On the behaviour at infinity of the solution of a second-order elliptic equation given on a cylinder

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In this note we consider the behaviour as $x_1 \to \infty$ of the solution of the equation

$$(1)\qquad \begin{cases} \mathcal{A}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x) = f(x), \\ u(x)|_{x_1=0} = \varphi(x), \end{cases}$$

given on the cylinder Q, the product of the half-line $(0, +\infty)$ and the torus T^{n-1}. In other words, the coefficients and the data of (1) are periodic in all variables except one, and we investigate the behaviour of solutions with the same properties. Throughout what follows we assume that the matrix $a_{ij}(x)$ is uniformly elliptic.

Similar questions for divergent equations have been studied in [1] and [2].

The most complete results have been obtained in the case when the coefficients of (1) are also periodic in the variable x_1. Let $a_{ij}(x)$ and $b_i(x)$ be sufficiently smooth functions that are periodic in all the variables $x_1, \dots, x_n$. Let $\mathcal{A}^*$ denote the operator formally adjoint to $\mathcal{A}$. The auxiliary problem

$$\mathcal{A}^* p = 0$$

has a unique periodic solution up to a factor, which we denote by $p(x)$. This solution is of constant sign (see [3]), so we assume that it is positive. We define the quantity

$$\beta = \int_{T^n} \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_{1i}(x) + b_1(x) \right) p(x)\, dx.$$

Theorem 1. *Suppose that the above conditions are satisfied by the operator $\mathcal{A}$, that the right-hand side $f(x)$ decreases exponentially: $|f(x)| \leqslant c_1 e^{-\alpha x_1}$, and that the boundary function $\varphi(x)$ is bounded: $|\varphi(x)| \leqslant c_2$.*

Then the condition $\beta > 0$ is necessary and sufficient for the existence of a solution of (1) *stabilizing to μ at infinity for each real μ. For each μ this solution is unique and converges exponentially:*

$$|u(x) - \mu| \leqslant c e^{-\gamma x_1}, \qquad \gamma > 0.$$

For $\beta > 0$, there are no other bounded solutions of (1) *except those that are ultimately constant.*

The condition $\beta \leqslant 0$ is necessary and sufficient for (1) *to have a unique bounded solution. This solution stabilizes to some constant μ exponentially:*

$$|u(x) - \mu| \leqslant c e^{-\gamma x_1}, \qquad \gamma > 0.$$

In both cases γ depends only on $\mathcal{A}$ and α. The constant c depends on c_1, c_2, $\mathcal{A}$, and for $\beta > 0$ also on μ.

Note also that the exponential decrease of the first l derivatives of $f(x)$ in (1) leads to the decrease of the first $(l+1)$ derivatives of the solution.

In the general case, without periodicity in the first variable x_1, the following result holds.

Theorem 2. *Suppose that the coefficients of $\mathcal{A}$ are sufficiently smooth functions uniformly bounded on Q and that the right-hand side $f(x)$ has compact support in Q. Then one of the following two possibilities occurs: either for each real μ there exists a unique solution of* (1) *tending to μ as $x_1 \to \infty$, and all bounded solution of* (1) *are of this form; or there exists a unique bounded solution $u(x)$. This solution stabilizes at infinity to some constant μ.*

The periodicity condition on x_1 in Theorem 1 can be relaxed considerably.

***Theorem* 3.** *Suppose that the coefficients of* (1) *are measurable and bounded and that on Q there is a solution of $\mathcal{A}^*q(x) = 0$ such that everywhere on Q*

$$0 < \varkappa_0 \leqslant q(x) \leqslant \varkappa_1 < \infty.$$

Then firstly, for each $k > 0$ the limit

$$\Lambda^k = \lim_{N\to\infty} \int_N^{N+k} dx_1 \int_{T^{n-1}} \left(b_1(x)\, q(x) - \sum_{i=1}^n a_{i1}(x) \frac{\partial}{\partial x_i} q(x) \right) dx_2 \ldots dx_n$$

exists and $\Lambda^k = k\Lambda^1$, and secondly, all the assertions of Theorem 1 *are valid if β is replaced by Λ^1 in the conditions.*

Remark 1. In all the theorems the torus T^{n-1} as a section of the cylinder Q can be replaced by an arbitrary smooth compact manifold.

Remark 2. If in the conditions of Theorems 1 and 3 we require exponential decrease of the right-hand side in the norm $H^{-1}(T^{n-1})$ with respect to the section, then in both theorems the estimates in the norm $L^\infty(T^{n-1})$ in the section must be replaced by the analogous estimates in the norm $L^2(T^{n-1})$.

Theorems 1 and 2 are proved by probabilistic methods, and Theorem 3 with the aid of energy estimates.

In conclusion we consider the more general equation

$$(2) \quad \begin{cases} \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x)\, u(x) = f(x), \\ u(x)|_{x_1=0} = \varphi(x) \end{cases}$$

on the same cylinder Q, the product of a half-line and a smooth compact manifold. We assume that the coefficients of (2) are sufficiently smooth functions periodic in x_1 and uniformly bounded on Q. We denote the operator on the left-hand side of (2) by $\mathcal{B}$ and consider the problem $\mathcal{B}p = \lambda p$ on the manifold $\mathcal{L} \times T^1$, where $\mathcal{L}$ is a manifold lying in a section of Q. It is known that for this problem the eigenvalue with maximal real part is real and simple, and the corresponding eigenfunction can be taken to be positive. We assume that this eigenvalue is zero. Then by the substitution $v(x) = p_0^{-1}(x)u(x)$ for the unknown function an equation of the form (2) reduces to one of form (1); here $p_0(x)$ is the eigenfunction mentioned above. Hence, the solution of (2) reduces at infinity to a periodic function proportional to $p_0(x)$.

References

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