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HOMOGENIZATION OF LÉVY-TYPE OPERATORS WITH OSCILLATING COEFFICIENTS*

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Abstract. The paper deals with homogenization of Lévy-type operators with rapidly oscillating coefficients. We consider cases of periodic and random statistically homogeneous micro-structures and show that in the limit we obtain a Lévy-operator. In the periodic case we study both symmetric and non-symmetric kernels whereas in the random case we only investigate symmetric kernels. We also address a nonlinear version of this homogenization problem.

Key words. homogenization, Levy-type operator, jump process

AMS subject classifications. 45E10, 60J75, 35B27, 45M05

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1. Introduction.

The paper deals with a homogenization problem for Lévy-type operators in $L^2(\mathbb{R}^d)$. We consider operators of the form

$$L^{\varepsilon}u(x) = \int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,\Lambda^{\varepsilon}(x,y)\,dy \qquad (x \in \mathbb{R}^d)\,, \tag{1.1}$$

where $\alpha \in (0,2)$ is fixed and $\varepsilon > 0$ is a small parameter. We will study various assumptions on the function $(x,y) \mapsto \Lambda^{\varepsilon}(x,y)$. Throughout the article we assume

$$\gamma^{-1} \le \Lambda^{\varepsilon}(x,y) \le \gamma \qquad (x,y \in \mathbb{R}^d) \tag{1.2}$$

for some $\gamma > 1$, which can be seen as an ellipticity assumption. Particular cases that we cover include $\Lambda^{\varepsilon}(x,y) = \Lambda\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$, resp., $\Lambda^{\varepsilon}(x,y) = \Lambda\left(\frac{x}{\varepsilon},y\right) + \Lambda\left(x,\frac{y}{\varepsilon}\right)$, where $(\xi,\eta) \mapsto \Lambda(\xi,\eta)$ is symmetric and periodic both in ξ and in η. Note that we also deal with some classes of nonsymmetric kernels and of random symmetric kernels. Moreover, the approach allows us to treat nonlinear nonlocal operators such as the fractional p-Laplace operator.

Given $\varepsilon > 0$, we first introduce a positive self-adjoint extension of the operator $-L^{\varepsilon}$ and then study the following homogenization problem:

Find an operator L^0 such that for any $m > 0$ and for any $f \in L^2(\mathbb{R}^d)$ the solutions u^{ε} of the equations $-L^{\varepsilon}u^{\varepsilon} + mu^{\varepsilon} = f$ converge, as $\varepsilon \to 0$, to the solution of the equation $-L^0u + mu = f$.

Given $\varepsilon > 0$, the operator L^{ε} describes a jump process in a nonhomogeneous medium with a periodic microstructure. For $\Lambda^{\varepsilon} = 1$ this operator coincides, up to a multiplicative constant, with the fractional Laplacian $(-\Delta)^{\alpha/2}$, which is the infinitesimal generator of the rotationally symmetric α-stable process [34, section

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6.31]. As we show in this work, the computation of the homogenization limit for a nonlocal operator of fractional order α of differentiability is rather different from the corresponding object for differential operators. In the symmetric case, it turns out that the effective jump rate is given as a simple average, whereas this is easily seen to be false for differential operators. In the nonsymmetric case treated in Theorem 1.4, however, we face similar phenomena as in the case of local differential operators.

Let us explain our motivation for the current study. Problems related to homogenization of diffusion-type operators, i.e., partial differential operators of second order, have been studied for several decades. We do not explicitly mention the fundamental works in this direction, but we refer the interested reader to the monographs [23, 10, 5, 6, 9, 38]. One important feature in this field is that the effective diffusion constants in the limit do not appear as the mean, resp., average of the diffusion coefficients in the ε-problems. In this work we show that this is different when studying integro-differential operators of order $\alpha \in (0,2)$ as considered in (1.1). This suggests that the order of the two limits $\alpha \to 2-$ (after suitable normalization) and $\varepsilon \to 0+$ might make a difference. This observation provides one motivation for the current work.

Another motivation comes from the fact that the stochastic processes generated by operators of the form of (1.1) have recently been used to model phenomena with long-range interaction, i.e., to models with a dominating nondiffusive behavior. This applies to models in mathematical finance [13, 29, 11, 12] as well as to models in natural sciences [40, 27, 42]. Some theoretical and applied literature has suggested that predators/foragers perform Lévy flights when searching for prey. The disputes about this theory seem to have stimulated interesting research about the usage of stochastic processes in biology and ecology [41, 14, 22]. Finally, let us mention that the fractional Laplace operator, resp., the corresponding α-stable jump process appear naturally when studying generalized Dirichlet-to-Neumann mappings [28, 8]. The present work studies the main mathematical object in the aforementioned areas if the environment is not homogeneous and/or isotropic. In this case one should study operators with variable coefficients. In periodic and statistically homogeneous media this leads to homogenization problems for such operators, which form the subject of our study.

Let us formulate our main results. We consider three different settings. Note that throughout the paper we deal with bilinear forms, resp., weak solutions because the expression $L^\varepsilon u(x)$ might not exist pointwisely, even for $u \in C_0^\infty(\mathbb{R}^d)$ and Λ^ε as in the aforementioned example. Some additional regularity of Λ^ε at the diagonal $x = y$ would be needed otherwise. Finally, let us explain a possible extension of our work. We study particular nonlocal operators L^ε given in (1.1). Rather than studying operators with the kernel $|x-y|^{-d-\alpha}$, one could allow for a class of more general kernels. One would then need to adjust the function spaces. In the end, analogous proofs would lead to similar results.

Let us now present the three settings of our study.

(I) *Symmetrizable and symmetric periodic kernels:* Here we assume that Λ^ε is a positive function satisfying one of the following two conditions.

(P1) Product structure: We assume

$$\Lambda^\varepsilon(x,y) = \lambda\left(\frac{x}{\varepsilon}\right)\mu\left(\frac{y}{\varepsilon}\right) \tag{1.3}$$

with λ and μ being 1-periodic in each coordinate direction and satisfying

$$\gamma^{-1} \le \lambda(\xi) \le \gamma, \quad \gamma^{-1} \le \mu(\eta) \le \gamma. \tag{1.4}$$

(P2) Symmetric locally periodic kernels: We assume

$$\Lambda^\varepsilon(x,y) = \Lambda\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \tag{1.5}$$

with a function $\Lambda(x,y,\xi,\eta)$ that is continuous in (x,y), measurable in (ξ,η) for all (x,y), and periodic in ξ and in η with period 1 in each coordinate direction and satisfies the following conditions:

$$\left.\begin{aligned} \Lambda(x,y,\xi,\eta) = \Lambda(y,x,\eta,\xi) \\ \gamma^{-1} \le \Lambda(x,y,\xi,\eta) \le \gamma \end{aligned}\right\} \text{ for all } x,\ y,\ \xi,\ \eta \in \mathbb{R}^d .$$

In order to characterize the limit behavior of u^ε we introduce an operator

$$L^0 u(x) = \int_{\mathbb{R}^d} \Lambda^{\text{eff}}(x,y) \frac{\big(u(y)-u(x)\big)}{|y-x|^{d+\alpha}}\, dy\,, \tag{1.6}$$

where

$$\Lambda^{\text{eff}}(x,y) = \begin{cases} \left(\displaystyle\int_{[0,1]^d} \frac{\mu(\xi)}{\lambda(\xi)} d\xi \right)^{-1} \left(\displaystyle\int_{[0,1]^d} \mu(\xi) d\xi \right)^2 & \text{in case (P1)}\,, \\ \displaystyle\int_{[0,1]^d} \int_{[0,1]^d} \Lambda(x,y,\xi,\eta)\, d\xi d\eta & \text{in case (P2)}\,. \end{cases} \tag{1.7}$$

Theorem 1.1. *Assume that one of the conditions* (P1), (P2) *holds true. Let* $m > 0$. *Then for every* $f \in L^2(\mathbb{R}^d)$ *the solution* u^ε *of*

$$(L^\varepsilon - m) u^\varepsilon = f \tag{1.8}$$

converges strongly in $L^2(\mathbb{R}^d)$ *and weakly in* $H^{\alpha/2}(\mathbb{R}^d)$ *to the solution* u^0 *of*

$$(L^0 - m) u^0 = f\,. \tag{1.9}$$

Remarks.

(i) Case (P2) contains the particular case of pure periodic coefficients, which we have mentioned above. If one assumes $\Lambda^\varepsilon(x,y) = \Lambda\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ with a function $\Lambda(\xi,\eta)$ that is periodic both in ξ and in η and satisfies for all ξ, $\eta \in \mathbb{R}^d$ the conditions $\Lambda(\xi,\eta) = \Lambda(\eta,\xi)$ and $\gamma^{-1} \le \Lambda(\xi,\eta) \le \gamma$, then this case is covered by (P2).

(ii) In case (P1) the function Λ^{eff} is constant; i.e., the operator L^0 is invariant under translations.

(iii) In case (P2) we can choose $\Lambda^\varepsilon(x,y) = a(x,y)\Lambda\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ with a function $a : \mathbb{R}^d \times \mathbb{R}^d \to [a_0, a_1] \subset (0,\infty)$. In this case

$$\Lambda^{\text{eff}}(x,y) = a(x,y) \int_{[0,1]^d} \int_{[0,1]^d} \Lambda(\xi,\eta)\, d\xi d\eta\,;$$

i.e., the limit operator L^0 is a nonlocal operator with bounded and measurable coefficients.

Theorem 1.1 deals with linear nonlocal operators. The methods of its proof can be applied to nonlinear problems, too. Let us provide a nonlinear analog of Theorem 1.1. Assume that $p > 1$ and $\alpha \in (0,p)$. Given $\varepsilon > 0$, define a nonlinear version L^ε_p of L^ε by

$$L^\varepsilon_p u(x) = \int_{\mathbb{R}^d} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{d+\alpha}} \Lambda^\varepsilon(x,y)\, dy \qquad (x \in \mathbb{R}^d)\,. \tag{1.10}$$

THEOREM 1.2. *Assume that one of the conditions* (P1), (P2) *holds true. Let* $m > 0$, $p > 1$, $0 < \alpha < p$, *and* $p' = \frac{p-1}{p}$. *For any* $f \in L^{p'}(\mathbb{R}^d)$ *the solution* u^ε *of*

$$L_p^\varepsilon u - m|u|^{p-2}u = f \tag{1.11}$$

converges strongly in $L^p(\mathbb{R}^d)$ *and weakly in* $W^{\frac{\alpha}{p},p}(\mathbb{R}^d)$, *as* $\varepsilon \to 0$, *to the solution* u^0 *of* $L_p^0 u^0 - m|u^0|^{p-2}u^0 = f$, *where*

$$L_p^0 u(x) = \int_{\mathbb{R}^d} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{d+\alpha}} \Lambda^{\mathrm{eff}}(x,y)\, dy$$

and $\Lambda^{\mathrm{eff}}(x,y)$ *is as in* (1.7).

Obviously, Theorem 1.2 contains Theorem 1.1 because we could choose $p = 2$. Since the proof of Theorem 1.2 does not require any new idea, we provide the proof of Theorem 1.1 in full detail. In subsection 2.3 we explain how to derive Theorem 1.2.

(II) *Symmetric random kernels:*

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space and $(T_y)_{y\in\mathbb{R}^d}$ be a d-dimensional ergodic dynamical system in Ω; see section 3 for a detailed definition. As in the case of deterministic symmetrizable kernels we consider two different setups.

(Q1) Product structure: We assume (1.3), where $\lambda(\xi)$ and $\mu(\xi)$ are realizations of statistically homogeneous ergodic fields in $\mathbb{R}^d$. Let $\omega \mapsto \widehat{\lambda}(\omega)$ and $\omega \mapsto \widehat{\mu}(\omega)$ be random variables such that for some $\gamma > 0$ and for almost every $\omega \in \Omega$

$$\gamma^{-1} \le \widehat{\lambda}(\omega) \le \gamma, \qquad \gamma^{-1} \le \widehat{\mu}(\omega) \le \gamma\,. \tag{1.12}$$

Set

$$\lambda(\xi) = \lambda(\xi,\omega) = \widehat{\lambda}(T_\xi\omega), \qquad \mu(\xi) = \mu(\xi,\omega) = \widehat{\mu}(T_\xi\omega)\,.$$

The limit operator takes the form (1.6) with

$$\Lambda^{\mathrm{eff}} = \left(\mathbf{E}\Big\{\frac{\widehat{\mu}(\cdot)}{\widehat{\lambda}(\cdot)}\Big\}\right)^{-1} \big\{\mathbf{E}\widehat{\mu}(\cdot)\big\}^2.$$

(Q2) Symmetric random structure: Here, we additionally assume some topological structure. We assume that Ω is a metric compact space. Assume $\mathcal{F}$ is the Borel σ-algebra of Ω. We further assume that the group $T_\cdot$ is continuous, that $\Lambda = \Lambda(x,y,\omega_1,\omega_2)$ is a continuous function on $\mathbb{R}^d \times \mathbb{R}^d \times \Omega \times \Omega$, and that the following symmetry condition is fulfilled: $\Lambda(x,y,\omega_1,\omega_2) = \Lambda(y,x,\omega_2,\omega_1)$. In this case we set

$$\Lambda^{\mathrm{eff}}(x,y) = \int_\Omega \int_\Omega \Lambda(x,y,\omega_1,\omega_2) d\mathbf{P}(\omega_1) d\mathbf{P}(\omega_2)\,. \tag{1.13}$$

THEOREM 1.3. *Assume that one of the conditions* (Q1), (Q2) *holds true. Let* $m > 0$. *Almost surely for any* $f \in L^2(\mathbb{R}^d)$ *the solution* u^ε *of* (1.8) *converges strongly in* $L^2(\mathbb{R}^d)$ *and weakly in* $H^{\alpha/2}(\mathbb{R}^d)$ *to the solution* u^0 *of* (1.9).

(III) *Nonsymmetric kernels:*

One important feature of our approach is that we can allow for certain nonsymmetric kernels in (1.1). In this case we assume $0 < \alpha < 1$. We assume that Λ^ε is a positive function satisfying $\Lambda^\varepsilon(x,y) = \Lambda\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ for a function $\Lambda(\xi,\eta)$ that is periodic both in ξ and in η and satisfies the following conditions:

(i) There is $\gamma > 1$ such that $\gamma^{-1} \le \Lambda(\zeta, \eta) \le \gamma$ for all ζ and η.
(ii) $\Lambda(\zeta, \eta)$ is a Lipschitz continuous function of (ζ, η) that is

$$\big|\Lambda(\zeta_1, \eta_1) - \Lambda(\zeta_2, \eta_2)\big| \le C(|\zeta_1 - \zeta_2| + |\eta_1 - \eta_2|) \quad \text{for all } \zeta_1,\ \eta_1,\ \zeta_2,\ \eta_2 \in \mathbb{R}^d.$$

As we explain in section 4, under these conditions the map $v \mapsto Lv$ with

$$Lv(\zeta) = \int_{\mathbb{R}^d} \frac{\Lambda(\zeta, \eta)\big(v(\eta) - v(\zeta)\big)}{|\zeta - \eta|^{d+\alpha}}\, d\eta$$

defines an unbounded linear operator L in $L^2(\mathbb{T}^d)$, whose adjoint is given by

$$L^* q(\zeta) = \int_{\mathbb{R}^d} \frac{\big(\Lambda(\eta, \zeta) q(\eta) - \Lambda(\zeta, \eta) q(\zeta)\big)}{|\zeta - \eta|^{d+\alpha}}\, d\eta \qquad (q \in L^2(\mathbb{T}^d))\,.$$

THEOREM 1.4. *For any $f \in L^2(\mathbb{R}^d)$ the solution u^ε of* (1.8) *converges strongly in $L^2(\mathbb{R}^d)$ and weakly in $H^{\alpha/2}(\mathbb{R}^d)$ to the solution u^0 of* (1.9). *Here, the effective jump kernel is given by $\Lambda^{\text{eff}} = \langle p_0 \rangle^{-1} \langle \Lambda p_0 \rangle$, where p_0 is the principal eigenfunction of the operator L^* on $\mathbb{T}^d$ and $\langle \Lambda p_0 \rangle = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \Lambda(\xi, \eta) p_0(\xi)\, d\xi d\eta$.*

Let us discuss related articles that deal with homogenization problems for Lévy-type operators, resp., jump processes.

A probabilistic approach to the homogenization problem for nonlocal operators in nondivergence form is developed in [21], [39], and [20]. An approach based on PDE methods and viscosity solutions can be found in [1, 2]. The PDE method has also been extended to several classes of nonlinear problems; see [3, 36, 37]. All these approaches, like ours, deal with approximations of the same differentiability order as the limit operator, resp., limit equation. Since one can approximate diffusions through much simpler objects such as random walks or Markov chains, it is not surprising that there are also homogenization models for jump processes that generate a diffusion in the limit; see [33] or [30].

The scaling limit of a solution to stochastic differential equations with stationary coefficients driven by Poisson random measures and Brownian motions has been studied in [32]. It was shown in [32, Theorem 5.3] that, under natural integrability conditions on the Poisson random measure, the limit exhibits a diffusive behavior with respect to the measure averaged over the realizations of the medium (annealed measure). As in our approach, no corrector appears. Convergence in law of jump processes with periodic jump intensities is also studied in [19], and [17, 18] focus on homogenization of processes with variable order. Aperiodic fractional obstacle problems are studied in [16].

The recent papers [15, 7] address problems which, to a certain extent, are related to the problems that we consider in the present work. In these papers the authors focus on the problem of H-compactness of a family of uniformly elliptic nonlocal operators and describe a possible structure of any limit point of this family. Our goal is to show that for the operators with (locally) periodic and statistically homogeneous coefficients the whole family of the rescaled operators G-converges and to compute the coefficients of the effective nonlocal operators. The results of [15, 7] imply that in our case there is a nontrivial set of the limit operators with known ellipticity bounds but leave open the question of their precise shape. Furthermore, we also provide a quenched convergence result for random kernels, and we treat some nonsymmetric cases. Finally, apart from the Gamma-convergence techniques, our proofs are rather different.

In the recent preprint [35] problems are studied that are related to ours. The results are limited to the case of deterministic symmetric coefficients. The authors prove Mosco-convergence of the corresponding quadratic forms and apply this result to the convergence of the underlying stochastic processes.

The organization of the article is simple. Section 2 contains the proofs of Theorems 1.1 and 1.2. We treat the cases (P1), (P2), resp., (Q1), (Q2) in separate subsections because the product structure of the kernels allows for a very short proof. In subsection 2.3 we explain how to prove Theorem 1.2. Sections 3 and 4 contain the proofs of Theorem 1.3 and 1.4, resp.

2. Symmetric, resp., symmetrizable periodic coefficients. In this section we provide the proof of Theorem 1.1. We provide two different proofs: one for case (P1) and a separate one for case (P2). Both proofs can be adapted for the remaining case, resp., but since the effective equation has a special form under (P1) and some proofs are shorter, we decide to look at this case separately. Let us start with some general observations.

For $0 < \alpha < 2$ we consider Lévy-type operators L^ε of the form (1.1), where $\varepsilon > 0$ is a small positive parameter. Our assumptions in cases (P1) and (P2) guarantee that Λ^ε satisfies

$$\gamma^{-1} \le \Lambda^\varepsilon(x,y) \le \gamma \qquad (x, y \in \mathbb{R}^d) \tag{2.1}$$

for some $\gamma > 1$ that does not depend on ε. Condition (2.1) can be seen as an ellipticity condition. As explained below, it guarantees coercivity of the corresponding bilinear form in Sobolev spaces of fractional order.

For each $\varepsilon > 0$ the operator L^ε is symmetric on $C_0^\infty(\mathbb{R}^d)$ in the weighted space $L^2(\mathbb{R}^d, \nu^\varepsilon)$, where $\nu^\varepsilon(x) = \nu(x/\varepsilon)$ and $\nu(z)$ equals $\mu(z)/\lambda(z)$ in case (P1) and $\nu(z)$ equals 1 in case (P2). Indeed, in case (P1) for u, $v \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}(L^\varepsilon u, v)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu^\varepsilon(x)\mu^\varepsilon(y)}{|x-y|^{d+\alpha}}\big(u(y)-u(x)\big)v(x)dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu^\varepsilon(x)\mu^\varepsilon(y)}{|x-y|^{d+\alpha}}\big(u(y)v(x)-u(y)v(y)\big)dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu^\varepsilon(x)\mu^\varepsilon(y)}{|x-y|^{d+\alpha}}\big(v(x)-v(y)\big)u(y)dxdy = (L^\varepsilon v, u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}\,.\end{aligned}$$

Here and in what follows we denote $\mu^\varepsilon(x) = \mu(x/\varepsilon)$. In case (P2) the symmetry can be checked in the same way.

Moreover, the quadratic form $(-L^\varepsilon u, v)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}$ is positive on $C_0^\infty(\mathbb{R}^d)$.

The inequality $(L^\varepsilon u, u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} \le 0$ follows from the relation

$$(L^\varepsilon u, u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\Lambda^\varepsilon(x,y)}{|x-y|^{d+\alpha}}\big(u(y)-u(x)\big)^2\nu^\varepsilon(x)dydx.$$

The quadratic form $a_{L^\varepsilon}(u,u) = (L^\varepsilon u, u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}$, $u \in C_0^\infty(\mathbb{R}^d)$, is closable; we keep the notation a_{L^ε} for its closure. The closed form a_{L^ε} has the domain $H^{\alpha/2}(\mathbb{R}^d)$. Indeed, due to (1.2), for $u \in C_0^\infty(\mathbb{R}^d)$ this quadratic form is comparable to the quadratic form

$$a_{\widehat{L}}(u,u) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}dydx.$$

We have $\gamma^{-1}a_{\widehat{L}}(u,u) \le a_{L^\varepsilon}(u,u) \le \gamma a_{\widehat{L}}(u,u)$. The fact that $a_{\widehat{L}}(u,u)$ with the domain $H^{\alpha/2}(\mathbb{R}^d)$ is closed is well known; the expression $\|u\|^2_{L^2(\mathbb{R}^2)} - a_{\widehat{L}}(u,u)$ defines the squared norm in $H^{\alpha/2}(\mathbb{R}^d)$ (see, for instance, [26, section 1.10.2]).

It should also be noted that the quadratic form $-a_{\widehat{L}}(u,u)$ corresponds to the fractional Laplacian $(-\Delta)^{\alpha/2}$

$$((-\Delta)^{\alpha/2}u,u) = \frac{c(d,\alpha)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}dydx$$

for any $u \in H^\alpha(\mathbb{R}^d)$; here $c(d,\alpha) > 0$ is a normalizing constant (see, e.g., [34, section 32]).

For the unique self-adjoint operator corresponding to the quadratic form a_{L^ε} (see [31, Theorem X.23]) we keep the notation L^ε; its domain is denoted $\mathcal{D}(L^\varepsilon)$. This operator is self-adjoint and negative in the weighted space $L^2(\mathbb{R}^d,\nu^\varepsilon)$. Moreover, $\mathcal{D}(L^\varepsilon) \subset H^{\alpha/2}(\mathbb{R}^d)$, and for $u \in \mathcal{D}(L^\varepsilon)$ we have $a_{L^\varepsilon}(u,u) = -(L^\varepsilon u,u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}$.

For a given constant $m > 0$ consider the resolvent $(m-L^\varepsilon)^{-1}$. Since L^ε is negative and self-adjoint in $L^2(\mathbb{R}^d,\nu^\varepsilon)$, we have

$$\|(m-L^\varepsilon)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d,\nu^\varepsilon),L^2(\mathbb{R}^d,\nu^\varepsilon))} \le \frac{1}{m}.$$

In view of the properties of λ and μ this yields

$$\|(m-L^\varepsilon)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} \le \frac{\gamma^2}{m}. \tag{2.2}$$

For a given $f \in L^2(\mathbb{R}^d)$ consider a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions to (1.8). Due to (2.2) for each $\varepsilon > 0$ this equation has a unique solution; moreover, $\|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \frac{\gamma^2}{m}\|f\|_{L^2(\mathbb{R}^d)}$.

As mentioned above, we provide two proofs of Theorem 1.1. In subsection 2.1 we provide a proof based on Γ-convergence. This proof is carried out assuming (P1). Second, we assume (P2) and prove Theorem 1.1 using compactness arguments in subsection 2.2. Note that either proof works well in any of our cases.

2.1. First proof of Theorem 1.1. Assuming (P1) we provide a proof of the theorem based on Γ-convergence. Consider the functional

$$F^\varepsilon(u) = a_{L^\varepsilon}(u,u) + m(u,u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} - 2(f,u)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}$$

for $u \in H^{\alpha/2}(\mathbb{R}^d)$. We extend this functional to the whole $L^2(\mathbb{R}^d)$ letting $F^\varepsilon(u) = +\infty$ for $u \in L^2(\mathbb{R}^d) \setminus H^{\alpha/2}(\mathbb{R}^d)$.

It is straightforward to check that for each $\varepsilon > 0$ the functional F^ε is continuous on $H^{\alpha/2}(\mathbb{R}^d)$ and strictly convex. Thus, it attains its minimum at a unique point. We denote this point by u^ε. It is straightforward to see that u^ε belongs to $\mathcal{D}(L^\varepsilon)$ and that u^ε is a solution of (1.8).

We denote $L^2_w(\mathbb{R}^d)$ the space of square integrable functions equipped with the topology of weak convergence. Here is our main auxiliary result.

THEOREM 2.1. *The family of functionals F^ε Γ-converges with respect to the $L^2_{\mathrm{loc}}(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d)$ topology to the functional defined by*

$$F^0(u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\bar{\mu}^2\frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}dydx + \overline{\mu/\lambda}\int_{\mathbb{R}^d}\{m(u(x))^2 - 2f(x)u(x)\}dx$$

for $u \in H^{\alpha/2}(\mathbb{R}^d)$ *and* $F^0(u) = +\infty$ *for* $u \in L^2(\mathbb{R}^d) \setminus H^{\alpha/2}(\mathbb{R}^2)$*, where*

$$\bar{\mu} = \int_{[0,1]^d} \mu(y)dy, \qquad \overline{\mu/\lambda} = \int_{[0,1]^d} (\mu(y))/(\lambda(y))dy.$$

Proof of Theorem 2.1. We begin with the Γ-lim inf inequality. Let $v \in H^{\alpha/2}(\mathbb{R}^d)$, and assume that a sequence $v^\varepsilon \in L^2(\mathbb{R}^d)$ converges to v in $L^2_{\rm loc}(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d)$ topology.

Denote

$$Q^\varepsilon(v) := \begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mu^\varepsilon(x)\mu^\varepsilon(y)\frac{(v(y)-v(x))^2}{|x-y|^{d+\alpha}}dydx, & v \in H^{\alpha/2}(\mathbb{R}^d),\\ +\infty, & v \in L^2(\mathbb{R}^d) \setminus H^{\alpha/2}(\mathbb{R}^d), \end{cases}$$

and

$$Q^0(v) := \begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\mu}^2\frac{(v(y)-v(x))^2}{|x-y|^{d+\alpha}}dydx, & v \in H^{\alpha/2}(\mathbb{R}^d),\\ +\infty, & v \in L^2(\mathbb{R}^d) \setminus H^{\alpha/2}(\mathbb{R}^d). \end{cases}$$

From the definition of F^ε and F^0 it easily follows that

$$c(m,f)\big(\|v\|_{H^{\alpha/2}(\mathbb{R}^d)} - 1\big) \le F^\varepsilon(v) \le C(m,f)\big(\|v\|_{H^{\alpha/2}(\mathbb{R}^d)} + 1\big),$$
$$c(m,f)\big(\|v\|_{H^{\alpha/2}(\mathbb{R}^d)} - 1\big) \le F^0(v) \le C(m,f)\big(\|v\|_{H^{\alpha/2}(\mathbb{R}^d)} + 1\big),$$

with strictly positive constants $c(m,f)$ and $C(m,f)$ that do not depend on ε.

Assume first that $F^0(v) = +\infty$. Then the Γ-lim inf inequality is trivial. Indeed, in this case $\|v\|_{H^{\alpha/2}(\mathbb{R}^d)} = +\infty$, and therefore $\liminf_{\varepsilon\to0} \|v^\varepsilon\|_{H^{\alpha/2}(\mathbb{R}^d)} = +\infty$ for any sequence $v^\varepsilon \in L^2(\mathbb{R}^d)$ that converges to v in $L^2_{\rm loc}(\mathbb{R}^d)$. This yields the desired Γ-lim inf inequality. Assume now that $F^0(v) < +\infty$. One can prove that

$$\begin{aligned}\liminf_{\varepsilon\to0}\{(v^\varepsilon, v^\varepsilon)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} - 2(v^\varepsilon, f)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}\}\\ \ge \overline{\mu/\lambda}\big((v,v)_{L^2(\mathbb{R}^d)} - 2(v,f)_{L^2(\mathbb{R}^d)}\big)\end{aligned} \tag{2.3}$$

for any sequence $v^\varepsilon \in L^2(\mathbb{R}^d)$ that converges to v in $L^2_{\rm loc}(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d)$ topology. Indeed, for any sequence v^ε that converges to v in $L^2_{\rm loc}(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d)$ we have

$$\begin{aligned} 0 &\le \liminf_{\varepsilon\to0}(v - v^\varepsilon, v - v^\varepsilon)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}\\ &= \liminf_{\varepsilon\to0}\big((v^\varepsilon, v^\varepsilon)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} + (v,v)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} - 2(v, v^\varepsilon)_{L^2(\mathbb{R}^d,\nu^\varepsilon)}\big).\end{aligned}$$

Since $\nu^\varepsilon v^\varepsilon$ converges to $(\overline{\mu/\lambda})v$ weakly in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, the last inequality implies that

$$0 \le \liminf_{\varepsilon\to0}(v^\varepsilon, v^\varepsilon)_{L^2(\mathbb{R}^d,\nu^\varepsilon)} - \overline{\mu/\lambda}(v,v)_{L^2(\mathbb{R}^d)}.$$

This yields (2.3).

Therefore, it suffices to show that

$$\liminf_{\varepsilon\to0} Q^\varepsilon(v^\varepsilon) \ge Q^0(v). \tag{2.4}$$

To this end we divide the integration area into three subsets as

$$\mathbb{R}^d \times \mathbb{R}^d = G_1^\delta \cup G_2^\delta \cup G_3^\delta \tag{2.5}$$

with

$$G_1^\delta = \{(x,y) \,:\, |x-y| \ge \delta,\, |x|+|y| \le \delta^{-1}\}, \tag{2.6}$$

$$G_2^\delta = \{(x,y) \,:\, |x-y| \le \delta,\, |x|+|y| \le \delta^{-1}\}, \quad G_3^\delta = \{(x,y) \,:\, |x|+|y| \ge \delta^{-1}\}. \tag{2.7}$$

Since the integral

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(v(y)-v(x)\big)^2}{|x-y|^{d+\alpha}}dydx$$

converges, for any $\kappa>0$ there exists $\delta>0$ such that

$$\int_{G_2^\delta\cup G_3^\delta} \overline{\mu}^2 \frac{\big(v(y)-v(x)\big)^2}{|x-y|^{d+\alpha}}dydx \le \kappa\,. \tag{2.8}$$

Obviously,

$$\liminf_{\varepsilon\to 0}\int_{G_2^\delta\cup G_3^\delta} \mu^\varepsilon(y)\mu^\varepsilon(x)\frac{\big(v^\varepsilon(y)-v^\varepsilon(x)\big)^2}{|x-y|^{d+\alpha}}dydx \ge 0. \tag{2.9}$$

In the domain G_1^δ we have

$$0 < c_1(\delta) \le \frac{\mu^\varepsilon(y)\mu^\varepsilon(x)}{|x-y|^{d+\alpha}} \le C_1(\delta),$$

and v^ε converges to v in $L^2(G_1^\delta)$. Therefore, as $\varepsilon\to 0$,

$$\int_{G_1^\delta} \mu^\varepsilon(y)\mu^\varepsilon(x)\frac{(v^\varepsilon(x))^2}{|x-y|^{d+\alpha}}dydx \longrightarrow \int_{G_1^\delta} \bar\mu^2 \frac{(v(x))^2}{|x-y|^{d+\alpha}}dydx$$

and

$$\int_{G_1^\delta} \mu^\varepsilon(y)\mu^\varepsilon(x)\frac{v^\varepsilon(y)v^\varepsilon(x)}{|x-y|^{d+\alpha}}dydx \longrightarrow \int_{G_1^\delta} \bar\mu^2 \frac{v(y)v(x)}{|x-y|^{d+\alpha}}dydx.$$

This yields

$$\lim_{\varepsilon\to 0}\int_{G_1^\delta} \mu^\varepsilon(y)\mu^\varepsilon(x)\frac{(v^\varepsilon(y)-v^\varepsilon(x))^2}{|x-y|^{d+\alpha}}dydx = \int_{G_1^\delta} \bar\mu^2 \frac{(v(y)-v(x))^2}{|x-y|^{d+\alpha}}dydx. \tag{2.10}$$

Combining (2.3)–(2.10) we conclude that

$$\liminf_{\varepsilon\to 0} F^\varepsilon(v^\varepsilon) \ge F^0(v) - \kappa.$$

Since κ is an arbitrary positive number, the desired Γ-lim inf inequality follows.

We turn to the Γ-lim sup inequality. It suffices to set $v^\varepsilon = v$. It is straightforward to check that $F^\varepsilon(v)\to F^0(v)$. This completes the proof of Theorem 2.1. ∎

We can finally provide the proof of our main result in case (P1).

Proof of Theorem 1.1. As a consequence of Theorem 2.1 any limit point of $\{u^\varepsilon\}$ is a minimizer of F^0; see [6, Theorem 1.21]. Since the minimizer of F^0 is unique, the whole family $\{u^\varepsilon\}$ converges, as $\varepsilon \to 0$, to $u = \mathrm{argmin} F^0$ in $L^2_w(\mathbb{R}^d) \cap L^2_{\mathrm{loc}}(\mathbb{R}^d)$ topology; here the subindex w indicates the weak topology.

It remains to show that u^ε converges to u in $L^2(\mathbb{R}^d)$. If we assume that u^ε does not converge to u in $L^2(\mathbb{R}^d)$, then, for a subsequence, for any $n \in \mathbb{N}$ there exists $\varepsilon(n) > 0$ such that for any $\varepsilon < \varepsilon(n)$ we have

$$\|u^\varepsilon\|_{L^2(\mathbb{R}^d \setminus G(n))} \geq C_2, \tag{2.11}$$

where $C_2 > 0$ is a constant that does not depend on n and $G(n)$ stands for the ball of radius n centered at the origin. Indeed, in this case there is a subsequence $\varepsilon \to 0$ such that $\|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \geq \tilde{C}$ for some $\tilde{C} > 0$. On the other hand, for each n there exists $\varepsilon(n) > 0$ such that for all $\varepsilon < \varepsilon(n)$ we have $\|u^\varepsilon - u\|_{L^2(G(n))} \leq \frac{\tilde{C}}{2}$. Combining the last two estimates and considering the fact that $\|u\|_{L^2(\mathbb{R}^d \setminus G(n))} \to 0$, as $n \to \infty$, we obtain the desired inequality (2.11) for sufficiently large n. Since $\|u^\varepsilon\|_{L^2(\mathbb{R}^d \setminus G(n))}$ is decreasing in n, (2.11) is also fulfilled for smaller n.

Let ψ be a $C_0^\infty(\mathbb{R})$ function such that $0 \leq \psi \leq 1$, $\psi(s) = 1$ for $s \in [0, \frac{1}{2}]$ and $\psi(s) = 0$ for $s \geq 1$. Denote $\psi_n(x) = \psi(\frac{|x|}{n})$. A straightforward computation shows that

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \big| a_{L^\varepsilon}(\psi_n u^\varepsilon, \psi_n u^\varepsilon) - a_{L^\varepsilon}(u^\varepsilon, u^\varepsilon) \big| = 0$$

and

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} f(x)\big(\psi_n(x) - 1\big) u^\varepsilon(x) dx = 0.$$

If inequality (2.11) is fulfilled, then for sufficiently small ε and sufficiently large n we have

$$m \int_{\mathbb{R}^d} \big(\psi_n(x) u^\varepsilon(x)\big)^2 \nu^\varepsilon(x)\, dx \leq m \int_{\mathbb{R}^d} \big(u^\varepsilon(x)\big)^2 \nu^\varepsilon(x)\, dx - \frac{1}{4} m C_2 \|\nu\|_{L^\infty},$$

and hence $F^\varepsilon(\psi_n u^\varepsilon) < F^\varepsilon(u^\varepsilon)$. This contradicts the fact that u^ε is a minimizer of F^ε. Thus, u^ε converges in $L^2(\mathbb{R}^d)$.

The minimizer u satisfies

$$-\bar{\mu}^2(-\Delta)^{\alpha/2} u - \overline{\mu/\lambda} m u = \overline{\mu/\lambda} f.$$

Dividing it by $\overline{\mu/\lambda}$ we arrive at (1.9). Theorem 1.1 is proved. ∎

2.2. Second proof of Theorem 1.1. In this section we give the second proof of Theorem 1.1. Here we assume that condition (P2) holds. This proof can be easily adapted to case (P1).

Second proof of Theorem 1.1. Here we consider an operator L^ε of the form

$$L^\varepsilon u(x) = \int_{\mathbb{R}^d} \frac{\Lambda\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\big(u(y) - u(x)\big)}{|y - x|^{d+\alpha}}\, dy \tag{2.12}$$

with a continuous in (x, y) and periodic measurable in ζ and η function $\Lambda(x, y, \zeta, \eta)$ such that

$$\Lambda(x, y, \zeta, \eta) = \Lambda(y, x, \eta, \zeta), \qquad \gamma^{-1} \leq \Lambda(x, y, \zeta, \eta) \leq \gamma.$$

Our assumptions on the setup ensure that Λ is a Carathéodory function and Λ^ε is well defined.

As was explained above, L^ε is a positive self-adjoint operator in $L^2(\mathbb{R}^d)$ whose domain $\mathcal{D}(L^\varepsilon)$ belongs to $H^{\alpha/2}(\mathbb{R}^d)$.

Multiplying $-L^\varepsilon u^\varepsilon + mu^\varepsilon = f$ by u^ε and integrating the resulting relation over $\mathbb{R}^d$ we conclude

$$\|u^\varepsilon\|_{H^{\alpha/2}(\mathbb{R}^d)} \le C$$

with a constant C that does not depend on ε. Therefore, for a subsequence, u^ε converges to some function $u \in H^{\alpha/2}(\mathbb{R}^d)$, weakly in $H^{\alpha/2}(\mathbb{R}^d)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. In order to characterize this limit function we multiply $-L^\varepsilon u^\varepsilon + mu^\varepsilon = f$ by a test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ and integrate the obtained relation over $\mathbb{R}^d$. After simple rearrangements this yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\Lambda^\varepsilon(x,y)(u^\varepsilon(y)-u^\varepsilon(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy + \int_{\mathbb{R}^d}(u^\varepsilon\varphi - f\varphi)dx = 0,$$

where $\Lambda^\varepsilon(x,y)$ stands for $\Lambda\big(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)$. Clearly, the second integral converges to the integral $\int_{\mathbb{R}^d}(u\varphi - f\varphi)dx$. Our goal is to pass to the limit in the first one. To this end we divide the integration area $\mathbb{R}^d\times\mathbb{R}^d$ into three parts in the same way it was done in (2.5), (2.6), and (2.7). The integral over $G_2^\delta \cup G_3^\delta$ admits the following estimate:

$$\begin{aligned}\Big|\int_{G_2^\delta\cup G_3^\delta} &\frac{\Lambda^\varepsilon(x,y)(u^\varepsilon(y)-u^\varepsilon(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy\Big|\\ &\le C\left(\int_{G_2^\delta\cup G_3^\delta}\frac{(u^\varepsilon(y)-u^\varepsilon(x))^2}{|x-y|^{d+\alpha}}\,dxdy\right)^{\frac12}\left(\int_{G_2^\delta\cup G_3^\delta}\frac{(\varphi(y)-\varphi(x))^2}{|x-y|^{d+\alpha}}\,dxdy\right)^{\frac12}\\ &\le C_1\left(\int_{G_2^\delta\cup G_3^\delta}\frac{(\varphi(y)-\varphi(x))^2}{|x-y|^{d+\alpha}}\,dxdy\right)^{\frac12}.\end{aligned}$$

The last integral tends to zero, as $\delta\to 0$. Similarly,

$$\Big|\int_{G_2^\delta\cup G_3^\delta}\frac{\overline{\Lambda}(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy\Big| \longrightarrow 0,$$

as $\delta\to 0$.

According to [43, Lemma 3.1] the family Λ^ε converges weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^d)$ to the function $\overline{\Lambda}$ with $\overline{\Lambda}(x,y) = \Lambda^{\mathrm{eff}}(x,y)$. Since u^ε converges to u in $L^2(G_1^\delta)$ and Λ^ε converges to $\overline{\Lambda}$ weakly on any bounded domain, we conclude

$$\begin{aligned}\int_{G_1^\delta}&\frac{\Lambda^\varepsilon(x,y)(u^\varepsilon(y)-u^\varepsilon(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy\\ &\underset{\varepsilon\to0}{\longrightarrow}\int_{G_1^\delta}\frac{\overline{\Lambda}(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy\,.\end{aligned}$$

Combining the above relations, we arrive at the conclusion that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\overline{\Lambda}(x,y)(u(y)-u(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy + \int_{\mathbb{R}^d}(u\varphi - f\varphi)dx = 0.$$

Since φ is an arbitrary C_0^∞ function, this implies that u is a solution of $-L^0u+mu=f$. Due to the uniqueness of a solution of this equation, the whole family u^ε converges to u, as $\varepsilon\to 0$.

It remains to justify the convergence in $L^2(\mathbb{R}^d)$. We have

$$0 \le (-L^\varepsilon(u^\varepsilon-u), u^\varepsilon-u) = -(L^\varepsilon u^\varepsilon, u^\varepsilon) + 2(L^\varepsilon u^\varepsilon, u) - (L^\varepsilon u, u).$$

Passing to the limit yields

$$\liminf_{\varepsilon\to 0}\big\{-(L^\varepsilon u^\varepsilon, u^\varepsilon)\big\} \ge -(L^0u,u). \tag{2.13}$$

Now the strong convergence of u^ε in $L^2(\mathbb{R}^d)$ can be obtained by the standard lower semicontinuity arguments. Indeed, multiplying $-L^\varepsilon u^\varepsilon + mu^\varepsilon = f$ by u^ε, integrating the resulting relation over $\mathbb{R}^d$, and passing to the limit as $\varepsilon\to 0$ we have

$$\lim_{\varepsilon\to 0}\big((-L^\varepsilon u^\varepsilon, u^\varepsilon) + m(u^\varepsilon, u^\varepsilon)\big) = (f,u).$$

If u^ε does not converge strongly in $L^2(\mathbb{R}^d)$, then for a subsequence $\lim_{\varepsilon\to 0} m(u^\varepsilon,u^\varepsilon) > m(u,u)$. Combining this with (2.13) for the same subsequence we obtain

$$\lim_{\varepsilon\to 0}\big((-L^\varepsilon u^\varepsilon, u^\varepsilon) + m(u^\varepsilon, u^\varepsilon)\big) > -(L^0u,u) + m(u,u) = (f,u).$$

The last relation here follows from the limit equation $-L^0u+mu=f$. We arrive at a contradiction. Thus, u^ε converges to u in norm. ∎

2.3. Proof of Theorem 1.2. Let us comment on the proof of Theorem 1.2. As mentioned above, the proof does not require any new idea but just an adjustment of the setting. For every $\varepsilon>0, m>0$, (1.11) possesses a unique solution $u^\varepsilon \in W^{\frac{\alpha}{p},p}(\mathbb{R}^d)$. It minimizes the variational functional

$$v \mapsto J(v) = \frac{1}{p}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(y)-v(x)|^p}{|x-y|^{d+\alpha}}\Lambda^\varepsilon(x,y)\,dydx + \frac{m}{p}|v|^p + \int_{\mathbb{R}^d} fv\,.$$

In order to establish bounds that are uniform in ε, we multiply (1.11) by u^ε, integrate the resulting relation over $\mathbb{R}^d$, and exploit the equality

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &\frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))u(x)}{|x-y|^{d+\alpha}}\Lambda^\varepsilon(x,y)\,dydx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(y)-u(x)|^p}{|x-y|^{d+\alpha}}\Lambda^\varepsilon(x,y)\,dydx\,.\end{aligned}$$

Then we easily deduce the estimate

$$\|u^\varepsilon\|_{W^{\frac{\alpha}{p},p}(\mathbb{R}^d)} \le C\|f\|_{L^{p'}(\mathbb{R}^d)} \tag{2.14}$$

with a constant C that does not depend on ε. Thus, there is a weakly convergent subsequence and a limit u^0. From here, the proof is the same as that of Theorem 1.1.

3. Symmetric random kernels. Let us first explain the notion of an ergodic dynamical system. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a standard probability space, and assume that $(T_y)_{y\in\mathbb{R}^d}$ is a d-dimensional ergodic dynamical system in this probability space, i.e., a collection of measurable maps $T_y:\Omega\mapsto\Omega$ such that

- $T_{y_1}T_{y_2}=T_{y_1+y_2}$ for all y_1 and y_2 in $\mathbb{R}^d$; $T_0=\mathrm{Id}$;
- $\mathbf{P}(T_yA)=\mathbf{P}(A)$ for all $A\in\mathcal{F}$ and all $y\in\mathbb{R}^d$;
- $T_\cdot:\mathbb{R}^d\times\Omega\mapsto\Omega$ is a measurable map. Here $\mathbb{R}^d\times\Omega$ is equipped with the σ-algebra $\mathcal{B}\times\mathcal{F}$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}^d$.

We say that $T_\cdot$ is ergodic if for any $A\in\mathcal{F}$ such that $T_yA=A$ for all $y\in\mathbb{R}^d$ we have either $\mathbf{P}(A)=0$ or $\mathbf{P}(A)=1$.

Let us first make some remarks. We study the limit behavior of operator L^ε defined in (1.1), as $\varepsilon\to0$. Clearly, estimate (2.2) remains valid in the random case. Therefore, for any given $f\in L^2(\mathbb{R}^d)$ the sequence of equations

$$(L^\varepsilon-m)u^\varepsilon=f \tag{3.1}$$

is well posed. Moreover, for any $\varepsilon>0$ a solution u^ε is uniquely defined, and $\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}\le\frac{\gamma^2}{m}\|f\|_{L^2(\mathbb{R}^d)}$.

3.1. First proof of Theorem 1.3. Now we are in the position to prove Theorem 1.3 in case (Q1).

Proof of Theorem 1.3 *in case* (Q1). In the same way as in the proof of Theorem 1.1 for any $f\in L^2(\mathbb{R}^d)$ we obtain the estimate

$$\|u^\varepsilon\|_{H^{\alpha/2}(\mathbb{R}^d)}\le C$$

with a deterministic constant C that does not depend on ε. Therefore, for each $\omega\in\Omega$ there is a subsequence that converges to a function $u^0\in H^{\alpha/2}(\mathbb{R}^d)$ weakly in $H^{\alpha/2}(\mathbb{R}^d)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Abusing slightly the notation we keep for this subsequence the same name u^ε.

Multiplying (3.1) by $\mu(\frac{x}{\varepsilon})(\lambda(\frac{x}{\varepsilon}))^{-1}\varphi(x)$ with $\varphi\in C_0^\infty(\mathbb{R}^d)$ and integrating the resulting equality over $\mathbb{R}^d$ after simple rearrangements we arrive at the following relation:

$$0=\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\mu^\varepsilon(y)\mu^\varepsilon(x)(u^\varepsilon(y)-u^\varepsilon(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy$$
$$+\int_{\mathbb{R}^d}\frac{\mu^\varepsilon(x)}{\lambda^\varepsilon(x)}(u^\varepsilon\varphi-f\varphi)dx\,.$$

Here, $\mu^\varepsilon(x)$ and $\lambda^\varepsilon(x)$ stand for $\mu(\frac{x}{\varepsilon})$ and $\lambda(\frac{x}{\varepsilon})$, resp. By the Birkhoff ergodic theorem the function $\boldsymbol{\mu}^\varepsilon$ with $\boldsymbol{\mu}^\varepsilon(x,y)=\mu^\varepsilon(y)\mu^\varepsilon(x)$ converges a.s., as $\varepsilon\to0$, to $\big(\mathbf{E}\{\widehat{\mu}(\cdot)\}\big)^2$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^d)$. In order to justify this convergence we consider a cube $Q_N=[-N,N]^d\times[-N,N]^d$ in $\mathbb{R}^d\times\mathbb{R}^d$ with an arbitrary $N>0$. For any $\varphi_1\in L^2([-N,N]^d)$ and $\varphi_2\in L^2([-N,N]^d)$ by the Birkhoff ergodic theorem we have

$$\int_{Q_N}\mu^\varepsilon(y)\mu^\varepsilon(x)\varphi_1(x)\varphi_2(y)\,dxdy=(\mu^\varepsilon,\varphi_1)_{L^2([-N,N]^d)}\,(\mu^\varepsilon,\varphi_2)_{L^2([-N,N]^d)}$$
$$\underset{\varepsilon\to0}{\longrightarrow}\big(\mathbf{E}\{\widehat{\mu}(\cdot)\}\big)^2\int_{Q_N}\varphi_1(x)\varphi_2(y)dxdy.$$

Since the sequence $\{\mu^\varepsilon(y)\mu^\varepsilon(x),\,(x,y)\in Q_N\}$ is bounded in $L^2(Q_N)$ and the set of finite linear combinations of product functions $\{\varphi_1(x)\varphi_2(y),\, (x,y)\in Q_N\}$ is dense in $L^2(Q_N)$, the required convergence follows. Similarly, $\frac{\mu^\varepsilon(x)}{\lambda^\varepsilon(x)}$ converges a.s. to $\mathbf{E}\{\frac{\widehat\mu(\cdot)}{\widehat\lambda(\cdot)}\}$ in $L^2_{\rm loc}(\mathbb{R}^d)$.

Following the line of the second proof of Theorem 1.1 we obtain

$$0=\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\big(\mathbf{E}\{\widehat\mu(\cdot)\}\big)^2(u^0(y)-u^0(x))(\varphi(y)-\varphi(x))}{|x-y|^{d+\alpha}}\,dxdy$$
$$+\,\mathbf{E}\Big\{\frac{\widehat\mu(\cdot)}{\widehat\lambda(\cdot)}\Big\}\int_{\mathbb{R}^d}(u^0\varphi-f\varphi)dx\,.$$

This yields the desired relation (1.9). The fact that the whole family $\{u^\varepsilon\}$ converges to u^0 a.s. follows from the uniqueness of a solution of (1.9). Finally, the convergence $\lim_{\varepsilon\to0}\|u^\varepsilon-u^0\|_{L^2(\mathbb{R}^d)}=0$ can be justified in the same way as in the second proof of Theorem 1.1. ∎

3.2. Second proof of Theorem 1.3. Next, we explain how to establish Theorem 1.3 in case (Q2). The proof will follow in a straightforward way once we have established the following auxiliary result.

LEMMA 3.1. *For any bounded measurable set $Q\subset\mathbb{R}^d\times\mathbb{R}^d$ the following limit relation holds a.s.:*

$$\int_Q\Lambda(x,y,T_{\frac{x}{\varepsilon}}\omega,T_{\frac{y}{\varepsilon}}\omega)\,dxdy\longrightarrow\int_Q\Lambda^{\rm eff}(x,y)\,dxdy\qquad as\ \varepsilon\to0,\tag{3.2}$$

where

$$\Lambda^{\rm eff}(x,y)=\int_\Omega\int_\Omega\Lambda(x,y,\omega_1,\omega_2)d\mathbf{P}(\omega_1)d\mathbf{P}(\omega_2).$$

Proof. We first recall that under assumption (Q2) the set $\overline{Q}\times\Omega\times\Omega$ is compact, and the function $\Lambda(x,y,\omega_1,\omega_2)$ is continuous on $\overline{Q}\times\Omega\times\Omega$. Therefore, $\Lambda(x,y,\omega_1,\omega_2)$ is equicontinuous on this set, and for any $\delta>0$ there exists $\varkappa>0$ such that

$$|\Lambda(x',y',\omega_1,\omega_2)-\Lambda(x'',y'',\omega_1,\omega_2)|\le\delta\quad\text{for all }\omega_1,\ \omega_2,$$

if $|(x',y')-(x'',y'')|\le\varkappa$ and $(x',y')\in\overline{Q}$, $(x'',y'')\in\overline{Q}$. Then

$$|\Lambda^{\rm eff}(x',y')-\Lambda^{\rm eff}(x'',y'')|$$
$$\le\int_{\Omega\times\Omega}|\Lambda(x',y',\omega_1,\omega_2)-\Lambda(x'',y'',\omega_1,\omega_2)|d\mathbf{P}(\omega_1)d\mathbf{P}(\omega_2)\le\delta.$$

Thus, $\Lambda^{\rm eff}$ is a continuous function.

Consider a partition $\{B_j\}_{j=1}^{N(\delta)}$ of Q that has the following properties:

(i) $\overline{Q}=\bigcup\overline{B}_j,\quad B_j\cap B_k=\emptyset$ if $j\ne k$.

(ii) $\mathrm{diam}(B_j)\le\varkappa$.

(iii) The inequality

$$\Big|\int_Q\Lambda^{\rm eff}(x,y)\,dxdy-\sum_{j=1}^N\Lambda^{\rm eff}(x_j,y_j)|B_j|\Big|\le\delta$$

holds, where $\{(x_j,y_j)\}_{j=1}^N$ is a set of points in Q such that $(x_j,y_j)\in B_j$.

By the Stone–Weierstrass theorem for each $j = 1,\dots,N$ there exist a finite set of continuous functions $\{\varphi_k^j(\omega),\, \psi_k^j(\omega)\}_{k=1}^L$ such that

$$\Big|\Lambda(x_j,y_j,\omega_1,\omega_2) - \sum_{k=1}^{L} \varphi_k^j(\omega_1)\psi_k^j(\omega_2)\Big| \le \delta.$$

This implies in particular that

$$\Big|\Lambda^{\mathrm{eff}}(x_j,y_j) - \sum_{k=1}^{L} \mathbf{E}\varphi_k^j \mathbf{E}\psi_k^j\Big| \le \delta. \tag{3.3}$$

Then we have a.s.

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_Q \Lambda(x,y,T_{\frac{x}{\varepsilon}}\omega, T_{\frac{y}{\varepsilon}}\omega)\, dxdy \\
&\quad\le \limsup_{\varepsilon\to 0} \sum_{j=1}^{N} \int_{B_j} \Lambda(x_j,y_j,T_{\frac{x}{\varepsilon}}\omega, T_{\frac{y}{\varepsilon}}\omega)\, dxdy + \delta|Q| \\
&\quad\le \limsup_{\varepsilon\to 0} \sum_{j=1}^{N} \int_{B_j} \sum_{k=1}^{L} \varphi_k^j(T_{\frac{x}{\varepsilon}}\omega)\psi_k^j(T_{\frac{y}{\varepsilon}}\omega)\, dxdy + 2\delta|Q| \\
&\quad= \sum_{j=1}^{N} \int_{B_j} \sum_{k=1}^{L} \mathbf{E}\varphi_k^j \mathbf{E}\psi_k^j\, dxdy + 2\delta|Q| \\
&\quad\le \sum_{j=1}^{N} \int_{B_j} \Lambda^{\mathrm{eff}}(x_j,y_j)\, dxdy + 3\delta|Q| \le \int_Q \Lambda^{\mathrm{eff}}(x,y)\, dxdy + \delta(3|Q|+1)\,.
\end{aligned}$$

The third relation here follows from the Birkhoff ergodic theorem and the fourth one from estimate (3.3). Similarly,

$$\liminf_{\varepsilon\to 0} \int_Q \Lambda(x,y,T_{\frac{x}{\varepsilon}}\omega, T_{\frac{y}{\varepsilon}}\omega)\, dxdy \ge \int_Q \Lambda^{\mathrm{eff}}(x,y)\, dxdy - \delta(3|Q|+1).$$

Since $\delta > 0$ can be chosen arbitrarily, this implies the desired relation (3.2). ∎

As a straightforward consequence of Lemma 3.1, the functions Λ^ε a.s. converge to Λ^{eff} $*$-weakly in $L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, as $\varepsilon\to 0$, where $\Lambda^\varepsilon(x,y)$ stands for $\Lambda(x,y,T_{\frac{x}{\varepsilon}}\omega, T_{\frac{y}{\varepsilon}}\omega)$.

With the help of this convergence result, the proof of Theorem 1.3 is immediate.

4. Nonsymmetric kernels. The aim of this section is to prove Theorem 1.4. We split the proof into three different steps. In subsection 4.1 we investigate the adjoint operator L^* and its principal eigenfunction. Subsection 4.2 provides uniform bounds on the functions u^ε. Finally, we consider the limit $\varepsilon\to 0$ in subsection 4.3.

4.1. Auxiliary periodic problems. We recall that the period of $\Lambda = \Lambda(x,y)$ both in x and in y variables is $[0,1]^d$. We deal here with an auxiliary (cell) problem defined in the space of periodic functions $L^2(\mathbb{T}^d)$. Notice that in this case the operator

$$Lv(\zeta) = \int_{\mathbb{R}^d} \frac{\Lambda(\zeta,\eta)\big(v(\eta)-v(\zeta)\big)}{|\zeta-\eta|^{d+\alpha}}\, d\eta \tag{4.1}$$

is an unbounded linear operator in $L^2(\mathbb{T}^d)$; here and in what follows we identify functions defined on the torus $\mathbb{T}^d$ with the corresponding periodic functions in $\mathbb{R}^d$. In order to introduce a domain of L we represent $\Lambda(\zeta,\eta)$ as $\Lambda(\zeta,\eta) = \Lambda^s(\zeta,\eta) + \Lambda^a(\zeta,\eta)$ with

$$\Lambda^s(\zeta,\eta) = \frac{1}{2}(\Lambda(\zeta,\eta) + \Lambda(\eta,\zeta)), \qquad \Lambda^a(\zeta,\eta) = \frac{1}{2}(\Lambda(\zeta,\eta) - \Lambda(\eta,\zeta)). \tag{4.2}$$

Then we represent the operator L as follows:

$$\begin{aligned} Lv(\zeta) &= \int_{\mathbb{R}^d} \frac{\Lambda^s(\zeta,\eta)\big(v(\eta)-v(\zeta)\big)}{|\zeta-\eta|^{d+\alpha}}\,d\eta + \int_{\mathbb{R}^d} \frac{\Lambda^a(\zeta,\eta)\big(v(\eta)-v(\zeta)\big)}{|\zeta-\eta|^{d+\alpha}}\,d\eta \\ &= L^s v(\zeta) + L^a v(\zeta). \end{aligned}$$

Since Λ^s is symmetric and satisfies the estimate $\gamma^{-1} \le \Lambda^s \le \gamma$, the operator L^s is self-adjoint in $L^2(\mathbb{T}^d)$, and its domain $\mathcal{D}(L^s)$ is dense in $H^{\alpha/2}(\mathbb{T}^d)$. In fact, for Lipschitz continuous Λ we have $\mathcal{D}(L^s) = H^\alpha(\mathbb{T}^d)$. Observe that for Lipschitz continuous Λ the function $\Lambda^a(\zeta,\eta)$ is also Lipschitz continuous, and $\Lambda^a(\zeta,\zeta) = 0$ for all ζ. It is then straightforward to check that the kernel $\Lambda^a(\zeta,\eta)\,|\zeta-\eta|^{-d-\alpha}$ admits the following upper bound:

$$|\Lambda^a(\zeta,\eta)|\,|\zeta-\eta|^{-d-\alpha} \le C\min(|\zeta-\eta|^{-d-\alpha+1}, |\zeta-\eta|^{-d-\alpha}).$$

Since the function $\min(|z|^{-d-\alpha+1}, |z|^{-d-\alpha})$ is integrable on $\mathbb{R}^d$, the operator L^a is bounded in $L^2(\mathbb{T}^d)$. Then the operator L with the domain $\mathcal{D}(L^s)$ is a closed operator in $L^2(\mathbb{T}^d)$, and its adjoint $L^* = L^s + (L^a)^*$ has the same domain $\mathcal{D}(L^s)$.

Direct computations show that the adjoint operator takes the form

$$L^* q(\zeta) = \int_{\mathbb{R}^d} \frac{\big(\Lambda(\eta,\zeta)q(\eta) - \Lambda(\zeta,\eta)q(\zeta)\big)}{|\zeta-\eta|^{d+\alpha}}\,d\eta. \tag{4.3}$$

THEOREM 4.1. *There exists a continuous positive function $p_0(\cdot) \in \mathcal{D}(L^*)$ such that $L^* p_0 = 0$ in $L^2(\mathbb{T}^d)$ and $p_0(\xi) \ge p^-$ for all $\xi \in \mathbb{T}^d$ for some constant $p^- > 0$.*

The remainder of this subsection is dedicated to the proof of Theorem 4.1, which itself uses several auxiliary results.

First we are going to show that the kernel of L^* in $L^2(\mathbb{T}^d)$ contains a continuous positive function; we denote it p_0. The uniqueness will be justified later on. To prove the existence of such a function p_0 we first introduce the operators that correspond to (4.1) and (4.3) in the space $C(\mathbb{T}^d)$. This will allow us to use the results from the theory of positive operators.

We represent the operator L^* in the form

$$\begin{aligned} -L^* q(\zeta) &= \Lambda(\zeta,\zeta) \int_{\mathbb{R}^d} \frac{\big(q(\zeta)-q(\eta)\big)}{|\zeta-\eta|^{d+\alpha}}\,d\eta + \int_{\mathbb{R}^d} \frac{\big(\Lambda(\zeta,\zeta)-\Lambda(\eta,\zeta)\big)q(\eta)}{|\zeta-\eta|^{d+\alpha}}\,d\eta \\ &-q(\zeta)\int_{\mathbb{R}^d} \frac{\big(\Lambda(\zeta,\zeta)-\Lambda(\zeta,\eta)\big)}{|\zeta-\eta|^{d+\alpha}}\,d\eta =: \Lambda(\zeta,\zeta)\big[\mathcal{L}^s q(\zeta) + \mathcal{L}^1 q(\zeta) + \mathcal{L}^2 q(\zeta)\big]\,. \end{aligned}$$

Since $\Lambda(\zeta,\eta)$ is a Lipschitz continuous function and $\alpha \in (0,1)$, the kernel of the operator $\mathcal{L}^1$ is integrable in η uniformly in ζ. Considering the fact that this kernel is continuous on the complement of the set $\{(\zeta,\eta) : \zeta = \eta\}$, we conclude that $\mathcal{L}^1$ is a bounded operator in $C(\mathbb{T}^d)$. The function $\int_{\mathbb{R}^d} \frac{(\Lambda(\eta,\zeta)-\Lambda(\zeta,\zeta))}{\Lambda(\zeta,\zeta)|\zeta-\eta|^{d+\alpha}}\,d\eta$ is continuous and periodic. Therefore, the operator $\mathcal{L}^2$ is also bounded in $C(\mathbb{T}^d)$.

The operator $\mathcal{L}^s$ is well defined for $u \in C^\infty(\mathbb{T}^d)$. From Proposition 4.3 below it follows that this operator is closable in $C(\mathbb{T}^d)$ and that any $\lambda > 0$ belongs to the resolvent set of its closure (still denoted by $\mathcal{L}^s$). Moreover, as a consequence of Proposition 4.3, the domain $\mathcal{D}_C(\mathcal{L}^s)$ of $\mathcal{L}^s$ in $C(\mathbb{T}^d)$ is a subset of $\mathcal{D}(\mathcal{L}^s)$, and on $\mathcal{D}_C(\mathcal{L}^s)$ these operators coincide.

LEMMA 4.2. *There exists $\beta > 0$ such that for any $\lambda > 0$ the resolvent $(\mathcal{L}^s + \lambda\mathbf{I})^{-1}$ is a bounded operator from $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$. Moreover, the following estimate holds:*

$$\|(\mathcal{L}^s + \lambda\mathbf{I})^{-1}\|_{C(\mathbb{T}^d)\to C(\mathbb{T}^d)} \le \lambda^{-1}. \tag{4.4}$$

Proof. Estimate (4.4) follows directly from the maximum principle. It suffices to obtain this estimate for a dense set in $C(\mathbb{T}^d)$. For $u \in C^\infty(\mathbb{T}^d)$ denote by ζ_0 a maximum point of $|u|$. Without loss of generality we assume that $u(\zeta_0) \ge 0$. Then $\mathcal{L}^s u(\zeta_0) \ge 0$. Therefore,

$$\lambda\|u\|_{C(\mathbb{T}^d)} = \lambda u(\zeta_0) \le \mathcal{L}^s u(\zeta_0) + \lambda u(\zeta_0) \le \|\mathcal{L}^s u + \lambda u\|_{C(\mathbb{T}^d)}.$$

This yields (4.4). ∎

We reformulate the first statement of Lemma 4.2 as a separate result.

PROPOSITION 4.3. *Let $f \in C(\mathbb{T}^d), \lambda > 0$. There is a constant $c \ge 1$ such that for every function $u \in H^\alpha(\mathbb{T}^d)$ satisfying*

$$(-\Delta)^{\frac{\alpha}{2}} u + \lambda u = f \quad \text{in } \mathbb{T}^d \tag{4.5}$$

the following estimate holds:

$$\|u\|_{C^\alpha(\mathbb{T}^d)} \le c\|f\|_{C(\mathbb{T}^d)}\,. \tag{4.6}$$

Proof. There are several ways to prove this result. One option would be to apply embedding results for the Riesz potential. Another option would be to use the Harnack inequality. Here, we give a proof based on the corresponding heat equation and the representation of solutions with the help of the fundamental solution. Let (P_t) denote the contraction semigroup of the operator $\partial_t + (-\Delta)^{\frac{\alpha}{2}}$ in $(0,\infty)\times\mathbb{R}^d$ that acts in the space $C_b(\mathbb{R}^d)$ of bounded continuous functions equipped with the norm $\|v\|_{C_b(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |v(x)|$. Notice that this semigroup need not be strongly continuous in $C_b(\mathbb{R}^d)$. It is known that for $f \in C_b(\mathbb{R}^d)$ the function $P_t f$ belongs to $C^\infty(\mathbb{R}^d)$ for any $t > 0$ and satisfies

$$|\nabla P_t f(x)| \le c_1 t^{-1/\alpha}\|f\|_{C_b(\mathbb{R}^d)} \qquad \text{for all } x \in \mathbb{R}^d \tag{4.7}$$

with some contant $c_1 \ge 1$ independent of x. This is proved in several works, e.g., in [4, Theorem 3.2]. In order to prove (4.6), let u be a solution to (4.5) and $x, y \in \mathbb{R}^d$. We only need to consider the case $|x-y| \le 1$. Assume $\rho \in (0,1)$. Then

$$\begin{aligned} |u(x)-u(y)| &\le \int_0^\infty e^{-\lambda t}|P_t f(x) - P_t f(y)|dt \\ &\le \int_0^\rho e^{-\lambda t}|P_t f(x) - P_t f(y)|dt + \int_\rho^\infty e^{-\lambda t}|P_t f(x) - P_t f(y)|dt\,. \end{aligned}$$

The first integral is estimated from above as follows:

$$\int_0^\rho e^{-\lambda t}|P_t f(x) - P_t f(y)|dt \le 2\|f\|_\infty \int_0^\rho e^{-\lambda t}dt$$
$$= 2\|f\|_\infty \frac{1}{\lambda}(1 - e^{-\lambda\rho}) \le 2\rho\|f\|_\infty \,.$$

For the estimate of the second integral we apply (4.7) and obtain

$$\int_\rho^\infty e^{-\lambda t}|P_t f(x) - P_t f(y)|dt \le c_1\|f\|_\infty |x-y| \int_\rho^\infty e^{-\lambda t} t^{-1/\alpha}dt.$$

Note that for $\alpha < 1$ we have

$$\int_\rho^\infty e^{-\lambda t} t^{-1/\alpha}dt \le \int_\rho^1 t^{-1/\alpha}dt + \int_1^\infty e^{-\lambda t}dt \le \frac{\alpha-1}{\alpha}(1-\rho^{\frac{\alpha-1}{\alpha}}) + \frac{1}{\lambda}e^{-\lambda}$$
$$\le c_2(\alpha)\max\{1, \rho^{\frac{\alpha-1}{\alpha}}\} + c_3(\lambda)\,.$$

Hence, we obtain for $\alpha < 1$

$$\int_\rho^\infty e^{-\lambda t}|P_t f(x) - P_t f(y)|dt \le c_1|x-y|\|f\|_\infty \Big(c_2 \max\{1, \rho^{\frac{\alpha-1}{\alpha}}\} + c_3\Big)\,.$$

Now we choose $\rho = |x-y|^\alpha$. Combining the estimates of the two integrals, we obtain the desired result. ∎

LEMMA 4.4. *There exist $\lambda_0 > 0$ and $\beta > 0$ such that for all $\lambda \ge \lambda_0$ the resolvent $(\mathcal{L}^s + \mathcal{L}^1 + \mathcal{L}^2 + \lambda\mathbf{I})^{-1}$ is a bounded operator from $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$.*

Proof. We have

$$(\mathcal{L}^s + \mathcal{L}^1 + \mathcal{L}^2 + \lambda\mathbf{I})^{-1} = \big([\mathbf{I} + (\mathcal{L}^1 + \mathcal{L}^2)(\mathcal{L}^s + \lambda\mathbf{I})^{-1}](\mathcal{L}^s + \lambda\mathbf{I})\big)^{-1}$$
$$= (\mathcal{L}^s + \lambda\mathbf{I})^{-1}[\mathbf{I} + (\mathcal{L}^1 + \mathcal{L}^2)(\mathcal{L}^s + \lambda\mathbf{I})^{-1}]^{-1}\,.$$

Letting $\lambda_0 = 2\|\mathcal{L}^1 + \mathcal{L}^2\|_{C(\mathbb{T}^d)\to C(\mathbb{T}^d)}$ one can easily check that $[\mathbf{I} + (\mathcal{L}^1 + \mathcal{L}^2)(\mathcal{L}^s + \lambda\mathbf{I})^{-1}]^{-1}$ is a bounded operator in $C(\mathbb{T}^d)$ for any $\lambda > \lambda_0$. Combining this with the first statement of Theorem 4.2, we obtain the required statement. ∎

Considering the properties of the function $\Lambda(\zeta,\zeta)$ and the representation $-L^*q(\zeta) = \Lambda(\zeta,\zeta)(\mathcal{L}^s q(\zeta) + \mathcal{L}^1 q(\zeta) + \mathcal{L}^2 q(\zeta)$ it is straightforward to see that, for sufficiently large λ, $(-L^* + \lambda)^{-1}$ is a bounded operator from $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$. Indeed, denoting $\mathcal{N} = \mathcal{L}^s + \mathcal{L}^1 + \mathcal{L}^2$ and $\Lambda^- = \min \Lambda(\zeta,\zeta)$ and taking $\lambda > \lambda_0 \max\big(1, \max\Lambda(\xi,\xi)\big)$ we have

$$-L^* + \lambda = \Lambda(\zeta,\zeta)\left(\mathcal{N} + \frac{\lambda}{\Lambda^-} + \frac{\lambda}{\Lambda(\zeta,\zeta)} - \frac{\lambda}{\Lambda^-}\right)$$
$$(4.8) \qquad = \Lambda(\zeta,\zeta)\left(\mathbf{I} + \left(\frac{\lambda}{\Lambda(\zeta,\zeta)} - \frac{\lambda}{\Lambda^-}\right)\left(\mathcal{N} + \frac{\lambda}{\Lambda^-}\right)^{-1}\right)\left(\mathcal{N} + \frac{\lambda}{\Lambda^-}\right).$$

Since

$$(\mathcal{N} + \lambda)^{-1} = (\mathcal{L}^s + \lambda)^{-1}\big(\mathbf{I} + (\mathcal{L}^1 + \mathcal{L}^2)(\mathcal{L}^s + \lambda)^{-1}\big)^{-1},$$

then

$$\|(\mathcal{N}+\lambda)^{-1}\|_{\mathcal{L}(C(\mathbb{T}^d),C(\mathbb{T}^d))} \le \frac{1}{\lambda - \frac{\lambda_0}{2}}$$

and

$$\left\| \left(\frac{\lambda}{\Lambda(\zeta,\zeta)} - \frac{\lambda}{\Lambda^-} \right) \left(\mathcal{N} + \frac{\lambda}{\Lambda^-} \right)^{-1} \right\|_{\mathcal{L}(C(\mathbb{T}^d),C(\mathbb{T}^d))} < 1.$$

From this estimate, using the representation of $(-L^*+\lambda)$ in (4.8), we obtain that $(-L^*+\lambda)^{-1}$ is a bounded operator from $C(\mathbb{T}^d)$ to $C(\mathbb{T}^d)$. Moreover, combining (4.8) with the statement of Proposition 4.3, we conclude that $(-L^*+\lambda)^{-1}$ is also a bounded operator from $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$.

The fact that $(-L+\lambda)^{-1}$ is a bounded operator from $C(\mathbb{T}^d)$ to $C^\beta(\mathbb{T}^d)$ can be justified in exactly the same way. This implies in particular that both $(-L+\lambda)^{-1}$ and $(-L^*+\lambda)^{-1}$ are compact operators in $C(\mathbb{T}^d)$.

From the maximum principle it follows that $\|(-L+\lambda)^{-1}\|_{\mathcal{L}(C(\mathbb{T}^d),C(\mathbb{T}^d))} \le \lambda^{-1}$. Also, by the standard maximum principle arguments, the operator $(L+\lambda)^{-1}$ maps the set of nonnegative continuous nonzero functions on $\mathbb{T}^d$ to the set of strictly positive continuous functions on $\mathbb{T}^d$. Therefore, the Krein–Rutman theorem applies to the operator $(-L+\lambda)^{-1}$. For the reader's convenience we formulate here this theorem.

THEOREM 4.5 ([25], Theorem 6.3). *Let X be a real ordered Banach space with a positive cone $K \subset X$ that has a nontrivial interior $K^o \ne \emptyset$. Assume that $T \in \mathcal{L}(X)$ is a compact linear positive operator on X, $TK \subset K$, that improves positivity. Then*

(a) *the operator T has a unique eigenvector $x_0 \in K^o$: $Tx_0 = \lambda_0 x_0$;*

(b) *the adjoint operator $T^\star$ has a unique eigenvector $x_0^\star \in K^\star$: $T^\star x_0^\star = \lambda_0 x_0^\star$, where $x_0^\star$ is a strongly positive functional that is $x_0^\star(x) > 0$ for any $x \in K\backslash\{0\}$;*

(c) *the corresponding eigenvalue $\lambda_0 > 0$ is simple and greater than the absolute value of any other eigenvalue of T.*

Denote by $(-L+\lambda)^{-1})^t$ the operator adjoint to $(-L+\lambda)^{-1})$ in the space $C(\mathbb{T}^d)$. It is a bounded operator in the space of generalized measures of finite total variation.

It is easy to check that $v=1$ is the principal eigenfunction of $(-L+\lambda)^{-1}$ and that the corresponding eigenvalue is equal to λ^{-1}.

By the Krein–Rutman theorem the adjoint operator $\big((-L+\lambda)^{-1}\big)^t$ maps the cone of nonnegative measures into itself, its principal eigenvalue is λ^{-1}, and the corresponding eigenmeasure is positive; we denote this eigenmeasure by μ_0.

By the definition of an adjoint operator, taking into account the properties of the operator $(-L^*+\lambda)^{-1}$, we derive that for any absolutely continuous measure $\mu(d\zeta) = g(\zeta)d\zeta$ on $\mathbb{T}^d$ with a continuous g we have

$$\big((-L+\lambda)^{-1}\big)^t \mu(d\zeta) = (-L^*+\lambda)^{-1} g\, d\zeta.$$

Since an absolutely continuous measure is nonnegative if and only if its density is nonnegative, the operator $(-L^*+\lambda)^{-1}$ maps the cone of continuous nonnegative functions into itself.

Applying one more time the maximum principle arguments we conclude that for any nontrivial continuous nonnegative function v on $\mathbb{T}^d$ the function $(-L^*+\lambda)^{-1}v$ is strictly positive. Hence, the Krein–Rutman theorem applies to the operator $(-L^*+$

$\lambda)^{-1}$. Denote by $p_0 \in C(\mathbb{T}^d)$ the principal eigenfunction of this operator. This function is strictly positive on $\mathbb{T}^d$. By the uniqueness of a positive eigenmeasure of $\big((-L+\lambda)^{-1}\big)^t$ (see [25], [24, Chapter 4]) we obtain $\mu_0 = p_0(\zeta)\,d\zeta$ and $(-L^*+\lambda)^{-1})p_0 = \lambda^{-1}p_0$. Therefore, $-L^*p_0 = 0$.

The proof of Theorem 4.1 is complete.

4.2. A priori estimates. Our next goal is to obtain a priori estimates for the solution of (1.8).

PROPOSITION 4.6. *Assume that* (1.8) *holds true for some* $f \in L^2(\mathbb{R}^d)$ *and* $u^\varepsilon \in \mathcal{D}(L^\varepsilon)$. *Then there exists a constant* c *such that*

$$\|u^\varepsilon\|_{H^{\alpha/2}(\mathbb{R}^d)} \le c\left(1+\frac{1}{m}\right)\|f\|_{L^2(\mathbb{R}^d)},$$

$$\|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \frac{c}{m}\|f\|_{L^2(\mathbb{R}^d)}.$$

The constant c *does not depend on* ε *or on* m.

Proof. Multiplying (1.8) by $u^\varepsilon(x)p_0(\frac{x}{\varepsilon})$ and integrating the resulting relation over $\mathbb{R}^d$ yields

$$\begin{aligned}(4.9)\qquad &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)u^\varepsilon(x)}{|x-y|^{d+\alpha}}dydx - m\int_{\mathbb{R}^d}p_0\big(\frac{x}{\varepsilon}\big)(u^\varepsilon(x))^2dx\\ &\qquad = \int_{\mathbb{R}^d}p_0\big(\frac{x}{\varepsilon}\big)u^\varepsilon(x)f(x)dx\,.\end{aligned}$$

The first term here can be transformed as follows:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)u^\varepsilon(x)}{|x-y|^{d+\alpha}}dydx\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)u^\varepsilon(y)u^\varepsilon(x)-\Lambda\big(\frac{y}{\varepsilon},\frac{x}{\varepsilon}\big)p_0\big(\frac{y}{\varepsilon}\big)(u^\varepsilon(x))^2}{|x-y|^{d+\alpha}}dydx\\
(4.10)\qquad&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{y}{\varepsilon},\frac{x}{\varepsilon}\big)p_0\big(\frac{y}{\varepsilon}\big)(u^\varepsilon(x))^2-\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)(u^\varepsilon(x))^2}{|x-y|^{d+\alpha}}dydx\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)u^\varepsilon(y)u^\varepsilon(x)-\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)(u^\varepsilon(y))^2}{|x-y|^{d+\alpha}}dydx\\
&\quad=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)u^\varepsilon(y)}{|x-y|^{d+\alpha}}dydx;\end{aligned}$$

here we have used the fact that by Theorem 4.1 the integral in (4.10) is equal to zero. Considering these equalities one can rewrite relation (4.9) as follows:

$$\begin{aligned}(4.11)\qquad &\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)^2}{|x-y|^{d+\alpha}}dydx + m\int_{\mathbb{R}^d}p_0\big(\frac{x}{\varepsilon}\big)(u^\varepsilon(x))^2dx\\ &\qquad = -\int_{\mathbb{R}^d}p_0\big(\frac{x}{\varepsilon}\big)u^\varepsilon(x)f(x)dx\,.\end{aligned}$$

By Theorem 4.1 the function p_0 satisfies the estimates $0 < p_- \le p_0(z) \le p^+$. Therefore, we have

$$\|u^\varepsilon\|_{H^{\alpha/2}(\mathbb{R}^d)} \le c\left(1+\frac{1}{m}\right)\|f\|_{L^2(\mathbb{R}^d)}, \qquad \|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \frac{p^+}{mp_-}\|f\|_{L^2(\mathbb{R}^d)}.$$

This completes the proof of the proposition. ◻

From the estimates of Theorem 4.6 one can deduce that the set of positive real numbers belongs to the resolvent set of operator L^ε. We should show that for all $m > 0$ the operator $(L^\varepsilon - m)$ maps $\mathcal{D}(L^\varepsilon)$ onto $L^2(\mathbb{R}^d)$. To this end we use a representation $\Lambda(\zeta,\eta) = \Lambda^s(\zeta,\eta) + \Lambda^a(\zeta,\eta)$ with Λ^s and Λ^a defined in (4.2). Since by construction the function Λ^a is Lipschitz continuous and $\Lambda^a(\zeta,\zeta) = 0$ for all ζ, then

$$\frac{\Lambda^a\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)}{|x-y|^{d+\alpha}} \le \min\left(\frac{c}{\varepsilon}|x-y|^{-d-\alpha+1}, 2\|\Lambda\|_{L^\infty}|x-y|^{-d-\alpha}\right).$$

Considering the fact that the function $\min(\frac{c}{\varepsilon}|z|^{-d-\alpha+1}, 2\|\Lambda\|_{L^\infty}|z|^{-d-\alpha})$ is integrable in $\mathbb{R}^d$, we conclude that the operator $(L^a)^\varepsilon$ with the kernel $\Lambda^a\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)|x-y|^{-d-\alpha}$ is bounded in $L^2(\mathbb{R}^d)$. We have $\|(L^a)^\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} < C(\varepsilon)$.

The operator $(L^s)^\varepsilon$ with a domain $\mathcal{D}(L^\varepsilon)$ is self-adjoint and negative in $L^2(\mathbb{R}^d)$. Therefore, $\|((L^s)^\varepsilon - m)^{-1}\| \le \frac{1}{m}$. Consequently, for $m \ge m_0$ with $m_0 = (C(\varepsilon))^{-1}$ the operator

$$(L^\varepsilon - m)^{-1} = ((L^s)^\varepsilon - m + (L^a)^\varepsilon)^{-1} = ((L^s)^\varepsilon + m)^{-1}\big(\mathbf{I} + (L^a)^\varepsilon((L^s)^\varepsilon + m)^{-1}\big)^{-1}$$

is bounded in $L^2(\mathbb{R}^d)$.

By Proposition 4.6 the norm of $(L^\varepsilon - m)^{-1}$ does not exceed $\frac{c}{m}$. Therefore, for $m > m_1 = m_0(1 - \frac{1}{c})$ the operator

$$(L^\varepsilon - m)^{-1} = (L^\varepsilon - m_0)^{-1}\big(\mathbf{I} + (m - m_0)(L^\varepsilon - m_0)^{-1}\big)^{-1}$$

is bounded in $L^2(\mathbb{R}^d)$. Iterating this step we conclude that for any $m > 0$ the operator $(L^\varepsilon - m)^{-1}$ is bounded.

In particular, (1.8) is well-posed, and it has a unique solution $u^\varepsilon \in \mathcal{D}(L^\varepsilon)$.

4.3. Passage to the limit. According to the estimates of Theorem 4.6 the family u^ε converges for a subsequence, as $\varepsilon \to 0$, to a function $u^0 \in H^{\alpha/2}(\mathbb{R}^d)$, weakly in $H^{\alpha/2}(\mathbb{R}^d)$. Furthermore, $u^\varepsilon \to u^0$ strongly in L^2 on any compact set in $\mathbb{R}^d$.

In order to characterize the function u^0 we multiply (1.8) by a test function $p_0(\frac{x}{\varepsilon})\varphi(x)$ with $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and integrate the resulting relation in $\mathbb{R}^d$. We have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\Lambda\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)p_0\left(\frac{x}{\varepsilon}\right)\big(u^\varepsilon(y) - u^\varepsilon(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx - m\int_{\mathbb{R}^d} p_0\big(\frac{x}{\varepsilon}\big)u^\varepsilon(x)\varphi(x)dx \\ = \int_{\mathbb{R}^d} p_0\big(\frac{x}{\varepsilon}\big)\varphi(x)f(x)dx\,.\end{aligned} \tag{4.12}$$

In the same way as in the proof of Theorem 4.6 one can show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx$$
$$=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(\varphi(y)-\varphi(x)\big)u^\varepsilon(y)}{|x-y|^{d+\alpha}}dydx.$$

We represent $\mathbb{R}^d\times\mathbb{R}^d$ as the union of two sets

$$\mathbb{R}^d\times\mathbb{R}^d=G_4^\delta\cup G_5^\delta \tag{4.13}$$

with

$$G_4^\delta=\{(x,y)\,:\,|x-y|\le\delta\},\quad G_5^\delta=\{(x,y)\,:\,|x-y|>\delta\}. \tag{4.14}$$

Denote

$$\mathcal{K}_\delta^\varepsilon(x,y)=\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(\varphi(y)-\varphi(x)\big)}{|x-y|^{d+\alpha}}\ \mathbf{1}_\delta(x-y),$$

where $\mathbf{1}_\delta(z)$ is the indicator function of the ball $\{z\in\mathbb{R}^d\,:\,|z|\le\delta\}$. It is easy to check that

$$0\le\big|\mathcal{K}_\delta^\varepsilon(x,y)\big|\le C_\varphi|x-y|^{1-d-\alpha}\mathbf{1}_\delta(x-y).$$

Since the integral

$$\int_{\mathbb{R}^d}|z|^{1-d-\alpha}\mathbf{1}_\delta(z)dz$$

tends to zero, as $\delta\to0$, we have

$$\int_{\mathbb{R}^d}dx\bigg(\int_{\mathbb{R}^d}\mathcal{K}_\delta^\varepsilon(x,y)u^\varepsilon(y)dy\bigg)^2\le C(\delta)\|u^\varepsilon\|^2_{L^2(\mathbb{R}^d)}, \tag{4.15}$$

where $C(\delta)\to0$, as $\delta\to0$. On the set G_5^δ the kernel is bounded. Therefore,

$$\int_{G_5^\delta}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(\varphi(y)-\varphi(x)\big)u^\varepsilon(y)}{|x-y|^{d+\alpha}}dydx\ \to\ \int_{G_5^\delta}\frac{\langle\Lambda p_0\rangle\big(\varphi(y)-\varphi(x)\big)u^0(y)}{|x-y|^{d+\alpha}}dydx,$$

where

$$\langle\Lambda p_0\rangle=\int_{\mathbb{T}^d\times\mathbb{T}^d}\Lambda(\zeta,\eta)p_0(\zeta)d\zeta d\eta.$$

Combining this convergence with (4.15) we conclude that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(\varphi(y)-\varphi(x)\big)u^\varepsilon(y)}{|x-y|^{d+\alpha}}dydx$$
$$\longrightarrow\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\langle\Lambda p_0\rangle\big(\varphi(y)-\varphi(x)\big)u^0(y)}{|x-y|^{d+\alpha}}dydx,$$

as $\varepsilon\to0$. Therefore,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)p_0\big(\frac{x}{\varepsilon}\big)\big(u^\varepsilon(y)-u^\varepsilon(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx$$
$$\longrightarrow\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\langle\Lambda p_0\rangle\big(u^0(y)-u^0(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx\,.$$

Passing to the limit in (4.12) yields

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\langle \Lambda p_0\rangle\big(u^0(y)-u^0(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx - m\int_{\mathbb{R}^d}\langle p_0\rangle u^0(x)\varphi(x)dx$$
$$= \int_{\mathbb{R}^d}\langle p_0\rangle f(x))\varphi(x)dx.$$

It remains to divide this equation by $\langle p_0\rangle$ and denote $\Lambda^{\mathrm{eff}} = \langle p_0\rangle^{-1}\langle \Lambda p_0\rangle$. Then the limit equation takes the form

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\Lambda^{\mathrm{eff}}\big(u^0(y)-u^0(x)\big)\varphi(x)}{|x-y|^{d+\alpha}}dydx - m\int_{\mathbb{R}^d} u^0(x)\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

Finally, we can complete the proof of Theorem 1.4. The weak convergence in $H^{\alpha/2}(\mathbb{R}^d)$ has already been proved. The convergence in $L^2(\mathbb{R}^d)$ can be shown in exactly the same way as in the proof of Theorem 1.1.

REFERENCES

[1] M. Arisawa, *Homogenization of a class of integro-differential equations with Lévy operators*, Comm. Partial Differential Equations, 34 (2009), pp. 617–624, https://doi.org/10.1080/03605300902963518.

[2] M. Arisawa, *Homogenizations of integro-differential equations with Lévy operators with asymmetric and degenerate densities*, Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), pp. 917–943, https://doi.org/10.1017/S0308210510001897.

[3] G. Barles, E. Chasseigne, A. Ciomaga, and C. Imbert, *Large time behavior of periodic viscosity solutions for uniformly parabolic integro-differential equations*, Calc. Var. Partial Differential Equations, 50 (2014), pp. 283–304, https://doi.org/10.1007/s00526-013-0636-2.

[4] R. F. Bass, *Regularity results for stable-like operators*, J. Funct. Anal., 257 (2009), pp. 2693–2722, https://doi.org/10.1016/j.jfa.2009.05.012.

[5] V. Berdichevsky, V. Jikov, and G. Papanicolaou, eds., *Homogenization*, Series on Advances in Mathematics for Applied Sciences 50, World Scientific, River Edge, NJ, 1999, https://doi.org/10.1142/9789812812919, in memory of Serguei Kozlov.

[6] A. Braides, *Gamma-Convergence for Beginners*, Oxford Lecture Ser. Math. Appl., Oxford University Press, Oxford, 2005, https://books.google.no/books?id=RYupPwAACAAJ.

[7] L. Bălilescu, A. Ghosh, and T. Ghosh, *Homogenization for Non-Local Elliptic Operators in Both Perforated and Non-Perforated Domains*, https://arxiv.org/pdf/1805.06264, 2018.

[8] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32 (2007), pp. 1245–1260, https://doi.org/10.1080/03605300600987306.

[9] G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev, *Homogenization*, Trans. Math. Monogr. 234, American Mathematical Society, Providence, RI, 2007, Methods and Applications, translated from the 2007 Russian original by Tamara Rozhkovskaya.

[10] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford Lecture Ser. Math. Appl. 17, Clarendon Press, Oxford University Press, New York, 1999.

[11] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[12] G. Di Nunno, B. Ø. Ksendal, and F. Proske, *Malliavin Calculus for Lévy Processes with Applications to Finance*, Universitext, Springer-Verlag, Berlin, 2009, https://doi.org/10.1007/978-3-540-78572-9.

[13] E. Eberlein, *Application of generalized hyperbolic Lévy motions to finance*, in Lévy Processes, Birkhäuser Boston, Boston, MA, 2001, pp. 319–336.

[14] A. M. Edwards, R. A. Phillips, N. W. Watkins, M. P. Freeman, E. J. Murphy, V. Afanasyev, S. V. Buldyrev, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, *Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer*, Nature, 449 (2007), pp. 1044–1048.

[15] J. Fernández Bonder, A. Ritorto, and A. M. Salort, *H-convergence result for nonlocal elliptic-type problems via Tartar's method*, SIAM J. Math. Anal., 49 (2017), pp. 2387–2408, https://doi.org/10.1137/16M1080215.

[16] M. Focardi, *Aperiodic fractional obstacle problems*, Adv. Math., 225 (2010), pp. 3502–3544, https://doi.org/10.1016/j.aim.2010.06.014.
[17] B. Franke, *The scaling limit behaviour of periodic stable-like processes*, Bernoulli, 12 (2006), pp. 551–570, https://doi.org/10.3150/bj/1151525136.
[18] B. Franke, *Correction to: "The scaling limit behaviour of periodic stable-like processes"* [Bernoulli, 12 (2006), pp. 551–570; mr2232732], Bernoulli, 13 (2007), p. 600, https://doi.org/10.3150/07-BEJ5127.
[19] B. Franke, *A functional non-central limit theorem for jump-diffusions with periodic coefficients driven by stable Lévy-noise*, J. Theoret. Probab., 20 (2007), pp. 1087–1100, https://doi.org/10.1007/s10959-007-0099-5.
[20] T. Fujiwara and M. Tomisaki, *Martingale approach to limit theorems for jump processes*, Stochastics Stochastics Rep., 50 (1994), pp. 35–64, https://doi.org/10.1080/17442509408833927.
[21] M. Horie, T. Inuzuka, and H. Tanaka, *Homogenization of certain one-dimensional discontinuous Markov processes*, Hiroshima Math. J., 7 (1977), pp. 629–641, http://projecteuclid.org/euclid.hmj/1206135757.
[22] N. Humphries and D. Sims, *Optimal foraging strategies: Lévy walks balance searching and patch exploitation under a very broad range of conditions*, J. Theoret. Biol., 358 (2014), pp. 179–193.
[23] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994, https://doi.org/10.1007/978-3-642-84659-5, translated from the Russian by G. A. Yosifian [G. A. Iosifyan].
[24] M. A. Krasnosel'skij, J. A. Lifshits, and A. V. Sobolev, *Positive Linear Systems*, Sigma Series in Applied Mathematics 5, Heldermann Verlag, Berlin, 1989, The Method of Positive Operators, translated from the Russian by Jürgen Appell.
[25] M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl., 1950 (1950), p. 128.
[26] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1, Grundlehren Math. Wiss., Springer, Berlin, 1972.
[27] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep., 339 (2000), p. 77, https://doi.org/10.1016/S0370-1573(00)00070-3.
[28] S. A. Molčanov and E. Ostrovskiĭ, *Symmetric stable processes as traces of degenerate diffusion processes*, Teor. Veroyatn. Primen., 14 (1969), pp. 127–130.
[29] D. Nualart and W. Schoutens, *Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance*, Bernoulli, 7 (2001), pp. 761–776, https://doi.org/10.2307/3318541.
[30] A. Piatnitski and E. Zhizhina, *Periodic homogenization of nonlocal operators with a convolution-type kernel*, SIAM J. Math. Anal., 49 (2017), pp. 64–81, https://doi.org/10.1137/16M1072292.
[31] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* II. *Fourier Analysis, Self-Adjointness*, Academic Press [Harcourt Brace Jovanovich], New York, 1975.
[32] R. Rhodes and V. Vargas, *Scaling limits for symmetric Itô-Lévy processes in random medium*, Stochastic Process. Appl., 119 (2009), pp. 4004–4033, https://doi.org/10.1016/j.spa.2009.10.004.
[33] N. Sandrić, *Homogenization of periodic diffusion with small jumps*, J. Math. Anal. Appl., 435 (2016), pp. 551–577, https://doi.org/10.1016/j.jmaa.2015.10.054.
[34] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Stud. Adv. Math. 68, Cambridge University Press, Cambridge, 2013, translated from the 1990 Japanese original, revised edition of the 1999 English translation.
[35] R. Schilling and T. Uemura, *Homogenization of Symmetric Lévy Processes on* $\mathbb{R}^d$, https://arxiv.org/pdf/1808.01667 (2018).
[36] R. W. Schwab, *Periodic homogenization for nonlinear integro-differential equations*, SIAM J. Math. Anal., 42 (2010), pp. 2652–2680, https://doi.org/10.1137/080737897.
[37] R. W. Schwab, *Stochastic homogenization for some nonlinear integro-differential equations*, Comm. Partial Differential Equations, 38 (2013), pp. 171–198, https://doi.org/10.1080/03605302.2012.741176.
[38] L. Tartar, *The General Theory of Homogenization*, Lecture Notes of the Unione Matematica Italiana 7, Springer-Verlag, Berlin; UMI, Bologna, 2009, https://doi.org/10.1007/978-3-642-05195-1, personalized introduction.
[39] M. Tomisaki, *Homogenization of càdlàg processes*, J. Math. Soc. Japan, 44 (1992), pp. 281–305, https://doi.org/10.2969/jmsj/04420281.

[40] V. V. Uchaikin and V. M. Zolotarev, *Chance and stability*, Modern Probability and Statistics, VSP, Utrecht, 1999, https://doi.org/10.1515/9783110935974, Stable Distributions and Their Applications, with a foreword by V. Yu. Korolev and Zolotarev.

[41] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, *Optimizing the success of random searches*, Nature, 401 (1999), pp. 911–914.

[42] W. A. Woyczyński, *Lévy processes in the physical sciences*, in Lévy Processes, Birkhäuser Boston, Boston, MA, 2001, pp. 241–266.

[43] V. V. Zhikov, *On two-scale convergence*, Tr. Semin. im. I. G. Petrovskogo, (2003), pp. 149–187, 410, https://doi.org/10.1023/B:JOTH.0000016052.48558.b4, translation in J. Math. Sci. (N.Y.), 120 (2004), pp. 1328–1352.