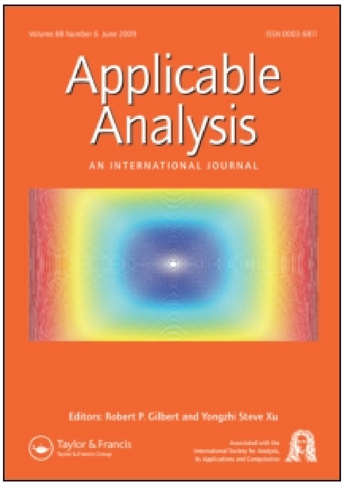

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Homogenization of Boundary-Value Problem in a Locally Periodic Perforated Domain

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Abstract

We consider a model homogenization problem for the Poisson equation in a locally periodic perforated domain with the smooth exterior boundary, the Fourier boundary condition being posed on the boundary of the holes. In the paper we construct the leading terms of formal asymptotic expansion. Then, we justify the asymptotics obtained and estimate the residual.

AMS: 35B27, 35B40

KEY WORDS: Homogenization, locally periodic perforation.

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In memory of Professor
Landis Evgeni Mikhailovich

Introduction.

Recent years many mathematical works were devoted to the asymptotic analysis of problems in perforated domains. Various homogenization results have been achieved for periodic, almost periodic and random structures. We mention here the general frameworks [15], [16], [21], [24], [27], [28], where the detail bibliography can be found.

In the paper we consider perforated media with locally periodic microstructure in the presence of a small dissipation at the boundary of the cavities. Corresponding mathematical description involves the Fourier boundary condition with a small parameter ε^α, characterizing the dissipation. The effective characteristics of the media depend essentially on the value of α. Earlier, similar problems for purely periodic structures were investigated in the works [6], [10], [11], [12], where the general convergence results were obtained for various values of α; namely, the case $-1 < \alpha < 1$ was considered in [10], the case $\alpha \leq -1$ in [11] and the case $\alpha \geq 1$ in [12]. The Stokes and Navier-Stokes systems in perforated domains were studied in [13]. Also there is an interesting work [14], devoted to the problem in domains with "small" cavities. It should be noted that the case of Neumann homogeneous boundary conditions were primely studied in [9], [22], [30].

When studying a locally periodic perforation, we encounter an additional difficulty: the fact that the geometry of the cavities is not fixed. One can apply the compensated compactness method [23] or the two-scale convergence method [1] to obtain the limit problem, but these methods do not allow to estimate the residual. Previously, locally periodic perforated structures have been studied in [20], [8], where, by means of the two-scale convergence method, the homogenized problem has been constructed and the weak convergence of solutions has been proved. In [26] another approach was used for study the problems in perforated domains with an arbitrary density of cavities. In the present paper we use the asymptotic expansion technique [2], [3] that requires the regularity of data but gives the estimates of the rate of convergence.

In the section 1 we introduce necessary notation, construct the family of domains, depending on a small positive parameter ε and pose the problem to be studied.

The sections 2 and 5 deal with a formal interior asymptotic expansion of the solution for $\alpha = 1$ and $\alpha > 1$, respectively.

The technical results obtained in the section 3, allow to justify the asymptotic expansion and to estimate the discrepancy.

Theorem 1 proved in the section 4, states that for $\alpha = 1$ two terms of the interior asymptotic expansion provide the precision of order $\sqrt{\varepsilon}$ in H^1-norm.

Theorem 2 proved in section 6, states that for $\alpha > 1$ two terms of the interior asymptotic expansion provide the precision of order $\max(\sqrt{\varepsilon}, \varepsilon^{\alpha-1})$ in H^1-norm.

Theorem 3 from the last section states that in the case $\alpha < 1$ the uniform estimate of solution is of order $\max(\sqrt{\varepsilon}, \varepsilon^{1-\alpha})$ in H^1-norm.

1 Statement of the problem.

First we define a perforated domain. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a smooth bounded domain. Denote

$$J^\varepsilon = \{j \in \mathbb{Z}^d : \text{dist}(\varepsilon j, \partial\Omega) \geq \varepsilon\sqrt{d}\}, \quad \square \equiv \{\xi : -\frac{1}{2} < \xi_j < \frac{1}{2}, \ j = 1, \dots, d\}.$$

Given an 1-periodic in ξ smooth function $F(x,\xi)$ such that $F(x,\xi)\big|_{\xi\in\partial\square} \geq \text{const} > 0$, $F(x,0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \square\setminus\{0\}$, we set

$$Q_j^\varepsilon = \{x \in \varepsilon(\square + j) \mid F(x, \frac{x}{\varepsilon}) \leq 0\}$$

and introduce the perforated domain as follows:

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{j\in J^\varepsilon} Q_j^\varepsilon.$$

We also use the following notation $\Omega_1^\varepsilon = \Omega\setminus \bigcup_{j\in J^\varepsilon} (\varepsilon(\square + j))$. Afterwards, we will often interprete 1-periodic in ξ functions as functions defined on d-dimensional torus $T^d \equiv \{\xi : \xi \in \mathbb{R}^d/\mathbb{Z}^d\}$.

According to the above construction the boundary $\partial\Omega^\varepsilon$ consists of $\partial\Omega$ and the boundary of the cavities $S_\varepsilon \subset \Omega$, $S_\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$.

We investigate the asymptotic behavior of solution $u_\varepsilon(x)$ as $\varepsilon \to 0$ of the following boundary-value problem in the domain Ω^ε :

$$\begin{cases} -\Delta u_\varepsilon = f(x) & \text{in} \quad \Omega^\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \\ \dfrac{\partial u_\varepsilon}{\partial n_\varepsilon} + \varepsilon^\alpha q\left(x, \dfrac{x}{\varepsilon}\right) u_\varepsilon = 0 & \text{on} \quad S_\varepsilon, \end{cases} \tag{1}$$

where n_ε is the internal normal to the boundary of "holes", $q(x,\xi)$ is a sufficiently smooth 1-periodic in ξ function.

Definition 1 *Function $u_\varepsilon \in H^1(\Omega^\varepsilon, \partial\Omega)$ is a solution of problem (1), if the following integral identity*

$$\int_{\Omega^\varepsilon} \nabla u_\varepsilon(x)\,\nabla v(x)\,dx + \varepsilon^\alpha \int_{S_\varepsilon} q\left(x, \frac{x}{\varepsilon}\right) u_\varepsilon(x) v(x)\,ds = \int_{\Omega^\varepsilon} f(x)\,v(x)\,dx \tag{2}$$

holds true for any function $v \in H^1(\Omega^\varepsilon, \partial\Omega)$.

Here we use the standard notation $H^1(\Omega^\varepsilon, \partial\Omega)$ for the closure of the set of $C^\infty(\overline{\Omega}^\varepsilon)$-functions vanishing in a neighborhood of $\partial\Omega$, by the $H^1(\Omega^\varepsilon)$ norm.

In what follows we show that $\alpha = 1$ is a critical value for problem (1); the dissipation dominates if $\alpha < 1$ and is neglectable if $\alpha > 1$.

Part I
The case $\alpha = 1$.

2 The formal homogenization procedure.

In this section we construct the leading "locally periodic" terms of the formal asymptotic expansion and, then, find the limit problem. To this end we represent the solution $u_\varepsilon(x)$ to problem (1) in the form of asymptotic series

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \dots \tag{3}$$

Substituting expression (3) in equation (1) and taking into account an evident relation

$$\frac{\partial}{\partial x}\zeta(x, \frac{x}{\varepsilon}) = \left(\frac{\partial}{\partial x}\zeta(x,\xi) + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi}\zeta(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}, \tag{4}$$

we obtain after simple transformations the formal equality

$$\begin{aligned}
-f(x) = \Delta_x u_\varepsilon(x) \cong \Delta_x u_0(x) + \varepsilon\left(\Delta_x u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\left(\nabla_x, \nabla_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \\
+\frac{1}{\varepsilon}\left(\Delta_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^2\left(\Delta_x u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\varepsilon\left(\nabla_x, \nabla_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \\
+\left(\Delta_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^3\left(\Delta_x u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \\
+2\varepsilon^2\left(\nabla_x, \nabla_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon\left(\Delta_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \dots
\end{aligned} \tag{5}$$

Similarily, substituting (3) in the boundary conditions in (1), we get the relation

$$\begin{aligned}
0 = \frac{\partial u_\varepsilon}{\partial n_\varepsilon} + \varepsilon q\left(x, \frac{x}{\varepsilon}\right)u_\varepsilon \cong (\nabla_x u_0, n_\varepsilon) + \varepsilon q\left(x, \frac{x}{\varepsilon}\right)u_0 + \varepsilon(\nabla_x u_1, n_\varepsilon) + \\
+\left(\nabla_\xi u_1|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) + \varepsilon^2 q\left(x, \frac{x}{\varepsilon}\right)u_1 + \varepsilon^2(\nabla_x u_2, n_\varepsilon) + \varepsilon\left(\nabla_\xi u_2|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) + \\
+\varepsilon^3 q\left(x, \frac{x}{\varepsilon}\right)u_2 + \varepsilon^3(\nabla_x u_3, n_\varepsilon) + \varepsilon^2\left(\nabla_\xi u_3|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) + \varepsilon^4 q\left(x, \frac{x}{\varepsilon}\right)u_3 + \dots
\end{aligned} \tag{6}$$

to be satisfied on S_ε.

Note that the normal vector n_ε depends on x and $\frac{x}{\varepsilon}$ in Ω^ε. Considering, as usually, x and $\xi = \frac{x}{\varepsilon}$ as independent variables, we represent n_ε in Ω^ε in the following form:

$$n_\varepsilon(x, \frac{x}{\varepsilon}) = \tilde{n}(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon n'_\varepsilon(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}, \tag{7}$$

where $\tilde{n}$ is a normal to $S(x) = \{\xi \mid F(x,\xi) = 0\}$,

$$n'_\varepsilon = n' + O(\varepsilon).$$

It is obvious, that

$$n_\varepsilon(x,\frac{x}{\varepsilon}) \equiv \frac{\nabla_x F(x,\frac{x}{\varepsilon})}{|\nabla_x F(x,\frac{x}{\varepsilon})|} = \left(\frac{\nabla_\xi F(x,\xi)}{|\nabla_\xi F(x,\xi)|} + \varepsilon\frac{\nabla_x F(x,\xi)}{|\nabla_\xi F(x,\xi)|} - \right.$$

$$\left. - \varepsilon\nabla_\xi F(x,\xi)\frac{(\nabla_x F(x,\xi),\nabla_\xi F(x,\xi))}{|\nabla_\xi F(x,\xi)|^3} + O(\varepsilon^2)\right)\Bigg|_{\xi=\frac{x}{\varepsilon}}.$$

Consequently, $\tilde{n}(x,\xi) = \dfrac{\nabla_\xi F(x,\xi)}{|\nabla_\xi F(x,\xi)|}$,

$$n'(x,\xi) = \frac{\nabla_x F(x,\xi)}{|\nabla_\xi F(x,\xi)|} - \nabla_\xi F(x,\xi)\frac{(\nabla_x F(x,\xi),\nabla_\xi F(x,\xi))}{|\nabla_\xi F(x,\xi)|^3}.$$

Collecting all the terms of order ε^{-1} in (5) and of order ε^0 in (6), we obtain the following auxiliary problem (see Fig.1):

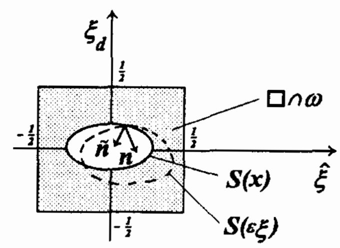

Figure 1: Cell of periodicity

$$\begin{cases} \Delta_\xi u_1(x,\xi) = 0 \quad \text{in} \quad \omega, \\ \dfrac{\partial u_1(x,\xi)}{\partial\tilde{n}} = -(\nabla_x(u_0(x)),\tilde{n}) \quad \text{on} \quad S(x), \end{cases} \tag{8}$$

to be solved in the space of 1-periodic in ξ functions; here x is a parameter, $\omega := \{\xi \in T^d \,|\, F(x,\xi) > 0\}$. This is the standard "cell" problem appearing in case of Neumann conditions on the boundary of holes. The solvability condition

$$\int_S (\nabla_x u_0(x),\tilde{n}(\xi))\ d\sigma = 0$$

for problem (8) is clearly satisfied, and its solution forms the first "internal" corrector in (3).

At the next step we collect all the terms of order ε^0 in (5) and of order ε^1 in (6). This yields

$$\begin{cases} \Delta_\xi u_2(x,\xi) = -f(x) - \Delta_x u_0(x) - 2(\nabla_\xi,\nabla_x u_1(x,\xi)) \text{ in } \omega, \\ \dfrac{\partial u_2(x,\xi)}{\partial\tilde{n}} = -(\nabla_x u_1(x,\xi),\tilde{n}) - (\nabla_\xi u_1(x,\xi),n') - \\ \qquad -(\nabla_x u_0(x),n') - q(x,\xi)u_0(x) \quad \text{on} \quad S(x). \end{cases} \tag{9}$$

The 1-periodic in ξ solution of the latter problem is the second term of the internal asymptotic expansion of $u_\varepsilon(x)$.

It is natural to represent the solution $u_1(x,\xi)$ of problem (8) in the form:

$$u_1(x,\xi) = (\nabla_x u_0(x), M(x,\xi)), \tag{10}$$

where 1-periodic vector–function $M(x,\xi) = (M_1(x,\xi),\dots,M_d(x,\xi))$ solves the problem

$$\begin{cases} \Delta_\xi M_i(x,\xi) = 0 & \text{in} \quad \omega, \\ \dfrac{\partial M_i(x,\xi)}{\partial \tilde n} = -\tilde n_i & \text{on} \quad S(x). \end{cases} \tag{11}$$

Now, (9) can be rewritten as follows

$$\begin{cases} \Delta_\xi u_2(x,\xi) = -f(x) - \Delta_x u_0(x) - 2\displaystyle\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j} - \\ \qquad -2\displaystyle\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial^2 M_i(x,\xi)}{\partial \xi_j \partial x_j} \quad \text{in} \quad \omega, \\ \dfrac{\partial u_2(x,\xi)}{\partial \tilde n} = -\displaystyle\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} M_i(x,\xi)\tilde n_j - \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial x_j}\,\tilde n_j - \\ \qquad -q(x,\xi)u_0(x) - \displaystyle\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j}\,n'_j - \\ \qquad -\displaystyle\sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\,n'_i \quad \text{on} \quad S(x). \end{cases} \tag{12}$$

Writing down the compatibility condition in the last problem, we get the following equation:

$$\begin{aligned} &\int_{\square\cap\omega} \Bigg(f(x) + \Delta_x u_0(x) + 2\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j} + \\ &+2\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial^2 M_i(x,\xi)}{\partial \xi_j \partial x_j}\Bigg) d\xi = \int_S \Bigg(\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} M_i(x,\xi)\tilde n_j + \\ &+\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial x_j}\,\tilde n_j + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j}\,n'_j + \\ &+\sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\,n'_i + q(x,\xi)u_0(x)\Bigg) d\sigma. \end{aligned} \tag{13}$$

¿From (13) by the Stokes formula we derive the equation

$$\begin{aligned} &|\square\cap\omega|\Delta_x u_0(x) + \sum_{i,j=1}^d \left\langle \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} + \\ &+\sum_{i,j=1}^d \left\langle \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} + |\square\cap\omega| f(x) = Q(x)u_0(x) + \sum_{i=1}^d U_i(x)\frac{\partial u_0(x)}{\partial x_i}, \end{aligned} \tag{14}$$

to be the limit equation in Ω. Here $< \cdot >$ means the integral over the set $\square \cap \omega$, $Q(x) = \int\limits_S q(x,\xi)\, d\sigma$, and $U_i(x) = \int\limits_S \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} n'_j + n'_i\right) d\sigma$.

Let us study in detail the functions $U_i(x)$. Fortunatily, it is not necessary to calculate $U_i(x)$. Instead, taking into account the selfadjointness of the operators of the initial problems and the convergence of the corresponding belinear forms, we obtain that the G-limit operator is necessary selfadjoint. Hence, the limit equation (14) takes the form:

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) + |\square \cap \omega| f(x) = Q(x) u_0(x) \qquad (15)$$

and, consequently,

$$U_i(x) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left\langle \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle - \sum_{j=1}^{d}\left\langle \frac{\partial^2 M_i(x,\xi)}{\partial x_j \partial \xi_j}\right\rangle. \qquad (16)$$

Clearly $\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle$ is a smooth matrix, moreover, arguing like in [24] one can verify that this matrix is positively defined.

So, we find the homogenized problem:

$$\left\{\begin{array}{rcl} \sum\limits_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) - & & \\ -Q(x) u_0(x) & = & -|\square \cap \omega| f(x) \ \text{ in } \Omega, \\ u_0(x) & = & 0 \quad \text{on} \quad \partial\Omega. \end{array}\right. \qquad (17)$$

The integral identity for problem (17) takes the form:

$$\int\limits_\Omega \left(\sum_{i,j=1}^{d}\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + Q(x) u_0(x) v(x)\right) dx = \qquad (18)$$

$$= \int\limits_\Omega |\square \cap \omega|\, f(x)\, v(x)\, dx$$

for any function $v \in \overset{\circ}{H^1}(\Omega)$.

Remark 1 *It should be noted that $M_i(x, \frac{x}{\varepsilon})$ are not defined in the whole Ω. Applying the technique of the symmetric extension [19] allows to extend $M(x,\xi)$ into the interior of the "holes" retaining the regularity of these functions. We keep the same notation for the extended functions.*

The limit behavior of the solution of problem (1) is described by the following statement.

Theorem 1 *Suppose that $f(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small ε problem (1) has the unique solution and the following estimate*

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le K_1\sqrt{\varepsilon} \tag{19}$$

takes place, where u_0 and u_1 are solutions of problems (17) and (8) respectively, and K_1 does not depend on ε.

Remark 2 *In fact, in the formulation of Theorem 1 the condition $q(x,\xi) \ge 0$ can be replaced by the weaker condition $Q(x) \ge 0$.*

3 Preliminary lemmas.

This section is devoted to various technical assertions, which will be used in the further analysis. Some of these assertions have been proved in [5], [7] (see also [4]). We omit their proofs.

Lemma 1 *Under the conditions of Theorem 1 the Friederichs type inequality*

$$\int\limits_{\Omega^\varepsilon} |\nabla v|^2 dx + \varepsilon \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon}) v^2 ds \ge C_1 \|v\|^2_{H^1(\Omega^\varepsilon, \partial\Omega)}$$

holds for any $v \in H^1(\Omega^\varepsilon, \partial\Omega)$, where C_1 does not depend on ε.

The next assertion is, in fact, a modified version of Lemma 5 from [7].

Lemma 2 *If*

$$\frac{1}{|\square \cap \omega|} \int\limits_{\square\cap\omega} Q(x)\ d\xi - \int\limits_S q(x,\xi)\ d\sigma \equiv 0, \tag{20}$$

then the following inequality

$$\left| \frac{1}{|\square \cap \omega|} \int\limits_{\Omega^\varepsilon} Q(x)\, v(x)\ dx - \varepsilon \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon})\, v(x)\ ds \right| \le C_3 \varepsilon \|v\|_{H^1(\Omega^\varepsilon)} \tag{21}$$

holds for any $v(x) \in H^1(\Omega^\varepsilon, \partial\Omega)$; the constant C_3 does not depend on ε.

Proof. By (20) the problem

$$\begin{cases} \Delta_\xi \Psi(x,\xi) = \frac{1}{|\square\cap\omega|} Q(x) & \text{in} \quad \omega, \\ \dfrac{\partial \Psi}{\partial n} = q(x,\xi) & \text{on} \quad S \end{cases} \tag{22}$$

has 1-periodic in ξ solution. Moreover, the solution is unique up to an additive constant. Let us multiply the equation (22) by the function $v(x) \in H^1(\Omega^\varepsilon, \partial\Omega)$ and integrate it over the domain Ω^ε. Integrating by parts the left-hand side of the obtained formula gives

$$\left|\frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon} Q(x)\,v(x)\,dx - \varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon})\,v(x)\,ds\right| = \left|\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon} \Delta_\xi\Psi(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx -\right.$$

$$\left. -\varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon})\,v(x)\,ds + \frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon} Q(x)v(x)dx\right| =$$

$$= \left|\varepsilon\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}\left(\nabla_x\left[\nabla_\xi\Psi(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}\right] - ((\nabla_x,\nabla_\xi)\Psi(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}\right) v(x)\,dx -\right.$$

$$\left. -\varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon})\,v(x)\,ds + \frac{1}{|\square\cap\omega|}\int_{\Omega_1^\varepsilon} Q(x)v(x)dx\right| \le \tag{23}$$

$$\le \varepsilon\left|\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}\left((\nabla_\xi\Psi(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}, \nabla_x v(x)\right) dx\right| +$$

$$+\varepsilon\left|\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}((\nabla_x,\nabla_\xi)\Psi(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx\right| +$$

$$+\varepsilon\left|\int_{\partial(\Omega\setminus\overline{\Omega_1^\varepsilon})}(\nabla_\xi\Psi,\mathcal{N})v\,ds\right| + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon} +$$

$$+\varepsilon^2\left|\int_{S_\varepsilon}\left((\nabla_\xi\Psi(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}, n'_\varepsilon\right) v(x)\,ds\right| \le C_3\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

Here $\mathcal{N}$ is the unit normal to $\partial(\Omega\setminus\overline{\Omega_1^\varepsilon})$. The lemma is proved.

The following lemma allows to neglect the right-hand side of the equation (1) in the thin layer Ω_1^ε without deterioration of the estimate. Proof of this lemma is similar to the proof of Lemma 8 from [5].

Lemma 3 *Suppose that y_ε is the solution of the problem*

$$\begin{cases} -\Delta y_\varepsilon = h^\varepsilon(x) & in \quad \Omega^\varepsilon, \\ y_\varepsilon = 0 \quad on \quad \Omega, \\ \frac{\partial y_\varepsilon}{\partial n_\varepsilon} + \varepsilon q(x,\frac{x}{\varepsilon})y_\varepsilon = 0 & on \quad S_\varepsilon, \end{cases} \tag{24}$$

where $h^\varepsilon(x) = f(x)$ for $x\in\Omega_1^\varepsilon$ and 0 otherwise. Then

$$\|y_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_4\varepsilon. \tag{25}$$

The following assertion can be proved in the same way as Lemma 5 from [7].

Lemma 4 *Suppose $w^\varepsilon(x) \in L_\infty(\Omega)$, and let Π^ε belong to $\{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) \le C_0\varepsilon\}$. Then the following inequality*

$$\left| \int_{\Pi^\varepsilon} w^\varepsilon(x)\Big|_{\xi=\frac{x}{\varepsilon}} \nabla_x u_0(x)\, v(x)\, dx \right| \le C_5 \varepsilon^{\frac{3}{2}} \|w\|_{L_\infty(\Omega)} \|v\|_{H^1(\Omega^\varepsilon)} \tag{26}$$

holds for any $v(x) \in H^1(\Omega^\varepsilon, \partial\Omega)$; the constant C_5 does not depend on ε.

4 The basic estimate.

Proof of Theorem 1. We are going to estimate the H^1-norm of the residual:

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

To this end we extend the functions $M_i(x, \xi)$ in the layer Ω_1^ε (see Remark 1 above) and substitute the expression

$$z_\varepsilon(x, \frac{x}{\varepsilon}) = u_0(x) + \varepsilon \chi^\varepsilon(\frac{x}{\varepsilon}) u_1(x, \frac{x}{\varepsilon}) - u_\varepsilon(x)$$

in the equation (1). Here we denote by $\chi^\varepsilon(\frac{x}{\varepsilon})$ a smooth cut-off function $0 \le \chi^\varepsilon(\frac{x}{\varepsilon}) \le 1$, such that $\chi^\varepsilon(\frac{x}{\varepsilon}) = 0$ if $x \in \Omega_1^\varepsilon$ and $\chi^\varepsilon(\frac{x}{\varepsilon}) = 1$ if $\mathrm{dist}(x, \Omega_1^\varepsilon) \ge \mathrm{dist}(S_\varepsilon, \Omega_1^\varepsilon)$, moreover $|\nabla_\xi \chi^\varepsilon(\xi)|$ and $|\Delta_\xi \chi^\varepsilon(\xi)|$ are uniformly bounded. This yields

$$\begin{aligned} \Delta_x \left(z_\varepsilon(x, \frac{x}{\varepsilon}) \right) &= \Delta_x u_0(x) + \varepsilon \chi^\varepsilon(\xi) \Delta_x u_1(x, \xi)\Big|_{\xi=\frac{x}{\varepsilon}} + 2 \left(\chi^\varepsilon(\xi) \nabla_x, \nabla_\xi u_1(x, \xi) \right)\Big|_{\xi=\frac{x}{\varepsilon}} + \\ &+ 2 \left(\nabla_x u_1(x, \xi), \nabla_\xi \chi^\varepsilon(\xi) \right)\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{1}{\varepsilon} (\chi^\varepsilon(\xi) \Delta_\xi u_1(x, \xi))\Big|_{\xi=\frac{x}{\varepsilon}} + \\ &+ \frac{1}{\varepsilon} (u_1(x, \xi) \Delta_\xi \chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{2}{\varepsilon} (\nabla_\xi \chi^\varepsilon(\xi), \nabla_\xi u_1(x, \xi))\Big|_{\xi=\frac{x}{\varepsilon}} - \Delta_x u_\varepsilon(x). \end{aligned} \tag{27}$$

Taking into account the relations

$$\begin{aligned} &\Delta_\xi u_1(x, \xi) = 0 \quad \forall\, x \in \Omega^\varepsilon \backslash \Omega_1^\varepsilon, \qquad \Delta_x u_\varepsilon(x) = -f(x) \quad \text{in} \quad \Omega^\varepsilon, \\ &2 \left(\nabla_x, \chi^\varepsilon(\xi) \nabla_\xi u_1(x, \xi) \right) = 2\chi^\varepsilon(\xi) \sum_{i,j=1}^d \frac{\partial M_i(x, \xi)}{\partial \xi_j} \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_j} + \\ &+ 2\chi^\varepsilon(\xi) \sum_{i,j=1}^d \frac{\partial^2 M_i(x, \xi)}{\partial x_j\, \partial \xi_j} \frac{\partial u_0(x)}{\partial x_i} \end{aligned} \tag{28}$$

and

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) - Q(x)u_0(x) = \tag{29}$$
$$= -|\square \cap \omega| f(x) \quad \text{in} \quad \Omega,$$

we can transform (27) in the domain $\Omega^\varepsilon \backslash \Omega_1^\varepsilon$ as follows:

$$\Delta_x\left(z_\varepsilon(x,\frac{x}{\varepsilon})\right) = \varepsilon\chi^\varepsilon(\xi)\Delta_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+2(\nabla_x u_1(x,\xi), \nabla_\xi \chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}} + 2\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=\frac{x}{\varepsilon}} + \Delta_x u_0(x) - \tag{30}$$
$$-\frac{1}{|\square\cap\omega|}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right) + \frac{1}{|\square\cap\omega|}Q(x)u_0(x) +$$
$$+\frac{1}{\varepsilon}(u_1(x,\xi)\Delta_\xi\chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{2}{\varepsilon}(\nabla_\xi\chi^\varepsilon(\xi), \nabla_\xi u_1(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}.$$

Similarly, on S_ε we have:

$$\frac{\partial z_\varepsilon(x,\frac{x}{\varepsilon})}{\partial n_\varepsilon} = -(\nabla_x u_\varepsilon(x), n_\varepsilon) + (\nabla_x u_0(x), n_\varepsilon) +$$
$$+\varepsilon\left(\nabla_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) + \left(\nabla_\xi u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) =$$
$$= \varepsilon q(x,\frac{x}{\varepsilon})u_\varepsilon(x) + (\nabla_x u_0(x), n_\varepsilon) + \varepsilon\left(\nabla_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) +$$
$$+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}, \tilde{n}^j(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon\left(\nabla_\xi u_1(x,\xi), n'_\varepsilon(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}.$$

At last, on $\partial\Omega$ we clearly have the trivial boundary condition:

$$z_\varepsilon(x,\frac{x}{\varepsilon}) \equiv 0.$$

Now, multiplying the equation (30) by $v(x)$ and integrating over Ω^ε, we get

$$\int_{\Omega^\varepsilon}\Delta_x\left(z_\varepsilon(x,\frac{x}{\varepsilon})\right)v(x)\,dx = \varepsilon\int_{\Omega^\varepsilon}\chi^\varepsilon(\xi)\Delta_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx +$$
$$+2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx +$$
$$+2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx + \int_{\Omega^\varepsilon}\Delta_x u_0(x)v(x)\,dx + \tag{31}$$

$$+\int_{\Omega^\varepsilon}\frac{1}{\varepsilon}(u_1(x,\xi)\Delta_\xi\chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v\,dx+\int_{\Omega^\varepsilon}\frac{2}{\varepsilon}(\nabla_\xi\chi^\varepsilon(\xi),\nabla_\xi u_1(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v\,dx-$$

$$-\frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)v(x)\,dx+$$

$$+\frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}Q(x)\,u_0(x)\,v(x)\,dx+\int_{\Omega_1^\varepsilon}f(x)v(x)\,dx+$$

$$+\int_{\Omega^\varepsilon}(\nabla_x u_1(x,\xi),\nabla_\xi\chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v\,dx.$$

On the other hand, with the help of the Green formula one can transform the left-hand side of (31) as follows

$$\int_{\Omega^\varepsilon}\Delta_x\left(z_\varepsilon(x,\frac{x}{\varepsilon})\right)v(x)\,dx=\int_{S_\varepsilon}\frac{\partial z_\varepsilon}{\partial n_\varepsilon}v(x)\,ds-\int_{\Omega^\varepsilon}\nabla z_\varepsilon\,\nabla v(x)\,dx=$$

$$=\varepsilon\int_{S_\varepsilon}q(x,\frac{x}{\varepsilon})u_\varepsilon(x)\,v(x)\,ds+\int_{S_\varepsilon}\frac{\partial u_0(x)}{\partial n_\varepsilon}v(x)\,ds+$$

$$+\varepsilon\int_{S_\varepsilon}\left(\nabla_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}},n_\varepsilon\right)v(x)\,ds+\varepsilon\int_{S_\varepsilon}(\nabla_\xi u_1(x,\xi),n'_\varepsilon(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds+$$

$$+\int_{S_\varepsilon}\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial\xi_j},\tilde n^j(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds-\int_{\Omega^\varepsilon}\nabla z_\varepsilon(x,\frac{x}{\varepsilon})\,\nabla v(x)\,dx.\qquad(32)$$

¿From (31) and (32) we derive

$$\int_{\Omega^\varepsilon}\nabla z_\varepsilon(x,\frac{x}{\varepsilon})\,\nabla v(x)\,dx=\varepsilon\int_{S_\varepsilon}q(x,\frac{x}{\varepsilon})u_\varepsilon(x)\,v(x)\,ds+\int_{S_\varepsilon}\frac{\partial u_0(x)}{\partial n_\varepsilon}v(x)\,ds+$$

$$+\varepsilon\int_{S_\varepsilon}\left(\nabla_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}},n_\varepsilon\right)v(x)\,ds+\varepsilon\int_{S_\varepsilon}(\nabla_\xi u_1(x,\xi),n'_\varepsilon(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds+$$

$$+\int_{S_\varepsilon}\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial\xi_j},\tilde n^j(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds-\qquad(33)$$

$$-\varepsilon\int_{\Omega^\varepsilon}\chi^\varepsilon(\xi)\Delta_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx--2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial M_i(x,\xi)}{\partial\xi_j}\frac{\partial^2u_0(x)}{\partial x_i\,\partial x_j}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx-$$

$$-2\int_{\Omega^\varepsilon}\sum_{i,j=1}^{d}\chi^\varepsilon(\xi)\frac{\partial^2M_i(x,\xi)}{\partial x_j\,\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx-\int_{\Omega^\varepsilon}\Delta_x u_0(x)\,v(x)\,dx-$$

$$-\int_{\Omega^\varepsilon} \frac{1}{\varepsilon}(u_1(x,\xi)\Delta_\xi \chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}} v\,dx - \int_{\Omega^\varepsilon} \frac{2}{\varepsilon}(\nabla_\xi \chi^\varepsilon(\xi), \nabla_\xi u_1(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}} v\,dx +$$

$$+\frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) v(x)\,dx -$$

$$-\frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon} Q(x)u_0(x)\,v(x)\,dx - \int_{\Omega_1^\varepsilon} f(x)v(x)\,dx.$$

In view of the evident relation

$$\operatorname{div}_\xi\left(\frac{\partial}{\partial x_j}\left(M_i(x,\xi)\frac{\partial u_0(x)}{\partial x_i}\right)\right)\Big|_{\xi=\frac{x}{\varepsilon}} = \varepsilon\operatorname{div}_x\left(\frac{\partial}{\partial x_j}\left(M_i(x,\xi)\frac{\partial u_0(x)}{\partial x_i}\right)\Big|_{\xi=\frac{x}{\varepsilon}}\right) -$$

$$-\varepsilon\operatorname{div}_x\left(\frac{\partial}{\partial x_j}\left(M_i(x,\xi)\frac{\partial u_0(x)}{\partial x_i}\right)\right)\Big|_{\xi=\frac{x}{\varepsilon}} \tag{34}$$

the Stokes formula gives

$$\int_{\Omega^\varepsilon} \sum_{i,j=1}^{d} \chi^\varepsilon(\xi)\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j} + \frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\frac{\partial u_0(x)}{\partial x_i}\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx =$$

$$= \varepsilon\int_{S_\varepsilon}\left(\nabla_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right) v(x)\,ds + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}; \tag{35}$$

here we also used the fact that all the integrals containing the derivatives of χ^ε, are of order ε. Now using (33) and the boundary condition in (17), we estimate the following expression

$$\left|\int_{\Omega^\varepsilon} \nabla z_\varepsilon(x,\frac{x}{\varepsilon})\,\nabla v(x)\,dx + \varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon}) z_\varepsilon(x,\frac{x}{\varepsilon})\,v(x)\,ds\right| \le$$

$$\le \varepsilon\left|\varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon})u_1(x,\frac{x}{\varepsilon})\,v(x)\,ds\right| +$$

$$+\left|\varepsilon\int_{S_\varepsilon} q(x,\frac{x}{\varepsilon})u_0(x)\,v(x)\,ds - \frac{1}{|\square\cap\omega|}\int_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon} Q(x)u_0(x)\,v(x)\,dx\right| +$$

$$+\left|\int_{\Omega_1^\varepsilon} \Delta_x u_0(x)\,v(x)\,dx\right| + \left|\varepsilon\int_{\Omega^\varepsilon} \chi^\varepsilon(\xi)\Delta_x u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx\right| + \tag{36}$$

$$+\left|\int_{S_\varepsilon}\left(\frac{\partial u_0(x)}{\partial n_\varepsilon} + \sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}, \tilde n^j(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}\right) v(x)\,ds\right| +$$

$$+\left|\int\limits_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}\sum_{i,j=1}^{d}\left(\frac{1}{|\Box\cap\omega|}\frac{\partial}{\partial x_j}\left[\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right]-\right.\right.$$

$$\left.-\frac{\partial}{\partial x_j}\left[\left(\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right)\frac{\partial u_0(x)}{\partial x_i}\right]\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx-$$

$$\left.-\varepsilon\int\limits_{S_\varepsilon}(\nabla_\xi u_1(x,\xi),n'_\varepsilon(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds\right|+\left|\int\limits_{\Omega_1^\varepsilon}f(x)v(x)\,dx\right|+$$

$$+\left|\int\limits_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}\sum_{i,j=1}^{d}(\chi^\varepsilon(\xi)-1)\frac{\partial}{\partial x_j}\Big(\frac{\partial M_i(x,\xi)}{\partial\xi_j}\frac{\partial u_0(x)}{\partial x_i}\Big)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx\right|+$$

$$+\left|\int\limits_{\Omega^\varepsilon}\frac{1}{\varepsilon}(u_1(x,\xi)\Delta_\xi\chi^\varepsilon(\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v\,dx+\int\limits_{\Omega^\varepsilon}\frac{2}{\varepsilon}(\nabla_\xi\chi^\varepsilon(\xi),\nabla_\xi u_1(x,\xi))\Big|_{\xi=\frac{x}{\varepsilon}}v\,dx\right|+$$

$$+O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9+O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}.$$

Let us estimate the term I_2. According to Lemma 2, we have

$$I_2=\left|\varepsilon\int\limits_{S_\varepsilon}q(x,\frac{x}{\varepsilon})u_0(x)\,v(x)\,ds-\frac{1}{|\Box\cap\omega|}\int\limits_{\Omega^\varepsilon\setminus\Omega_1^\varepsilon}Q(x)u_0(x)\,v(x)\,dx\right|\le$$

$$\le C_6\,\varepsilon\,\|u_0\|_{H^1(\Omega^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)}.$$

The terms I_1 and I_4 clearly satisfy the estimate

$$|I_1|+|I_4|\le C_7\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

The identity $I_5\equiv 0$ follows from the boundary condition of problem (8). Let us estimate the integral I_6. Considering (16) it is easy to verify that

$$\int\limits_{\Box\cap\omega}\left[\frac{1}{|\Box\cap\omega|}\frac{\partial}{\partial x_j}\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle-\frac{\partial}{\partial x_j}\left(\delta_{ij}-\frac{\partial M_i(x,\xi)}{\partial\xi_j}\right)\right]\,d\xi-$$

$$-\bar{U}_i(x)=0.$$

Applying the technique of the proof of Lemma 2, one can show that the latter relation implies the inequality

$$|I_6|\le C_8\varepsilon\left\|\frac{\partial^2u_0(x)}{\partial x_i\,\partial x_j}\right\|_{H^1(\Omega^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)};$$

here we used the C^1-smoothness of $f(x)$. By Lemma 3 one can assume that the function $f(x)$ is equal to 0 in the layer Ω_1^ε. Then $I_7=0$. The term I_3 can obviously be estimated as follows:

$$I_3\le C_9\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}.$$

Finally, due to the properties of $\chi^\varepsilon(\xi)$ one can apply Lemma 4 in order to estimate I_8 and I_9. This gives

$$|I_8| + |I_9| \le C_{11}\sqrt{\varepsilon}\|v\|_{H^1(\Omega^\varepsilon)}.$$

Substituting $v = u_0 + \varepsilon\chi^\varepsilon u_1 - u_\varepsilon$ in (36) and taking into account all the estimates above, Lemma 1 and the evident relation $\|\varepsilon u_1(1-\chi^\varepsilon)\|_{H^1(\Omega^\varepsilon)} \le C_{12}\sqrt{\varepsilon}$, we obtain (19). The theorem is proved.

Part II

The case $\alpha > 1$.

5 The formal homogenization procedure.

This section deals with problem (1) in the case $\alpha > 1$. Substituting the expression

$$u_\varepsilon(x) = u_0(x) + \varepsilon^{\alpha-1}u_{1,-1}\left(x,\frac{x}{\varepsilon}\right) + \varepsilon u_{0,1}\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^\alpha u_{1,0}\left(x,\frac{x}{\varepsilon}\right) + \tag{37}$$
$$+\varepsilon^2 u_{0,2}(x,\frac{x}{\varepsilon}) + \varepsilon^{\alpha+1}u_{1,1}(x,\frac{x}{\varepsilon}) + \ldots + \varepsilon^{k\alpha+l}u_{k,l}(x,\frac{x}{\varepsilon}) + \ldots$$

in equation (1) and taking into account an evident relation (4), we obtain after simple transformations the following formal equality

$$-f(x) = \Delta_x u_\varepsilon(x) \cong$$
$$\cong \Delta_x u_0(x) + \varepsilon^{\alpha-1}\left(\Delta_x u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\varepsilon^{\alpha-2}\left(\nabla_x, \nabla_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+\varepsilon^{\alpha-3}\left(\Delta_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon\left(\Delta_x u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\left(\nabla_x, \nabla_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+\frac{1}{\varepsilon}\left(\Delta_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^\alpha\left(\Delta_x u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\varepsilon^{\alpha-1}\left(\nabla_x, \nabla_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+\varepsilon^{\alpha-2}\left(\Delta_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^2\left(\Delta_x u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon 2\left(\nabla_x, \nabla_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+\left(\Delta_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^{\alpha+1}\left(\Delta_x u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \tag{38}$$
$$+2\varepsilon^\alpha\left(\nabla_x, \nabla_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \varepsilon^{\alpha-1}\left(\Delta_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \ldots +$$
$$+\varepsilon^{k\alpha+l}\left(\Delta_x u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + 2\varepsilon^{k\alpha+l-1}\left(\nabla_x, \nabla_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} +$$
$$+\varepsilon^{k\alpha+l-2}\left(\Delta_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \ldots$$

Similarily, on S_ε we get

$$0 = \frac{\partial u_\varepsilon}{\partial n_\varepsilon} + \varepsilon^\alpha q\left(x,\frac{x}{\varepsilon}\right)u_\varepsilon \cong \left(\nabla_x u_0, n_\varepsilon\right) + \varepsilon^\alpha q\left(x,\frac{x}{\varepsilon}\right)u_0 + \varepsilon^{\alpha-1}\left(\nabla_x u_{1,-1}, n_\varepsilon\right) +$$

$$
\begin{gathered}
+\varepsilon^{\alpha-2}\left(\nabla_\xi u_{1,-1}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+\varepsilon^{2\alpha-1}q\left(x,\frac{x}{\varepsilon}\right)u_{1,-1}+\varepsilon\left(\nabla_x u_{0,1}, n_\varepsilon\right)+\\
+\left(\nabla_\xi u_{0,1}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+\varepsilon^{\alpha+1}q\left(x,\frac{x}{\varepsilon}\right)u_{0,1}+\varepsilon^{\alpha}\left(\nabla_x u_{1,0}, n_\varepsilon\right)+\\
+\varepsilon^{\alpha-1}\left(\nabla_\xi u_{1,0}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+ \qquad (39)\\
+\varepsilon^{2\alpha}q\left(x,\frac{x}{\varepsilon}\right)u_{1,0}+\varepsilon^2\left(\nabla_x u_{0,2}, n_\varepsilon\right)+\varepsilon\left(\nabla_\xi u_{0,2}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+\varepsilon^{\alpha+2}q\left(x,\frac{x}{\varepsilon}\right)u_{0,2}+\\
+\varepsilon^{\alpha+1}\left(\nabla_x u_{1,1}, n_\varepsilon\right)+\varepsilon^{\alpha}\left(\nabla_\xi u_{1,1}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+\varepsilon^{2\alpha+1}q\left(x,\frac{x}{\varepsilon}\right)u_{1,1}+\ldots+\\
+\varepsilon^{k\alpha+l}\left(\nabla_x u_{k,l}, n_\varepsilon\right)+\varepsilon^{k\alpha+l-1}\left(\nabla_\xi u_{k,l}|_{\xi=\frac{x}{\varepsilon}}, n_\varepsilon\right)+\varepsilon^{(k+1)\alpha+l}q\left(x,\frac{x}{\varepsilon}\right)u_{k,l}+\ldots
\end{gathered}
$$

Keeping in mind (7) and collecting all the terms with like powers of ε in (38) and (39), we arrive at the following auxiliary problems:

$$
\left\{\begin{array}{lll}
\Delta_\xi u_{1,-1}(x,\xi)=0 & \text{in} & \omega,\\
\dfrac{\partial u_{1,-1}(x,\xi)}{\partial\tilde{n}}=0 & \text{on} & S(x),
\end{array}\right. \qquad (40)
$$

$$
\left\{\begin{array}{l}
\Delta_\xi u_{1,0}(x,\xi)=-2\left(\nabla_\xi,\nabla_x u_{1,-1}(x,\xi)\right)\ \text{in}\ \omega,\\
\dfrac{\partial u_{1,0}(x,\xi)}{\partial\tilde{n}}=-\left(\nabla_x u_{1,-1}(x,\xi),\tilde{n}\right)\quad\text{on}\quad S(x).
\end{array}\right. \qquad (41)
$$

and problem (8) for $u_{0,1}(x,\xi)$, to be solved in the space of 1-periodic in ξ functions.

It follows from (40) that $u_{1,-1}$ does not depend on ξ. In fact, for our purposes it suffices to put $u_{1,-1}\equiv 0$. Then $u_{1,0}\equiv 0$ solves (41).

At the next step we collect all the terms of order ε^0 in (38) and of order ε^1 in (39). This yields

$$
\left\{\begin{array}{rcl}
\Delta_\xi u_{0,2}(x,\xi) & = & -f(x)-\Delta_x u_0(x)-2\left(\nabla_\xi,\nabla_x u_{0,1}(x,\xi)\right)\ \text{in}\ \omega,\\
\dfrac{\partial u_{0,2}(x,\xi)}{\partial\tilde{n}} & = & -\left(\nabla_x u_{0,1}(x,\xi),\tilde{n}\right)-\left(\nabla_\xi u_{0,1}(x,\xi),n'\right)-\\
 & & -\left(\nabla_x u_0(x),n'\right)\quad\text{on}\quad S(x).
\end{array}\right. \qquad (42)
$$

If we represent $u_{0,1}(x,\xi)=\left(\nabla_x u_0(x),M(x,\xi)\right)$, where 1-periodic vector–function $M(x,\xi)=(M_1(x,\xi),\ldots,M_d(x,\xi))$ solves problem (11), then (42) takes the form

$$
\left\{\begin{array}{rcl}
\Delta_\xi u_{0,2}(x,\xi) & = & -f(x)-\Delta_x u_0(x)-2\displaystyle\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\,\frac{\partial M_i(x,\xi)}{\partial\xi_j}-\\
 & & -2\displaystyle\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial^2 M_i(x,\xi)}{\partial\xi_j\,\partial x_j}\qquad\text{in}\qquad\omega,\\
\dfrac{\partial u_{0,2}(x,\xi)}{\partial\tilde{n}} & = & -\displaystyle\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}M_i(x,\xi)\tilde{n}_j-\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial x_j}\,\tilde{n}_j-\\
 & & -\displaystyle\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial\xi_j}\,n'_j-\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,n'_i\ \text{on}\ S(x).
\end{array}\right. \qquad (43)
$$

Writing down the compatibility condition in the last problem, we get the following equation:

$$\int_{\square\cap\omega}\left(f(x)+\Delta_x u_0(x)+2\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\frac{\partial M_i(x,\xi)}{\partial \xi_j}+\right.$$
$$\left.+2\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\,\partial x_j}\right)d\xi=\int_S\left(\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}M_i(x,\xi)\tilde{n}_j+\right. \tag{44}$$
$$\left.+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial x_j}\tilde{n}_j+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M_i(x,\xi)}{\partial \xi_j}n'_j+\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}n'_i\right)d\sigma.$$

In the same way as in Section 2 we find the homogenized problem:

$$\begin{cases}\displaystyle\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right) = -|\square\cap\omega|f(x) \text{ in } \Omega,\\ u_0(x) = 0 \quad \text{on} \quad \partial\Omega.\end{cases} \tag{45}$$

The integral identity for problem (45) reads

$$\int_\Omega\sum_{i,j=1}^{d}\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}=\int_\Omega|\square\cap\omega|\,f(x)\,v(x)\,dx \tag{46}$$

for any function $v\in\overset{\circ}{H}{}^1(\Omega)$.

The limit behavior of the solution of problem (1) is described by the following statement.

Theorem 2 *Suppose that $f(x)\in C^1(\mathbb{R}^d)$, and let $q(x,\xi)$ be a smooth nonnegative function. Then, for any sufficiently small ε problem (1) has the unique solution and the following estimate*

$$\|u_0+\varepsilon u_{0,1}-u_\varepsilon\|_{H^1(\Omega^\varepsilon)}\le K_2\ \max(\varepsilon^{\alpha-1},\sqrt{\varepsilon}) \tag{47}$$

takes place, where u_0 and $u_{0,1}$ are solutions of problems (45) and (8) respectively, and K_2 does not depend on ε.

The proof is similar to that of Theorem 1 and relies on the following two simple assertions:

Lemma 5 *Under the conditions of Theorem 2 the inequality*

$$\int_{\Omega^\varepsilon}|\nabla v|^2dx+\varepsilon^\alpha\int_{S_\varepsilon}q(x,\frac{x}{\varepsilon})v^2ds\ge C_{13}\|v\|^2_{H^1(\Omega^\varepsilon)}$$

holds for any $v\in H^1(\Omega^\varepsilon,\partial\Omega)$.

Lemma 6 *For any $v \in H^1(\Omega^\varepsilon)$*

$$\left| \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon}) u_0(x)\, v(x)\; ds \right| \le C_{14}\, \varepsilon^{-1}\, \|u_0\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}.$$

We omit their proof.

Part III
The case $\alpha < 1$.

6 The homogenization theorem.

In the case $\alpha < 1$ the limit behavior of the solution of problem (1) is described by the following statement.

Theorem 3 *Suppose that $f(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough strictly positive function. Then, for any sufficiently small ε problem (1) has the unique solution and the following estimate*

$$\|u_\varepsilon\|_{L_2(\Omega^\varepsilon)} \le K_3\; \max(\varepsilon^{\frac{1-\alpha}{2}}, \sqrt{\varepsilon}) \tag{48}$$

takes place, K_3 being independent of ε.

Proof of Theorem 3. First let us note that Lemma 5 still holds under the conditions of Theorem 3. Writing down the integral identity for problem (1), by the Cauchy–Schwartz–Bunyakovskii inequality, we obtain the uniform boundedness of $u_\varepsilon(x)$ in $H^1(\Omega^\varepsilon)$. Indeed,

$$\|u_\varepsilon(x)\|^2_{H^1(\Omega^\varepsilon)} \le C_{15} \left| \int\limits_{\Omega^\varepsilon} |\nabla u_\varepsilon(x)|^2\, dx + \varepsilon^\alpha \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon}) u_\varepsilon^2(x)\, ds \right| =$$

$$= \left| \int\limits_{\Omega^\varepsilon} f(x) u_\varepsilon(x)\, dx \right| \le \|f(x)\|_{L_2(\Omega^\varepsilon)}\, \|u_\varepsilon(x)\|_{H^1(\Omega^\varepsilon)}. \tag{49}$$

Hence,

$$\|u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_{13}. \tag{50}$$

Let us recall the notation $Q(x) = \int\limits_S q(x,\xi)\; d\sigma$. Under the assumptions of the theorem the function $Q(x)$ is uniformly positive and the estimate holds

$$\int\limits_{\Omega^\varepsilon} u_\varepsilon^2\; dx \le \frac{C_{16}}{|\Box \cap \omega|} \int\limits_{\Omega^\varepsilon} Q(x) u_\varepsilon^2(x)\; dx = C_{16} \Big\{ \varepsilon \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon}) u_\varepsilon^2(x)\; ds +$$

$$+\frac{1}{|\square \cap \omega|}\int\limits_{\Omega^\varepsilon} Q(x)u_\varepsilon^2(x)\,dx - \varepsilon\int\limits_{S_\varepsilon} q(x,\frac{x}{\varepsilon})u_\varepsilon^2(x)\,ds\Big\} \le$$

$$\le C_{16}\Big\{\varepsilon\int\limits_{S_\varepsilon} q(x,\frac{x}{\varepsilon})u_\varepsilon^2(x)\,ds + \varepsilon\|u_\varepsilon\|^2_{H^1(\Omega^\varepsilon)}\Big\};$$

the last inequality here can be proved in the same way as Lemma 2. On the other hand, from the integral identity we have

$$\left|\varepsilon\int\limits_{S_\varepsilon} q(x,\frac{x}{\varepsilon})u_\varepsilon^2(x)\,ds\right| = \varepsilon^{1-\alpha}\left|\int\limits_{\Omega^\varepsilon} f(x)u_\varepsilon(x)\,dx - \int\limits_{\Omega^\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx\right| \le$$

$$\le \varepsilon^{1-\alpha}\|f(x)\|_{L_2(\Omega^\varepsilon)}\,\|u_\varepsilon(x)\|_{L_2(\Omega^\varepsilon)} + O(\varepsilon^{1-\alpha}).$$

Combining the preceeding estimates and keeping in mind (50), we immediatly get (48). The theorem is proved.

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