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On the asymptotic behaviour of the kernel of an adjoint convection-diffusion operator in a long cylinder

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Abstract. This paper studies the asymptotic behaviour of the principal eigenfunction of the adjoint Neumann problem for a convection diffusion operator defined in a long cylinder. The operator coefficients are 1-periodic in the longitudinal variable. Depending on the sign of the so-called longitudinal drift (a weighted average of the coefficients), we prove that this principal eigenfunction is equal to the product of a specified periodic function and of an exponential, up to the addition of fast decaying boundary layer terms.

1. Introduction

We study the asymptotic behavior, for $\varepsilon > 0$ going to 0, of the solution p^ε of the following boundary value problem

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla p^\varepsilon\big) - \mathrm{div}\big(b(y)p^\varepsilon\big) = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla p^\varepsilon \cdot n + b(y)\cdot n p^\varepsilon = 0 & \text{on } \partial Q_\varepsilon, \end{cases} \tag{1.1}$$

where $Q_\varepsilon = (0, 1/\varepsilon)\times G$ is a long cylinder in the direction e_1 of cross section G. The above problem is the adjoint of the Neumann problem for the standard convection diffusion operator $Au = -\mathrm{div}\big(a(y)\nabla u\big) + b(y)\cdot\nabla u$, which admits 0 as a first eigenvalue with the corresponding constant first eigenfunction. Therefore, by the Krein–Rutman theorem, there exists a unique solution p^ε of (1.1), up to a multiplicative constant (see Lemma 2.1 below).

Our main results (Theorems 3.6 and 4.5) can be summarized as follows. The asymptotic behavior of p^ε depends on the sign of the so-called longitudinal effective drift $\bar{b}_1$, which is a kind of weighted average of the velocity field $b(y)$, in the axial direction e_1, defined by (2.7). Denote by $y_1 = y\cdot e_1$ the longitudinal variable.

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If $\overline{b}_1 > 0$, then, under a proper normalization, there exists a constant $\theta_0 > 0$ and a 1-periodic in the variable y_1 function $p_{\theta_0}(y) > 0$ such that

$$p^\varepsilon(y) \approx e^{-\theta_0 y_1} p_{\theta_0}(y),$$

where the approximation is up to the addition of boundary layer terms concentrating at both extremities of the cylinder and decaying faster to zero than the main limit $e^{-\theta_0 y_1} p_{\theta_0}(y)$. If $\overline{b}_1 = 0$, then the same holds true with $\theta_0 = 0$. If $\overline{b}_1 < 0$, a symmetric situation occurs with $\theta_0 < 0$.

There are many motivations to study the asymptotic behavior of (1.1). First, it appears as a simplified model of reaction-diffusion equations with asymmetric potentials as studied in [18], [19], [16]. The simplification is that (1.1) is a scalar equation (representing a single species instead of two), but the addition of the convective term makes it non trivial (clearly, if $b(y) = 0$, then $p^\varepsilon(y)$ is a constant). The fact that, asymptotically as ε goes to 0, the solution p^ε concentrates at one end of the cylinder, depending on the sign of the exponent θ_0, or equivalently of the drift $\overline{b}_1$, is a manifestation of the so-called motor effect. This phenomenon was first studied by homogenization methods in [18]: their result was weaker (albeit more general) in the sense that it gives an asymptotic behavior for the logarithm of the solution, namely

$$\log p^\varepsilon(y) \approx -\theta_0\, y_1.$$

The key tool in [18] was the homogenization of a Hamilton–Jacobi equation, obtained by a logarithmic change of unknowns. The homogenization techniques for Hamilton–Jacobi type equations with (locally-) periodic coefficients were developed in [11], [12].

A second motivation is the homogenization of convection-diffusion-reaction equations in periodic heterogeneous media. There are many applications such as transport in porous media [3], [5] or nuclear reactor physics [7]. Indeed, by rescaling the space variable as $x = \varepsilon y$, (1.1) is equivalent to

$$(1.2) \qquad \begin{cases} -\mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon}\Big)\nabla p^\varepsilon\Big) - \dfrac{1}{\varepsilon}\,\mathrm{div}\Big(b\Big(\dfrac{x}{\varepsilon}\Big)\,p^\varepsilon\Big) = 0 \quad \text{in } \varepsilon\, Q_\varepsilon, \\ a\Big(\dfrac{x}{\varepsilon}\Big)\nabla p^\varepsilon\cdot n + \dfrac{1}{\varepsilon}\,b\Big(\dfrac{x}{\varepsilon}\Big)\cdot n p^\varepsilon = 0 \quad \text{on } \varepsilon\partial Q_\varepsilon, \end{cases}$$

where $\varepsilon Q_\varepsilon$ is now a cylinder of length 1 and small cross section εG. This geometrical setting is the usual one for homogenization since the cylinder has now a fixed length. The case of Dirichlet boundary conditions for (1.2) at both extremities of the cylinder is by now well known. Actually, in such a case, one can consider a more general domain Ω, not necessarily a thin cylinder. Of course, in the case of Dirichlet boundary conditions, the first eigenvalue is usually not zero. In any case, the asymptotic behavior of the first eigenfunction is completely understood, even for more complicated systems [6], [2], [4], [5]. The case of Neumann boundary conditions is far from being fully understood, and there are very few works which address it. All of them address merely the 1-d case or the present almost 1-d setting of a thin cylinder. Apart from the previously cited work [18], [19], [16], let us mention [1] which, being 1-d, heavily relies on methods of ordinary differential equations. Our present setting is more general than that of [1] since all operators are d-dimensional but, still, we consider only cylinders (and not general domains) in order to force the direction of the drift vector $\overline{b}_1$ along the cylinder axis. Nevertheless, the main difference with [1] is the presence of delicate boundary layer terms at the cylinder ends. Our present results in the Neumann case are quite different from that in the Dirichlet case, as explained in Remark 3.8.

It should also be noted that the principal eigenvalue of the problem studied in this paper is equal to zero. It follows from the fact that this problem is the adjoint to a homogeneous Neumann problem for a convection-diffusion operator. This makes a difference with [1], where a generic Fourier boundary condition is imposed at the end points of the interval. This might lead to a different behaviour of the solution.

A third motivation is the homogenization of the following "primal" parabolic problem:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \dfrac{1}{\varepsilon} b\Big(\dfrac{x}{\varepsilon}\Big)\nabla u^\varepsilon - \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big) = 0 \quad \text{in } \mathbb{R}^+ \times \varepsilon Q_\varepsilon, \\ a\Big(\dfrac{x}{\varepsilon}\Big)\nabla u^\varepsilon \cdot n = 0 \quad \text{on } \mathbb{R}^+ \times \varepsilon \partial Q_\varepsilon, \\ u^\varepsilon(0,x) = u^{\text{init}}(x) \quad \text{in } \varepsilon Q_\varepsilon. \end{cases} \tag{1.3}$$

Since the first eigenfunction of the primal problem is a constant, associated to the zero first eigenvalue, we know that for each $\varepsilon > 0$ the solution u^ε converges to a constant as t goes to ∞. However, the value of this constant depends precisely on the adjoint solution p^ε of (1.2), since we easily find by integration by parts that

$$\frac{d}{dt}\int_{\varepsilon Q_\varepsilon} u^\varepsilon(t,x)\, p^\varepsilon(x)\, dx = 0.$$

Therefore, in order to find the limit, as $\varepsilon \to 0$, of this constant, equal to

$$\int_{\varepsilon Q_\varepsilon} u^{\text{init}}(x)\, p^\varepsilon(x)\, dx,$$

one has to investigate the limit behaviour of p^ε. This is an additional motivation for studying the adjoint problem (1.1). In particular, only the behavior of the initial data close to the left hand $y_1 = 0$ of the cylinder will matter if $\overline{b}_1 > 0$ and conversely otherwise.

A fourth motivation comes from studying stochastic diffusion processes in the cylinder. Indeed, under proper normalization, the solution of problem (1.1), respectively of (1.2), coincides with the density of the invariant measure of a diffusion process ξ_t^ε with generator $A = -\operatorname{div}\big(a(y)\nabla\big) + b(y)\cdot\nabla$ (respectively, $A^\varepsilon = -\operatorname{div}\big(a(x/\varepsilon)\nabla\big) + \varepsilon^{-1}b(x/\varepsilon)\cdot\nabla$) and with reflection at the cylinder boundary, see [14] for further details. Furthermore, the time evolution of the law of non-stationary distribution of the said diffusion process is described by equation (1.3). The results of this work can be used for determining the limit behaviour of the effective covariance of additive functionals of $\xi^\varepsilon_\cdot$.

Finally we acknowledge that other authors have been studying the limit behaviour of solutions and eigenpairs of elliptic problems, stated in asymptotically long cylinders: see e.g. [8], [9] and [10].

The content of our paper is as follows. The next section 2 gives a precise description of problem (1.1), with all the necessary assumptions and definitions. Section 3 gives our main result (Theorem 3.6) in the case $\overline{b}_1 > 0$. Section 4 deals with the case $\overline{b}_1 = 0$ (see Theorem 4.5). Section 5 explains how our results can be extended to coefficients with minimal regularity. Section 6 gives some perspectives and open problems, while Section 7 is an appendix, recalling for the reader convenience a key technical result from [17], as well as some useful corollaries.

Notation. As usual, C denotes a constant which may vary from place to place but is always independent of ε, except otherwise mentioned.

2. Statement of the problem

Given a smooth bounded connected domain $G \subset \mathbb{R}^{d-1}$ and a small positive parameter ε, we define a cylinder

$$Q_\varepsilon = \{y \in \mathbb{R}^d \,:\, 0 < y_1 < 1/\varepsilon\,,\, y' := (y_2, \dots, y_d) \in G\}.$$

Let A be the linear convection-diffusion operator defined in Q_ε, with a symmetric matrix a, and Neumann boundary conditions

$$Au = -\mathrm{div}\big(a(y)\nabla u\big) + b(y)\cdot\nabla u, \quad a(y)\nabla u\cdot n = 0 \quad \text{on } \partial Q_\varepsilon, \tag{2.1}$$

and its adjoint A^* defined by

$$A^*u = -\mathrm{div}\big(a(y)\nabla u\big) - \mathrm{div}(b(y)u), \quad a(y)\nabla u\cdot n + b(y)\cdot nu = 0 \quad \text{on } \partial Q_\varepsilon. \tag{2.2}$$

We consider the corresponding Neumann problem

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla u\big) + b(y)\cdot\nabla u = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla u\cdot n = 0 & \text{on } \partial Q_\varepsilon, \end{cases} \tag{2.3}$$

and its adjoint problem

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla p^\varepsilon\big) - \mathrm{div}\big(b(y)p^\varepsilon\big) = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla p^\varepsilon\cdot n + b(y)\cdot np^\varepsilon = 0 & \text{on } \partial Q_\varepsilon. \end{cases} \tag{2.4}$$

Here and in what follows $n = n(y)$ stands for the external normal on ∂Q_ε and $v_1 \cdot v_2$ denotes the inner product of vectors v_1 and v_2 in $\mathbb{R}^d$.

We assume that the coefficients of A satisfy the following properties.

(A1) Uniform ellipticity. The matrix a_{ij} is real, symmetric, positive definite: there exists $\Lambda > 0$ such that

$$\|a_{ij}\|_{L^\infty(Q_\varepsilon)} \le \Lambda^{-1},\ 1 \le i,j \le d, \quad \|b_i\|_{L^\infty(Q_\varepsilon)} \le \Lambda^{-1},\ 1 \le i \le d,$$
$$a_{ij}(y)\,\xi_i\,\xi_j \ge \Lambda|\xi|^2 \quad \text{for all } y \in Q_\varepsilon \ \text{ and } \xi \in \mathbb{R}^d.$$

(A2) Periodicity. All the coefficients $a_{ij}(y)$ and $b_i(y)$ are bounded and 1-periodic in the axial variable y_1.

For presentation simplicity we also assume that all the coefficients are sufficiently regular. In Section 5 we show that this last assumption can be discarded. The symmetry of a is assumed just for presentational simplicity. Our approach also applies in the case of a non-symmetric matrix a. Moreover, if the entries of a are $W^{1,\infty}$ regular, the non-symmetric case is reduced to the symmetric one.

Lemma 2.1. *For each $\varepsilon > 0$, problem (2.4) has a unique, up to a multiplicative constant, solution. Under a proper normalization, this solution is positive in Q_ε.*

Proof. By the maximum principle, any solution of problem (2.3) is equal to a constant. Consider the spectral problem related to problem (2.3) and obtained by replacing 0 on the right-hand side of the equation in (2.3) with λu. By the Krein–Rutman theorem (see [15]), $\lambda = 0$ is the eigenvalue of this operator with the smallest real part. By the same theorem, for each $\varepsilon > 0$ problem (2.4) has a unique, up to a multiplicative constant, solution. This solution does not change sign. This implies the desired statement. □

We now introduce several auxiliary problems and definitions. Denoting $Y = (0,1) \times G$ and $\partial_l Y = [0,1] \times \partial G$, we consider the following problem:

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla p_0(y)\big) - \mathrm{div}\big(b(y)\,p_0(y)\big) = 0 & \text{in } Y, \\ a(y)\nabla p_0(y)\cdot n(y) + b(y)\cdot n(y)\,p_0(y) = 0 & \text{on } \partial_l Y, \\ p_0 \ \text{ is 1-periodic in } y_1. \end{cases} \tag{2.5}$$

Using the Krein–Rutman theorem one can show (see [17]) that this problem has a unique up to a multiplicative constant solution. Moreover, this solution does not change sign. In truth, p_0 is the first eigenfunction corresponding to the first eigenvalue $\lambda_0 = 0$ of the cell spectral problem for the adjoint operator A^* defined by (2.2). In order to fix the normalization, we assume from now on that

$$\int_Y p_0(y)\,dy = 1. \tag{2.6}$$

Next, we define the effective drift which governs the asymptotic behavior of problem (2.4) (see [20]).

Definition 2.2. For the operator A, defined by (2.1), we introduce its so-called longitudinal effective drift, given by

$$\bar{b}_1 = \int_Y \big(a\nabla p_0 + bp_0\big)\cdot e_1\,dy, \tag{2.7}$$

where p_0 is the first adjoint eigenfunction, solution of (2.5) and normalized by (2.6), and e_1 is the first coordinate vector in $\mathbb{R}^d$.

Note that, in Definition 2.2, we take advantage of the fact that the first eigenfunction of the cell spectral problem for the operator A is constant, equal to 1. If p_0 were not normalized by (2.6), then $\overline{b}_1$ should be divided by $\int_Y p_0(y)\,dy$.

In the sequel, we consider separately two cases, namely $\overline{b}_1 \neq 0$ and $\overline{b}_1 = 0$. In the first case we assume for the sake of definiteness that $\overline{b}_1 > 0$. The opposite case is reduced to this one by replacing y_1 with $-y_1$.

3. Main results for positive effective drift $\overline{b}_1 > 0$

In this section we formulate our main result when $\overline{b}_1 > 0$.

Lemma 3.1. *Let $\overline{b}_1 > 0$. Then, under the normalization condition*

$$\max_{Q_\varepsilon} p^\varepsilon(y) = 1, \tag{3.1}$$

the following limit relation holds:

$$\lim_{\varepsilon\to 0}\ \max_{y'\in G} p^\varepsilon\left(\varepsilon^{-1}, y'\right) = 0. \tag{3.2}$$

Furthermore,

$$p^\varepsilon(y) \le C\, e^{-\varkappa y_1}, \quad y \in Q_\varepsilon, \tag{3.3}$$

for some $\varkappa > 0$ and $C > 0$ that do not depend on ε.

Proof. In a first step, we prove a uniform local Harnack inequality for p^ε, using a reflection argument. Denote by $Q_{r,s}$ a finite cylinder $\{y \in \mathbb{R}^d : r < y_1 < s,\ y' \in G\}$ and by G_s the cross section $\{y \in \mathbb{R}^d : y_1 = s,\ y' \in G\}$.

We then introduce the functions

$$\tilde{a}^\varepsilon(y) = \begin{cases} a(y), & \text{in } Q_\varepsilon\,,\\ a(-y_1, y'), & \text{in } Q_{-\varepsilon^{-1},0}\,; \end{cases} \qquad \tilde{p}^\varepsilon(y) = \begin{cases} p^\varepsilon(y), & \text{in } Q_\varepsilon\,,\\ p^\varepsilon(-y_1, y'), & \text{in } Q_{-\varepsilon^{-1},0}\,; \end{cases}$$

$$\tilde{b}^\varepsilon(y) = \begin{cases} b(y), & \text{in } Q_\varepsilon\,,\\ (-b_1(-y_1, y'), b'(-y_1, y')), & \text{in } Q_{-\varepsilon^{-1},0}\,, \end{cases}$$

and extend them periodically in the infinite cylinder $Q_{-\infty,\infty}$. with the period $2\varepsilon^{-1}$ in y_1. The function $\tilde{p}^\varepsilon$ satisfies the equation

$$\begin{aligned} -\mathrm{div}\big(\tilde{a}^\varepsilon(y)\nabla\tilde{p}^\varepsilon(y)\big) - \mathrm{div}\big(\tilde{b}^\varepsilon(y)\tilde{p}^\varepsilon(y)\big) = 0 \quad &\text{in } Q_{-\infty,\infty},\\ \tilde{a}^\varepsilon(y)\nabla\tilde{p}^\varepsilon(y)\cdot n(y) + \tilde{b}^\varepsilon(y)\cdot n(y)\tilde{p}^\varepsilon(y) = 0 \quad &\text{on } \partial Q_{-\infty,\infty}. \end{aligned}$$

Making one more reflection with respect to ∂G we may assume that $\tilde{p}^\varepsilon$ satisfies the equation

$$-\mathrm{div}\big(\tilde{a}^\varepsilon(y)\nabla\tilde{p}^\varepsilon(y)\big) - \mathrm{div}\big(\tilde{b}^\varepsilon(y)\tilde{p}^\varepsilon(y)\big) = 0$$

in a larger cylinder $(-\infty, +\infty) \times \widehat{G}$ with $\overline{G} \subset \widehat{G}$. Therefore (see Corollary 8.21 in [13]), $\tilde{p}^\varepsilon$ satisfies the Harnack inequality uniformly in ε. So does p^ε. This means that, for any $r \in [0, \varepsilon^{-1} - 1]$, the inequality

$$\max_{\overline{Q}_{r,r+1}} p^\varepsilon \leq C \min_{\overline{Q}_{r,r+1}} p^\varepsilon$$

holds with a constant C that does not depend on ε, nor on r.

In a second step, we prove the asymptotic decay of p^ε by a contradiction argument. We represent p^ε as a sum of two functions $p^\varepsilon = p^{-,\varepsilon} + p^{+,\varepsilon}$, where $p^{-,\varepsilon}$ and $p^{+,\varepsilon}$ solve the following problems:

$$\begin{cases} -\operatorname{div}\big(a(y)\nabla p^{-,\varepsilon}(y)\big) - \operatorname{div}\big(b(y)p^{-,\varepsilon}(y)\big) = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla p^{-,\varepsilon}(y)\cdot n(y) + b(y)\cdot n(y)p^{-,\varepsilon}(y) = 0 & \text{on } \partial_l Q_\varepsilon, \\ p^{-,\varepsilon} = p^\varepsilon \quad \text{on } G_0, \qquad p^{-,\varepsilon} = 0 & \text{on } G_{1/\varepsilon}, \end{cases} \tag{3.4}$$

and

$$\begin{cases} -\operatorname{div}\big(a(y)\nabla p^{+,\varepsilon}(y)\big) - \operatorname{div}\big(b(y)p^{+,\varepsilon}(y)\big) = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla p^{+,\varepsilon}(y)\cdot n(y) + b(y)\cdot n(y)p^{+,\varepsilon}(y) = 0 & \text{on } \partial_l Q_\varepsilon, \\ p^{+,\varepsilon} = 0 \quad \text{on } G_0, \qquad p^{+,\varepsilon} = p^\varepsilon & \text{on } G_{1/\varepsilon}, \end{cases} \tag{3.5}$$

with $\partial_l Q_\varepsilon = \partial Q_\varepsilon \setminus (G_0 \cup G_{1/\varepsilon})$ being the lateral boundary of Q_ε. Due to the fact that Dirichlet conditions are imposed on the cylinder bases, both problems (3.4) and (3.5) are well posed for each $\varepsilon > 0$, so that the functions $p^{\pm,\varepsilon}$ are uniquely defined. The reduction to problems (3.4) and (3.5) with Dirichlet boundary conditions allows us to use some previous results of [17].

We now use factorization techniques (see [2] and references therein) to simplify the above equations. It amounts to factorize the unknown by $p_0(y)$ and to multiply the equations by the primal first eigenfunction (which, in the case of (2.1), is equal to 1). Defining $q^{\pm,\varepsilon}$ by the identity

$$p^{\pm,\varepsilon}(y) = p_0(y)q^{\pm,\varepsilon}(y) \tag{3.6}$$

and using equation (2.5) for p_0, (3.4) and (3.5) become, after straightforward rearrangements,

$$\begin{cases} -\operatorname{div}\big(p_0(y)a(y)\nabla q^{-,\varepsilon}(y)\big) - \check{b}(y)\nabla q^{-,\varepsilon}(y) = 0 & \text{in } Q_\varepsilon, \\ p_0(y)a(y)\nabla q^{-,\varepsilon}(y)\cdot n(y) = 0 & \text{on } \partial_l Q_\varepsilon, \\ q^{-,\varepsilon} = p^\varepsilon (p_0)^{-1} \quad \text{on } G_0, \qquad q^{-,\varepsilon} = 0 & \text{on } G_{1/\varepsilon}, \end{cases} \tag{3.7}$$

and

$$\begin{cases} -\operatorname{div}\big(p_0(y)a(y)\nabla q^{+,\varepsilon}(y)\big) - \check{b}(y)\nabla q^{+,\varepsilon}(y) = 0 & \text{in } Q_\varepsilon, \\ p_0(y)a(y)\nabla q^{+,\varepsilon}(y)\cdot n(y) = 0 & \text{on } \partial_l Q_\varepsilon, \\ q^{+,\varepsilon} = 0 \quad \text{on } G_0, \qquad q^{+,\varepsilon} = p^\varepsilon (p_0)^{-1} & \text{on } G_{1/\varepsilon}, \end{cases} \tag{3.8}$$

with $\check{b}(y) = a(y)\nabla p_0(y) + b(y)p_0(y)$. By the definition of p_0 we have

$$\text{div}\, \check{b}(y) = 0, \quad \check{b}(y) \cdot n = 0 \text{ on } \partial_l Q_\varepsilon, \quad \int_Y \check{b}(y) \cdot e_1 dy = \overline{b}_1, \tag{3.9}$$

where $\overline{b}_1$ is precisely the longitudinal effective drift, introduced in Definition 2.2. Let us denote by $\check{A}$ the operator appearing in (3.7) and (3.8), namely

$$\check{A}u = -\text{div}\big(p_0(y)a(y)\nabla u\big) - \check{b}(y) \cdot \nabla u \quad \text{in } Y,$$
$$p_0(y)a(y)\nabla u(y) \cdot n = 0 \quad \text{on } \partial_l Y,$$

with its adjoint $\check{A}^*$

$$\check{A}^*u = -\text{div}\big(p_0(y)a(y)\nabla u\big) + \check{b}(y) \cdot \nabla u(y) \quad \text{in } Y,$$
$$p_0(y)a(y)\nabla u(y) \cdot n = 0 \quad \text{on } \partial_l Y.$$

It is easy to check that the kernel of $\check{A}^*$ in the unit cell Y, with 1-periodic boundary conditions in y_1, is equal to a constant. Considering our normalization for the kernel of adjoint operator and recalling Definition 2.2 of the longitudinal effective drift, we conclude after simple computations that the effective longitudinal drift of $\check{A}$ is $-|Y|^{-1}\,\overline{b}_1$. Under our standing assumptions this drift is negative.

By contradiction with (3.2), assume now that, for a subsequence, $\max_{G_{1/\varepsilon}}(p^\varepsilon)$ does not go to zero as $\varepsilon \to 0$. Then, by the Harnack inequality,

$$0 < C \le p^\varepsilon(\varepsilon^{-1}, y') \le 1, \quad 0 < C \le q^{+,\varepsilon}(\varepsilon^{-1}, y').$$

Because the effective drift of $\tilde{A}$ is negative, as a consequence of Theorem 6.1 and Lemma 6.3 in [17], and by Corollary 7.2 in the Appendix, there are constants C_ε, $0 < C \le C_\varepsilon \le C_1$, and $\varkappa > 0$ such that

$$|q^{-,\varepsilon}| \le e^{-\varkappa/\varepsilon}, \quad |q^{+,\varepsilon} - C_\varepsilon| \le e^{-\varkappa/\varepsilon} \quad \text{in } Q_{\frac{1}{3\varepsilon},\frac{2}{3\varepsilon}}.$$

Considering the definition of $q^{\pm,\varepsilon}$ and $p^{\pm,\varepsilon}$, we derive from that last inequalities that

$$|p^\varepsilon(y) - C_\varepsilon p_0(y)| \le e^{-\varkappa/\varepsilon} \quad \text{in } Q_{\frac{1}{3\varepsilon},\frac{2}{3\varepsilon}}. \tag{3.10}$$

By local elliptic estimates the last inequality implies

$$\|p^\varepsilon - C_\varepsilon p_0\|_{H^1(Q_{s,s+1})} \le Ce^{-\varkappa/\varepsilon}, \quad \frac{1}{3\varepsilon} \le s \le \frac{2}{3\varepsilon} - 1. \tag{3.11}$$

On the other hand, integrating (2.4) on $Q_{0,r}$ we get

$$\int_{G_r} \big(a(y)\nabla p^\varepsilon(y) \cdot n + b(y) \cdot np^\varepsilon(y)\big)\, dy' = 0,$$

while integrating (2.5) on $Q_{s,r}$ shows that the following surface integral:

$$\int_{G_r} \big(a(y)\nabla p_0(y) \cdot n + b(y) \cdot np_0(y)\big)\, dy' = \overline{b}_1 > 0 \tag{3.12}$$

is constant for all $r \in [0, 1/\varepsilon]$. Since $C_\varepsilon \geq C > 0$, the last two relations contradict (3.10), (3.11). Thus, (3.2) holds true.

The assumption that (3.3) does not hold leads to a contradiction in exactly the same way. This completes the proof. □

One of the key ingredients of our study is the following auxiliary problem stated in a semi-infinite cylinder $Q_\infty = (0, \infty) \times G$:

$$\begin{cases} -\operatorname{div}\big(a(y)\nabla p^\infty(y)\big) - \operatorname{div}\big(b(y)p^\infty\big) = 0 & \text{in } Q_\infty, \\ a(y)\nabla p^\infty(y)\cdot n(y) + b(y)\cdot n(y)p^\infty(y) = 0 & \text{on } \partial Q_\infty. \end{cases} \tag{3.13}$$

The boundary condition at $+\infty$ reads

$$\lim_{y_1\to\infty} p^\infty(y) = 0. \tag{3.14}$$

We also widely use the exponential, or so-called Gelfand, transformation of the operators A and A^* defined on Y by

$$A_\theta v(y) = e^{-\theta y_1}A(e^{\theta y_1}v(y)), \quad A^*_\theta v(y) = e^{\theta y_1}A^*(e^{-\theta y_1}v(y)), \quad \theta \in \mathbb{R},$$

with the corresponding Neumann-type boundary conditions on $\partial_l Y$. Denote by $\lambda(\theta)$ the principal eigenvalue of A_θ and A^*_θ on Y in the space of 1-periodic in y_1 functions. By the Krein–Rutman theorem, $\lambda(\theta)$ is real and simple for each $\theta \in \mathbb{R}$. Moreover, according to [6], $\lambda(\theta)$ is a smooth strictly concave function of θ that tends to $-\infty$, as $\theta \to \pm\infty$.

Under our standing assumptions $\lambda(0) = 0$. It can also be checked (see [6]) that $\lambda'(0) > 0$ if and only if $\overline{b}_1 > 0$. Therefore, there is a unique $\theta_0 > 0$ such that $\lambda(\theta_0) = 0$. We denote by p_{θ_0} the corresponding periodic in y_1 eigenfunction of $A^*_{\theta_0}$ which is normalized in such a way that $\int_Y p_{\theta_0}(y)\,dy = 1$.

Lemma 3.2. *Let $\overline{b}_1 > 0$. Then problem (3.13)–(3.14) has a unique up to a multiplicative constant bounded solution p^∞. This solution decays exponentially, as $y_1 \to \infty$. Moreover, p^∞ admits the following representation:*

$$p^\infty(y) = e^{-\theta_0 y_1}p_{\theta_0}(y) + p^-_{\text{bl}}(y), \tag{3.15}$$

where, for some $\theta_1 > \theta_0$,

$$|p^-_{\text{bl}}(y)| \leq Ce^{-\theta_1 y_1}.$$

Remark 3.3. In representation (3.15), the function p^-_{bl} is a boundary layer which decays exponentially faster than the main term as y_1 goes to $+\infty$. Notice that, according to Lemma 3.2, in the case $\overline{b}_1 > 0$ problem (3.13) has a unique $L^2(Q_\infty)$ eigenfunction related to the eigenvalue 0.

If we replace in (3.13) the Neumann boundary condition at the cylinder base with the Dirichlet condition, then the modified problem reads

$$\begin{aligned} &-\operatorname{div}\big(a(y)\nabla p(y)\big) - \operatorname{div}\big(b(y)p\big) = 0 \quad \text{in } Q_\infty, \\ &p = 0 \quad \text{on } G_0, \quad a(y)\nabla p(y)\cdot n(y) + b(y)\cdot n(y)p(y) = 0 \quad \text{on } \partial_l Q_\infty. \end{aligned}$$

Although 0 still belongs to the spectrum of this problem, there is no localized eigenfunction related to 0. The only solution of this problem with an additional condition (3.14) is the function identically equal to zero.

Proof of Lemma 3.2. Consider the function p^ε introduced in (2.4) on the cylinder Q_ε. From (3.1) and Lemma 3.1, there exists a constant $C > 0$, which does not depend on ε, such that $0 < C \le \max_{G_0} p^\varepsilon \le 1$. Indeed, due to (3.3), the maximum of p^ε is attained in a finite cylinder that does not depend on ε. Then the lower bound follows from the Harnack inequality. Since the coefficients in (2.4) do not depend on ε, then, according to [13], p^ε are uniformly in ε Hölder continuous functions in the whole domain Q_ε. Passing to the limit $\varepsilon \to 0$ in the family p^ε (up to a subsequence), we obtain a function $\mathtt{p}^\infty$ which solves problem (3.13)–(3.14) and satisfies estimate (3.3) for all $y \in Q_\infty$. Indeed, the fact that $\mathtt{p}^\infty$ satisfies the equation (3.13) in Q_∞ and the boundary condition on the lateral boundary and on G_0 is evident. It is also clear that $\max_{Q_\infty} \mathtt{p}^\infty = 1$.

Let us show that with a properly chosen constant c the function $c\mathtt{p}^\infty$ admits representation (3.15). To this end we notice that the function $\mathtt{p}^\infty$ coincides with a solution to the following problem:

$$
\begin{cases} A^* v = 0 \quad \text{in } Q_\infty, \\ a\nabla v \cdot n + b \cdot n v = 0 \quad \text{on } \partial Q_\infty \backslash G_0, \\ v(0, y') = \mathtt{p}^\infty(0, y'), \\ \lim_{y_1 \to \infty} \; v = 0. \end{cases} \tag{3.16}
$$

Consider the operator defined on $H^{1/2}(G)$ that maps the Dirichlet boundary condition on G_0 into the trace on G_1 of the solution of (3.16). We denote this operator by $\mathcal{S}$, so that

$$
v(1, y') = \mathcal{S}\mathtt{p}^\infty(0, y').
$$

Due to smoothing properties of elliptic equations, the operator $\mathcal{S}$ is well defined and compact in the space of continuous functions on G. It also follows by the maximum principle that $\mathcal{S}$ maps the cone of positive functions into itself. Then according to [15] the principal eigenvalue, μ_1 say, of $\mathcal{S}$ is real simple and positive, and all other points of the spectrum belong to the ball of radius $\bar{\mu}$ with $\bar{\mu} < \mu_1$. Denote by v_1 the eigenfunction corresponding to μ_1. Since $\mathcal{S}^n v$ tends to zero, as $n \to \infty$, for any solution v of (3.16), we have $\mu_1 < 1$. It is then easy to check that $\theta_0 = -\log \mu_1$, and that $v_1(y') = p_{\theta_0}(0, y')$. Letting $\theta_1 = -\log \bar{\mu}$, we obtain from [15] that

$$
\mathcal{S}^n \mathtt{p}^\infty(0, y') = c_0 \, e^{\theta_0 n} v_1(y') + \tilde{v}(n, y')
$$

with $c_0 > 0$ and $|\tilde{v}(n, y')| \le C e^{-\theta_1 n}$. This implies the representation

$$
\mathtt{p}^\infty(y) = c_0 \, e^{-\theta_0 y_1} p_{\theta_0}(y) + \mathtt{p}_{\mathrm{bl}}^-(y) \tag{3.17}
$$

with $|\mathtt{p}_{\mathrm{bl}}^-(y)| \le C e^{-\theta_1 y_1}$. Dividing this relation by c_0 yields (3.15).

We proceed with the uniqueness. Suppose that in addition to p^∞ there is another solution $p^{1,\infty}$ of problem (3.13), (3.14). Denote by v^1 a solution to problem (3.16) with $\mathtt{p}^\infty(0,y')$ replaced with $p^{1,\infty}(0,y')$. Then $v^1 = p^{1,\infty}$, and $v^1(n,y') = \mathcal{S}^n p^{1,\infty}(0,y')$. Therefore, this solution also admits representation (3.17) with some constant c_0^1 which need not be positive. We set $q(y) = p^{1,\infty}(y)(p^\infty(y))^{-1}$. Due to (3.15) and the Hölder continuity of $p^{1,\infty}(p^\infty)^{-1}$, $q(y)$ satisfies the estimate $|q| \le C_2$. Moreover, $q(y)$ converges to a constant as $y_1 \to \infty$. We denote this constant by $q_{\inf}$. It is easy to check that q solves in Q_∞ the following problem:

$$-\mathrm{div}\big(\hat a(y)\nabla q(y)\big) + \hat b(y)\nabla q(y) = 0 \quad \text{in } Q_\infty\,,$$
$$\hat a(y)\nabla q(y)\cdot n(y) = 0 \quad \text{on } \partial Q_\infty\,,$$

with $\hat a = (p^\infty)^2 a$ and $\hat b = (p^\infty)^2 b$. It readily follows from the Harnack inequality that the coefficients $\hat a$ and $\hat b$ are locally uniformly bounded, and $\hat a$ is locally uniformly elliptic. Denote by $M(r)$ and $m(r)$ respectively the maximum and the minimum of q over the cross section G_r. We have $\lim_{r\to\infty} M(r) = \lim_{r\to\infty} m(r) = q_{\inf}$. If $q \ne \mathrm{const}$, then either $M(r) > q_{\inf}$, or $m(r) < q_{\inf}$ for some r. This contradicts the maximum principle. □

Lemma 3.4. *Let $\overline{b}_1 > 0$. There exists a constant c_ε such that*

$$p^\varepsilon(y) = c_\varepsilon\big(p_\varepsilon^-(y) + e^{-\theta_0 y_1} p_{\theta_0}(y) + p_\varepsilon^+(y)\big), \tag{3.18}$$

where

$$|p_\varepsilon^-(y)| \le c\, e^{-\theta_1 y_1}, \quad |p_\varepsilon^+(y)| \le c\left(e^{-\theta_0/\varepsilon} e^{\theta_2(y_1-\varepsilon^{-1})} + e^{-\theta_1/\varepsilon}\right) \tag{3.19}$$

with constants $\theta_1 > \theta_0$ and $\theta_2 > 0$. Moreover, as $\varepsilon \to 0$,

$$c_\varepsilon \to c_0, \quad p^\varepsilon \to c_0\, p^\infty \quad \textit{uniformly in } Q_\infty,$$

with c_0 defined in (3.17).

Remark 3.5. In formula (3.18), the functions p_ε^- and p_ε^+ are boundary layers which are exponentially smaller than the main term $e^{-\theta_0 y_1} p_{\theta_0}(y)$ for $1 \ll y_1 \ll \varepsilon^{-1}$.

Proof. We represent p^ε as the sum of solutions to the following two problems:

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla \widehat{p}^{\,-,\varepsilon}(y)\big) - \mathrm{div}\big(b(y)\widehat{p}^{\,-,\varepsilon}(y)\big) = 0 & \text{in } Q_\infty, \\ \ a(y)\nabla \widehat{p}^{\,-,\varepsilon}(y)\cdot n(y) + b(y)\cdot n(y)\widehat{p}^{\,-,\varepsilon}(y) = 0 & \text{on } \partial_l Q_\infty, \\ \widehat{p}^{\,-,\varepsilon} = p^\varepsilon \quad \text{on } G_0, \qquad \lim\limits_{y_1\to\infty} \widehat{p}^{\,-,\varepsilon} = 0, \end{cases} \tag{3.20}$$

and

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla \widehat{p}^{\,+,\varepsilon}(y)\big) - \mathrm{div}\big(b(y)\widehat{p}^{\,+,\varepsilon}(y)\big) = 0 & \text{in } Q_\varepsilon, \\ \ a(y)\nabla \widehat{p}^{\,+,\varepsilon}(y)\cdot n(y) + b(y)\cdot n(y)\widehat{p}^{\,+,\varepsilon}(y) = 0 & \text{on } \partial_l Q_\varepsilon, \\ \widehat{p}^{\,+,\varepsilon} = 0 \quad \text{on } G_0, \qquad \widehat{p}^{\,+,\varepsilon} = p^\varepsilon - \widehat{p}^{\,-,\varepsilon} & \text{on } G_{1/\varepsilon}. \end{cases} \tag{3.21}$$

In exactly the same way as in the proof of Lemma 3.2, one can show that

$$\widehat{p}^{\,-,\varepsilon} = c_\varepsilon\big(p_\varepsilon^-(y) + e^{-\theta_0 y_1} p_{\theta_0}(y)\big) \quad \text{in } Q_\infty, \tag{3.22}$$

where $|p_\varepsilon^-(y)| \le ce^{-\theta_1 y_1}$ with $\theta_1 > \theta_0$, as defined in the proof of Lemma 3.2. Moreover, since $p^\varepsilon(0,y')$ converges to $\mathtt{p}^\infty(0,y')$, we have $c_\varepsilon \to c_0$ and $p_\varepsilon^- \to (p^\infty - e^{-\theta_0 y_1} p_{\theta_0}) = p_{\rm bl}^-$. It follows from (3.22) and the standard elliptic estimates that

$$\Big| \int_{G_r} \big(a(y)\nabla \widehat{p}^{-,\varepsilon}(y)\cdot n + b(y)\cdot n \widehat{p}^{-,\varepsilon}(y)\big)\, dy' \Big|$$
$$= \lim_{z\to\infty} \Big| \int_{G_z} \big(a(y)\nabla \widehat{p}^{-,\varepsilon}(y)\cdot n + b(y)\cdot n \widehat{p}^{-,\varepsilon}(y)\big)\, dy' \Big| = 0.$$

In the same way as in the proof of Lemma 3.1, this implies that

$$\min p^\varepsilon(\varepsilon^{-1},\cdot) < \max \widehat{p}^{-,\varepsilon}(\varepsilon^{-1},\cdot), \quad \min \widehat{p}^{-,\varepsilon}(\varepsilon^{-1},\cdot) < \max p^\varepsilon(\varepsilon^{-1},\cdot).$$

Making the same factorization as in (3.6) and applying the results from [17], see also Theorem 7.1 and Corollary 7.2 in the Appendix, one can check that there exist constants $C > 0$ and $\hat\theta > 0$ such that

$$|\widehat{p}^{+,\varepsilon}(y) - C_\varepsilon p_0(y)| \le C(e^{-\theta_0 y_1} + e^{\hat\theta(y_1 - 1/\varepsilon)}) \|p^\varepsilon(\varepsilon^{-1},\cdot) - \widehat{p}^{-,\varepsilon}(\varepsilon^{-1},\cdot)\|_{L^\infty}, \tag{3.23}$$

with a constant C_ε that satisfies the inequalities

$$\min(p^\varepsilon(\varepsilon^{-1},\cdot) - \widehat{p}^{-,\varepsilon}(\varepsilon^{-1},\cdot)) \le C_\varepsilon \le \max(p^\varepsilon(\varepsilon^{-1},\cdot) - \widehat{p}^{-,\varepsilon}(\varepsilon^{-1},\cdot)).$$

From the last three relations and (3.22) we obtain

$$\widehat{p}^{+,\varepsilon}(y) \le C\, e^{-\varepsilon^{-1}\theta_0}\, e^{-\theta_2(y_1 - \varepsilon^{-1})}$$

with $\theta_2 > 0$. Combining the last estimate with (3.22) yields the desired representation of p^ε. Other statements are straightforward consequences of the uniqueness of a solution to problem (3.13). □

Consider the scaled and shifted functions $P^\varepsilon = e^{\theta_0/\varepsilon} p^\varepsilon(y_1 + 1/\varepsilon, y')$. These functions are defined in the cylinder $Q_{-1/\varepsilon,0} = (-1/\varepsilon, 0) \times G$. We assume first that $1/\varepsilon$ is integer. Then the coefficients with shifted argument coincide with the original coefficients. It follows from the previous lemma and the standard elliptic estimates (see [13]) that

$$0 < C \le P^\varepsilon(0,y') \le C_1;$$
$$\big|P^\varepsilon(y) - c_\varepsilon e^{-\theta_0 y_1} p_{\theta_0}(y)\big| \le C_1\big(e^{\theta_2 y_1} + e^{\theta_0 \varepsilon^{-1}} e^{-\theta_1(\varepsilon^{-1} + y_1)}\big) \quad \text{in } Q_{-1/\varepsilon,0},$$

where $0 < c \le c_\varepsilon \le c_1$, the constants c, C, c_1 and C_1 do not depend on ε. Moreover, P^ε is uniformly in ε Hölder continuous in any finite cylinder $Q_{-L,0}$. Therefore P^ε converges for a subsequence, as $\varepsilon \to 0$, locally uniformly and weakly in $H^1_{\rm loc}$ to a function $\mathtt{P}^\infty$ such that

$$0 < C \le \mathtt{P}^\infty(0,y') \le C_1; \quad \big|\mathtt{P}^\infty(y) - c_0 e^{-\theta_0 y_1} p_{\theta_0}(y)\big| \le C_1 e^{\theta_2 y_1} \text{ in } Q_{-\infty,0}. \tag{3.24}$$

Passing to the limit in the integral identity of problem

$$
(3.25)\qquad \begin{aligned} -\mathrm{div}\big(a(y)\nabla P^\varepsilon(y)\big) - \mathrm{div}\big(b(y)P^\varepsilon(y)(y)\big) &= 0 \quad \text{in } Q_{-1/\varepsilon,0},\\ a(y)\nabla P^\varepsilon(y)(y)\cdot n(y) + b(y)\cdot n(y)P^\varepsilon(y)(y) &= 0 \quad \text{on } \partial Q_{-1/\varepsilon,0}, \end{aligned}
$$

we conclude that $\mathtt{P}^\infty$ satisfies the equation

$$
(3.26)\qquad \begin{aligned} -\mathrm{div}\big(a(y)\nabla \mathtt{P}^\infty(y)\big) - \mathrm{div}\big(b(y)\mathtt{P}^\infty(y)(y)\big) &= 0 \quad \text{in } Q_{-\infty,0},\\ a(y)\nabla \mathtt{P}^\infty(y)(y)\cdot n(y) + b(y)\cdot n(y)\mathtt{P}^\infty(y)(y) &= 0 \quad \text{on } \partial Q_{-\infty,0}. \end{aligned}
$$

In the same way as in the proof of Lemma 3.2, one can show that a solution of problem (3.26) that satisfies the estimate

$$
\mathtt{P}^\infty(y) = c_0\, e^{-\theta_0 y_1}\, p_{\theta_0}(y)\big(1+o(1)\big) \quad \text{in } Q_{-\infty,0}.
$$

is unique. Furthermore, taking into account (3.24) one can check that $\mathtt{P}^\infty(y) = c_0 e^{-\theta_0 y_1} p_{\theta_0}(y) + c_0 p^+_{\mathrm{bl}}(y)$, where $|p^+_{\mathrm{bl}}(y)| \le c\, e^{\theta_2 y_1}$.

This implies that $e^{\theta_0/\varepsilon} p^+_\varepsilon(y)$ converges to $p^+_{\mathrm{bl}}(y_1 - 1/\varepsilon, y')$ uniformly in Q_ε.

We summarize the results of this section in the following statement.

Theorem 3.6. *Let conditions* **(A1)**–**(A2)** *be fulfilled, and assume that* $\overline{b}_1 > 0$. *Then, under a proper normalization, the solution of problem* (2.4) *admits the following representation:*

$$
(3.27)\qquad p^\varepsilon(y) = e^{-\theta_0 y_1} p_{\theta_0}(y) + p^-_\varepsilon(y) + p^+_\varepsilon(y),
$$

where, for some constants $\theta_1 > \theta_0$ *and* $\theta_2 > 0$,

$$
(3.28)\qquad |p^-_\varepsilon(y)| \le C e^{-\theta_1 y_1}, \quad |p^+_\varepsilon(y)| \le C\big(e^{-\theta_1/\varepsilon} + e^{-\theta_0 \varepsilon} e^{\theta_2 (y_1 - 1/\varepsilon)}\big).
$$

Moreover, p^-_ε *converges to* p^-_{bl} *uniformly in* Q_ε, *and* $e^{\theta_0/\varepsilon} p^+_\varepsilon(y)$ *converges to* $p^+_{\mathrm{bl}}(y_1 - 1/\varepsilon, y')$ *uniformly in* Q_ε.

Proof. It suffices to introduce a new normalization of p^ε dividing it by the constant c_ε defined in Lemma 3.4. Then, dividing relation (3.18) by c_ε and considering estimates (3.19) in Lemma 3.4, one concludes that, under the new normalization, p^ε satisfies (3.27)–(3.28), and the announced convergence of p^-_ε and $e^{\theta_0/\varepsilon} p^+_\varepsilon(y)$ holds. □

Remark 3.7. In formula (3.27), the functions p^-_ε and p^+_ε are boundary layers which are exponentially smaller than the main term $e^{-\theta_0 y_1} p_{\theta_0}(y)$ for $1 \ll y_1 \ll \varepsilon^{-1}$. Notice that (3.27) holds under a normalization of p^ε that differs from that in (3.1). More precisely, we have to divide p^ε by the constant c_ε defined in Lemma 3.4.

Remark 3.8. If in problem (2.4) we consider Dirichlet boundary condition at both ends G_0 and $G_{1/\varepsilon}$ of the cylinder (still keeping the lateral Neumann boundary conditions on $\partial_l Q_\varepsilon$), then the asymptotic behavior, predicted by Theorem 3.6, changes completely. Of course, in such a case, the first eigenvalue λ^ε is not zero anymore and, denoting the first eigenfunction $p^\varepsilon_{\rm Dir}(y)$, (2.4) becomes

$$
\left\{
\begin{array}{l}
-\mathrm{div}\big(a(y)\nabla p^\varepsilon_{\rm Dir}\big) - \mathrm{div}\big(b(y) p^\varepsilon_{\rm Dir}\big) = \lambda^\varepsilon p^\varepsilon_{\rm Dir} \quad \text{in } Q_\varepsilon,\\
a(y)\nabla p^\varepsilon_{\rm Dir}\cdot n + b(y)\cdot n p^\varepsilon_{\rm Dir} = 0 \quad \text{on } \partial_l Q_\varepsilon,\\
p^\varepsilon_{\rm Dir} = 0 \quad \text{on } G_0 \cup G_{1/\varepsilon}.
\end{array}
\right.
\tag{3.29}
$$

Indeed, after some simple adaptation, the results of [6], [7] show that the solution $p^\varepsilon_{\rm Dir}(y)$ of (3.29) satisfies

$$
p^\varepsilon_{\rm Dir}\left(\frac{x}{\varepsilon}\right) \approx e^{-\theta_0 x_1/\varepsilon}\, p_{\theta_0}\left(\frac{x}{\varepsilon}\right) p^1(x_1),
$$

where $p^1(x_1)$ is the first eigenfunction of an homogenized problem in the segment $(0,1)$ (which is the limit of the rescaled cylinder $\varepsilon Q_\varepsilon$) with Dirichlet boundary condition. Typically p^1 is a cosine function. Furthermore, the approximation is not merely up to the addition of boundary layers; rather, homogenization correctors have to be added to improve the approximation. The absence of homogenized problem for the Neumann case studied in the present paper is thus in sharp contrast with the Dirichlet case of [6], [7].

4. Main result for vanishing effective drift $\overline{b}_1 = 0$

In the case $\overline{b}_1 = 0$, we shall prove (see Theorem 4.5) that the function p^ε is exponentially close, in the interior part of the cylinder, to the periodic eigenfunction p_0, solution of (2.5). In the vicinity of the cylinder bases the difference between p^ε and p_0 is an exponential boundary layer.

The construction of the boundary layers relies on the following statement.

Lemma 4.1. *Let* $\overline{b}_1 = 0$. *Then problem* (3.13) *has a unique, up to a multiplicative constant, bounded solution. Moreover, there are constants* $\vartheta > 0$, $C > 0$ *and* c *such that*

$$
|p^\infty - c p_0| \le C e^{-\vartheta y_1}.
\tag{4.1}
$$

Proof. Consider a sequence of problems (2.4) and the corresponding solutions p^ε normalized in such a way that

$$
\max_{Q_\varepsilon} p^\varepsilon = 1.
$$

Denote

$$
\hat a(y) = p_0(y) a(y), \qquad \hat b(y) = a(y)\nabla p_0(y) + p_0(y) b(y).
$$

Representing $p^\varepsilon(y) = p_0(y) q^{*,\varepsilon}(y)$, we arrive at the following problem:

$$
\left\{
\begin{array}{ll}
-\mathrm{div}\big(\hat a(y)\nabla q^{*,\varepsilon}(y)\big) - \mathrm{div}\big(\hat b(y) q^{*,\varepsilon}(y)\big) = 0 & \text{in } Q_\varepsilon,\\
\hat a(y)\nabla q^{*,\varepsilon}(y)\cdot n(y) + \hat b(y)\cdot n(y) q^{*,\varepsilon}(y) = 0 & \text{on } \partial_l Q_\varepsilon,\\
\hat a(y)\nabla q^{*,\varepsilon}(y)\cdot n(y) + \hat b(y)\cdot n(y) q^{*,\varepsilon}(y) = 0 & \text{on } G_0 \cup G_{1/\varepsilon}.
\end{array}
\right.
\tag{4.2}
$$

Observe that by the definition of p_0 we have

$$\operatorname{div}\big(\hat{b}(y)q^{*,\varepsilon}(y)\big) = \hat{b}(y)\nabla q^{*,\varepsilon}(y) \ \text{ in } Q_\varepsilon, \quad \hat{b}(y)\cdot n(y) = 0 \ \text{ on } \partial_l Q_\varepsilon. \tag{4.3}$$

Therefore,

$$\max_{Q_\varepsilon} q^{*,\varepsilon} = \max_{G_0\cup G_{1/\varepsilon}} q^{*,\varepsilon}, \quad \min_{Q_\varepsilon} q^{*,\varepsilon} = \min_{G_0\cup G_{1/\varepsilon}} q^{*,\varepsilon}. \tag{4.4}$$

Indeed, due to (4.3), the equation in (4.2) takes the form

$$-\operatorname{div}\big(\hat{a}(y)\nabla q^{*,\varepsilon}(y)\big) - \hat{b}(y)\nabla q^{*,\varepsilon}(y) = 0, \quad y\in Q_\varepsilon,$$
$$\hat{a}(y)\nabla q^{*,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon.$$

Since $q^{*,\varepsilon}$ satisfies homogeneous Neumann condition on the lateral boundary, $q^{*,\varepsilon}$ cannot attain its maximum (or minimum) in the interior of Q_ε nor on the lateral boundary, unless $q^{*,\varepsilon}$ is a constant.

Lemma 4.2. *The following inequalities hold true:*

$$\max_{G_0} q^{*,\varepsilon} \geq \min_{G_{1/\varepsilon}} q^{*,\varepsilon}, \quad \min_{G_0} q^{*,\varepsilon} \leq \max_{G_{1/\varepsilon}} q^{*,\varepsilon}.$$

Proof. Assume that $\min_{G_0} q^{*,\varepsilon} > \max_{G_{1/\varepsilon}} q^{*,\varepsilon}$. Then there is $\varkappa\in\mathbb{R}$ such that

$$\min_{G_0} q^{*,\varepsilon} > \varkappa > \max_{G_{1/\varepsilon}} q^{*,\varepsilon}. \tag{4.5}$$

Consider an auxiliary problem

$$\begin{cases} -\operatorname{div}\big(\hat{a}(y)\nabla q^{\varkappa,\varepsilon}(y)\big) - \operatorname{div}\big(\hat{b}(y)q^{\varkappa,\varepsilon}(y)\big) = 0 \ \text{ in } Q_\varepsilon, \\ -\hat{a}(y)\nabla q^{\varkappa,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ q^{\varkappa,\varepsilon}(y) = q^{*,\varepsilon}(y) \quad \text{on } G_0, \\ q^{\varkappa,\varepsilon}(y) = \varkappa \quad \text{on } G_{1/\varepsilon}. \end{cases} \tag{4.6}$$

By the maximum principle and due to (4.5), the minimum of $q^{\varkappa,\varepsilon}$ over Q_ε is attained on $G_{1/\varepsilon}$, and furthermore

$$\hat{a}(y)\nabla q^{\varkappa,\varepsilon}\cdot n < 0 \quad \text{on } G_{1/\varepsilon}.$$

Integrating this relation over $G_{1/\varepsilon}$ and considering the fact that

$$\int_{G_{1/\varepsilon}} \hat{b}(y)\cdot n\,dy' = 0,$$

we get

$$\int_{G_{1/\varepsilon}} \big(\hat{a}(y)\nabla q^{\varkappa,\varepsilon}\cdot n - \hat{b}\cdot n q^{\varkappa,\varepsilon}\big)dy' < 0.$$

Therefore,

$$\int_{G_{1/\varepsilon}} \big(\hat a(y)\nabla(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\cdot n-\hat b\cdot n(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\big)dy'<0 \quad \text{on } G_{1/\varepsilon}. \tag{4.7}$$

On the other hand, the function $(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})$ has its minimum at G_0, and thus, by the strong maximum principle,

$$a(y)\,\frac{\partial}{\partial n}(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})<0 \quad \text{on } G_0. \tag{4.8}$$

Integrating equations (4.2) and (4.6) over Q_ε, taking the difference of the resulting relations and integrating by parts, we obtain

$$\begin{aligned}0=&-\int_{G_{1/\varepsilon}} \big(\hat a(y)\nabla(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\cdot n-\hat b(y)\cdot n(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\big)dy'\\ &-\int_{G_0} \big(\hat a(y)\nabla(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\cdot n-\hat b(y)\cdot n(q^{\varkappa,\varepsilon}-q^{*,\varepsilon})\big)dy'<0.\end{aligned}$$

We arrive at a contradiction. This completes the proof of Lemma 4.2. □

It follows from our normalization condition for p^ε, the definition of $q^{*,\varepsilon}$ and the properties of p_0 that $C\le\max_{Q_\varepsilon} q^{*,\varepsilon}\le C^{-1}$. Combining these estimates with Lemma 4.2 and the Harnack inequality yields

$$C\le\min_{Q_\varepsilon} q^{*,\varepsilon}\le\max_{Q_\varepsilon} q^{*,\varepsilon}\le C^{-1}$$

for a positive constant C that does not depend on ε. Passing to the limit in (4.2), as $\varepsilon\to 0$, we obtain a solution of the following problem:

$$\begin{cases}-\mathrm{div}\big(\hat a(y)\nabla q^{*,0}(y)\big)-\mathrm{div}\big(\hat b(y)q^{*,0}(y)\big)=0 & \text{in } Q_\infty,\\ \hat a(y)\nabla q^{*,0}(y)\cdot n(y)=0 & \text{on } \partial_l Q_\infty,\\ \hat a(y)\nabla q^{*,0}(y)\cdot n(y)+\hat b(y)\cdot n(y)q^{*,0}(y)=0 & \text{on } G_0,\end{cases} \tag{4.9}$$

such that $C\le\inf_{Q_\infty} q^{*,0}\le\sup_{Q_\infty} q^{*,0}\le C^{-1}$. This proves the existence of a positive bounded solution. Estimate (4.1) follows from Theorem 6.1 and Lemma 6.3 in [17]. The uniqueness can be proved in the same way as in the previous section. □

Lemma 4.3. *For each $\varepsilon>0$ there is a unique constant $\varkappa=\varkappa(\varepsilon)$ such that for the solution of problem (4.6) the following relation is fulfilled:*

$$J_\varkappa:=\int_{G_0}\big(-\hat a(y)\nabla q^{\varkappa,\varepsilon}\cdot n-\hat b(y)\cdot n q^{\varkappa,\varepsilon}\big)dy'=0. \tag{4.10}$$

Proof. In the same way as in the proof of Lemma 4.2 one can show that $J_\varkappa>0$ if $\varkappa>\max_{G_0} q^{\varkappa,\varepsilon}$, and $J_\varkappa<0$ if $\varkappa<\max_{G_0} q^{\varkappa,\varepsilon}$. Since $J_\varkappa$ is a continuous function of $\varkappa$, the existence of desired $\varkappa$ follows. The uniqueness is straightforward. □

Lemma 4.4. *As $\varepsilon \to 0$, the sequence $q^{\varkappa(\varepsilon),\varepsilon}$ converges to $q^{*,0}$.*

Proof. By the definition of $q^{\varkappa,\varepsilon}$ we have $q^{\varkappa(\varepsilon),\varepsilon}(0,y') = q^{*,\varepsilon}(0,y')$. Passing to the limit one can easily check that the limit function $\tilde{q}^{*,0}$ is a bounded solution to the following problem:

$$\begin{cases} -\mathrm{div}\big(\hat{a}(y)\nabla\tilde{q}^{*,0}(y)\big) - \mathrm{div}\big(\hat{b}(y)\tilde{q}^{*,0}(y)\big) = 0 \ \text{ in } Q_\infty, \\ \hat{a}(y)\nabla\tilde{q}^{*,0}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\infty, \\ \tilde{q}^{*,0}(y) = q^{*,0}(y) \quad \text{on } G_0. \end{cases}$$

The desired statement is now a consequence of the uniqueness result obtained in [17]. □

We now turn to the main result of this section. Let p^∞ be a bounded solution of problem (3.13) such that $|p^\infty - p_0| \le c\,e^{-\vartheta y_1}$, $\vartheta > 0$. In addition to p^∞, we also introduce a function P^∞_γ as a bounded solution to the following problem:

$$\begin{aligned} -\mathrm{div}\big(a(y)\nabla P^\infty_\gamma(y)\big) - \mathrm{div}\big(b(y)P^\infty_\gamma(y)\big) = 0 \quad & \text{in } Q_{-\infty,\gamma}\,, \\ -a(y)\nabla P^\infty_\gamma(y)\cdot n(y) - b(y)\cdot n(y)P^\infty_\gamma(y) = 0 \quad & \text{on } \partial Q_{-\infty,\gamma}\,, \end{aligned} \tag{4.11}$$

with $Q_{-\infty,\gamma} = (-\infty,\gamma)\times G$. By Lemma 4.1 such a solution exists and is unique up to a multiplicative constant. Due to periodicity of the coefficients, $P^\infty_\gamma(y_1+1,y') = P^\infty_{\gamma+1}(y)$. As we did with p^∞, we normalize P^∞_γ in such a way that $(P^\infty_\gamma - p_0) \to 0$ as $y_1 \to -\infty$.

Theorem 4.5. *Let $\overline{b}_1 = 0$. Then, under a proper normalization, there exists $\vartheta > 0$ such that*

$$|p^\varepsilon(y) - (p^\infty(y) + P^\infty_{1/\varepsilon}(y) - p_0(y))| \le C_\varepsilon(e^{-\vartheta y_1} + e^{\vartheta(y_1 - 1/\varepsilon)}),$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$, so that

$$\|p^\varepsilon - (p^\infty + P^\infty_{1/\varepsilon} - p_0)\|_{L^\infty(Q_\varepsilon)} \longrightarrow 0.$$

Remark 4.6. Theorem 4.5 states that p^ε is equal to the 1-periodic eigenfunction p_0, solution of (2.5), up to the addition of boundary layers which are exponentially small for $1 \ll y_1 \ll \varepsilon^{-1}$. The boundary layers are precisely $(p^\infty - p_0)$ on the left and $(P^\infty_{1/\varepsilon} - p_0)$ on the right of the cylinder.

Proof. In addition to problem (4.6), we also consider the problem

$$\begin{cases} -\mathrm{div}\big(\hat{a}(y)\nabla q^{\varkappa_1,\varepsilon}_-(y)\big) - \mathrm{div}\big(\hat{b}(y)q^{\varkappa_1,\varepsilon}_-(y)\big) = 0 \ \text{ in } Q_\varepsilon, \\ -\hat{a}(y)\nabla q^{\varkappa_1,\varepsilon}_-(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ q^{\varkappa_1,\varepsilon}_-(y) = q^{*,\varepsilon}(y) \quad \text{on } G_{1/\varepsilon}, \\ q^{\varkappa_1,\varepsilon}_-(y) = \varkappa_1 \quad \text{on } G_0. \end{cases} \tag{4.12}$$

By Lemma 4.3 there is a constant $\varkappa_1 = \varkappa_1(\varepsilon)$ such that

$$J_{1,\varkappa_1} := \int_{G_{1/\varepsilon}} \big(\hat{a}(y)\nabla q_-^{\varkappa_1,\varepsilon} \cdot n - \hat{b}(y) \cdot n q_-^{\varkappa_1,\varepsilon}\big) dy' = 0. \tag{4.13}$$

Choosing now the constants $\varkappa$ and $\varkappa_1$ in such a way that relations (4.10) and (4.13) hold true, it is straightforward to check that the function

$$\check{q}^\varepsilon(y) = q^{*,\varepsilon}(y) - q^{\varkappa(\varepsilon),\varepsilon}(y) - q_-^{\varkappa_1(\varepsilon),\varepsilon}(y)$$

solves the following problem:

$$\begin{cases} -\mathrm{div}\big(\hat{a}(y)\nabla\check{q}^\varepsilon(y)\big) - \mathrm{div}\big(\hat{b}(y)\check{q}^\varepsilon(y)\big) = 0 \ \text{ in } Q_\varepsilon, \\ -\hat{a}(y)\nabla\check{q}^\varepsilon(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ \check{q}^\varepsilon(y) = -\varkappa \quad \text{on } G_{1/\varepsilon}, \\ \check{q}^\varepsilon(y) = -\varkappa_1 \quad \text{on } G_0, \end{cases} \tag{4.14}$$

and satisfies the relation

$$\int_{G_0} \big(\hat{a}(y)\nabla\check{q}^\varepsilon \cdot n - \hat{b}(y)\cdot n\check{q}^\varepsilon\big) dy' = 0. \tag{4.15}$$

By the same arguments as in the proof of Lemma 4.2, we conclude that $\varkappa_1 = \varkappa$.

Choosing now a normalization condition in such a way that $\varkappa = 1$, we see that

$$q^{*,\varepsilon}(y) = q^{\varkappa,\varepsilon}(y) + q_-^{\varkappa,\varepsilon}(y) - 1$$

and

$$p^\varepsilon(y) = q^{\varkappa,\varepsilon}(y)p_0(y) + q_-^{\varkappa,\varepsilon}(y)p_0(y) - p_0(y).$$

Consider a bounded solution of the problem

$$\begin{cases} -\mathrm{div}\big(\hat{a}(y)\nabla q_0^{\infty,\varepsilon}(y)\big) - \mathrm{div}\big(\hat{b}(y)q_0^{\infty,\varepsilon}(y)\big) = 0 \ \text{ in } Q_\infty, \\ \hat{a}(y)\nabla q_0^{\infty,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\infty, \\ q_0^{\infty,\varepsilon}(y) = q^{*,\varepsilon}(y) \quad \text{on } G_0. \end{cases}$$

By the arguments used in the proof of Lemma 4.2 and the maximum principle, one can deduce that, for some $\vartheta > 0$,

$$\|q_0^{\infty,\varepsilon} - q^{\varkappa,\varepsilon}\|_{L^\infty(Q_\varepsilon)} \le c\, e^{-\vartheta/\varepsilon},$$

and, since $\varkappa = 1$, this yields

$$|q^{\varkappa,\varepsilon}(y) - 1| \le c\, e^{-\vartheta y_1}, \quad |q_-^{\varkappa,\varepsilon}(y) - 1| \le c\, e^{\vartheta(y_1-(1/\varepsilon))}.$$

Sending the length of the cylinder to ∞, we obtain

$$|q^{*,0}(y) - 1| \le c\, e^{-\vartheta y_1}, \quad |q_-^{*,0}(y_1 - 1, y') - 1| \le c\, e^{\vartheta(y_1-(1/\varepsilon))}$$

Taking into account the relations $p^\infty(y) = q^{*,0}(y)p_0(y)$ and $P^\infty_{1/\varepsilon}(y) = q_-^{*,0}(1/\varepsilon - y_1, y')p_0(y)$, we deduce the desired statements from the last three formulae. This completes the proof. ∎

5. Equations with non-smooth coefficients

In this section we show that the regularity assumption that was imposed in the previous sections can be discarded. We assume here that conditions **(A1)** and **(A2)** are fulfilled, and that the entries of the matrix $a(\cdot)$ and the components of the vector field $b(\cdot)$ are merely $L^\infty(Y)$ functions. Under these assumptions the proof of Lemma 2.1 remains unchanged.

5.1. The case $\overline{b}_1 > 0$

Lemma 5.1. *The statements of Lemma 3.1 remain valid.*

Proof. The proof of the uniform local Harnack inequality did not use any regularity of the coefficients. Thus, this inequality holds. We now change the factorization which lead to equations (3.7) and (3.8) in the proof of Lemma 3.1. We do so because of regularity issues (see the discussion in Remark 5.2). Letting

$$p^{\pm,\varepsilon}(y) = p_0(y)\, q^{\pm,\varepsilon}(y) \tag{5.1}$$

and multiplying the resulting equation by $p_0(y)$, after straightforward rearrangements we get

$$\begin{cases} -\mathrm{div}\big(p_0^2(y)a(y)\nabla q^{-,\varepsilon}(y)\big) - p_0^2(y)b(y)\nabla q^{-,\varepsilon}(y) = 0 \ \text{ in } Q_\varepsilon, \\ \qquad -p_0^2(y)a(y)\nabla q^{-,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ q^{-,\varepsilon} = p^\varepsilon (p_0)^{-1} \quad \text{on } G_0, \quad q^{-,\varepsilon} = 0 \quad \text{on } G_{1/\varepsilon}, \end{cases} \tag{5.2}$$

and

$$\begin{cases} -\mathrm{div}\big(p_0^2(y)a(y)\nabla q^{+,\varepsilon}(y)\big) - p_0^2(y)b(y)\nabla q^{+,\varepsilon}(y) = 0 \ \text{ in } Q_\varepsilon, \\ \qquad -p_0^2(y)a(y)\nabla q^{+,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ q^{+,\varepsilon} = 0 \quad \text{on } G_0, \quad q^{+,\varepsilon} = p_\varepsilon (p_0)^{-1} \quad \text{on } G_{1/\varepsilon}. \end{cases} \tag{5.3}$$

Let us denote by $\tilde{A}$ the following operator:

$$\tilde{A}u = -\mathrm{div}\big(p_0^2(y)a(y)\nabla u\big) - p_0^2(y)b(y)\cdot\nabla u,$$

$$-\, p_0^2(y)a(y)\nabla u(y)\cdot n(y) = 0 \text{ on } \partial_l Y,$$

with its adjoint $\tilde{A}^*$

$$\tilde{A}^*u = -\mathrm{div}\big(p_0^2(y)a(y)\nabla u\big) + \mathrm{div}(p_0^2(y)b(y)u),$$

$$-\, p_0^2(y)a(y)\nabla u(y)\cdot n(y) + p_0^2(y)b(y)\cdot n(y)u(u) = 0 \text{ on } \partial_l Y.$$

It is easy to check that the kernel of $\tilde{A}^*$ in the unit cell Y, with 1-periodic boundary conditions in y_1, is equal to $1/p_0$. Considering the normalized function

$$\Big(\int_Y \frac{1}{p_0(y)}\,dy\Big)^{-1}\frac{1}{p_0}$$

and recalling Definition 2.2 of the longitudinal effective drift, we conclude after simple computations that the effective longitudinal drift of $\tilde{A}$ (the operator appearing in (3.7) and (3.8)) is

$$-\Big(\int_Y \frac{1}{p_0(y)}\,dy\Big)^{-1}\bar{b}_1.$$

Under our standing assumptions, this drift is negative.

By contradiction with (3.2), assume now that, for a subsequence, $\max_{G_{1/\varepsilon}}(p^\varepsilon)$ does not go to zero as $\varepsilon \to 0$. Then, by the Harnack inequality,

$$0 < C \le p^\varepsilon(\varepsilon^{-1}, y') \le 1, \quad 0 < C \le q^{+,\varepsilon}(\varepsilon^{-1}, y').$$

According to [17] and Corollary 7.2, because the effective drift of $\tilde{A}$ is negative, there are constants C_ε, $0 < C \le C_\varepsilon \le C_1$, and $\varkappa > 0$ such that

$$|q^{-,\varepsilon}| \le e^{-\varkappa/\varepsilon}, \quad |q^{+,\varepsilon} - C_\varepsilon| \le e^{-\varkappa/\varepsilon} \quad \text{in } Q_{\frac{1}{3\varepsilon},\frac{2}{3\varepsilon}}.$$

Considering the definition of $q^{\pm,\varepsilon}$ and $p^{\pm,\varepsilon}$, we derive from that last inequalities that

$$|p^\varepsilon(y) - C_\varepsilon p_0(y)| \le e^{-\varkappa/\varepsilon} \quad \text{in } Q_{\frac{1}{3\varepsilon},\frac{2}{3\varepsilon}}. \tag{5.4}$$

By the local elliptic estimates the last inequality implies

$$\|p^\varepsilon - C_\varepsilon p_0\|_{H^1(Q_{s,s+1})} \le Ce^{-\varkappa/\varepsilon}, \quad \frac{1}{3\varepsilon} \le s \le \frac{2}{3\varepsilon} - 1. \tag{5.5}$$

On the other hand, integrating (2.4) on $Q_{0,r}$, we get

$$\int_{G_r} \big(a(y)\nabla p^\varepsilon(y)\cdot n + b(y)\cdot n p^\varepsilon(y)\big)\,dy' = 0,$$

while integrating (2.5) on $Q_{s,r}$ shows that the following surface integral,

$$\int_{G_r} \big(a(y)\nabla p_0(y)\cdot n + b(y)\cdot n p_0(y)\big)\,dy' = \bar{b}_1 > 0, \tag{5.6}$$

is constant for all $r \in [0, 1/\varepsilon]$. Since $C_\varepsilon \ge C > 0$, the last two relations contradict (5.4), (5.5). Thus, (3.2) holds true.

The assumption that (3.3) does not hold leads to a contradiction in exactly the same way. This completes the proof. □

Remark 5.2. It is a common practice to write down the factorized equations for $q^{-,\varepsilon}$ and $q^{+,\varepsilon}$ in the form (3.7) and (3.8). The advantage of this representation is the divergence-free structure of $\check{b} = a\nabla p_0 + bp_0$. Indeed, it satisfies $\operatorname{div}\check{b} = 0$ in Q_ε, and $\check{b}\cdot n = 0$ on $\partial_l Q_\varepsilon$. This simplifies the study of problems (3.7) and (3.8). However, there is an important disadvantage. If the original coefficients $a(y)$ and $b(y)$ are just measurable, then $\check{b}(y) = a(y)\nabla p_0(y) + b(y)p_0(y)$ need not belong to L^∞, while the coefficients in (5.2) and (5.3) remain bounded.

In the proofs of Lemma 3.2 and Lemma 3.4 we did not use regularity of the coefficients. Therefore, the statements of these lemmata hold under our standing assumptions. Then Theorem 3.6 also remains valid.

Theorem 5.3. *Let assumptions* **(A1)**–**(A2)** *be fulfilled, and assume that the coefficients of equations* (2.4) *are bounded measurable functions. Then all the statements of Theorem* 3.6 *hold true.*

5.2. The case $\overline{b}_1 = 0$

In the case of non-smooth coefficients we cannot use equation (4.2) any more because its coefficients need not be bounded. Instead, we write down the problem for $q^{*,\varepsilon}$ in the following form:

$$\begin{cases} -\mathrm{div}\big(p_0^2(y)a(y)\nabla q^{*,\varepsilon}(y)\big) - p_0^2(y)b(y)\nabla q^{*,\varepsilon}(y) = 0 \ \text{ in } Q_\varepsilon, \\ \qquad p_0^2(y)a(y)\nabla q^{*,\varepsilon}(y)\cdot n(y) = 0 \text{ on } \partial_l Q_\varepsilon, \\ q^{*,\varepsilon} = p^\varepsilon(p_0)^{-1} \text{ on } G_0, \quad q^{*,\varepsilon} = p^\varepsilon(p_0)^{-1} \text{ on } G_{1/\varepsilon}, \end{cases} \tag{5.7}$$

which is equivalent to (4.2) for smooth coefficients. This implies by the maximum principle relations (4.4).

The proof of Lemma 4.2 should be modified as follows. Assuming by contradiction that $\max_{G_{1/\varepsilon}} q^{*,\varepsilon} < \min_{G_0} q^{*,\varepsilon}$ and taking a constant $\varkappa$ that satisfies the inequality $\max_{G_{1/\varepsilon}} q^{*,\varepsilon} < \varkappa < \min_{G_0} q^{*,\varepsilon}$, we consider the auxiliary problem

$$\begin{cases} -\mathrm{div}\big(p_0^2(y)a(y)\nabla q^{\varkappa,\varepsilon}(y)\big) - p_0^2(y)b(y)\nabla q^{\varkappa,\varepsilon}(y) = 0 \ \text{ in } Q_\varepsilon, \\ \qquad p_0^2(y)a(y)\nabla q^{\varkappa,\varepsilon}(y)\cdot n(y) = 0 \quad \text{on } \partial_l Q_\varepsilon, \\ q^{\varkappa,\varepsilon} = p^\varepsilon(p_0)^{-1} \quad \text{on } G_0, \quad q^{\varkappa,\varepsilon} = \varkappa \quad \text{on } G_{1/\varepsilon}. \end{cases} \tag{5.8}$$

Subtract the equation in (5.8) from the equation in (5.7), multiply the difference by $(p_0(y))^{-1}$ and integrate the resulting relation over Q_ε. After integration by parts and straightforward rearrangements, this yields

$$\begin{aligned} &-\int_{G_0} \big[a\nabla\big(p_0(q^{*,\varepsilon} - q^{\varkappa,\varepsilon})\big)\cdot n + b\cdot n\, p_0(q^{*,\varepsilon} - q^{\varkappa,\varepsilon})\big]\,dy' \\ &\quad -\int_{G_{1/\varepsilon}} \big[a\nabla\big(p_0(q^{*,\varepsilon} - q^{\varkappa,\varepsilon})\big)\cdot n + b\cdot n\, p_0(q^{*,\varepsilon} - q^{\varkappa,\varepsilon})\big]\,dy' = 0. \end{aligned} \tag{5.9}$$

Since $q^{*,\varepsilon} - q^{\varkappa,\varepsilon} = 0$ on G_0 and $p_0(q^{*,\varepsilon} - q^{\varkappa,\varepsilon}) \le 0$ in Q_ε, the first term on the left-hand side of (5.9) is non-positive. By the definition of $q^{*,\varepsilon}$,

$$\int_{G_{1/\varepsilon}} \big[a\nabla(p_0 q^{*,\varepsilon})\cdot n + b\cdot n\, p_0 q^{*,\varepsilon}\big]\,dy' = 0.$$

We also have

$$\begin{aligned} &\int_{G_{1/\varepsilon}} \big[a\nabla\big(p_0 q^{\varkappa,\varepsilon}\big)\cdot n + b\cdot n\, p_0 q^{\varkappa,\varepsilon}\big]\,dy' \\ &\qquad = \int_{G_{1/\varepsilon}} p_0 a\nabla q^{\varkappa,\varepsilon}\cdot n\,dy' + \varkappa\overline{b}_1 = \int_{G_{1/\varepsilon}} p_0 a\nabla q^{\varkappa,\varepsilon}\cdot n\,dy'. \end{aligned}$$

Since $q^{\varkappa,\varepsilon} = \varkappa$ on $G_{1/\varepsilon}$ and $q^{\varkappa,\varepsilon} \geq \varkappa$ in Q_ε, the integral on the right-hand side here is non-negative, and, therefore, the second term on the left-hand side of (5.9) is non-positive.

Consider now two constants $\varkappa_1$ and $\varkappa_2$ such that

$$\max_{G_{1/\varepsilon}} q^{*,\varepsilon} < \varkappa_1 < \varkappa_2 < \min_{G_0} q^{*,\varepsilon}.$$

Writing down the equation for the difference $q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}$, multiplying this equation by $(p_0)^{-1}(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon})$ and integrating the resulting relation over Q_ε, after integration by parts and straightforward rearrangements we obtain

$$-\int_{G_0} p_0(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) a\nabla(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) \cdot n\, dy' \\ - \int_{G_{1/\varepsilon}} p_0(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) a\nabla(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) \cdot n\, dy' \\ + \int_{Q_\varepsilon} p_0 a\nabla(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) \cdot \nabla(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) dy = 0.$$

The first integral on the left-hand side is equal to zero because $q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon} = 0$ on G_0. Since $q^{\varkappa_1,\varepsilon} \neq q^{\varkappa_2,\varepsilon}$ in Q_ε, the third integral is strictly positive. Therefore,

$$-(\varkappa_1 - \varkappa_2) \int_{G_{1/\varepsilon}} p_0 a\nabla(q^{\varkappa_1,\varepsilon} - q^{\varkappa_2,\varepsilon}) \cdot n\, dy' < 0,$$

and for at least one of the constants $\varkappa_1$ and $\varkappa_2$ equality (5.9) is contradictory. This completes the proof of Lemma 4.2. Other statements in Section 4 can be justified in exactly the same way as in the smooth case. We arrive at the following result.

Theorem 5.4. *Let assumptions* **(A1)**–**(A2)** *be fulfilled, and assume that the coefficients of equations* (2.4) *are bounded measurable functions. Then all the statements of Theorem* 4.5 *hold true.*

6. Perspectives

In this short section we discuss possible generalizations of the results of this work.

Operators with locally periodic coefficients. Consider the problem

$$-\text{div}\big(a(x, \varepsilon^{-1}x)\nabla p^\varepsilon\big) - \frac{1}{\varepsilon}\,\text{div}\big(b(x, \varepsilon^{-1}x)p^\varepsilon\big) = 0 \quad \text{in } \varepsilon Q_\varepsilon,$$
$$-a(x, \varepsilon^{-1}x)\nabla p^\varepsilon \cdot n - b(x, \varepsilon^{-1}x) \cdot n p^\varepsilon = 0 \quad \text{on } \varepsilon\partial Q_\varepsilon.$$

Under the assumption that $a(x, y)$ and $b(x, y)$ are periodic in y_1 and a uniform ellipticity assumption one can study the logarithmic asymptotics of a solution of this problem as $\varepsilon \to 0$. Making the logarithmic transform of p^ε we reduce the above problem to homogenization problem for a perturbed Hamilton–Jacobi type equation. Then we can use the approaches developed in [12], [19]. Additional difficulties here are due to the fact that the homogenization is combined with the dimension reduction. We should also derive the effective boundary conditions at the end points of the interval where the limit equation is stated. The work on this problem is in progress.

Fourier boundary conditions on the cylinder bases. Instead of adjoint Neumann boundary conditions on the cylinder bases in (1.1) one can consider the spectral problem with arbitrary Fourier boundary conditions on the bases. In this case the principal eigenvalue need not be equal to zero any more. In the 1-d case this problem has been investigated in [1]. In the multidimensional case, making again a logarithmic transformation of the principal eigenfunction, one can reduce the studied spectral problem to an appropriate boundary value problem for the corresponding perturbed Hamilton–Jacobi type equation. The derivation of effective boundary conditions for the effective Hamilton–Jacobi equation is getting rather non-trivial in this case. This work is also in progress.

Elliptic systems. We believe that in the case of cooperative systems to which the maximum principle applies the results of this work hold true and can be proved by the same methods (but we did not check this). For more general elliptic systems the question is completely open.

7. Appendix

In this Appendix we formulate, for the reader convenience, the key results from [17] and provide a number of corollaries of these results.

Let, as in (3.13), $Q_\infty = (0,\infty)\times G$, and consider the following problem:

$$\begin{cases} -\mathrm{div}\big(a(y)\nabla v(y)\big) + b(y)\nabla v(y) = 0 & \text{in } Q_\infty, \\ a(y)\nabla v(y)\cdot n(y) = 0 & \text{on } (0,+\infty)\times\partial G, \\ v(y) = v_0(y) & \text{on } G_0; \end{cases} \tag{7.1}$$

here v_0 is a given function, $v_0 \in L^\infty(G_0)\cap H^{1/2}(G_0)$.

Theorem 7.1 (Theorem 6.1 in [17]). *If $\overline{b}_1 < 0$, then for any constant c there is a solution of (7.1) that converges to c as $y_1\to+\infty$. Such a solution (with a fixed limit c) is unique.*

If $\overline{b}_1 \geq 0$, then problem (7.1) has a unique bounded solution.

In both cases any bounded solution v of problem (7.1) converges to a constant at exponential rate that is there exist constants $\gamma>0$, c and C_0 such that

$$|v(y) - c| \leq C_0\, e^{-\gamma y_1},$$

and the constant γ does not depend on v_0.

In the case $\overline{b}_1 \geq 0$ we denote by $c(v_0)$ the unique constant to which the bounded solution converges at infinity.

Consider also in the cylinder Q_ε the problem

$$
(7.2) \qquad \begin{cases} -\mathrm{div}\big(a(y)\nabla v^\varepsilon(y)\big) + b(y)\nabla v^\varepsilon(y) = 0 & \text{in } Q_\varepsilon, \\ a(y)\nabla v^\varepsilon(y)\cdot n(y) = 0 & \text{on } (0,\varepsilon^{-1})\times\partial G, \\ v^\varepsilon(y) = v_0(y) & \text{on } G_0, \\ v^\varepsilon(y) = v_1(y) & \text{on } G_{\varepsilon^{-1}}. \end{cases}
$$

As a consequence of Theorem 7.1 we have:

Corollary 7.2. *Let $\overline{b}_1 > 0$. Then*

$$
|v^\varepsilon(y) - c(v_0)| \leq C\left(\|v_0\|_{L^\infty(G)} e^{-\gamma y_1} + \|v_1\|_{L^\infty(G)} e^{\gamma(y_1-\varepsilon^{-1})}\right),
$$

with a constant C that does not depend on v_0 and v_1.

Proof. Let v be a solution of problem (7.1) with Dirichlet boundary condition v_0 on G_0. Then by Theorem 7.1 we have $|v(y) - c(v_0)| \leq C\|v_0\|_{L^\infty(G)}e^{-\gamma y_1}$. In the cylinder $Q_{-\infty,\varepsilon^{-1}}$ consider the following problem:

$$
(7.3) \qquad \begin{cases} -\mathrm{div}\big(a(y)\nabla v_+^\varepsilon(y)\big) + b(y)\nabla v_+^\varepsilon(y) = 0 \quad \text{in } Q_{-\infty,\varepsilon^{-1}}, \\ a(y)\nabla v_+^\varepsilon(y)\cdot n(y) = 0 \quad \text{on } (-\infty,\varepsilon^{-1})\times\partial G, \\ v_+^\varepsilon(y) = v_1(y) - c(v_0) \quad \text{on } G_{\varepsilon^{-1}}, \\ v_+^\varepsilon(y) \to 0, \ \text{as } y_1 \to -\infty. \end{cases}
$$

By Theorem 7.1 this problem has a unique solution. Moreover,

$$
|v_+^\varepsilon(y)| \leq C(\|v_0\|_{L^\infty(G)} + \|v_1\|_{L^\infty(G)})e^{\gamma(y_1-\varepsilon^{-1})}.
$$

Clearly, the function $v + v_+^\varepsilon - v^\varepsilon$ satisfies the equation and the boundary condition on the lateral boundary in (7.2). On the bases of Q_ε we have

$$
\begin{aligned} |v + v_+^\varepsilon - v^\varepsilon|_{G_0} &\leq C\,(\|v_0\|_{L^\infty(G)} + \|v_1\|_{L^\infty(G)})e^{-\gamma\varepsilon^{-1}}, \\ |v + v_+^\varepsilon - v^\varepsilon|_{G_{\varepsilon^{-1}}} &\leq C\,\|v_0\|_{L^\infty(G)}\,e^{-\gamma\varepsilon^{-1}}. \end{aligned}
$$

Then, by the maximum principle,

$$
|v + v_+^\varepsilon - v^\varepsilon| \leq C\,(\|v_0\|_{L^\infty(G)} + \|v_1\|_{L^\infty(G)})\,e^{-\gamma\varepsilon^{-1}}
$$

in Q_ε. This yields the desired bound. □

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