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Spectrum of a convolution operator with potential

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Let $V = V(x)$ and $a = a(x)$ be functions on $\mathbb{R}^d$, where the function $a \in L_1(\mathbb{R}^d)$ is complex-valued and satisfies $a(-x) = \overline{a(x)}$, and V is such that its Fourier transform $\widehat{V}$ obeys the same conditions as a. The Fourier image of a is denoted by $\hat{a}$. Note that under these conditions the functions V and $\hat{a}$ are real-valued, bounded, and continuous.

We study the spectrum of the bounded linear self-adjoint operator

$$(\mathcal{L}u)(x) := \int_{\mathbb{R}^d} a(x-y)u(y)\,dy + V(x)u(x) \quad \text{in } L_2(\mathbb{R}^d). \tag{1}$$

Let $a_{\min} := \inf_{\mathbb{R}^d} \hat{a}$, $a_{\max} := \sup_{\mathbb{R}^d} \hat{a}$, $V_{\min} := \inf_{\mathbb{R}^d} V$, $V_{\max} := \sup_{\mathbb{R}^d} V$. By the conditions on a and V we have $a_{\min} \leqslant 0 \leqslant a_{\max}$ and $V_{\min} \leqslant 0 \leqslant V_{\max}$. Our fist result describes the essential spectrum and a possible location of the discrete spectrum of the operator $\mathcal{L}$.

Theorem 1. *The essential spectrum of the operator $\mathcal{L}$ coincides with the segment $[\mu_0, \mu_1]$, where $\mu_0 := \min\{a_{\min}, V_{\min}\}$ and $\mu_1 := \max\{a_{\max}, V_{\max}\}$. The points of the discrete spectrum of $\mathcal{L}$ can lie only in the intervals $[a_{\min} + V_{\min}, \mu_0)$ and $(\mu_1, a_{\max} + V_{\max}]$ and can accumulate only at μ_0 and μ_1.*

The rest of our note describes the discrete spectrum of the operator $\mathcal{L}$. The results are given only for the interval $[a_{\min} + V_{\min}, \mu_0)$, because the change of the operator $\mathcal{L}$ to $-\mathcal{L}$ swaps $[a_{\min} + V_{\min}, \mu_0)$ and $(\mu_1, a_{\max} + V_{\max}]$, and so the results that follow can be directly extended to the second interval. We also assume that $\mu_0 = V_{\min}$, for otherwise it suffices to apply the Fourier transform to the operator $\mathcal{L}$, which preserves its structure, but swaps functions a and V. Let $Q_r(x)$ denote the cube in $\mathbb{R}^d$ with centre at x and side length $2r$; for $x = 0$ we write $Q_r = Q_r(0)$.

Theorem 2. *Let $a_{\min} \geqslant V_{\min}$, let x_0 be a global minimum point of the function V, and let $\delta > 0$ be such that $\displaystyle\int_{Q_\delta(x_0)} \bigl(V(x) - V_{\min}\bigr)\,dx + \int_{Q_\delta \times Q_\delta} \operatorname{Re} a(x-y)\,dx\,dy < 0$. Then the discrete spectrum of the operator $\mathcal{L}$ is located inside $[a_{\min} + V_{\min}, \mu_0)$.*

We set $V_-(x) := -\min\{V(x), 0\}$ and $\hat{a}_-(\xi) := -\min\{\hat{a}(\xi), 0\}$.

Theorem 3. *Let $a_{\min} \geqslant V_{\min}$, let the functions V_-, $\hat{a}_-$ belong to $L_1(\mathbb{R}^d)$, and let the integrals $I_V := \displaystyle\int_{\mathbb{R}^d} \frac{V_-(x)\,dx}{V_-(x) + V_{\min}}$, $I_a := \dfrac{1}{(2\pi)^d} \displaystyle\int_{\mathbb{R}^d} \frac{\hat{a}_-(x)\,dx}{\hat{a}_-(x) + V_{\min}}$ be finite.*

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Then the number of eigenvalues of the operator $\mathcal{L}$ to the left of the point $V_{\min}$ is bounded above by $I_a I_V$.

Theorem 4. *Let $a_{\min} \geqslant V_{\min}$ and let $V(x) \equiv V_{\min}$ on some cube $Q_r(x_0)$. Assume that at least one of the following two conditions is satisfied*: 1) $a_{\min} < 0$ *and* $\hat{a}(\xi) \leqslant 0$ *for all* $\xi \in \mathbb{R}^d$; 2) *there exists* $\eta \leqslant r$ *such that* $\int_{Q_\eta} a(x) e^{(2\pi \mathrm{i}/\eta) m\cdot x}\, dx \leqslant 0$ *for all* $m \in \mathbb{Z}^d$ *and* $a(x) \not\equiv 0$ *in* Q_η. *Then the operator $\mathcal{L}$ has countably many discrete eigenvalues in the interval $[a_{\min} + V_{\min}, \mu_0)$, which accumulate at μ_0.*

Consider now the operator $\mathcal{L}$ in (1) with small integral term:

$$(\mathcal{L}_\varepsilon u)(x) := \varepsilon \int_{\mathbb{R}^d} a(x-y) u(y)\, dy + V(x) u(x) \quad \text{in } L_2(\mathbb{R}^d), \tag{2}$$

where $\varepsilon > 0$ is a small parameter, the functions a and V satisfy the assumptions of Theorem 1, and $\varepsilon a_{\min} > V_{\min}$ for all sufficiently small ε. The following result holds.

Theorem 5. *Let $d = 1$, let x_0 be a global minimum point for V, and let $V \in C^5(x_0 - \eta, x_0 + \eta)$ for some $\eta > 0$, $a \in W^5_\infty(\mathbb{R})$ and $V''(x_0) > 0$. Also let $a(0) < 0$. Then, for sufficiently small ε, the operator $\mathcal{L}_\varepsilon$ has precisely one isolated eigenvalue to the left of the point $V_{\min}$. Moreover, this eigenvalue is simple and behaves asymptotically as*

$$V_{\min} - \lambda_\varepsilon = \varepsilon^2 \pi^2 a^2(0) (V''(x_0))^{-2} + o(\varepsilon^3). \tag{3}$$

Let us briefly discuss the main results. Theorem 1 describes the location of the essential spectrum of the operator $\mathcal{L}$ and a possible location of the discrete spectrum. Theorem 2 provides sufficient conditions for the existence of a discrete spectrum in terms of the behaviour of the functions V and a near the points x_0 and 0, respectively. Theorem 3 is an analogue of the classical Birman–Schwinger theorem in the case of non-local operators, which gives a good upper bound for the number of points of the discrete spectrum for $d \geqslant 3$. (For $d = 1, 2$, this upper bound is finite under additional constraints on the behaviour of V near x_0. Note that these constraints are not satisfied under the conditions of Theorem 5.) It is worth pointing out that, under the hypotheses of Theorem 3, the relation $I_a I_V = +\infty$ does not necessarily mean that there exist infinitely many discrete eigenvalues. Theorem 4 describes sufficient conditions for the existence of a countable number of eigenvalues under the assumption that V attains its global minimum on a set of positive measure. Theorem 5 asserts that, in the case when $d = 1$ and the convolution kernel of a is small, there appears precisely one eigenvalue with asymptotics (3). We have also been able to find a lower estimate for the number of eigenvalues, but this result is quite bulky. We only note that it is formulated in terms of the Taylor coefficients of the convolution kernel at 0 and the behaviour of V near x_0.

Operators of type $\mathcal{L}$ appear in models of population dynamics based on birth-and-death processes. In particular, such operators control the evolution of the first correlation function in an inhomogeneous contact model in $\mathbb{R}^d$ which describes the spread of epidemics in a population. The first correlation function is the averaged density of configurations in the contact model, and hence the appearance of the discrete spectrum for the evolution operator indicates a qualitative change in the large time behavior of the model, when, with the presence of a positive discrete spectrum, the average density of the configuration points in bounded domains grows exponentially. Such operators also play an important role in the description of some types of porous medium.

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