Asymptotic Behavior of Solutions to a Boundary Value Problem with Small Parameter

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We study a mixed boundary value problem in an infinite strip consisting of two parts, one of which is periodically perforated. Our results include estimates of solutions and asymptotic formulas for them as the constant in the boundary condition tends to zero.

Dedicated to Mark Iosifovich Vishik on the occasion of his 75th anniversary

§1. INTRODUCTION

Numerous problems in mechanics, physics, and chemistry require studying the behavior of solutions to differential equations near a media interface for the case in which one of the media is micro-inhomogeneous and the characteristic inhomogeneity size is small. For instance, these problems arise in studying liquid and gas flows through filters containing a number of periodically arranged channels (adsorption problems), in the investigation of diffusion processes in a tissue, in studying wave diffraction on perforated surfaces (in acoustics), and so on. A specific exchange law between the outer media and the boundary of obstacles is characteristic for this kind of problems. In simple cases, this law can be expressed by a linear, local mixed boundary condition. In more complicated situations, the

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law can be expressed by nonlinear, nonlocal conditions at the boundary of the obstacle. Here we consider a boundary value problem for a function of boundary-layer type linking two solutions of a differential equation in two contacting domains, one of them being perforated. The exchange law at the obstacle boundary is formalized in the form of a mixed boundary condition.

Let us mention some papers dealing with these problems. In [1] a problem of this kind arising from the investigation of chemical reactions is studied; paper [2] is devoted to the construction of asymptotics for a boundary value problem in a half-perforated medium near the contact of the perforated and the nonperforated parts. We refer the interested reader to the bibliography therein.

We study the boundary value problem:

$$\begin{aligned} &\Delta u_\varepsilon(y) = f(y) \quad \text{in} \quad \Omega_\varepsilon, \quad u_\varepsilon(y) \quad \text{is 1-periodic in} \quad x, \\ &u_\varepsilon\big|_{t=-1} = u_\varepsilon\big|_{t=1} = 0, \\ &\frac{\partial u_\varepsilon}{\partial n} + b\Big(\frac{y}{\varepsilon}\Big)u_\varepsilon = 0 \quad \text{on} \quad \varepsilon\partial g \cap \Omega^+. \end{aligned} \tag{1.1}$$

Here we use the following notation:

$$y = (x,t), \quad x \in \mathbb{R}^n, \quad -1 \le t \le 1, \quad \Omega_\varepsilon = \Omega^- \cup \Gamma \cup \Omega_\varepsilon^+,$$

where

$$\begin{aligned} \Omega^- &= \{(x,t) | 0 < x_i < 1, \ i = 1,\dots,n, \ -1 < t < 0\}, \\ \Gamma &= \{(x,t) \,|\, 0 < x_i < 1, \ t = 0\}, \\ \Omega_\varepsilon^+ &= \Omega^+ \setminus \varepsilon g, \quad \Omega^+ = \{(x,t) \,|\, 0 < x_i < 1, \ 0 \le t \le 1\}, \\ g &= \bigcup_{m \in \mathbb{Z}^{n+1}} (Y + m), \end{aligned}$$

and the "inclusion" Y is a subdomain of the unit cube.

One can prove by using methods of [2] that the solution $u_\varepsilon(x)$ of this problem has the form

$$u_\varepsilon(y) = v_0(y) + \sqrt{\varepsilon} C v_1(y) + \varepsilon w\Big(\frac{y}{\varepsilon}\Big) \frac{\partial v_0}{\partial t}\bigg|_{t=-0} + o(\sqrt{\varepsilon}),$$

where $v_0(x)$ and $v_1(x)$ satisfy the following boundary problems:

$$\begin{aligned} &\Delta v_0 = f \text{ in } \Omega^-, \quad v_0 \quad \text{is 1-periodic in} \quad x, \\ &v_0\big|_{t=-1} = 0, \quad v_0\big|_{t=0} = 0; \\ &\Delta v_1 = f \text{ in } \Omega^-, \quad v_1 \quad \text{is 1-periodic in} \quad x, \\ &v_1\big|_{t=-1} = 0, \quad v_1|_{t=0} = -\frac{\partial v_0}{\partial t}\bigg|_{t=0}. \end{aligned}$$

One of the principal goals of this paper is to determine the constant C as well as the "boundary-layer" function w that describes the behavior of u_ε in the vicinity of the boundary Γ separating the perforated and the nonperforated media. More precisely, the function w is defined by

$$w = \begin{cases} u & \text{if} \quad t > 0 \\ u - C/\sqrt{\varepsilon} & \text{if} \quad t \le 0, \end{cases}$$

where u, in turn, is defined in the following by (2.1) with $a \equiv \varepsilon b$.

Our main result is a quite precise description of the function u and the constant C. Namely,

$$u = -\frac{1}{\sqrt{\varepsilon}} \exp(-\sqrt{\varepsilon}(C_+ t_+ + v + w) + C_-),$$

where the function v decays exponentially as $t \to \pm\infty$ uniformly with respect to $\varepsilon > 0$, the function w vanishes for $t < 0$ and is 1-periodic with respect to t for $t \geq 0$, and C_- and C_+ are constants. All functions involved are analytic in $\sqrt{\varepsilon}$. The constant C_{-0} corresponding to $\varepsilon = 0$ and the cited constant C are related by

$$C = \exp(C_{-0}).$$

The constants C_{-0} and C_{+0} can be found from the auxiliary boundary problem (3.11) in a single cell. More precisely, the constant C_{+0} is given by

$$C_{+0} = \sqrt{\int_{\omega_0} a_0 d\omega \Big/ \int_{\Omega_0} \left| \frac{\partial W}{\partial x} + e_t \right|^2 d\Omega},$$

where Ω_0 is the compact manifold with boundary obtained from the domain $\Omega^+ \setminus Y$ by the identification $(x,t) \sim (x',t')$ if $(x,t) - (x',t') \in \mathbb{Z}^{n+1}$, W is the solution of the nonhomogeneous Neumann problem

$$\Delta W(x,t) = 0, \quad (x,t) = y \in \Omega_0,$$
$$\frac{\partial W}{\partial n} = -n_t \quad \text{on the boundary} \quad \omega_0 = \partial\Omega_0 \ \text{ of } \ \Omega_0,$$

and e_t is the unit vector in direction t. The constant C_{-0} is given by

$$\exp(-C_{-0}) = \sqrt{\int_{\omega_0} a_0 d\omega \int_{\Omega_0} \left| \frac{\partial W}{\partial x} + e_t \right|^2 d\Omega}.$$

Thus, these constants, as well as C, depend only on the geometry of the inclusion Y.

It is noteworthy that our main results are obtained by using "delinearization" (see (3.1), (3.2)) and the implicit function theorem for smooth maps of Banach manifolds. By the methods developed here, one can construct the subsequent terms in the asymptotic expansion of u_ε.

§2. PRELIMINARY ESTIMATES

Consider the boundary value problem

$$\Delta u(x,t) = \delta(t), \quad (x,t) = y \in \Omega,$$
$$\frac{\partial u}{\partial n} + au = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \ \text{ of } \ \Omega. \tag{2.1}$$

Here x lies on the torus $T = T^n = \mathbb{R}^n/\mathbb{Z}^n$, $t \in \mathbb{R}$, δ is the Dirac delta-function, and all objects involved are 1-periodic with respect to t for $t \geq 0$; that is, $a(x, t+1) = a(x,t)$ if $t \geq 0$ and $(x, t+1) \in \Omega$ or ω if and only if $(x,t) \in \Omega$ or ω (again if $t \geq 0$).

The manifold Ω is naturally locally isomorphic to a subdomain in $\mathbb{R}^{n+1}$ and all differential operators involved must be understood in the usual Euclidean sense.

Suppose also that the domain $\Omega \cap \{t \leq 0\}$ coincides with the entire half-strip $T \times \{t \leq 0\}$. Moreover, we assume that Ω is a connected domain with Lipschitz boundary ω and the positive function a is bounded away from zero by a constant $\alpha > 0$, that is, $a \geq \alpha > 0$. Then it can be shown that (2.1) has a unique solution with finite Dirichlet integral

$$\int_\Omega \left|\frac{\partial u}{\partial y}\right|^2 d\Omega < \infty.$$

Heuristically, this arrangement corresponds to a single strip of width ϵ for the original problem (1.1). Our aim is to establish the following asymptotic estimate as $a \to 0$. Let $g(a)$ be the (constant with respect to x) limit value of $u(x,t)$ as $t \to -\infty$. Then $g(a) = O(1/\sqrt{\alpha})$. To this end, we treat (2.1) as a variational problem of the following kind:

$$\int_\Omega \frac{1}{2}\left|\frac{\partial u}{\partial y}\right|^2 d\Omega + \int_\omega \frac{1}{2} a u^2 d\omega + \int_{\{t=0\}} u dx \to \min. \tag{2.2}$$

Our task is readily reduced to establishing that the last term in (2.2) is $O(1/\sqrt{\alpha})$ if u is the extremal function. To begin the proof, we note that by the maximum principle u is negative. (This also follows directly from (2.2)). Second, note that the left-hand side of (2.2) should be ≤ 0 if u is extremal. Now we set $v = \sqrt{\alpha} u$. Then (2.2) is reduced to

$$J(v) = \frac{1}{\sqrt{\alpha}} \int_\Omega \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\Omega + \sqrt{\alpha} \int_\omega \frac{1}{2}\frac{a}{\alpha} v^2 d\omega + \int_{\{t=0\}} v dx \to \min, \tag{2.3}$$

and we must prove that $\int_{\{t=0\}} v dx = O(1)$. The remarks above imply

$$\left|\int_{\{t=0\}} v dx\right| \geq \frac{1}{\sqrt{\alpha}} \int_\Omega \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\Omega + \sqrt{\alpha} \int_\omega \frac{1}{2} v^2 d\omega. \tag{2.4}$$

Now we apply the extension theorem [4] to the connected Lipschitz domain Ω (see [4]) and conclude that the extremal function v can be extended to the entire strip $\overline{\Omega} = T \times \mathbb{R}$ in such a way that

$$\int_{\overline{\Omega}} \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\overline{\Omega} \leq C \int_\Omega \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\Omega$$

and

$$\int_{\overline{\Omega}} \frac{1}{2} v^2 d\overline{\Omega} \leq C\Big(\int_\Omega \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\Omega + \int_\omega \frac{1}{2} v^2 d\omega\Big)$$

for some absolute (i.e., depending only on Ω) constant C. Therefore, by resorting to the Cauchy inequality, we obtain

$$\sqrt{\int_{\{t=0\}} v^2 dx} \geq C\Big(\frac{1}{\sqrt{\alpha}} \int_{\overline{\Omega}} \frac{1}{2}\left|\frac{\partial v}{\partial y}\right|^2 d\overline{\Omega} + \sqrt{\alpha} \int_{\overline{\Omega}} \frac{1}{2} v^2 d\overline{\Omega}\Big), \tag{2.5}$$

where C is again an absolute constant. However, by using the elementary inequality

$$ab \leq \frac{1}{2}\Big(ca^2 + \frac{1}{c}b^2\Big),$$

from (2.5) we derive

$$\sqrt{\int_{\{t=0\}} v^2 dx} \geq C \int_{\overline{\Omega}} \left| v \frac{\partial v}{\partial y} \right| d\overline{\Omega},$$

and hence,

$$\sqrt{\int_{\{t=0\}} v^2 dx} \geq C \int_{\overline{\Omega}} \left| v \frac{\partial v}{\partial t} \right| d\overline{\Omega}. \tag{2.6}$$

On the other hand,

$$\int_{\{t=0\}} v^2 dx = -2 \int_{\overline{\Omega} \cap \{t \geq 0\}} v \frac{\partial v}{\partial t} d\overline{\Omega}, \tag{2.7}$$

and the direct comparison of (2.6) with (2.7) gives

$$\left| \int_{\overline{\Omega} \cap \{t \geq 0\}} v \frac{\partial v}{\partial t} d\overline{\Omega} \right| \leq C,$$

where C is an absolute constant. Thus,

$$\left| \int_{\{t=0\}} v dx \right| \leq C, \tag{2.8}$$

as desired.

Now suppose that the function a in the boundary condition (2.1) is also bounded above: $\alpha \leq a \leq C\alpha$, where C is an absolute constant. Then, in a similar manner one can prove the reverse inequality $g(a) \geq C(1/\sqrt{\alpha})$, or, in our further notation,

$$\left| \int_{\{t=0\}} v dx \right| \geq C$$

(we recall that C stands for an absolute constant, which can be different at different places). To this end, we first prove that the minimum of the functional (2.3) is bounded above by negative absolute constant $-C$. Taking this temporarily for granted, we conclude that

$$\left| \int_{\{t=0\}} v dx \right| \geq C$$

(exactly with the *preceding* constant C) because the quadratic part of the functional (2.3) is obviously positive. Now, to estimate the minimum of the functional J we evaluate $J(w)$ for

$$w(x,t) = \begin{cases} A & \text{if } t \leq 0, \\ A\exp(-\sqrt{\alpha}t) & \text{if } t \geq 0, \end{cases}$$

where A is still to be found. Simple calculations show that $J(w) \leq C_1 A^2 + C_2 A$, where C_i are positive absolute constants. Now, for a suitable negative A (for instance, $A = -C_2/2C_1$) we obtain $J(w) \leq -C$, where C is an absolute constant. The proof is complete.

§3. ASYMPTOTIC FORMULAS

Now we proceed to asymptotic formulas for the solution of problem (2.1) with $a = \varepsilon a_0$ as $\varepsilon \to 0$. We seek solutions of the form

$$u = -\frac{1}{\sqrt{\varepsilon}} \exp(-\sqrt{\varepsilon}\phi + C_-). \tag{3.1}$$

Then it is easy to see that the unknown function ϕ and the constant C_- satisfy the nonlinear boundary value problem

$$\begin{aligned} \Delta\phi(x,t) &= \delta(t)\exp(\sqrt{\varepsilon}\phi - C_-) + \sqrt{\varepsilon}\left|\frac{\partial\phi}{\partial y}\right|^2, \quad (x,t) = y \in \Omega, \\ \frac{\partial\phi}{\partial n} &= \sqrt{\varepsilon}a_0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega. \end{aligned} \tag{3.2}$$

Let us show that system (3.2) can be resolved by the more or less conventional power series approach. Specifically, the solution has the form

$$\phi = C_+ t_+ + v + w, \tag{3.3}$$

where

$$t_+(x,t) = \begin{cases} 0 & \text{if} \quad t \le 0, \\ t & \text{if} \quad t \ge 0, \end{cases}$$

C_+ is an unknown constant, v is a function such that its restrictions to the domains

$$\Omega_+ = \Omega \cap \{t \ge 0\} \quad \text{and} \quad \Omega_- = \Omega \cap \{t \le 0\}$$

belong to the corresponding Sobolev spaces $H^1(\Omega_+)$ and $H^1(\Omega_-)$, and w vanishes for $t < 0$, is 1-periodic for $t \ge 0$, and belongs to the Sobolev space $H^1(\Omega_0)$, where

$$\Omega_0 = \Omega \cap \{0 \le t \le 1\}/\{(x,0) \sim (x,1)\}.$$

Any variable involved, such as C_+, C_-, v, and w, can be expanded in a power series with respect to $\sqrt{\varepsilon}$, so that

$$\phi = \phi_0 + \sqrt{\varepsilon}\phi_1 + \cdots, \tag{3.4}$$

where the first approximation ϕ_0 satisfies the *linear* system

$$\begin{aligned} \Delta\phi_0(x,t) &= \delta(t)\exp(-C_{-0}), \quad (x,t) = y \in \Omega, \\ \frac{\partial\phi_0}{\partial n} &= 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega. \end{aligned} \tag{3.5}$$

More precisely, the following relations hold for the components of the function $\phi_0 = C_{+0}t_+ + w_0 + v_0$:

$$\begin{aligned} \Delta w_0(x,t) &= 0, \quad (x,t) = y \in \Omega_0, \\ \frac{\partial w_0}{\partial n} &= -C_{+0}n_t \quad \text{on the boundary} \quad \omega_0 = \partial\Omega_0 \quad \text{of} \quad \Omega_0, \end{aligned} \tag{3.6}$$

where n_t is the t-component of the outward normal vector, and

$$\begin{aligned} &\Delta v_0(x,t) = 0, \quad (x,t) = y \in \Omega_\pm, \\ &[v_0(x,0)] = -w_0(x,0), \quad \left[\frac{\partial v_0}{\partial t}(x,0)\right] = -\frac{\partial w_0}{\partial t}(x,0) + \exp(-C_{-0}) - C_{+0}, \\ &\frac{\partial v_0}{\partial n} = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega, \end{aligned} \tag{3.7}$$

where the square brackets stand for the jumps of the corresponding functions at $t = 0$. It readily follows from the compatibility conditions for (3.7) that

$$\exp(-C_{-0}) = C_{+0} + \int_T \frac{\partial w_0}{\partial t}(x,0)dx. \tag{3.8}$$

It is however clear that the solution to (3.5) is by no means unique. In particular, if we add any constant to the solution ϕ_0 of this Neumann problem, we still obtain a solution. Similarly, the function w_0 is not uniquely determined by (3.6). It turns out that to find ϕ_0 we must invoke the compatibility conditions for the second approximation. More precisely, the second approximation ϕ_1 satisfies the system

$$\begin{aligned} &\Delta\phi_1(x,t) = \delta(t)\exp(-C_{-0})\phi_0 + \left|\frac{\partial \phi_0}{\partial y}\right|^2, \quad (x,t) = y \in \Omega, \\ &\frac{\partial \phi_1}{\partial n} = a_0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega. \end{aligned} \tag{3.9}$$

Now the "periodic part" of the compatibility condition reads

$$\int_{\Omega_0} \left|\frac{\partial w_0}{\partial y} + C_{+0}\frac{\partial t_+}{\partial y}\right|^2 d\Omega = \int_{\omega_0} a_0 d\omega. \tag{3.10}$$

(Note that $\partial t_+/\partial y$ is just the unit vector e_t in the direction t.) Condition (3.10) allows us to determine the function w_0 modulo an additive constant and the constant C_{+0} uniquely. Indeed, let W be a (periodic) solution to

$$\begin{aligned} &\Delta W(x,t) = 0, \quad (x,t) = y \in \Omega_0, \\ &\frac{\partial W}{\partial n} = -n_t \quad \text{on the boundary} \quad \omega_0 = \partial\Omega_0 \quad \text{of} \quad \Omega_0. \end{aligned} \tag{3.11}$$

This function is determined uniquely modulo an additive constant, and its gradient is determined uniquely. Now it follows from (3.10) that

$$C_{+0}^2 \int_{\Omega_0} \left|\frac{\partial W}{\partial x} + e_t\right|^2 d\Omega = \int_{\omega_0} a_0 d\omega. \tag{3.12}$$

Thus, Eqs. (3.11) and (3.12) allow us to determine the (positive) constant C_{+0} uniquely and the function

$$w_0 = C_{+0}W \tag{3.13}$$

modulo an additive constant. Hence, relation (3.8) acquires the form

$$\exp(-C_{-0}) = C_{+0} \int_T \left(\frac{\partial W}{\partial t}(x,0) + 1\right)dx. \tag{3.14}$$

One can show that the integral in (3.14) is strictly positive, and therefore, one can determine the constant C_{-0} from (3.14). Indeed, the integral

$$\int_{\Omega_0 \cap \{t=\tau\}} \left(\frac{\partial W}{\partial t}(x,\tau) + 1\right)dx$$

does not depend on τ since the function $W + t_+$ is a solution of the homogeneous Neumann problem in Ω_0. It follows that

$$\int_T \Big(\frac{\partial W}{\partial t}(x,0)+1\Big)dx = \int_{\Omega_0} \Big(\frac{\partial W}{\partial t}(x,t)+1\Big)d\Omega.$$

However, the positivity of the latter integral is well known (see also (4.34)). Thus, the first approximation to the solution u of the third asymptotic boundary value problem (2.1) with $a = \varepsilon a_0$ has the form

$$u = -\frac{1}{\sqrt{\varepsilon}}\exp(-\sqrt{\varepsilon}\phi + C_-),$$

where the function ϕ tends to zero as $t \to -\infty$ and is close to a function of the form $C_+t_+ + w$, where w is a periodic function, as $t \to +\infty$. Moreover, $u \to -(1/\sqrt{\varepsilon})\exp(C_-)$ as $t \to -\infty$ and we have the approximate equality $\exp(-C_-) \cong C_+$, which becomes exact as $\sqrt{\varepsilon} \to 0$. All in all, we obtain a quite precise approximate solution of our original problem by solving some boundary value problem in a single cell Ω_0.

Before proceeding to the proof, let us make some remarks justifying ansatz (3.1). This is a kind of renormalization procedure for the solution of the ill-posed Neumann problem

$$\begin{gathered}\Delta u(x,t) = \delta(t), \quad (x,t) = y \in \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega,\end{gathered} \tag{3.15}$$

formally corresponding to $\varepsilon = 0$ in (2.1). This problem however has no solutions, at least in the class of functions with finite Dirichlet integral in Ω. Equation (3.1) essentially means that the situation can be improved by subtracting the "infinite constant" $\exp(C_-)/\sqrt{\varepsilon}$ from u since $-(1\sqrt{\varepsilon})\exp(-\sqrt{\varepsilon}\phi)$ is close to $-1/\sqrt{\varepsilon}+\phi$ for small ε. These arguments at the same time "explain" why Eq. (3.5) for the first approximation is linear.

To study the properties of the function $v(x)$ in (3.3) more carefully, let us restrict Eq. (2.1) to Ω_+ and consider the boundary value problem

$$\begin{gathered}\Delta z(x,t) = 0, \quad (x,t) = y \in \Omega_+, \\ z\big|_{t=0} = z_0(x), \quad \frac{\partial z}{\partial n} + az = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega.\end{gathered} \tag{3.16}$$

The solution bounded as $t \to \infty$ is unique. Let R be the transition operator that maps a boundary function $z_0(x)$ on T^n to the restriction $z(x,1)$ of the corresponding solution of (3.16) to $\{(x,t) : t = 1\}$. It is clear that $z(x,m) = R^m z_0$ for any positive integer m. By using the maximum principle and standard elliptic estimates, it is easy to see that the kernel $r(x,x')$ of this operator is a smooth positive function such that the inequality

$$0 < c < r(x,x') < c^{-1} \tag{3.17}$$

holds uniformly with respect to $\varepsilon > 0$. Indeed, let $r_0(x,t,x')$ be a solution of the problem

$$\begin{gathered}\Delta r_0(x,t,x') = 0, \quad y \in \Omega_+, \\ r_0|_{t=0} = \delta(x-x'), \quad \frac{\partial r_0}{\partial n} + ar_0 = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega.\end{gathered} \tag{3.18}$$

Then we have $r(x,x') = r_0(x,1,x')$. By the maximum principle,

$$r_D(y,x') \le r_0(y,x') \le r_N(y,x'),$$

where r_D and r_N are the solutions satisfying, respectively, the Dirichlet and the Neumann homogeneous boundary conditions at the boundary of the holes. The functions r_D and r_N are positive and independent of ε. Hence, (3.17) holds.

According to [6], the first eigenvalue λ_0 of R is simple and positive, and the other eigenvalues λ_k, $k \geq 1$, satisfy the estimate

$$|\lambda_k| < q\lambda_0, \quad q < 1, \tag{3.19}$$

where q is independent of ε. Moreover, if suitably normalized, the first eigenfunction $p_0(x)$ is also positive and even bounded below and above by uniform positive bounds. We choose $p_0(x)$ to satisfy the equality $\int_{T^n} p_0(x)dx = 1$. By the maximum principle, $\lambda_0 < 1$.

Let $p(y)$ be the solution of (3.16) with the boundary condition $p(y)|_{t=0} = p_0(x)$. It follows from the comparison of the results cited above with the representation (3.1), (3.3) that

$$\lambda_0 = \exp(-\sqrt{\varepsilon}C_+), \quad p(y) = (\text{const}) \exp(-\sqrt{\varepsilon}(C_+t + w(y))). \tag{3.20}$$

Now we consider another semigroup, R'^m, defined as follows:

$$R'^m z = p_0^{-1}\lambda_0^{-m}R^m(zp_0). \tag{3.21}$$

Its generator R' has the kernel $r'(x,x') = p_0(x')r(x,x')p_0^{-1}(x)\lambda_0^{-1}$. It is possible to show by using ideas from [7] (though we shall not deal with this question) that the operator R' is the transition operator for the boundary value problem

$$\begin{gathered} \Delta z(y) + 2\nabla \ln p(y) \cdot \nabla z(y) = 0, \quad y \in \Omega_+ \\ z|_{t=0} = z_0(x), \quad \frac{\partial z}{\partial n} = 0 \quad \text{on the boundary} \quad \omega = \partial\Omega \quad \text{of} \quad \Omega, \end{gathered} \tag{3.22}$$

to be solved in the class of bounded functions.

The fact that the function $z \equiv 1$ is the first eigenfunction of the operator R' implies the relation

$$\int_{T^n} r'(x,x')dx' \equiv 1 \tag{3.23}$$

for all $x \in T^n$. Then, by the same argument as above, $0 < c < r'(x,x') < c^{-1}$ uniformly in $\varepsilon > 0$. Thus,

$$\operatorname*{osc}_{\{t=1\}} z(y) < c_1 \operatorname*{osc}_{\{t=0\}} z(y), \quad c_1 < 1. \tag{3.24}$$

By iterating the last relation, we obtain

$$\operatorname*{osc}_{\{t=m\}} z(y) < c_1^m \operatorname*{osc}_{\{t=0\}} z(y) = \exp(-c_2 m) \operatorname*{osc}_{\{t=0\}} z(y), \quad c_2 > 0. \tag{3.25}$$

This means that $|v(y)| \leq c_3 \exp(-c_2 t)$ uniformly in $\varepsilon > 0$.

§4. MAIN THEOREM

Summarizing, we can now state the main theorem.

Theorem 1. *Problem* (2.1), *where* $a = \varepsilon a_0$, *for small* ε *admits a unique solution of the form*

$$u = -\frac{1}{\sqrt{\varepsilon}} \exp(-\sqrt{\varepsilon}(C_+ t_+ + v + w) + C_-),$$

where the "decreasing component" v *is a function such that its restrictions to the domains* $\Omega_+ = \Omega \cap \{t \geq 0\}$ *and* $\Omega_- = \Omega \cap \{t \leq 0\}$ *belong to the corresponding Sobolev spaces* $H^1(\Omega_+)$ *and* $H^1(\Omega_-)$ *and decay exponentially as* $t \to \infty$ *uniformly with respect to* $\varepsilon > 0$, *the "periodic component"* w *vanishes for* $t < 0$, *is a 1-periodic function for* $t \geq 0$, *and belongs to the Sobolev space* $H^1(\Omega_0)$, *and* C_- *and* C_+ *are constants. All variables involved are analytic in* $\sqrt{\varepsilon}$. *The "first-order approximation"* u_0 *to* u *behaves like the constant* $-(1/\sqrt{\varepsilon})\exp(C_{-0})$ *as* $t \to -\infty$ *and like* $-(1/\sqrt{\varepsilon})\exp(-\sqrt{\varepsilon}(C_{+0}t + w_0) + C_{-0})$ *as* $t \to \infty$, *where the "first-order approximations"* C_{-0}, C_{+0} *and* w_0 *are defined via relations* (3.11), (3.12), (3.13), *and* (3.14).

To establish these results, we use the implicit function theorem [3] for analytic maps of Banach manifolds. Let us choose a sufficiently large positive integer N (any $N > (n+1)/2$ will suffice). Consider the Banach space $\mathcal{B}$ consisting of H^1 functions ϕ in Ω of the form $\phi = Ct_+ + v + w$, where v belongs to the Sobolev spaces $H^{N+1}(\Omega_+)$ and $H^{N+1}(\Omega_-)$ and w is 1-periodic for $t \geq 0$, belongs to the Sobolev space $H^{N+1}(\Omega_0)$ (recall that $\Omega_0 = \Omega \cap \{0 \leq t \leq 1\}/\{(x,0) \sim (x,1)\}$), and vanishes as $t \leq 0$. The map Φ to be studied is defined by (3.2). Namely, consider the Banach space $\mathcal{C}$ of pairs (f,g), where $f = v' + w'$, v' belongs to the Sobolev spaces $H^{N-1}(\Omega_+)$ and $H^{N-1}(\Omega_-)$, w' is 1-periodic for $t \geq 0$, belongs to the Sobolev space $H^{N-1}(\Omega_0)$, and vanishes as $t \leq 0$, $g = v'' + w''$, v'' belongs to the Sobolev spaces $H^{N-1/2}(\omega)$, and w'' is a 1-periodic function and belongs to the Sobolev space $H^{N-1/2}(\omega_0)$. In what follows we sometimes regard the functions f and g as the pairs $f = (v', w')$ and $g = (v'', w'')$. This is justified by the uniqueness of the decompositions of these functions into sums of periodic and decreasing components. Let $F = F_\varepsilon : \mathbb{R} \times \mathcal{B} \to \mathcal{C}$ be the following map:

$$F_\varepsilon(C, v, w) = (f, g),$$

where

$$f = \Delta\phi(x,t) - \sqrt{\varepsilon}\left|\frac{\partial \phi}{\partial y}\right|^2, \quad (x,t) = y \in \Omega_\pm, \qquad g = \frac{\partial \phi}{\partial n} - \sqrt{\varepsilon} a_0. \tag{4.26}$$

Here $\phi = Ct_+ + v + w$, and it is not difficult to verify that $(f,g) \in \mathcal{C}$. To take the jump condition at $t = 0$ into account, we must impose the following constraint on ϕ:

$$[v(x,0)] = -w(x,0), \quad \left[\frac{\partial v}{\partial t}(x,0)\right] = -\frac{\partial w}{\partial t}(x,0) + \exp(-\sqrt{\varepsilon}\phi + C_-) - C, \tag{4.27}$$

where the square brackets stand for the jumps. Another necessary condition satisfied by any solution (C, v, w) of the equation $F_\varepsilon(C, v, w) = (0,0)$ is

$$\int_{\Omega_0} \left|\frac{\partial w}{\partial y} + C\frac{\partial t_+}{\partial y}\right|^2 d\Omega = \int_{\omega_0} a_0 d\omega. \tag{4.28}$$

(Recall that $\partial t_+/\partial y$ is just the unit vector e_t in the direction t.)

Now we introduce the Banach submanifolds $M \subset \mathbb{R}^2 \times \mathcal{B}$ and $N \subset \mathcal{C}$ such that the superposition $\Phi = F \circ p$, where p is the projection $p(C_-, C, v, w) = (C, v, w)$ and $F = F_\varepsilon$: $\mathbb{R} \times \mathcal{B} \to \mathcal{C}$ maps M into N. Namely, set

$$M = \{(C_-, C, v, w) \in \mathbb{R}^2 \times \mathcal{B} | \quad ((4.27) \quad \text{and} \quad (4.28) \quad \text{hold})\},$$

and

$$N = \left\{ (f,g) = (v', w', v'', w'') \in \mathcal{C} \,\Big|\, \int_{\Omega_0} w' d\Omega = \int_{\omega_0} w'' d\omega \right\}.$$

By virtue of the Sobolev embedding theorem, M is a Banach submanifold of $\mathbb{R} \times \mathcal{B}$, and it is easy to see that $\Phi = \Phi_\varepsilon : M \to N$. To apply the implicit function theorem, we must study the differential $d\Phi$ of the map Φ at the point $P = (C_{-0}, C_{+0}, v_0, w_0)$, $\varepsilon = 0$. Here C_{-0}, C_{+0}, v_0, and w_0 are defined in (3.12), (3.13), (3.11), and (3.8). Obviously, $\Phi_0(C_{-0}, C_{+0}, v_0, w_0) = 0$, and to verify the possibility of solving the equation $\Phi_\varepsilon = 0$ uniquely in the vicinity of P one must prove that the differential $d\Phi(P)$ is a linear isomorphism. This differential is given by the formulas

$$d\Phi(P)(C_-, C, v, w) = (f, g) = (v', w', v'', w''),$$

where

$$f = \Delta\phi(x,t), \quad (x,t) = y \in \Omega_\pm, \quad g = \frac{\partial \phi}{\partial n} \text{ on } \omega, \tag{4.29}$$

or, in other words,

$$\begin{aligned} v' &= \Delta v(x,t), \quad (x,t) = y \in \Omega_\pm, \quad v'' = \frac{\partial v}{\partial n} \text{ on } \omega, \\ w' &= \Delta w(x,t), \quad (x,t) = y \in \Omega_0, \quad w'' = \frac{\partial w}{\partial n} + Cn_t \text{ on } \omega_0, \end{aligned} \tag{4.30}$$

subject to the constraints

$$[v(x,0)] = -w(x,0), \quad \left[\frac{\partial v}{\partial t}(x,0)\right] = -\frac{\partial w}{\partial t}(x,0) + \exp(C_{-0})C_- - C, \tag{4.31}$$

and

$$\int_{\Omega_0} \left(\frac{\partial w_0}{\partial y} + C_{+0}e_t, \frac{\partial w}{\partial y} + Ce_t \right) d\Omega = 0. \tag{4.32}$$

(This last condition (4.32) stems from (4.28).) We must verify that for a given quadruple (v', w', v'', w'') such that

$$\int_{\Omega_0} w' d\Omega = \int_{\omega_0} w'' d\omega$$

there exists a unique pair (v, w), satisfying (4.31), (4.32), and (4.30). It is well known (e.g., see [5]) that the first pair of equations in (4.30) can be resolved under the constraint (4.31) provided that

$$\exp(C_{-0})C_- - C + \int_\Omega v' d\Omega = \int_\omega v'' d\omega + \int_T \frac{\partial w}{\partial t}(x,0) dx$$

and $w(x,0)$ in (4.31) is suitably disturbed by an additive constant (to ensure that $\int_T v(x,-0)dx = 0$). It is also well known that the second pair of equations in (4.30) can be solved for any given C *if we neglect condition* (4.32). We can fix the constant C and some solution w to the second pair of equations in (4.30). Then we can choose a suitable constant C_- and add another suitable constant to w so that these conditions are satisfied.

Now the surjectivity of the differential $d\Phi(P)$ is reduced to the inequality

$$\int_{\Omega_0} \left(\frac{\partial w_0}{\partial y} + C_{+0}e_t, e_t \right) d\Omega \neq 0. \tag{4.33}$$

In view of (3.13), this, in turn, follows from the inequality

$$\int_{\Omega_0} \left(\frac{\partial W}{\partial y} + e_t, e_t\right) d\Omega > 0 \tag{4.34}$$

for the solution W of (3.11). Here we sketch the proof of this important inequality, well-known in the homogenization theory. Indeed, the function W is a solution to the minimization problem

$$\int_{\Omega_0} \left|\frac{\partial W}{\partial y} + e_t\right|^2 d\Omega \to \min.$$

Therefore, it must satisfy the orthogonality relation

$$\int_{\Omega_0} \left(\frac{\partial W}{\partial y} + e_t, \frac{\partial W}{\partial y}\right) d\Omega = 0,$$

which implies that

$$\int_{\Omega_0} \left(\frac{\partial W}{\partial y} + e_t, e_t\right) d\Omega = \int_{\Omega_0} \left|\frac{\partial W}{\partial y} + e_t\right|^2 d\Omega > 0.$$

This completes the proof of (4.34) and, thus, the surjectivity of the differential $d\Phi(P)$ is established. To ensure its injectivity, it suffices (again by the implicit function theorem) to verify that problem (4.26), (4.28), where $f = 0$ and $g = 0$, does not admit more than one solution. However, any solution of (4.26), (4.28) provides a solution u with finite Dirichlet integral

$$\int_{\Omega} |\partial u/\partial y|^2 \, d\Omega$$

to the boundary value problem (2.1), where $a = \varepsilon a_0$. But the latter solution is well known to be unique. This completes the proof of all our asymptotic formulas.

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