\ & + e_l A(y)(\nabla_y + 2i\pi\theta_0)\chi_{kj} + (\operatorname{div}_y + 2i\pi\theta_0)\left(A(y) e_l \chi_{kj}\right) \\ & + e_k A(y) e_l \zeta_j + e_j A(y) e_l \zeta_k + e_k A(y) e_j \zeta_l. \end{aligned} \tag{101}$$

There exists a unique solution of (101), up to the addition of a multiple of ψ_1. Indeed, the right-hand side of (101) satisfies the required compatibility condition (i.e., it is orthogonal to ψ_1) because all derivatives of $\lambda_1(\theta)$, up to third order, are zero at $\theta = \theta_0$.

We perform a time renormalization by introducing a new unknown,

$$\tilde{u}_\varepsilon(t, x) = e^{\frac{\lambda_1(\theta_0)t}{\varepsilon^4}} u_\varepsilon(t, x), \tag{102}$$

which satisfies

$$\begin{aligned} &\frac{\partial \tilde{u}_\varepsilon}{\partial t} - \varepsilon^{-2}\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_\varepsilon\right) \\ &+\frac{c(\frac{x}{\varepsilon})-\lambda_1(\theta_0)}{\varepsilon^4}\tilde{u}_\varepsilon + d\left(x, \tfrac{x}{\varepsilon}\right)\tilde{u}_\varepsilon = 0 \qquad \text{in } \Omega \times (0,T), \\ &\tilde{u}_\varepsilon = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ &\tilde{u}_\varepsilon(t=0,x) = u^0_\varepsilon(x) \quad \text{in } \Omega. \end{aligned} \tag{103}$$

As usual we obtain the following *a priori* estimate:

$$\|\tilde{u}_\varepsilon\|_{L^\infty((0,T);L^2(\Omega)^K)} + \varepsilon\|\nabla\tilde{u}_\varepsilon\|_{L^2((0,T)\times\Omega)^{N\times K}} \leqq C\|u^0_\varepsilon\|_{L^2(\Omega)^K},$$

where the constant $C > 0$ does not depend on ε.

Theorem 8. *Assume that the initial data $u^0_\varepsilon \in L^2(\Omega)^K$ is of the form*

$$u^0_\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^0(x), \tag{104}$$

with $v^0 \in L^2(\Omega)$. The solution of (97) *can be written as*

$$u_\varepsilon(t,x) = e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^4}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x)\right), \tag{105}$$

where r_ε is a remainder term such that

$$\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)^K} = 0 \text{ for any compact set } \omega \subset \mathbb{R}^N,$$

and v_ε converges weakly in $L^2\left((0,T); H^2(\Omega)\right)$ to the solution v of the scalar fourth-order homogenized problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + \mathrm{div}\,\mathrm{div}\left(A^*\nabla\nabla v\right) = 0 \qquad \text{in } \Omega \times (0,T), \\ &\frac{\partial v}{\partial n} = v = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ &v(t=0,x) = v^0(x) \quad \text{in } \Omega, \end{aligned} \tag{106}$$

with $A^* = \frac{1}{(2\pi)^4 4!}\nabla_\theta\nabla_\theta\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$.

To prove Theorem 8 we need the following generalization of Lemma 3.

Lemma 10. *Let u_ε be a bounded sequence in $L^2(\mathbb{R}^N)^K$. Assume that there exists a finite constant C such that*

$$\int_{\mathbb{R}^N}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \cdot \nabla\overline{u}_\varepsilon + \frac{c\left(\frac{x}{\varepsilon}\right) - \lambda_1(\theta_0)}{\varepsilon^2} u_\varepsilon \cdot \overline{u}_\varepsilon\right) dx \leqq C\varepsilon^2. \tag{107}$$

Then, under assumption (98)*,*

$$u_\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(x) + r_\varepsilon(x), \tag{108}$$

where v_ε is a bounded scalar sequence in $H^2(\mathbb{R}^N)$ and $\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2(\omega)^K} = 0$ for any compact set $\omega \subset \mathbb{R}^N$.

Proof. Introducing the rescaled Bloch decomposition (16) of $u_\varepsilon(x)$ with $\eta = \frac{\theta-\theta_0}{\varepsilon}$,

$$u_\varepsilon(x) = \sum_{k\geqq 1} \int_{\varepsilon^{-1}\mathbb{T}^N} \alpha_k^\varepsilon(\eta)\psi_k\left(\frac{x}{\varepsilon}, \theta_0 + \varepsilon\eta\right) e^{2i\pi\eta\cdot x} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} d\eta,$$

the same arguments as those in the proof of Lemma 3 and the estimate

$$\lambda_1(\theta) - \lambda_1(\theta_0) \geqq C|\theta-\theta_0|^4 \quad \forall\theta \in \mathbb{T}^N,$$

show that

$$\int_{\varepsilon^{-1}\mathbb{T}^N} |\eta|^4 |\alpha_1^\varepsilon(\eta)|^2 d\eta \leqq C.$$

Defining $v_\varepsilon(x)$ as the inverse Fourier transform of $\alpha_1^\varepsilon(\eta)$, we deduce that v_ε is uniformly bounded in $H^2(\mathbb{R}^N)$. □

Proof of Theorem 8. The proof is similar to that of Theorem 2. The first step is identical: the function $w_\varepsilon(t,x) = u_\varepsilon(t,x)e^{-2i\pi\frac{\theta_0\cdot x}{\varepsilon}}$ two-scale converges to a limit $v(t,x)\psi_1(y,\theta_0)$. In the second step, we multiply (103) by the complex conjugate of

$$\Psi_\varepsilon = e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}\left(\psi_1\left(\frac{x}{\varepsilon},\theta_0\right)\phi(t,x) + \varepsilon\sum_{k=1}^N \frac{\partial\phi}{\partial x_k}(t,x)\zeta_k\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\sum_{k,l=1}^N \frac{\partial^2\phi}{\partial x_k\partial x_l}(t,x)\chi_{kl}\left(\frac{x}{\varepsilon}\right) + \varepsilon^3\sum_{j,k,l=1}^N \frac{\partial^3\phi}{\partial x_j\partial x_k\partial x_l}(t,x)\xi_{jkl}\left(\frac{x}{\varepsilon}\right)\right), \tag{109}$$

where $\phi(t,x)$ is a smooth, compactly supported, test function defined from $[0,T)\times\Omega$ into $\mathbb{R}$, $\zeta_k(y)$ is the solution of (14), $\chi_{kl}(y)$ is the solution of (15), and $\xi_{jkl}(y)$ is the solution of (101). After some tedious algebra we find that

$$\begin{aligned}
&\int_\Omega |\psi_1^\varepsilon|^2 v^0\overline{\phi}(0)\,dx - \int_0^T\int_\Omega w_\varepsilon\cdot\overline{\psi}_1^\varepsilon\frac{\partial\overline{\phi}}{\partial t}\,dt\,dx \\
&- \int_0^T\int_\Omega A^\varepsilon w_\varepsilon\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\cdot e_k\overline{\chi}_{jl}^\varepsilon\,dt\,dx \\
&- \int_0^T\int_\Omega A^\varepsilon w_\varepsilon\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\cdot(\varepsilon\nabla - 2i\pi\theta_0)\overline{\eta}_{jkl}^\varepsilon\,dt\,dx \\
&+ \int_0^T\int_\Omega A^\varepsilon\overline{\eta}_{jkl}^\varepsilon(\varepsilon\nabla + 2i\pi\theta_0)w_\varepsilon\cdot\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\,dt\,dx \\
&+ \int_0^T\int_\Omega d^\varepsilon w_\varepsilon\cdot\overline{\psi}_1^\varepsilon\overline{\phi}\,dt\,dx = \mathcal{O}(\varepsilon).
\end{aligned} \tag{110}$$

Passing to the two-scale limit in each term of (110) gives

$$\int_\Omega\int_{\mathbb{T}^N} |\psi_1|^2 v^0\overline{\phi}(0)\,dx\,dy - \int_0^T\int_\Omega\int_{\mathbb{T}^N} |\psi_1|^2 v\frac{\partial\overline{\phi}}{\partial t} dt\,dx\,dy$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega\int_{\mathbb{T}^N} A\psi_1 v\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\cdot e_k\overline{\chi}_{jl}\,dt\,dx\,dy\\
&-\int_0^T\int_\Omega\int_{\mathbb{T}^N} A\psi_1 v\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\cdot(\nabla_y-2i\pi\theta_0)\overline{\eta}_{jkl}\,dt\,dx\,dy\\
&+\int_0^T\int_\Omega\int_{\mathbb{T}^N} A\overline{\eta}_{jkl}(\nabla_y+2i\pi\theta_0)\psi_1 v\cdot\nabla\frac{\partial^3\overline{\phi}}{\partial x_j\partial x_k\partial x_l}\,dt\,dx\,dy\\
&+\int_0^T\int_\Omega\int_{\mathbb{T}^N} d\psi_1 v\cdot\overline{\psi}_1\overline{\phi}\,dt\,dx\,dy=0.
\end{aligned}
\tag{111}
$$

Recalling the normalization $\int_{\mathbb{T}^N}|\psi_1|^2dy=1$, and introducing

$$
\begin{aligned}
A^*_{jklm}=\int_{\mathbb{T}^N}\Big(&-A\psi_1 e_m\cdot e_k\overline{\chi}_{jl}-A\psi_1 e_m\cdot(\nabla_y-2i\pi\theta_0)\overline{\eta}_{jkl}\\
&+A\overline{\eta}_{jkl}(\nabla_y+2i\pi\theta_0)\psi_1\cdot e_m\Big)dy
\end{aligned}
\tag{112}
$$

(which has to be symmetrized), and $d^*(x)=\int_{\mathbb{T}^N}d(x,y)\psi_1(y)\cdot\overline{\psi}_1(y)\,dy$, we find that (111) is equivalent to

$$
\int_\Omega v^0\overline{\phi}(0)\,dx-\int_0^T\int_\Omega\left(v\frac{\partial\overline{\phi}}{\partial t}-A^*v\cdot\nabla\nabla\nabla\nabla\overline{\phi}-d^*(x)v\overline{\phi}\right)dt\,dx=0
$$

which is a very weak form of the homogenized equation (106). To recover the Dirichlet boundary condition, we use Lemma 10 which implies that $v\in H^2(\mathbb{R}^N)$ and $v=0$ in any compact set $\omega\subset\left(\mathbb{R}^N\setminus\Omega\right)$. Thus v belongs to $H^2_0(\Omega)$.

The compatibility condition of the equation giving the fourth derivative of ψ_1 shows that the tensor A^*, defined by (112), is indeed equal to $\frac{1}{(2\pi)^4 4!}\nabla_\theta\nabla_\theta\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$, and thus is real, symmetric, positive definite by assumption (98). Therefore, the homogenized problem (106) is well posed. By uniqueness of the solution, the entire sequence v_ε converges to v. □

9. Homogenization of fourth-order equations

Our method also applies to fourth-order problems. Although systems of equations can be treated, for simplicity we focus on the case of a single equation, without loss of generality since there is no maximum principle for fourth-order elliptic equations. Let us introduce the following symmetric fourth-order operator

$$
\mathcal{A}^\varepsilon=\operatorname{div}\operatorname{div}\left(\Theta\left(\frac{x}{\varepsilon}\right)\nabla\nabla\right)-\frac{1}{\varepsilon^2}\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla\right)+\frac{1}{\varepsilon^4}c\left(\frac{x}{\varepsilon}\right)+d\left(x,\frac{x}{\varepsilon}\right),
\tag{113}
$$

with periodic coefficients $\Theta(y)=\{\Theta_{ijkl}(y)\}$, $A(y)=\{A_{ij}(y)\}$ and $c(y)$ which are real periodic functions in $L^\infty(\mathbb{T}^N)$. Furthermore, Θ and A are symmetric tensors, and Θ is uniformly elliptic (A need not be positive). The locally periodic term $d(x,y)$ belongs to $L^\infty\left(\Omega;C(\mathbb{T}^N)\right)$.

Under these assumptions the Bloch decomposition for (113) is basically the same as that for second order operators. On the torus $\mathbb{T}^N$ we introduce the Bloch operators

$$\begin{aligned}\mathcal{A}(\theta)\psi(y) &= e^{-2i\pi y\cdot\theta}\mathcal{A}e^{2i\pi y\cdot\theta}\psi(y)\\ &= (\nabla_y + 2i\pi\theta)(\nabla_y + 2i\pi\theta)\cdot(\Theta(y)(\nabla_y + 2i\pi\theta)(\nabla_y + 2i\pi\theta))\psi(y)\\ &\quad -(\nabla_y + 2i\pi\theta)\cdot(A(y)(\nabla_y + 2i\pi\theta)\psi(y) + c(y)\psi(y),\end{aligned}$$

with $\mathcal{A} = \operatorname{div}\operatorname{div}(\Theta(y)\nabla\nabla) - \operatorname{div}(A(y)\nabla) + c(y)$. Then, the Bloch spectral cell problem

$$\mathcal{A}(\theta)\psi_n = \lambda_n(\theta)\psi_n \quad \text{in} \quad L^2(\mathbb{T}^N)$$

has a discrete spectrum $\lambda_1(\theta) \leqq \lambda_2(\theta) \leqq \cdots \leqq \lambda_n(\theta) \to +\infty$. Moreover, all the statements of Lemma 1 (and its rescaled version) remain valid.

It is quite natural to make assumption (98) which implies $\nabla_\theta\nabla_\theta\lambda_1(\theta_0) = 0$. For example, (98) is easily seen to be satisfied with $\theta_0 = 0$ if there are no zero- and second-order terms in (113), i.e., $A \equiv 0$, $c \equiv 0$.

We begin with the parabolic Cauchy problem

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} + \mathcal{A}^\varepsilon u_\varepsilon = 0 && \text{in } \Omega\times(0,T),\\ &u_\varepsilon = 0,\quad \frac{\partial u_\varepsilon}{\partial n} = 0 && \text{on } \partial\Omega\times(0,T),\\ &u_\varepsilon(t=0,x) = u_\varepsilon^0(x) && \text{in } \Omega.\end{aligned} \tag{114}$$

Theorem 9. *Assume* (98). *Let* $u_\varepsilon(t,x)$ *be a solution of* (114) *with* $\mathcal{A}^\varepsilon$ *given by* (113)*, and* $u_\varepsilon^0 \in L^2(\Omega)$ *be an initial data of the form*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon},\theta_0\right)e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}v^0(x),$$

with $v^0 \in L^2(\Omega)$. *Then* u_ε *can be written as*

$$u_\varepsilon(t,x) = e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^4}}\left(\psi_1\left(\frac{x}{\varepsilon},\theta_0\right)e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}}v_\varepsilon(t,x) + r_\varepsilon(t,x)\right),$$

where the remainder term r_ε *satisfies the relation*

$$\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)} = 0 \textit{ for any compact set } \omega\subset\mathbb{R}^N,$$

and v_ε *converges weakly in* $L^2\left((0,T);H^2(\Omega)\right)$ *to the solution* v *of*

$$\begin{aligned} &\frac{\partial v}{\partial t} + \mathcal{A}^* v = 0 && \textit{in } \Omega\times(0,T),\\ &v = 0,\quad \frac{\partial v}{\partial n} = 0 && \textit{on } \partial\Omega\times(0,T),\\ &v(t=0,x) = v^0(x) && \textit{in } \Omega,\end{aligned}$$

with the homogenized operator

$$\mathcal{A}^* = \operatorname{div}\operatorname{div}\left(\Theta^*\nabla\nabla\right) + d^*(x) \tag{115}$$

and $\Theta^* = \frac{1}{(2\pi)^4 4!}\nabla_\theta\nabla_\theta\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$, $d^*(x) = \int_{\mathbb{T}^N} d(x,y)|\psi_1(y,\theta_0)|^2 dy$.

The proof of Theorem 9 is very similar to that of Theorem 8. Upon defining $\tilde{u}^\varepsilon(t,x) = e^{\frac{\lambda_1(\theta_0)t}{\varepsilon^4}} u^\varepsilon(t,x)$, the *a priori* estimates are

$$\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)^N} + \varepsilon^2\|\nabla\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)^{N^2}} \leqq C,$$

which, up to a subsequence, implies the following two-scale convergences for $w^\varepsilon = e^{-2i\pi x\cdot\theta_0/\varepsilon}\tilde{u}^\varepsilon(t,x)$:

$$\begin{aligned} w^\varepsilon &\overset{2s}{\rightharpoonup} v(t,x)\psi_1(y,\theta_0), \\ \varepsilon\nabla w^\varepsilon &\overset{2s}{\rightharpoonup} v(t,x)\nabla_y\psi_1(y,\theta_0), \\ \varepsilon^2\nabla\nabla w^\varepsilon &\overset{2s}{\rightharpoonup} v(t,x)\nabla_y\nabla_y\psi_1(y,\theta_0), \end{aligned}$$

where $v(t,x)$ is a limit point of a sequence v^ε, bounded in $L^2\left((0,T); H^2(\mathbb{R}^N)\right)$, introduced in a variant of Lemmas 3 and 10. Eventually, we use the same test function defined in (109). We safely leave the details to the reader.

We then study the Dirichlet spectral problem

$$\mathcal{A}^\varepsilon u_n^\varepsilon = \lambda_n^\varepsilon u_n^\varepsilon, \quad u_n^\varepsilon \in H_0^2(\Omega)$$

stated in a bounded domain $\Omega \subset \mathbb{R}^N$, which, under the standing ellipticity assumptions, admits a discrete spectrum, $\lambda_n^\varepsilon \to +\infty$ as $n \to +\infty$, with corresponding normalized eigenfunctions denoted by u_n^ε.

Theorem 10. *Assume* (98). *Then, for any $n \geqq 1$,*

$$\lambda_n^\varepsilon = \frac{\lambda_1(\theta_0)}{\varepsilon^4} + \mu_n + o(1) \qquad as\ \varepsilon \to 0$$

and the corresponding eigenfunction $u_n^\varepsilon(x)$ admits the representation

$$u_n^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon},\theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_n^\varepsilon(x) + r_n^\varepsilon(x), \tag{116}$$

where

$$\lim_{\varepsilon\to 0}\|r_n^\varepsilon\|_{L^2(\Omega)} = 0, \quad \|v_n^\varepsilon\|_{H^2(\Omega)} \leqq C, \quad \lim_{\varepsilon\to 0}\|v_n^\varepsilon\|_{L^2(\Omega)} = 1, \tag{117}$$

and the family v_n^ε is relatively compact in $L^2(\Omega)$. Moreover, any limit point v_n^0, as $\varepsilon \to 0$, of the sequence v_n^ε is a normalized eigenfunction associated with the n^{th} eigenvalue μ_n of the scalar homogenized spectral problem

$$\mathcal{A}^* v = \mu v \qquad in\ \Omega, \qquad v \in H_0^2(\Omega),$$

with $\mathcal{A}^*$ defined by (115). *If μ_n is a simple eigenvalue of the latter problem, the entire sequence v_n^ε converges to the homogenized eigenfunction v_n.*

The proof is a combination of those of Theorems 3 and 8. The crucial point is to obtain a uniform estimate for the energy $(\mathcal{A}^\varepsilon u_n^\varepsilon, u_n^\varepsilon)$. To this end we use a test function of the type (109).

Finally, for the hyperbolic system

$$\begin{aligned} \frac{\partial^2 u_\varepsilon}{\partial t^2} + \mathcal{A}^\varepsilon u_\varepsilon &= 0 && \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0, \quad \frac{\partial u_\varepsilon}{\partial n} &= 0 && \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(0,x) &= u_\varepsilon^0(x) && \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial t}(0,x) &= u_\varepsilon^1(x) && \text{in } \Omega, \end{aligned} \tag{118}$$

we obtain different homogenized limits according to the sign of $\lambda_1(\theta_0)$.

Theorem 11. *Let* (98) *be fulfilled, and assume that* $\lambda_1(\theta_0) = 0$ *and the initial data are*

$$\begin{aligned} u_\varepsilon^0(x) &= \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} v^0(x) \in H_0^2(\Omega), \\ u_\varepsilon^1(x) &= \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} v^1(x) \in L^2(\Omega), \end{aligned}$$

with $v^0 \in H_0^2(\Omega)$ *and* $v^1 \in L^2(\Omega)$. *The solution of* (118), *with* $\mathcal{A}^\varepsilon$ *given by* (113), *can be written as*

$$u_\varepsilon(t,x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi \frac{\theta_0 \cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x),$$

where the remainder term r_ε *satisfies the relation*

$$\lim_{\varepsilon \to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)} = 0 \textit{ for any compact set } \omega \subset \mathbb{R}^N,$$

and v_ε *converges weakly in* $L^2\left((0,T); H^2(\Omega)\right)$ *to the solution* v *of*

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} + \mathcal{A}^* v &= 0 && \textit{in } \Omega \times (0,T), \\ v = 0, \quad \frac{\partial v}{\partial n} &= 0 && \textit{on } \partial\Omega \times (0,T), \\ v(t=0,x) &= v^0(x) && \textit{in } \Omega, \\ \tfrac{\partial v}{\partial t}(t=0,x) &= v^1(x) && \textit{in } \Omega, \end{aligned}$$

with $\mathcal{A}^*$ *defined by* (115).

The proof is the same as that of Theorem 4. If $\lambda_1(\theta_0) \neq 0$, then we need to look at a different time scaling. Instead of (118), we now consider

$$\begin{aligned} \varepsilon^4 \frac{\partial^2 u_\varepsilon}{\partial t^2} + \mathcal{A}^\varepsilon u_\varepsilon &= 0 && \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0, \quad \frac{\partial u_\varepsilon}{\partial n} &= 0 && \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(0,x) &= u_\varepsilon^0(x) && \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial t}(0,x) &= u_\varepsilon^1(x) && \text{in } \Omega, \end{aligned} \tag{119}$$

Theorem 12. *Let* (98) *be fulfilled, and assume that the initial data are*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v^0(x) \in H_0^2(\Omega),$$

with $v^0 \in H_0^2(\Omega)$*, and that* $\varepsilon^4 u_\varepsilon^1(x)$ *is bounded in* $L^2(\Omega)$ *while* $\varepsilon^4\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) u_\varepsilon^1(x)$ *converges weakly to* 0 *in* $L^2(\Omega)$*.*

If $\lambda_1(\theta_0) < 0$*, the solution of* (119) *can be written as*

$$u_\varepsilon(t,x) = e^{\frac{\sqrt{-\lambda_1(\theta_0)}t}{\varepsilon^4}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x) + r_\varepsilon(t,x)\right),$$

where $\lim_{\varepsilon\to 0}\|r_\varepsilon\|_{L^2((0,T)\times\omega)} = 0$ *for any compact set* $\omega \subset \mathbb{R}^N$*, and* v_ε *converges weakly in* $L^2\left((0,T); H^2(\Omega)\right)$ *to the solution* v *of*

$$\begin{aligned} 2\sqrt{-\lambda_1(\theta_0)}\frac{\partial v}{\partial t} + \mathcal{A}^* v = 0 \qquad & \text{in } \Omega\times(0,T),\\ v = 0, \quad \frac{\partial v}{\partial n} = 0 \qquad & \text{on } \partial\Omega\times(0,T),\\ v(t=0,x) = \tfrac{1}{2}v^0(x) \quad & \text{in } \Omega. \end{aligned}$$

If $\lambda_1(\theta_0) > 0$ *the solution of* (119) *satisfies*

$$u_\varepsilon(t,x) = e^{i\frac{\sqrt{\lambda_1(\theta_0)}t}{\varepsilon^4}} e^{2i\pi\frac{\theta_0\cdot x}{\varepsilon}} v_\varepsilon(t,x),$$

where v_ε *two-scale converges to* $\psi_1(y,\theta_0)v(t,x)$ *and* $v \in L^2\left((0,T); H_0^2(\Omega)\right)$ *is the solution of*

$$\begin{aligned} 2i\sqrt{\lambda_1(\theta_0)}\frac{\partial v}{\partial t} + \mathcal{A}^* v = 0 \qquad & \text{in } \Omega\times(0,T),\\ v = 0, \quad \frac{\partial v}{\partial n} = 0 \qquad & \text{on } \partial\Omega\times(0,T),\\ v(t=0,x) = \tfrac{1}{2}v^0(x) \quad & \text{in } \Omega, \end{aligned}$$

with $\mathcal{A}^*$ *defined by* (115).

Again the proof is similar to those of Theorems 5 and 6.

Assumption (98) is not the only possible one. In particular, it may happen that $\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$ does not vanish at the minimum point θ_0. Therefore, we now make assumption (9), i.e., $\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$ is positive definite instead of (98).

Remark 19. We give an explicit example where (9) is satisfied rather than (98). Consider an arbitrary periodic, symmetric, uniformly elliptic operator $\mathcal{B}$ of the form $\mathcal{B} = -\mathrm{div}_y(B(y)\nabla_y) + c(y)$ and its Bloch spectrum $\mu_1(\theta) \leqq \mu_2(\theta) \leqq \cdots$. Adding, if necessary, a sufficiently large positive constant to c, we can assume that $\mu_1(\theta) \geqq C > 0$. Considering the relation

$$\left(e^{-2i\pi y\cdot\theta}\mathcal{B}^2 e^{2i\pi y\cdot\theta}\right)\psi = \left(e^{-2i\pi y\cdot\theta}\mathcal{B}e^{2i\pi y\cdot\theta}\right)\left(e^{-2i\pi y\cdot\theta}\mathcal{B}e^{2i\pi y\cdot\theta}\right)\psi$$

we conclude that the Bloch spectrum of the operator $\mathcal{A} = \mathcal{B}^2$ is $\left(\lambda_n(\theta) = \mu_n^2(\theta)\right)_{n\geq 1}$. According to Remark 1, the unique minimum point of μ_1 is attained at $\theta_0 = 0$ and the matrix $\nabla_\theta \nabla_\theta \mu_1(0)$ is positive definite. Since $\mu_1(\theta)$ is strictly positive, the function $\lambda_1(\theta) = \mu_1^2(\theta)$ also has a unique minimum point at $\theta_0 = 0$ and its Hessian at 0 is positive definite.

Under assumption (9) we need to change the scaling of (113) and consider instead the new operator

$$\mathcal{A}^\varepsilon = \varepsilon^2 \operatorname{div}\operatorname{div}\left(\Theta\left(\frac{x}{\varepsilon}\right)\nabla\nabla\right) - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla\right) + \frac{1}{\varepsilon^2}c\left(\frac{x}{\varepsilon}\right) + d\left(x, \frac{x}{\varepsilon}\right). \tag{120}$$

Then, the homogenization of the parabolic equation is given by a result similar to Theorem 2.

Theorem 13. *Assume* (9). *Let $u_\varepsilon(t, x)$ be a solution of the parabolic equation* (114) *with $\mathcal{A}^\varepsilon$ given by* (120)*, and $u_\varepsilon^0 \in L^2(\Omega)$ be initial data of the form*

$$u_\varepsilon^0(x) = \psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2\pi i \frac{\theta_0 \cdot x}{\varepsilon}} v^0(x),$$

with $v^0 \in L^2(\Omega)$. Then u_ε can be written as

$$u_\varepsilon(t, x) = e^{-\frac{\lambda_1(\theta_0)t}{\varepsilon^2}}\left(\psi_1\left(\frac{x}{\varepsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\varepsilon}} v_\varepsilon(t, x) + r_\varepsilon(t, x)\right),$$

where the remainder term r_ε satisfies

$$\lim_{\varepsilon\to 0} \|r_\varepsilon\|_{L^2((0,T)\times\omega)} = 0$$

on any compact set $\omega \subset \mathbb{R}^N$, and v_ε converges weakly in $L^2\left((0, T); H^1(\Omega)\right)$ to the solution v of the scalar homogenized problem

$$\begin{aligned} \frac{\partial v}{\partial t} - \operatorname{div}\left(A^*\nabla v\right) + d^*(x)\, v &= 0 && \text{in } \Omega \times (0, T),\\ v &= 0 && \text{on } \partial\Omega \times (0, T),\\ v(0, x) &= v^0(x) && \text{in } \Omega, \end{aligned}$$

with $A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\lambda_1(\theta_0)$ and $d^*(x) = \int_{\mathbb{T}^N} d(x, y)|\psi_1(y, \theta_0)|^2 dy$.

The proof of Theorem 13 relies on the same test function as that in the proof of Theorem 2. It should be noted that although $u^\varepsilon(t, x)$ belongs to $L^2\left((0, T); H_0^2(\Omega)\right)$, the sequence v^ε, defined in Theorem 13, is only bounded in $L^2\left((0, T); H^1(\mathbb{R}^N)\right)$, uniformly with respect to ε. This is due to assumption (9) which allows us to prove Lemma 3 but not Lemma 10.

Of course, similar results can be obtained for the spectral problem and for the hyperbolic equation: in both cases the homogenized operator is of second order in space as in Theorem 13.

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