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EFFECTIVE DIFFUSION FOR A PARABOLIC OPERATOR WITH PERIODIC POTENTIAL*

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Abstract. The asymptotic behaviour of effective diffusion for a parabolic equation in R^n with periodic potential and small initial diffusion is discussed. The potential is assumed to be localized in periodic islands—sets around points of the integer lattice in R^n, where the density of diffusing particles increases. Off these islands the particles are annihilated. Logarithmic asymptotics of the effective diffusion are found when the initial diffusion tends to zero in terms of the geometrical characteristics of the given potential. These results rely on the large deviation technique for the diffusion of respective particles. For symmetric islands, the logarithmic asymptotics of the effective diffusion depend only on the distance between the islands.

Key words. homogenization, effective diffusion, large deviations

AMS(MOS) subject classification. 35B27

Introduction. We consider here a diffusion model that allows the creation and annihilation of particles. We assume that the particles are in a hostile medium with periodic inclusions of favourable islands. We discuss the behaviour of a population of particles in such media for a long time. This question, as it was discovered in more general framework in [1], is related to a so-called automodel homogenization theory for parabolic equations (see [2]). According to that approach, the long-time behaviour of such systems could be described in terms of effective diffusion, which, in our model, depends on the geometry of the potential and the initial diffusivity. This dependence is quite inexplicit, and the aim of this paper is to find the asymptotics of effective diffusion assuming that the initial diffusivity vanishes. We show that these asymptotics are defined by the following typical behaviour of the particle: It spends an exponentially long time inside the island and then makes a quick jump to one of the nearest islands.

The respective parabolic equation for the density of the particles is

$$\frac{\partial}{\partial t} u = (\mu^2\Delta + v(x))u, \qquad u\big|_{t=0} = u_0(x), \tag{0.1}$$

where we suppose that the potential is given by

$$v(x) = \begin{cases} 1, & \textit{if } \exists k \in Z^n\colon x \in (Q+k); \\ 0, & \text{otherwise,} \end{cases} \tag{0.2}$$

where Q is a simply connected bounded domain that is diffeomorphic to the ball and such that dist $(Q, (Q+k)) > 0$ for any $k \in Z^n \backslash \{0\}$ and Z^n is integer lattice. In (0.1), μ^2 is the initial molecular diffusivity, u is the density of the particles, and u_0 is the initial density. For fixed μ, the long-time behaviour of the solution u is described in terms of the first eigenvalue and eigenfunction of the periodic eigenvalue problem for the Schrödinger operator

$$(\mu^2\Delta + v(x))p = \lambda p. \tag{0.3}$$

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We normalize this eigenfunction $p > 0$ by the condition

$$\langle p \rangle = 1, \tag{0.4}$$

where $\langle \cdot \rangle$ means the average of a periodic function over the period cube. The diffusion properties of the solution of (0.1) are described in terms of so-called effective or homogenized diffusion. Let us define for $i = 1, 2, \dots, n$ the periodic functions ψ_i from the equations

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} p^2 \frac{\partial}{\partial x_j} (\psi_i + x_i) = 0. \tag{0.5}$$

Then the effective diffusion matrix $\sigma(\mu)$ is defined as

$$\sigma(\mu) = \langle (I + \nabla\psi)^T p^2 (I + \nabla\psi) \rangle / \langle p^2 \rangle, \tag{0.6}$$

where $\nabla\psi = ((\partial/\partial x_i)\psi_j)$, I is the unit matrix, and the symbol T means transposition of the matrix. Now, for the solution (0.1), we have the following asymptotics (see [1]) as $t \to \infty$ in the region $\{(x, t)\colon x^2 < c_1 t + c_2\}$:

$$u(x, t) \sim \exp(-\lambda t) p(x) \hat{u}(x, t)(1 + o(1)), \tag{0.7}$$

where $\hat{u}(x, t)$ satisfies the parabolic equation

$$\frac{\partial}{\partial t} \hat{u} = \nabla \cdot (\sigma \nabla \hat{u}), \qquad \hat{u}\big|_{t=0} = \langle p^2 \rangle u_0(x), \tag{0.8}$$

where $\sigma \equiv \sigma(\mu) = (\sigma(\mu)_{ij})$ is the effective diffusion matrix, given by (0.6). The main aim of this paper is the investigation of $\sigma(\mu)$ when $\mu \to 0$.

As is clear from formula (0.6), to this end it is necessary to find the behaviour of p for small μ and also to study (0.5). However, we will use another reduction of effective diffusion. Namely, we will transform our equation to an equation with potential drift and then use our previous results [3]. It is shown that for effective diffusion,

$$\lim_{\mu \to 0} \mu \ln \sigma(\mu) = -\hat{\Theta},$$

where $\hat{\Theta}$ is a positive matrix, whose coefficients are given below (see Theorems 1 and 2), in the terms of geometric characteristics of the domain Q and the logarithm is well defined for positive matrix. In the case of cubically symmetric sets Q, our theorem states that $\hat{\Theta} = dI$ where d is the shortest distance between the islands.

It should be noted that an alternative approach for the investigation of the effective characteristics for the equations with vanishing viscosity was proposed in [4] and [5], where full asymptotic expansion for the equation with potential drift was constructed. However, the last method exploits smoothness of the potential, at least near the saddle points, and that is not the case of this paper, where singularity at the saddle point occurs, since it is the caustic point for the respective flow.

1. Eigenvalue problem for periodic operator. Here we find the behaviour of the first eigenvalue and eigenfunction of (0.3). It is more convenient to introduce a potential $V = v - 1$ and divide (0.3) by μ^2. This leads to

$$\left(\Delta + \frac{1}{\mu^2} V\right) p = \lambda p, \tag{1.1}$$

where we keep the same notation for the obviously changed eigenvalue λ. Let $\kappa, \kappa_1, \kappa_2, \dots$ and $z(y), z_1(y), z_2(y), \dots$ be, respectively, the eigenvalues and eigenfunctions of the following Dirichlet problem in the domain Q:

$$\Delta z = \kappa z, \qquad z\big|_{\partial Q} = 0. \tag{1.2}$$

set $\{x \in Q: \operatorname{dist}(x, \partial Q) = \alpha\}$. Rescaling the variables $y = x/\mu$, we come to the equation $(\Delta + V(\mu y) - \mu^2\lambda)p(\mu y) = 0$ with bounded coefficients. In these coordinates, an integral of p^2 over the rescaled period torus is less than $c\mu^{-n}$. Then, using De Giorgi's estimate [7, Thm. 8.24], we find that $p(x) \leqq c\mu^{-n/2}$ for all x from T^n. Let us consider (1.1) in the domain $\{x \in T^n \setminus Q: \operatorname{dist}(x, \partial Q) < 2\alpha\}$. According to the last estimate and Proposition 2, we obtain

$$p|_{s=\alpha} \leqq c \exp\left(-\frac{\alpha}{2\mu}\right), \tag{1.14}$$

with coordinates (s, φ), which were defined in Proposition 1. Now let us consider a function $p(x)$ in Q_α and represent it as a sum $p = p^1 + p^2$, where $A^\mu p^i = 0$, $p^1|_{s=-\alpha} = p|_{s=-\alpha}$, $p^1|_{s=\alpha} = 0$, $p^2|_{s=-\alpha} = 0$, $p^2|_{s=\alpha} = p|_{s=\alpha}$. From Proposition 1, considering the uniform boundedness of p in μ on the set $\{x: s = -\alpha\}$, we have

$$p^1|_{\partial Q} \leqq c\mu. \tag{1.15}$$

Next, let us rewrite the equation for p^2 in the form $(-\Delta + \lambda)p^2 = (1/\mu^2)Vp^2$ and compare p^2 with a solution of a problem $(-\Delta + \lambda)p^3 = 0$, $p^3|_{s=\pm\alpha} = p^2|_{s=\pm\alpha}$. As we mentioned above, this problem is well posed, so that, by (1.14), we have $p^3 \leqq c \exp(-\alpha/2\mu)$. Then, according to the maximum principle and the positivity of $-(1/\mu^2)V(x)p^2(x)$, the estimate $p^2(x) \leqq p^3(x) \leqq c \exp(-\alpha/2\mu)$ holds. To complete the proof of Proposition 3, it is sufficient to compose that estimate with (1.15).

PROPOSITION 4. *The eigenfunction p satisfies the estimate $|\nabla p||_{\partial Q} \leqq c$, where c is independent of μ.*

Proof. This proposition is a simple consequence of standard elliptic estimates [7, Thm. 8.32]. Indeed, after rescaling the coordinates $y = x/\mu$ in a neighbourhood of a point on ∂Q and considering (1.13), we obtain $|\nabla p(\mu y)| \leqq c\mu$. Therefore, in x-coordinates we have the required estimate.

PROPOSITION 5. *Function $p(x)$ satisfies*

$$\int_{T^n \setminus Q} |\nabla p|^2 \, dx \leqq c\mu. \tag{1.16}$$

Proof. $p(x)$ minimizes the variational problem

$$-\lambda = \inf_{\|q\|_{L^2(T^n)}=1} \left(\int_Q |\nabla q|^2 \, dx + \int_{T^n \setminus Q} |\nabla q|^2 \, dx + \frac{1}{\mu^2} \int_{T^n \setminus Q} q^2 \, dx \right). \tag{1.17}$$

As in Proposition 1, using (1.13), we can prove the estimate

$$p(x) \leqq c\mu \exp\left(-\frac{1}{2\mu} \operatorname{dist}(x, Q)\right), \tag{1.18}$$

for all $x \in T^n \setminus Q$. So, for $q = p$, the third term on the right-hand side of (1.17) is less than $c\mu$ for some positive constant c. Let us suppose that (1.16) does not hold. In this case, we can define a function

$$p'(x) = \begin{cases} p(x), & x \in Q, \\ p(0, \varphi) \exp(-s/\mu)\phi(s), & x \in T^n \setminus Q, \end{cases}$$

where ϕ is C^∞-function, $0 \leqq \phi \leqq 1$, $\phi(s) = 1$ for $s < \alpha/2$ and $\phi(s) = 0$ for $s \geqq \alpha$. From Proposition 3, we have

$$\int_{T^n \setminus Q} |\nabla p'|^2 \, dx + \frac{1}{\mu^2} \int_{T^n \setminus Q} p'^2 \, dx \leqq c\mu. \tag{1.19}$$

Hence the function $p'(x)$ satisfies

$$\|p'\|_{L^2(T^n)} = 1 + O(\mu), \qquad \int_Q |\nabla p'|^2\,dx = \int_Q |\nabla p|^2\,dx.$$

Then our assumption that (1.16) is false contradicts the fact that $p(x)$ minimizes (1.17). To complete the proof of Lemma 1, let us introduce the function

$$p''(x) = \begin{cases} p(s,\varphi) - p(0,\varphi)\phi(-s)\exp(s/\mu), & x \in Q, \\ 0, & x \notin Q. \end{cases} \tag{1.20}$$

By Proposition 3 and (1.18),

$$\|p''\|_{L^2(T^n)} = \|p''\|_{L^2(Q)} = 1 + O(\mu). \tag{1.21}$$

From the definition of p'' and Propositions 4 and 5, it follows that

$$\left| \int_{T^n} |\nabla p''|^2\,dx - \int_{T^n} |\nabla p|^2\,dx \right| \leqq c\mu. \tag{1.22}$$

Now, we have

$$\begin{aligned} \kappa &\geqq -\frac{1}{\|p''\|^2_{L^2(Q)}} \int_Q |\nabla p''|^2\,dy = -(1+O(\mu)) \int_{T^n} |\nabla p''|^2\,dy \\ &\geqq -\int_{T^n} |\nabla p|^2\,dy - c\mu \geqq -\int_{T^n} |\nabla p|^2\,dy - \frac{1}{\mu^2}\int_{T^n\setminus Q} p^2\,dy - c\mu \geqq \lambda - c\mu. \end{aligned}$$

Here we use (1.21) and (1.22). Thus the lemma is proved.

Now, let us continue the eigenfunctions $z(x)$ and $z_i(x)$, $i = 1, 2, \ldots$ of the problem (1.2), which are defined only on Q into $T^n \setminus Q$ by 0, and denote these functions by $z(x)$ and $z_i(x)$, respectively.

LEMMA 2. *The eigenfunctions $p(x)$ and $z(x)$ satisfy*

$$\lim_{\mu\to 0} |p - z|_{C(T^n)} = 0.$$

Proof. In $T^n \setminus Q$, $z = 0$, therefore, considering (1.18), we obtain

$$|p - z|_{C(T^n\setminus Q)} \leqq c\mu. \tag{1.23}$$

Now, let us show that it suffices to prove the following estimate:

$$\lim_{\mu\to 0} \|p - z\|_{L^2(T^n)} = 0. \tag{1.24}$$

Indeed, in Q we have $(\Delta - \lambda)(p - z) = -(\Delta - \lambda)z = -(\Delta - \kappa)z - (\kappa - \lambda)z = -(\kappa - \lambda)z$. The function $z(x)$ is bounded and does not depend on μ. At the same time, by Lemma 1, $|\kappa - \lambda|$ is less than $c\mu$, so that $|(\kappa - \lambda)z| \leqq c\mu$. Then, by estimation of the solution of elliptic equations (cf. [7, Thm. 8.24]), we have in the domain $\tilde{Q}_\alpha = \{x \in Q\colon \operatorname{dist}(x, \partial Q) > \alpha\}$, $\alpha > 0$,

$$|p - z|_{C(\tilde{Q}_\alpha)} \leqq c(\alpha)\|p - z\|_{L^2(Q)} + c\mu, \tag{1.25}$$

where we also use the uniform boundness of λ with respect to μ. Then, for sufficiently small $\alpha > 0$, the Dirichlet problem for operator $(\Delta - \lambda)$ is well posed in a domain $Q \setminus \tilde{Q}_\alpha$. Therefore, the estimate

$$|p - z|_{C(Q\setminus\tilde{Q}_\alpha)} \leqq c(\alpha)(c\mu + c|p - z|_{C(\partial Q \cup \partial\tilde{Q}_\alpha)})$$

holds. Together with (1.23) and (1.25), this estimate leads to the required result.

To prove (1.24), let us recall the following definition of p'':

$$p''(x)=\begin{cases} p(s,\varphi)-p(0,\varphi)\phi(-s)\exp(s/\mu), & x\in Q,\\ 0, & x\notin Q.\end{cases}$$

It is clear that $\|p''\|_{L^2(Q)}=\|p''\|_{L^2(T^n)}=1+O(\mu)$, $\|p''-p\|_{L^2(T^n)}\to 0$ when $\mu\to 0$. The function $p''(x)$ can be expanded in a Fourier series

$$p''(x)=\alpha_0 z+\alpha_1 z_1+\cdots=\alpha_0 z+\tilde z,$$

where $\alpha_0=\langle p''z\rangle$, $\alpha_i=\langle p''z_i\rangle$, and

$$\alpha_0^2+\|\tilde z\|_{L^2}^2=\|p''\|_{L^2}^2=1+O(\mu), \tag{1.26}$$

$$-\langle\nabla z\cdot\nabla\tilde z\rangle=\langle\Delta z\tilde z\rangle=\kappa\langle z\tilde z\rangle=0. \tag{1.27}$$

According to (1.26), (1.27), Proposition 5, and the variational properties of κ_1, we obtain

$$\begin{aligned}-\lambda&=\int_{T^n}|\nabla p|^2\,dx+\frac{1}{\mu^2}\int_{T^n\setminus Q}p^2\,dx\geqq\int_Q|\nabla p''|^2\,dx-c\mu\\ &=\alpha_0^2\int_Q|\nabla z|^2\,dx+\int_Q|\nabla\tilde z|^2\,dx-c\mu\geqq\alpha_0^2(-\kappa)+(1-\alpha_0^2)(-\kappa_1)-c\mu.\end{aligned}$$

By Lemma 1, we have $-\kappa\geqq-\lambda$. After simple transformation, we can write $1-\alpha_0^2\leqq c\mu/|\kappa-\kappa_1|$. Therefore, $\alpha_0\to 1$ when $\mu\to 0$. This implies (1.24).

The behaviour of the next eigenvalue λ_1 of the problem (0.3) is described by the following lemma.

LEMMA 3. *We have the following limiting relation*:

$$\lim_{\mu\to 0}\lambda_1(\mu)=\kappa_1.$$

Proof. We can find κ_1 as the solution of the variational problem

$$\kappa_1=\sup_{\|\tilde z\|_{L^2}=1,\langle\tilde z z\rangle=0}\left(-\int_Q|\nabla\tilde z|^2\,dx\right).$$

The eigenfunction $z_1(x)$ gives the minimum of this problem. By Lemma 2,

$$\lim_{\mu\to 0}\langle pz_1\rangle=0.$$

Let us fix an arbitrary point x_0 inside Q and define a function

$$\theta(x)=\begin{cases} r_0^2-|x-x_0|^2, & |x-x_0|\leqq r_0,\\ 0, & \text{otherwise},\end{cases}$$

where $r_0=\operatorname{dist}(x_0,\partial Q)$. Considering the positivity of $z(x)$ and Lemma 2, we find that $\tilde\alpha=(\langle z_1p\rangle/\langle\theta p\rangle)\to 0$ when $\mu\to 0$. Therefore, for a function $\tilde z_1=z_1-\tilde\alpha\theta$, we obtain the following limit relations:

$$\lim_{\mu\to 0}\|\tilde z_1-z_1\|_{H^1(T^n)}=0,\qquad \lim_{\mu\to 0}\|\tilde z_1\|_{L^2(T^n)}=1.$$

Therefore,

$$\lim_{\mu\to 0}\left(-\int_{T^n}|\nabla\tilde z_1|^2\,dx\right)=\kappa_1.$$

Furthermore, according to the choice of $\tilde{\alpha}$, we have $\langle p\tilde{z}_1\rangle = 0$. Finally, from previous relations, we conclude that

$$\kappa_1 = \lim_{\mu\to 0} \int_{T^n} -|\nabla \tilde{z}_1|^2\, dx \leqq \liminf_{\mu\to 0} \sup_{\|\tilde{p}\|_{L^2}=1, \langle \tilde{p}p\rangle = 0} \left(-\int_{T^n} |\nabla \tilde{p}|^2\, dx \right) = \liminf_{\mu\to 0} \lambda_1 .$$

This inequality yields the uniform boundedness of λ_1 in μ.

Now the statements of Propositions 3-5 can be proved for λ_1 and $p_1(x)$ exactly as above. Then, the end of the proof of Lemma 3 is similar to the proof of Lemma 1.

COROLLARY 1. *It follows from Lemmas* 1 *and* 3 *that, uniformly in* μ,

$$|\lambda_1(\mu) - \lambda(\mu)| \geqq c > 0.$$

Now let us denote by Ω *the domain* $\{x \in R^n\colon \operatorname{dist}(x, Q) < \operatorname{dist}(x, \bigcup_{k\in Z^n\setminus\{0\}} (Q+k))\}$. *It is obvious that* $\bigcup_{k\in Z^n} (\bar{\Omega} + k) = R^n$.

LEMMA 4. *Uniformly in* $x \in \Omega\setminus Q$,

$$\lim_{\mu\to 0} \mu \ln p(x) = -\operatorname{dist}(x, \partial Q). \tag{1.28}$$

Proof. Let us first prove the estimate

$$p(x)|_{\partial Q} \geqq c\mu. \tag{1.29}$$

For this purpose, we construct a barrier function $\bar{u}(s)$ as a solution of the following auxiliary problem:

$$\frac{d^2}{ds^2}\bar{u} + 2a\frac{d}{ds}\bar{u} + \frac{1}{\mu^2} V(s)\bar{u} = 0,$$

$$\bar{u}(-\alpha) = 1, \qquad \bar{u}(\alpha) = 0,$$

where $a = 2\max|b_1|$. Using an explicit formula for the solution, as in Proposition 1, we can show that $\bar{u}(s)$ is a monotonic function that satisfies inequality

$$\bar{u}(0) \geqq c\mu. \tag{1.30}$$

By Lemma 2 and the positivity of $z(x)$, we have $p|_{s=-\alpha} \geqq c > 0$ uniformly in $\mu > 0$. Then, for some $c_1 > 0$, the difference $(p - c_1\bar{u})$ is positive on the set $\{x\colon s = \pm\alpha\}$ and satisfies the relation

$$\begin{aligned}
\left(\Delta + \frac{1}{\mu^2} V - \lambda\right)(p - c_1\bar{u}) &= -c_1\left(\Delta + \frac{1}{\mu^2} V - \lambda\right)\bar{u} \\
&= -c_1\left(\frac{d^2}{ds^2} + b_1(s,\varphi)\frac{d}{ds} + \frac{1}{\mu^2} V(s) - \lambda\right)\bar{u}(s) \\
&= -c_1\left(\frac{d^2}{ds^2} + 2a\frac{d}{ds} + \frac{1}{\mu^2} V(s)\right)\bar{u} \\
&\quad + c_1(2a - b_1(s,\varphi))\frac{d}{ds}\bar{u} + c_1\lambda\bar{u} \leqq 0.
\end{aligned}$$

Now (1.29) follows from (1.30) and the probabilistic representation for the solution of the last equation.

To estimate $p(x)$ from below in $T^n\backslash Q$, we rewrite (1.1) in the form $(\mu\Delta+(1/\mu)V-\mu\lambda)p=0$ and consider the corresponding diffusion process $\xi_t^x=\sqrt{2\mu}w_t^x$ starting from point x. According to [6, Chap. 4, Thm. 1.2], for any $\delta>0$, there exists $c(\delta)>0$, such that for all $x\in T^n\backslash Q$

$$\mathbf{P}\{\tau<T\}\geqq c(\delta)\exp\left(-\frac{\operatorname{dist}^2(x,\partial Q)/4T+\delta}{\mu}\right),$$

where τ is the exit time from $T^n\backslash Q$ for ξ_t^x. Substituting $T=\operatorname{dist}(x,\partial Q)/2$, we find that

$$\mathbf{P}\{\tau<\operatorname{dist}(x,\partial Q)/2\}\geqq c(\delta)\exp\left(-\frac{\operatorname{dist}(x,\partial Q)+\delta}{2\mu}\right).$$

Therefore, using the probabilistic representation of $p(x)$ and (1.29), we obtain

$$\begin{aligned}
p(x)&=\mathbf{M}\left(p(\xi_\tau^x)\exp\int_0^\tau\left(-\frac{1}{\mu}-\mu\lambda\right)dt\right)\\
&\geqq c_1\mu\mathbf{P}\{\tau<\operatorname{dist}(x,\partial Q)/2\}\exp\left(-\frac{\operatorname{dist}(x,\partial Q)}{2}\left(\frac{1}{\mu}+\mu\lambda\right)\right)\\
&\geqq c_1\mu c(\delta)\exp\left(-\frac{1}{\mu}\left[\frac{\operatorname{dist}(x,\partial Q)}{2}+\delta+\frac{\operatorname{dist}(x,\partial Q)}{2}+\mu^2\lambda\right]\right)\\
&\geqq c_1(\delta)\exp\left(-\frac{\operatorname{dist}(x,\partial Q)+2\delta}{\mu}\right).
\end{aligned}$$

This estimate implies that

$$\liminf_{\mu\to0}\mu\ln p(x)\geqq-\operatorname{dist}(x,\partial Q). \tag{1.31}$$

To prove the opposite estimate

$$\limsup_{\mu\to0}\mu\ln p(x)\leqq-\operatorname{dist}(x,\partial Q),$$

let us again write the solution $p(x)$ in probabilistic form

$$\begin{aligned}
p(x)&=\mathbf{M}\left(p(\xi_\tau^x)\exp\left(-\int_0^\tau\left(\frac{1}{\mu}+\mu\lambda\right)\mathrm{d}t\right)\right)\\
&\leqq c\mu\left(\mathbf{P}\left\{\tau<\frac{\operatorname{dist}(x,\partial Q)}{4}\right\}+\mathbf{P}\{\tau>\operatorname{dist}(x,\partial Q)\}\exp\left(-\frac{\operatorname{dist}(x,\partial Q)}{\mu}\right)\right.\\
&\qquad\left.+\int_{\operatorname{dist}(x,\partial Q)/4}^{\operatorname{dist}(x,\partial Q)}\frac{d}{ds}\mathbf{P}\{\tau<s\}\exp\left(-\frac{s}{\mu}\right)ds\right)\\
&\leqq c\mu\left(\mathbf{P}\{\tau<\operatorname{dist}(x,\partial Q)/4\}+\exp\left(-\frac{\operatorname{dist}(x,\partial Q)}{\mu}\right)\right.\\
&\qquad+\mathbf{P}\{\tau<s\}\exp\left(-\frac{s}{\mu}\right)\Big|_{s=\operatorname{dist}(x,\partial Q)/4}^{\operatorname{dist}(x,\partial Q)}\\
&\qquad\left.+\frac{1}{\mu}\int_{\operatorname{dist}(x,\partial Q)/4}^{\operatorname{dist}(x,\partial Q)}\mathbf{P}\{\tau<s\}\exp\left(-\frac{s}{\mu}\right)ds\right).
\end{aligned}$$

According to [6, Chap. 4, Thm. 1.2] for any $\delta>0$, uniformly in $s\in[\operatorname{dist}(x,\partial Q)/4, \operatorname{dist}(x,\partial Q)]$,

$$\mathbf{P}\{\tau<s\}\leqq c(\delta)\exp\left(-\frac{\operatorname{dist}^2(x,\partial Q)/4s-\delta}{\mu}\right).$$

This means that

$$\begin{aligned}p(x)&\leqq c\left(\exp\left(-\frac{\operatorname{dist}(x,\partial Q)-\delta}{\mu}\right)+\exp\left(-\frac{\operatorname{dist}(x,\partial Q)}{\mu}\right)\right.\\&\qquad\left.+\int_{\operatorname{dist}(x,\partial Q)/4}^{\operatorname{dist}(x,\partial Q)}\exp\left(-\frac{\operatorname{dist}^2(x,\partial Q)/4s+s-\delta}{\mu}\right)ds\right)\\&\leqq c\left(\exp\left(-\frac{\operatorname{dist}(x,\partial Q)-\delta}{\mu}\right)+\int_{\operatorname{dist}(x,\partial Q)/4}^{\operatorname{dist}(x,\partial Q)}\exp\left(-\frac{\operatorname{dist}(x,\partial Q)-\delta}{\mu}\right)ds\right)\\&\leqq c\exp\left(-\frac{\operatorname{dist}(x,\partial Q)-\delta}{\mu}\right).\end{aligned}$$

Thus the lemma is proved.

To study the limiting behaviour of $\nabla p(x)$ as μ tends to zero, we represent $p(x)$ in $\Omega\backslash Q$ in the form $p(x)=\exp(-s(x)/\mu)q(x)$, where $s(x)=\operatorname{dist}(x,\partial Q)$ and $q(x)$ is a new unknown function. Substituting it into (1.1), we find that

$$\Delta q-\frac{2}{\mu}\nabla s\cdot\nabla q-\frac{1}{\mu}\Delta s q-\lambda q=0. \tag{1.32}$$

This change of the unknown function is correct if and only if there are no caustic points in $\Omega\backslash Q$, i.e., if and only if *the rays that are extended out from ∂Q in the direction of the external normals to ∂Q do not intersect each other in Ω*. We henceforth suppose that this condition holds, and hence the function $s(x)$ is smooth. For instance, it is always true if the domain Q is convex.

From the definition of $q(x)$ and estimates of $p(x)$, we know that

$$c\mu\leqq q|_{\partial Q}\leqq c^{-1}\mu,\qquad c>0. \tag{1.33}$$

PROPOSITION 6. *Uniformly in each compact subset of $\Omega\backslash Q$,*

$$c\mu\leqq q(x)\leqq c^{-1}\mu,\qquad c>0. \tag{1.34}$$

Proof. By Lemma 4, for any $\delta>0$, there exists $c(\delta)$ such that

$$\max_{\partial\Omega} q(x)\leqq c(\delta)\exp(\delta/\mu).$$

Denote by $\tilde{\xi}_t^x$ the diffusion process corresponding to the operator $(\mu\Delta-\nabla s\cdot\nabla)$, and let τ be its exit time from $\Omega\backslash Q$. According to [6, Chap. 4, Thm. 1.2], there exist $t_0>0$ and $\gamma>0$ such that, uniformly in $x\in\Omega\backslash Q$,

$$\mathbf{P}\{\tau>t_0\}\leqq c\exp(-\gamma/\mu).$$

Considering the strong Markov property of $\tilde{\xi}_t^x$, we find that

$$\mathbf{P}\{\tau>kt_0\}\leqq c^k\exp(-k\gamma/\mu)$$

for any $x \in \Omega \backslash Q$. This implies, for all sufficiently small μ, the convergence of the integral $\mathbf{M}(q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x)+\mu\lambda)\, dt)$, which, in that case, represents the solution $q(x)$ of (1.32). Now let us write $q(x)$ in the form

$$\begin{aligned} q(x) = &\mathbf{M}\left(\chi_{\{\tau > t_0\}} q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x) + \mu\lambda)\, dt\right) \\ &+\mathbf{M}\left(\chi_{\{\tau < t_0\}} \chi_{\{\tilde{\xi}_\tau^x \notin \partial Q\}} q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x) + \mu\lambda)\, dt\right) \\ &+\mathbf{M}\left(\chi_{\{\tau < t_0\}} \chi_{\{\tilde{\xi}_\tau^x \in \partial Q\}} q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x) + \mu\lambda)\, dt\right). \end{aligned}$$

Here we denote by $\chi_{\{\cdot\}}$ the characteristic function. It follows from [6, Chap. 4, Lemma 2.1] that the second integral in the right-hand side is less than $c \exp(-\gamma_1/\mu) \exp(\delta/\mu)$ for some positive γ_1. Let us estimate the first integral as follows:

$$\begin{aligned} &\mathbf{M}\left(\chi_{\{\tau > t_0\}} q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x) + \mu\lambda)\, dt\right) \\ &\quad \leqq c(\delta) \exp(\delta/\mu) \sum_{k=1}^{\infty} c^k \exp(-k\gamma_1/\mu) \exp(c_2(k+1)) \leqq c_3 \exp\left(\frac{2\delta - \gamma_1}{\mu}\right). \end{aligned}$$

Choosing $\delta = \gamma_1/4$, we obtain

$$q(x) = \mathbf{M}\left(\chi_{\{\tau < t_0\}} \chi_{\{\tilde{\xi}_\tau^x \in \partial Q\}} q(\tilde{\xi}_\tau^x) \exp \int_0^\tau (\Delta s(\tilde{\xi}_t^x) + \mu\lambda)\, dt\right) + O(\exp(-\gamma_1/4\mu)).$$

To complete the proof of the proposition, it suffices to use (1.33). By (1.34) and standard Schauder estimates [7, Thm. 8.32], we have $|\nabla q(x)| \leqq c/\mu$. In fact, we have the following lemma.

LEMMA 5. *Uniformly on each compact subset of* $\Omega \backslash Q$,

$$|\nabla q(x)| = \mu o\left(\frac{1}{\mu}\right). \tag{1.35}$$

Proof. First, let us consider the following Dirichlet problem in the half-space $\{x \in R^n: x_1 > 0\}$:

$$\left(\Delta - \frac{\partial}{\partial x_1}\right) u = 0, \quad u|_{x_1=0} = \varphi(x'), \quad x' = (x_2, x_3, \cdots), \tag{1.36}$$

where $\varphi(x')$ is an arbitrary bounded $C^1(R^{n-1})$-function. This problem has a unique bounded solution $u(x)$.

PROPOSITION 7. *Uniformly in* $\varphi \in \{\varphi \in C^1(R^{n-1}): \sup_{R^{n-1}} |\varphi(x')| \leqq 1\}$

$$\lim_{x_1 \to \infty} |\nabla_{x'} u| = 0.$$

Proof. Let $\psi(z, t)$ be the bounded solution of the problem

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\right)\psi = 0,$$

$$\psi|_{t=0} = 0, \quad \psi|_{z=0} = 1, \quad \psi|_{z=\infty} = 0.$$

It is easy to see that

$$K(x, y) = \int_0^\infty \left(\frac{1}{2\pi t}\right)^{(n-1)/2} \exp\left(-\frac{|x'-y'|^2}{2t}\right) \frac{\partial}{\partial t} \psi(x_1, t)\, dt$$

is a Green's function for (1.36). The solution of the problem

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial z^2}\right)\tilde{u}=0,$$

$$\tilde{u}|_{t=0}=0, \quad \tilde{u}|_{z=0}=1, \quad \tilde{u}|_{z=\infty}=0$$

can be found explicitly as follows:

$$\tilde{u}(z,t)=\int_z^\infty \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)dy.$$

The function $\psi(z+t,t)$ obviously satisfies the equation $(\partial/\partial t-\partial^2/\partial z^2)\psi(z+t,t)=0$. According to the maximum principle, $\psi(z+t,t)|_{z=0}=\psi(t,t)<1$, so that $\psi(z,t)\leqq \tilde{u}(z-t,t)$ on the set $\{(z,t)\colon z>t,\ t>0\}$. Let us now estimate $(\partial/\partial x_2)u(x))$ as follows:

$$\begin{aligned}
\left|\frac{\partial}{\partial x_2}u(x_1,0)\right| &= \left|\int_{R^{n-1}}dx'\int_0^\infty\left(\frac{1}{2\pi t}\right)^{(n-1)/2}\left(\frac{\partial}{\partial x_2}\exp\left(-\frac{|x'|^2}{2t}\right)\right)\frac{\partial}{\partial t}\psi(x_1,t)\varphi(x')\,dt\right| \\
&= \left|\int_0^\infty dt\int_{R^{n-1}}\frac{x_2}{t}\left(\frac{1}{2\pi t}\right)^{(n-1)/2}\exp\left(-\frac{|x'|^2}{2t}\right)\frac{\partial}{\partial t}\psi(x_1,t)\varphi(x')\,dx'\right| \\
&\leqq c\left|\int_0^\infty\int_0^\infty\frac{x_2}{t}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x_2^2}{2t}\right)\frac{\partial}{\partial t}\psi(x_1,t)\,dx_2\,dt\right| \\
&= c\left|\int_0^\infty dt\int_0^\infty y\exp(-y^2)\frac{1}{\sqrt{t}}\frac{\partial}{\partial t}\psi(x_1,t)\,dy\,dt\right| \\
&\leqq c\left|\int_0^\infty\frac{1}{\sqrt{t}}\frac{\partial}{\partial t}\psi(x_1,t)\,dt\right| = c\int_0^\infty\left(\frac{1}{t}\right)^{3/2}\psi(x_1,t)\,dt \\
&\leqq c\int_0^{x_1/2}\left(\frac{1}{t}\right)^{3/2}\tilde{u}(x_1-t,t)\,dt+c\int_{x_1/2}^\infty\left(\frac{1}{t}\right)^{3/2}dt \\
&\leqq c\int_0^{x_1/2}\left(\frac{1}{t}\right)^{3/2}\exp\left(-\frac{x_1^2}{8t}\right)dt+c\frac{1}{\sqrt{x_1}}\leqq c\frac{1}{\sqrt{x_1}},
\end{aligned}$$

which proves the proposition.

PROPOSITION 8. *Under the same conditions as in Proposition* 7, *the following limit relation holds*:

$$\lim_{x_1\to\infty}\left|\frac{\partial}{\partial x_1}u\right|=0, \tag{1.38}$$

where the convergence is uniform for $\varphi\in\{\varphi\in C^1(R^{n-1})\colon \sup_{R^{n-1}}\varphi(x')\leqq 1\}$.

Proof. Let $\theta(t)$ be a C^∞-function such that $0\leqq\theta\leqq 1$, $\theta(t)=0$ when $t>2$ and $\theta(t)=1$ when $t<1$. The function $\tilde{u}(x)=(u(x)-u(M_1,0))\theta(M|x'|)$, where M and M_1 are positive constants, satisfies the equation

$$\begin{gathered}
\left(\Delta-\frac{\partial}{\partial x_1}\right)\tilde{u}=\nabla_{x'}u(x)\cdot\nabla_{x'}\theta(M|x'|)+(u(x)-u(M_1,0))\Delta\theta(M|x'|), \\
\tilde{u}|_{x_1=M_1}=(u(M_1,x')-u(M_1,0))\theta(M|x'|).
\end{gathered} \tag{1.39}$$

We denote the right-hand side and boundary condition of this problem by $\tilde{f}(x,M,M_1)$ and $\tilde{\varphi}(x,M,M_1)$, respectively. Considering the choice of $\theta(t)$, we can easily check that these two functions satisfy the following inequalities:

$$|\tilde{f}(x,M,M_1)|\leqq cM, \qquad |\tilde{\varphi}(x',M,M_1)|\leqq c\sup_{x_1=M_1}|\nabla_{x'}u|M^{-1}=M_2.$$

It is clear that function $(M_2 + cM(x_1 - M_1))$ is a barrier for $\tilde{u}(x)$ in the domain $\{x \in R^n: x_1 > M_1\}$, so that

$$|\tilde{u}(x)| \leqq M_2 + cM(x_1 - M_1). \tag{1.40}$$

According to Proposition 7, we can choose $M(M_1)$ in such a way that M and M_2 tend to zero when $M_1 \to \infty$. Finally, (1.38) follows from (1.40) and the standard Schauder estimate [7, Thm. 8.32].

To prove (1.35), let us fix an arbitrary point $x_0 \in \Omega \backslash \bar{Q}$ and introduce, in the neighbourhood of this point, a new orthonormal basis such that the first vector in it coincides with the normal ray to ∂Q passing through x_0. We denote coordinates in this basis by $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$. After rescaling the coordinates $z = \tilde{x}/\mu$, (1.32) takes the form

$$\Delta q - b(\mu z) \cdot \nabla q - \mu c(\mu z) q - \mu^2 \lambda q = 0, \tag{1.41}$$

where $b_i(\tilde{x})$ and $c(\tilde{x})$ are smooth bounded functions and $b_i(0) = \delta_{1i}$. By Proposition 6, for all sufficiently small μ, $q(z)/\mu$ is bounded in a ball of radius $2/\sqrt{\mu}$ with center in x_0/μ. Let $\theta(t)$ be the same function as in Proposition 8. Applying $(\Delta - \partial/\partial z_1)$ to the function $\tilde{q}(z) = (1/\mu) q(z) \theta(\sqrt{\mu}\,|z'|) \theta(\sqrt{\mu}\, z_1)$, we obtain

$$\left(\Delta - \frac{\partial}{\partial z_1}\right)\tilde{q} = O(\sqrt{\mu}), \qquad \tilde{q}|_{z_1=0} = O(1).$$

Now we can represent $\tilde{q}(z)$ as a sum $\tilde{q}(z) = \tilde{q}^1(z) + \tilde{q}^2(z)$, where

$$\left(\Delta - \frac{\partial}{\partial z_1}\right)\tilde{q}^1 = O(\sqrt{\mu}), \qquad \tilde{q}^1|_{z_1=0} = 0 \tag{1.42}$$

and

$$\left(\Delta - \frac{\partial}{\partial z_1}\right)\tilde{q}^2 = 0, \qquad \tilde{q}^2|_{z_1=0} = O(1).$$

According to Propositions 7 and 8, $|\nabla \tilde{q}^2|$ tends to zero when $z_1 \to \infty$. The linear function kz_1 with a suitable coefficient $k = O(\sqrt{\mu})$ is obviously the barrier function for $\tilde{q}^1(z)$ in (1.42). Therefore, on the set $\{z: z_1 = \mu^{-1/3}\}$, $\tilde{q}^1(z)$ tends to zero when $\mu \to 0$. Using the standard Schauder estimate, we conclude that $|\nabla \tilde{q}| = o(1)$ uniformly on the set $\{z: z_1 = \mu^{-1/3}\}$. Finally, in x-coordinates, we have $|\nabla \tilde{q}(x)| = o(1/\mu)$ on the set $\{x: \tilde{x}_1 = \mu^{2/3}\}$. To complete the proof, it is sufficient to note that $q(x) = \mu \tilde{q}(x)$ in the $\sqrt{\mu}$-neighbourhood of x_0.

In the remainder of this paper, the function $\nabla p/p$ plays a significant role. Let us consider its properties.

PROPOSITION 9. *The estimate*

$$\mu \left|\frac{\nabla p}{p}\right| \leqq c \tag{1.43}$$

holds uniformly for $x \in T^n$.

Proof. In the coordinates $y = x/\mu$, (1.1) has a form

$$(\Delta - V(\mu y) - \mu^2 \lambda) p = 0.$$

Consequently, $p(\mu y)$ satisfies the Harnack inequality [7, Thm. 8.20] in each unit ball. Together with the Schauder estimate [7, Thm. 8.32], this implies that

$$\left|\frac{\nabla_y p(\mu y)}{p(\mu y)}\right| \leqq c.$$

In x-coordinates, this gives (1.43).

PROPOSITION 10. *Uniformly on each compact subset of* Q,

$$\lim_{\mu\to 0} \mu \frac{\nabla p}{p} = 0. \tag{1.44}$$

Proof. The proof follows from Lemma 2, positivity of $z(x)$ in Q, and the Schauder estimate.

Now let us study the logarithmic derivative of $p(x)$ in $\Omega\backslash Q$. Here we have the following proposition.

PROPOSITION 11. *For any* $x \in \Omega\backslash\bar{Q}$,

$$\lim_{\mu\to 0} \mu \frac{\nabla p}{p} = -\nabla(\mathrm{dist}\,(x, \partial Q)),$$

where the convergence is uniform on each compact subset of $\Omega\backslash\bar{Q}$.

Proof. This proposition is a simple consequence of the definition of $q(x)$, Proposition 6, and Lemma 5. Indeed, it holds that

$$\mu\frac{\nabla p}{p} = -\nabla s + \mu\frac{\nabla q}{q} = \mu o\left(\frac{1}{\mu}\right) - \nabla(\mathrm{dist}\,(x, \partial Q))$$
$$= -\nabla(\mathrm{dist}\,(x, \partial Q)) + o(1).$$

2. Asymptotics of effective diffusion. In this section, we find the logarithmic asymptotics of the effective diffusion matrix of the problem (0.1). Using the properties of $p(x)$, which were obtained in the previous paragraph, we reduce this problem to another problem that was studied in [3].

Let us transform the operator $(A-\lambda) = (\Delta + (1/\mu^2)V(x) - \lambda)$ as follows:

$$(\tilde{A}-\lambda) = (p^{-1}Ap - \lambda) = p^{-1}(A-\lambda)p = \Delta + 2\frac{\nabla p}{p}\cdot\nabla.$$

We denote the last operator multiplied by μ as $\tilde{B}$. Since the vector field $2(\nabla p/p) = 2\nabla \ln p$ is the gradient of a periodic function, the homogenized operator has the form $\sigma_{ij}(\mu)(\partial/\partial x_i)(\partial/\partial x_j)$ for some strictly positive matrix $\sigma(\mu)$. It is clear that $\tilde{B}$ possesses the following properties:

P1. Its spectrum on the torus is $\{0, \mu(\lambda_1-\lambda), \mu(\lambda_2-\lambda), \dots\}$;

P2. The first eigenfunction of the adjoint operator is $cp^2(x)$, where $c = \langle p^2\rangle$ is a normalizing coefficient.

Further considerations are similar to those in [3]. The main difference is the asymptotic nonsmoothness of the vector field $\mu(\nabla p/p)$. To demonstrate the main ideas, let us first suppose that Q is a cubically symmetric domain. In that case, $\sigma_{ij}(\mu) = \sigma(\mu)I$ and the limit operator has the form $\sigma(\mu)\Delta$. Let ξ_t be a diffusion process on T^n corresponding to the operator $\tilde{B}$ with initial density $cp^2(x)$. We denote by s_0 the minimum of $s(x)$ over $\partial\Omega$. It is clear that $s_0 = (\mathrm{dist}\,(Q, \bigcup_{k\in Z^n\backslash\{0\}}(Q+k))/2$.

PROPOSITION 12. *For any* $\delta > 0$ *and* $T > 0$, *there exists* $c(\delta)$ *such that*

$$\mathbf{P}\{\xi_T \notin \Omega\} \leq c(\delta)\exp(-(2s_0-\delta)/\mu). \tag{2.1}$$

Proof. According to [6, Chap. 5, Thm. 3.2] and Proposition 11, the *action functional* of $\tilde{B}$ on any compact subset of $\Omega\backslash\bar{Q}$ is equal to the action functional of the operator $(\mu/2)\Delta - \nabla s(x)\cdot\nabla$. For any sufficiently small $\delta_1 > 0$, we can define the following sets: $S_1 = \{x \in \Omega\backslash Q\colon s(x) = \delta_1\}$, $S_2 = \{x\in\Omega\backslash Q\colon s(x) = 2\delta_1\}$, and $S_3 = \{x \in \Omega\colon \mathrm{dist}\,(x, \partial\Omega) = \delta_1\}$. Let ν_1 be the exit time from $R^{\delta_1} = \{x \in \Omega\backslash Q\colon \mathrm{dist}\,(x, \partial(\Omega\backslash Q)) > \delta_1\}$ for the process ξ_t^x starting from x. According to [4, Chap. 6, Lemma 2.1], there exists $t(\delta_1)$ such that, for all $x \in S_2$,

$$\mathbf{P}\{\nu_1 < t(\delta_1)\} \leqq c(\delta_1) \exp\left(-\frac{1}{\mu}\right). \tag{2.2}$$

Let us introduce the following sequence of Markov times:

$$\begin{aligned} \nu_1 &= \inf\{t > 0 \colon \xi_t^x \in S_1 \cup S_3\}, & \pi_1 &= \inf\{t > \nu_1 \colon \xi_t^x \in S_2\}, \\ \nu_2 &= \inf\{t > \pi_1 \colon \xi_t^x \in S_1 \cup S_3\}, & \pi_2 &= \inf\{t > \nu_2 \colon \xi_t^x \in S_2\}, \\ &\vdots \\ \nu_k &= \inf\{t > \pi_{k-1} \colon \xi_t^x \in S_1 \cup S_3\}, & \pi_k &= \inf\{t > \nu_k \colon \xi_t^x \in S_2\}. \end{aligned}$$

By [6], for any $\delta > 0$, there exists $c(\delta)$ such that

$$\mathbf{P}\{\xi_{\nu_1}^x \in S_3\} \leqq c(\delta) \exp\left(-2\frac{s_0 - s(x) - 2\delta_1 - \delta}{\mu}\right),$$

for any $x \in R^{\delta_1}$. Considering the strong Markov property of ξ_t and (2.2), we obtain

$$\begin{aligned} \mathbf{P}\{\xi_T^x \notin \Omega\} &\leqq \mathbf{P}\{\xi_{\nu_1}^x \in S_3\} \\ &\quad + \sum_{i=1}^{[T/t(\delta_1)]+1} [\mathbf{P}(\{\xi_{\nu_i}^x \in S_1\} \cap \{\xi_{\nu_{i+1}}^x \in S_3\}) + \mathbf{P}\{\nu_i < t(\delta_1)\}] \\ &\leqq \left(\frac{T}{t(\delta_1)} + 1\right) c(\delta) \left(\exp\left(-2\frac{s_0 - s(x) - 2\delta_1 - \delta}{\mu}\right) + \exp\left(-\frac{1}{\mu}\right)\right) \\ &= c(\delta_1, \delta) \exp\left(-2\frac{s_0 - s(x) - 2\delta_1 - \delta}{\mu}\right). \end{aligned}$$

Here we denote by $[\cdot]$ the integer part. Similarly, for $x \in \{x \in \Omega \colon \operatorname{dist}(x, Q) \leqq \delta_1\}$, we have

$$\mathbf{P}\{\xi_T^x \notin \Omega\} \leqq c(\delta, \delta_1) \exp\left(-2\frac{s_0 - 4\delta_1 - \delta}{\mu}\right).$$

Using the last two estimates, we can conclude that, for any $\delta > 0$ and $T > 0$, there exists $c(\delta)$ such that

$$\mathbf{P}\{\xi_T^x \notin \Omega\} \leqq c(\delta) \exp\left(-2\frac{s_0 - s(x) - \delta}{\mu}\right) \tag{2.3}$$

for $x \in \Omega \backslash Q$, and

$$\mathbf{P}\{\xi_T^x \notin \Omega\} \leqq c(\delta) \exp\left(-2\frac{s_0 - \delta}{\mu}\right) \tag{2.4}$$

for $x \in Q$. Finally, by Lemma 4 and Property P2 of $\tilde{B}$, the density $cp^2(x)$ of invariant measure of ξ_t satisfies the following relation uniformly in x:

$$\lim_{\mu \to 0} \mu \ln(cp^2(x)) = \begin{cases} 0, & x \in Q, \\ -2s(x), & x \in T^n \backslash Q. \end{cases} \tag{2.5}$$

Together with (2.3) and (2.4), this implies (2.1).

COROLLARY 2. *For any $\delta > 0$, there exists $c(\delta)$ such that*

$$\mathbf{P}\{\xi_{\exp(\delta/\mu)} \notin \Omega\} \leqq c(\delta) \exp\left(-2\frac{s_0 - 2\delta}{\mu}\right).$$

Proof. By (2.1), we find that

$$\mathbf{P}\{\xi_{\exp(\delta/\mu)}\notin\Omega\}\leqq\sum_{j=0}^{[\exp(\delta/\mu)/T]+1}\mathbf{P}(\{\xi_{jT}\in\Omega\}\cap\{\xi_{(j+1)T}\notin\Omega\})$$

$$\leqq(\exp(\delta/\mu)/T+1)c(\delta)\exp\left(-2\frac{s_0-\delta}{\mu}\right)\leqq c(\delta)\exp\left(-2\frac{s_0-2\delta}{\mu}\right).$$

Now let us estimate the probability of a jump of ξ_t into another period from below. It is clear that, for some integer vector $j^1\in Z^n\backslash\{0\}$, there exists $\tilde{x}\in\partial\Omega\cap\partial(\Omega+j^1)$ such that $\operatorname{dist}(\tilde{x},Q)=\operatorname{dist}(\tilde{x},Q+j^1)=s_0$.

PROPOSITION 13. *For any $\delta>0$, we can find $c(\delta)$ and $T(\delta)$ such that*

$$\mathbf{P}\{\operatorname{dist}(\xi_{T(\delta)},Q+j^1)<s_0/2\}\geqq c(\delta)\exp\left(-2\frac{s_0+\delta}{\mu}\right).$$

Proof. Since ξ_t is a strong Markov process, it is sufficient to prove the following two estimates:

(i) For any $\delta>0$, there exists $c(\delta)$ and $T(\delta)$ such that

$$\mathbf{P}\{|\xi_{T(\delta)}-\tilde{x}|<\delta\}\geqq c(\delta)\exp\left(-2\frac{s_0+\delta}{\mu}\right);$$

(ii) For any $\delta>0$, there exist $c(\delta)$ and $T(\delta)$ such that, uniformly in $x\in\{x\colon|x-\tilde{x}|<\delta\}$,

$$\mathbf{P}\{\operatorname{dist}(\xi^x_{T(\delta)},Q+j^1)<s_0/2\}\geqq c(\delta)\exp\left(-\frac{\delta_1}{\mu}\right),$$

where $\delta_1\to0$ when $\delta\to0$. The first is a simple consequence of [6], where we also use the structure of the action functional for $\tilde{B}$ in $\Omega\backslash Q$.

Let us prove (ii). The probability $\mathbf{P}\{\operatorname{dist}(\xi^x_t,Q+j^1)<s_0/2\}$ is the solution of the problem

$$\left(\frac{\partial}{\partial t}-\tilde{B}\right)\psi=0,\qquad\psi|_{t=0}=\phi(x),\tag{2.6}$$

where $\phi(x)$ is a characteristic function of the set $\{x\colon\operatorname{dist}(x,Q+j^1)<s_0/2\}$. Considering the choice of $\tilde{x}$, we can find a point $\bar{x}$ such that $|\tilde{x}-\bar{x}|=\delta$ and $\operatorname{dist}(\bar{x},Q+j^1)=s_0-\delta$. According to [6] and Proposition 11, for some $T(\delta)>0$,

$$\psi(T(\delta),\bar{x})=\mathbf{P}\{\operatorname{dist}(\xi^{\bar{x}}_{T(\delta)},Q+j^1)<s_0/2\}\geqq\tfrac{1}{2}.\tag{2.7}$$

In the coordinates $y=x/\mu$, $\tau=t/\mu^2$, (2.6) takes the form

$$\frac{\partial}{\partial\tau}\psi-\Delta\psi-2\mu\frac{\nabla p}{p}\cdot\nabla\psi=0.$$

By Proposition 9, $\mu(\nabla p/p)$ is uniformly bounded in μ and x, so that the Harnack inequality [8]

$$c\leqq\psi(\tau,y)/\psi(\tau,y')\leqq c^{-1}\tag{2.8}$$

holds when $|y-y'|\leqq1$ and $\tau\geqq1$. Let us cover the segment that connects an arbitrary point $x\in\{x\colon|x-\bar{x}|<2\delta\}$ and $\bar{x}$ by the sequence of balls of radius μ with centers on the segment. This can be done with the number of balls not exceeding $[4\delta/\mu+1]$. Iterating (2.8), we obtain

$$\psi(t,x)/\psi(t,\bar{x})\geqq c^{(4\delta/\mu)+1}\geqq\exp(-\delta_1/\mu),$$

where $\delta_1=-5\delta\ln c$. Together with (2.7), this inequality implies (ii).

LEMMA 1. *Eigenvalues λ and κ satisfy the inequality*

$$\kappa \leqq \lambda \leqq \kappa + c\mu, \tag{1.3}$$

where the constant c is independent of μ.

Proof. Let us use a variational representation for the first eigenvalue

$$\lambda = \sup_{\|u\|_{L^2(T^n)}=1} \left\{ -\int_{T^n} |\nabla u|^2\, dx - \frac{1}{\mu^2} \int_{T^n\setminus Q} u^2\, dx \right\}, \tag{1.4}$$

where $T^n = R^n/Z^n$. If we reduce the variational set in (1.4) to functions that are identically zero on $T^n\setminus Q$, we obtain the definition of κ, and hence $\kappa \leqq \lambda$. Let us prove the second inequality in (1.3). Denote by Q_α the set $\{x \in T^n\colon \operatorname{dist}(x, \partial Q) < \alpha\}$. It is quite easy to see that, for sufficiently small α, the domain Q_α has a smooth boundary, and the equation

$$\left(\Delta + \frac{1}{\mu^2} V(x)\right) u = \lambda u + f, \qquad u|_{\partial Q_\alpha} = \varphi, \tag{1.5}$$

has a unique solution for any $f \in C^1(Q_\alpha)$ and $\varphi \in C^1(\partial Q_\alpha)$. For such α, (1.5) is uniformly invertible with respect to μ, and its solution satisfies the maximum principle. To see this, we must use the inequality $\kappa \leqq \lambda < 0$. Under the same assumption, the solution of (1.5) has a probabilistic representation (see [6, Chap. 1]), shown below:

$$\begin{aligned} u(x) = &-M\left(\int_0^\tau f(w_t^x) \exp\left(\int_0^t \left(\frac{1}{\mu^2} V(w_s^x) - \lambda\right) ds\right) dt\right) \\ &+ M\left(\varphi(w_\tau^x) \exp\left(\int_0^\tau \left(\frac{1}{\mu^2} V(w_t^x) - \lambda\right) dt\right)\right), \end{aligned} \tag{1.6}$$

where w_t^x is the standard Wiener process starting from x, and τ is exit time from Q_α. From (1.6) it follows, in particular, that, for $f \equiv 0$ and positive φ, the solution u is positive. To prove the lemma, we need the following proposition.

PROPOSITION 1. *The solution u of the problem*

$$\begin{gathered} A^\mu u = \left(\Delta + \frac{1}{\mu^2} V(x) - \lambda\right) u = 0, \\ u|_{\{x\in Q:\, \operatorname{dist}(x,\partial Q)=\alpha\}} = 1, \qquad u|_{\{x\notin Q:\, \operatorname{dist}(x,\partial Q)=\alpha\}} = 0, \end{gathered} \tag{1.7}$$

satisfies on ∂Q the following inequality:

$$u|_{\partial Q} \leqq c\mu, \tag{1.8}$$

where c is independent of μ.

Proof. Let us introduce on Q_α the new coordinates $(s, \varphi_1, \varphi_2, \dots, \varphi_{n-1})$, where $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$ are local coordinates on ∂Q, and s is the distance from ∂Q taken with positive sign outside Q and with negative sign inside Q. In these coordinates, (1.7) becomes

$$\begin{gathered} \frac{\partial^2 u}{\partial s^2} + a_{ij}(s, \varphi) \frac{\partial}{\partial \varphi_i} \frac{\partial}{\partial \varphi_j} u + b_1(s, \varphi) \frac{\partial}{\partial s} u + b_i(s, \varphi) \frac{\partial}{\partial \varphi_i} u + \frac{1}{\mu^2} Vu - \lambda u = 0, \\ u(-\alpha, \varphi) = 1, \qquad u(\alpha, \varphi) = 0, \end{gathered} \tag{1.9}$$

where $a_{ij}(s, \varphi)$ and $b_i(s, \varphi)$ are smooth functions, and the $(n-1)\times(n-1)$ matrix $\{a_{ij}\}$ is uniformly elliptic. To estimate the solution of (1.9), let us consider an auxiliary equation for the function $\bar{u}$

$$\frac{d^2}{ds^2} \bar{u} - 2a \frac{d}{ds} \bar{u} + \frac{1}{\mu^2} V\bar{u} + \kappa \bar{u} = 0,$$

with the same boundary conditions and $a=\max(2|\kappa|, \max|b_1|)$. It can be shown that the function $b_1(s,\varphi)$ does not depend on the choice of local coordinates, but only on the point x, so that the $\max|b_1|$ is defined correctly. The last equation can be solved explicitly as follows:

$$\bar{u}=\begin{cases}\alpha_1\exp(\nu_1 s)+\alpha_2\exp(\nu_2 s), & s\leqq 0,\\ \beta_1\exp(\mu_1 s)+\beta_2\exp(\mu_2 s), & s\geqq 0,\end{cases}\tag{1.10}$$

where

$$\nu_{1,2}=a\mp\sqrt{a^2+\kappa},\qquad \mu_{1,2}=a\mp\sqrt{a^2+\kappa+\frac{1}{\mu^2}},$$

$$\begin{aligned}\beta_1&=\frac{\mu_1-\nu_2+(\nu_2-\mu_2)\exp(\alpha(\mu_1-\mu_2))}{\nu_1-\nu_2}\exp(-\nu_1\alpha)\\&\quad-\frac{\mu_1-\nu_1+(\nu_1-\mu_2)\exp(\alpha(\mu_1-\mu_2))}{\nu_1-\nu_2}\exp(-\nu_2\alpha),\\ \beta_2&=-\exp(\alpha(\mu_1-\mu_2)),\\ \alpha_1&=\frac{\mu_1-\mu_2\exp(\alpha(\mu_1-\mu_2))-\nu_2+\nu_2\exp(\alpha(\mu_1-\mu_2))}{\nu_1-\nu_2}\beta_1,\\ \alpha_2&=\frac{\mu_1-\mu_2\exp(\alpha(\mu_1-\mu_2))-\nu_1+\nu_1\exp(\alpha(\mu_1-\mu_2))}{\nu_2-\nu_1}\beta_1.\end{aligned}\tag{1.11}$$

Using these explicit formulae, we can obtain that $\bar{u}(0)\leqq c\mu$ and that $\bar{u}(s)$ is a monotonically decreasing function on $(-\alpha,\alpha)$ for $\alpha<(\ln\nu_2-\ln\nu_1)/2(\nu_1-\nu_2)$. The difference $(\bar{u}(s)-u(x))$ satisfies the equation

$$\begin{aligned}A^\mu(\bar{u}(s)-u(x))=A^\mu\bar{u}(s)&=\left(\frac{\partial^2}{\partial s^2}+b_1(s,\varphi)\frac{\partial}{\partial s}+\frac{1}{\mu^2}V-\lambda\right)\bar{u}(s)\\&=(2a-b_1)\frac{\partial}{\partial s}\bar{u}+(\kappa-\lambda)\bar{u}\leqq 0,\qquad (\bar{u}(s)-u(x))|_{\partial Q_\alpha}=0.\end{aligned}$$

Here we use the first inequality in (1.3) and monotonicity of $\bar{u}(s)$. Now by (1.6) we have $\bar{u}(s)>u(x)$ so $u(x)|_{\partial Q}\leqq c\mu$. The proposition is proved.

Now we will estimate $p(x)$ in $T^n\backslash Q$. Let us consider in a spherical layer $\{x: r_1<|x|<r_2\}$, $0<r_1<r_2$ in R^n, the solutions $u_1(x)$ and $u_2(x)$ of the equation $(\Delta-1/\mu^2+\lambda)u_i=0$ with boundary conditions $u_1|_{|x|=r_1}=0$, $u_1|_{|x|=r_2}=1$, and $u_2|_{|x|=r_1}=1$, $u_2|_{|x|=r_2}=0$, respectively. For these solutions, in the same manner as in Proposition 1, we have the following estimates.

PROPOSITION 2. *For each $\delta>0$, on the sphere $\{x: |x|=(r_1+r_2)/2\}$, the following inequality holds*:

$$u_{1,2}|_{|x|=(r_1+r_2)/2}\leqq c(\delta)\exp\left(-\frac{r_2-r_1}{2\mu}(1-\delta)\right).\tag{1.12}$$

PROPOSITION 3. *The first eigenfunction $p(x)$ of the problem* (0.3) *satisfies on ∂Q the following estimate*:

$$p(x)|_{\partial Q}\leqq c\mu.\tag{1.13}$$

Proof. From the standard Schauder inequality [7, Thm. 6.2] and the uniform boundedness of λ in μ, it follows that $p(x)$ and $\nabla p(x)$ are uniformly bounded on the

Now we can obtain the main result in the symmetrical case.

THEOREM 1. *The effective diffusion $\sigma(\mu)$ satisfies the limit relation*

$$\lim_{\mu\to 0} \mu \ln \sigma(\mu) = -2s_0.$$

Proof. Since this proof is parallel to the proof of [3, Thm. 1], we outline only the main steps of the proof. From Corollary 1, we deduce that ξ_t^x has uniform mixing properties with respect to μ. This means, in particular, that for random sequence

$$\eta_k = [\xi_{(K+1)T} - \xi_{KT}], \quad T = C(\delta)\, e^{\delta/\mu}, \quad K = 0, 1, 2, \dots,$$

where ξ_t is the diffusion in R^n with the generator $\tilde{B}$, and $[\cdot]$-denotes the integer part, the mixing coefficient could be chosen in the form $\varphi(K) = 2^{-K}$. Having that, we can apply general central limit theorem for the random sequences with strong mixing property (see, e.g., [9]). For $\sigma(\mu)$, we have the representation

$$\sigma(\mu) = E\eta_0 \otimes \eta_0 + \sum_{K=1}^{\infty} E(\eta_0 \otimes \eta_K + \eta_K \otimes \eta_0),$$

where $x \otimes y = \{x_i x_j\}$. Corollary 2 and Proposition 13 enable us to show the matrix estimates

$$C_2 E\eta_0 \otimes \eta_0 \leqq \sigma(\mu) \leqq C_1 E\eta_0 \otimes \eta_0, \qquad 0 < C_2 < C_1 < \infty,$$

and above that, to establish the following inequality:

$$C_2(\delta)\, e^{-2(s_0+\delta)/\mu} \leqq E\eta_0 \otimes \eta_0 \leqq C_1(\delta)\, e^{-2(s_0-\delta)/\mu}$$

for any $\delta > 0$. The last equation yields the statement of the Theorem 1. In the general nonsymmetric case, the matrix $\sigma_{ij}(\mu)$ is not isotropic, so we study the limiting behaviour of its eigenvalues and eigenfunctions.

Let $U(x)$ be a periodic continuous function that is equal to zero in Q and coincides with $s(x)$ in Ω. We introduce the following numbers and vectors:

$$s_1 = \min_{i \in Z^n \setminus \{0\}} \; \inf_{x(\cdot), x(0)=0, x(1)=i} \; \sup_t U(x(t)), \tag{2.9}$$

where $x(\cdot)$ is a continuous curve that connects 0 and i. Let i^1 be the vector minimizing (2.9) and $z^1 = i^1/|i^1|$. Denote by $\{e^1, e^2, \dots, e^n\}$, the linear space that is generated by vectors $e^1, e^2, \dots, e^n$. Then

$$s_2 = \min_{i \in Z^n \setminus \{z^1\}} \; \inf_{x(\cdot), x(0)=0, x(1)=i} \; \sup_t U(x(t)), \tag{2.10}$$

i^2 minimizes (2.10), and z^2 is a unit vector $z^2 \in \{i^1, i^2\}$, which is orthogonal to $\{i^1\}$. Continuing this process, on the kth step we find that

$$s_k = \min_{i \in Z^n \setminus \{z^1, z^2, \dots, z^{k-1}\}} \; \inf_{x(\cdot), x(0)=0, x(1)=i} \; \sup_t U(x(t)), \tag{2.11}$$

where i^k minimizes (2.11) and $z^k \in \{i^1, i^2, \dots, i^k\}$ is a unit vector orthogonal to $\{i^1, i^2, \dots, i^{k-1}\}$. Let Θ be a matrix, which in the basis $z^1, z^2, \dots, z^n$, is diagonal with eigenvalues $s_1, s_2, \dots, s_n$, respectively. Then, as a consequence of [3, Thm. 2] we have the following theorem.

THEOREM 2. *The matrix of effective diffusion satisfies the relation*

$$\lim_{\mu\to 0} \mu \ln \{\sigma_{ij}(\mu)\} = -2\Theta = -\hat{\Theta}.$$

Proof. We use the same approach as in the proof of Theorem 1. Then with the same means, we obtain instead of Corollary 2 and Proposition 13, the following bilateral inequality:

$$C_2 \exp\left(-2\frac{s_k+\delta}{\mu}\right) \leqq P(\eta_0 = i^k) \leqq C_1 \exp\left(-2\frac{s_k-\delta}{\mu}\right).$$

Therefore, fixing $\xi \in R^n$ and discussing the stationary scalar sequence (η_k, ξ) as above, we obtain

$$C_2\left(\exp\left(-2\left(\frac{\Theta+\delta}{\mu}\right)\right)\xi,\xi\right) \leqq (\sigma(\mu)\xi, \xi) \leqq C_1\left(\exp\left(-2\left(\frac{\Theta-\delta}{\mu}\right)\right)\xi,\xi\right),$$

and the theorem follows.

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REFERENCES

[1] S. M. Kozlov, *Reducibility of quasiperiodic operators and homogenization*, Trans. Moscow Math. Soc., 46 (1983), pp. 99–123.

[2] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[3] S. M. Kozlov and A. L. Piatnitski, *Homogenization and vanishing viscosity*, Math. Sbornik, 181 (1990), pp. 813–832. (In Russian.)

[4] S. M. Kozlov, *Geometric aspects of homogenization*, Russian Math. Surveys, 44 (1989), pp. 91–144.

[5] ———, *Asymptotics of Laplace–Dirichlet Integrals*, Funktsional Anal. i Ego Prilozhen., 24 (1990), pp. 115–125.

[6] A. D. Wentzell and M. I. Freidlin, *Fluctuations in Dynamical Systems Caused by Small Random Perturbations*, Nauka, Moscow, 1979. (In Russian.)

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[8] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Nauka, Moscow, 1967. (In Russian.)

[9] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968.