ISSN 1061-9208, Russian Journal of Mathematical Physics, 2024, Vol. 31, No. 1, pp. 137–145. © Pleiades Publishing, Ltd., 2024.

# On the Homogenization of Nonlocal Convolution Type Operators

**A. Piatnitski**[*,**,***,1], **V. Sloushch**[****,2], **T. Suslina**[****,3], **and E. Zhizhina**[*,**,4]

[*] *The Arctic University of Norway, Campus Narvik, P.O. Box 385, Narvik, 8505 Norway*
[**] *Institute of Information Transmission Problems of the Russian Academy of Sciences, Bolshoi Karetnyi 19, Moscow, 127051 Russia*
[***] *People's Friendship University of Russia, Miklukho-Maklaya str. 6, Moscow, 117198 Russia*
[****] *St. Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034 Russia*

[1] *E-mail:* apiatnitski@gmail.com
[2] *E-mail:* v.slouzh@spbu.ru
[3] *E-mail:* t.suslina@spbu.ru
[4] *E-mail:* elena.jijina@gmail.com

Received November 24, 2023; revised November 24, 2023; accepted November 24, 2023

**Abstract**— In $L_2(\mathbb{R}^d)$, we consider a self-adjoint bounded operator $\mathbb{A}_\varepsilon$, $\varepsilon > 0$, of the form

$$(\mathbb{A}_\varepsilon u)(\mathbf{x}) = \varepsilon^{-d-2} \int_{\mathbb{R}^d} a((\mathbf{x}-\mathbf{y})/\varepsilon)\mu(\mathbf{x}/\varepsilon, \mathbf{y}/\varepsilon)\,(u(\mathbf{x}) - u(\mathbf{y}))\, d\mathbf{y}.$$

It is assumed that $a(\mathbf{x})$ is a nonnegative function such that $a(-\mathbf{x}) = a(\mathbf{x})$ and $\int_{\mathbb{R}^d}(1+|\mathbf{x}|^4)a(\mathbf{x})\,d\mathbf{x} < \infty$; $\mu(\mathbf{x},\mathbf{y})$ is $\mathbb{Z}^d$-periodic in each variable, $\mu(\mathbf{x},\mathbf{y}) = \mu(\mathbf{y},\mathbf{x})$ and $0 < \mu_- \leqslant \mu(\mathbf{x},\mathbf{y}) \leqslant \mu_+ < \infty$. For small $\varepsilon$, we obtain an approximation of the resolvent $(\mathbb{A}_\varepsilon + I)^{-1}$ in the operator norm on $L_2(\mathbb{R}^d)$ with an error of order $O(\varepsilon^2)$.

**DOI** 10.1134/S106192084010114

We consider the homogenization problem for a periodic nonlocal convolution type operator with an integrable kernel. Our goal is to estimate the rate of convergence in the operator norm. The present paper is a further development of the research started in [8]. The method is a modification of the operator-theoretic approach (see [1–3]).

## 1. CONVOLUTION TYPE OPERATORS

In $L_2(\mathbb{R}^d)$, we study a nonlocal operator $\mathbb{A}_\varepsilon$ defined by

$$\mathbb{A}_\varepsilon u(\mathbf{x}) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\Big)\mu\Big(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}\Big)\big(u(\mathbf{x}) - u(\mathbf{y})\big)\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d, \quad u \in L_2(\mathbb{R}^d); \tag{1.1}$$

here $\varepsilon$ is a small positive parameter. We impose the following conditions on the coefficients $a(\mathbf{x})$ and $\mu(\mathbf{x},\mathbf{y})$:

$$a(\mathbf{x}) \geqslant 0, \quad a \in L_1(\mathbb{R}^d), \quad \|a\|_{L_1(\mathbb{R}^d)} > 0, \quad a(-\mathbf{x}) = a(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d; \tag{1.2}$$

$$0 < \mu_- \leqslant \mu(\mathbf{x},\mathbf{y}) \leqslant \mu_+ < +\infty, \tag{1.3}$$

$$\mu(\mathbf{x},\mathbf{y}) = \mu(\mathbf{y},\mathbf{x}), \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d; \tag{1.4}$$

$$\mu(\mathbf{x}+\mathbf{m},\mathbf{y}+\mathbf{n}) = \mu(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, \quad \mathbf{m},\mathbf{n} \in \mathbb{Z}^d. \tag{1.5}$$

Under these assumptions, the operator $\mathbb{A}_\varepsilon$ is bounded, self-adjoint, and nonnegative. Write

$$M_k(a) := \int_{\mathbb{R}^d} |\mathbf{x}|^k a(\mathbf{x})\, d\mathbf{x}, \quad k \in \mathbb{N}.$$

Then, due to the condition $0 < \int_{\mathbb{R}^d} a(\mathbf{x})\,d\mathbf{x} < \infty$, the finiteness of the moment $M_k(a)$ implies the finiteness of the moments $M_1(a), \dots, M_{k-1}(a)$. In what follows, we assume the finiteness of $M_k(a)$ with $k \leqslant 4$ that might change from one statement to another.

Operators of the form (1.1) arise in models of mathematical biology and population dynamics, they have been actively studied recent years; see [4–7] and references therein. Homogenization problem for such operators was studied in [6], where under the condition $M_2(a) < \infty$, it was shown that the resolvent $(\mathbb{A}_\varepsilon+I)^{-1}$ strongly converges to the resolvent $(\mathbb{A}^0 + I)^{-1}$ of the effective operator, as $\varepsilon \to 0$. The effective operator is a second-order elliptic differential operator $\mathbb{A}^0 = -\operatorname{div} g^0\nabla$ with constant coefficients. Thus, an interesting effect is observed in this problem: for a bounded nonlocal operator $\mathbb{A}_\varepsilon$, the effective operator is unbounded local operator $\mathbb{A}^0$. For operators with a nonsymmetric kernel, similar problems were studied in [7], where, for the corresponding parabolic equations, the homogenization result is valid in moving coordinates. The problem in a periodically perforated domain was investigated by variational methods in [4].

## 2. THE EFFECTIVE OPERATOR. MAIN RESULT

Let us describe the homogenized operator. As usual in homogenization theory, in order to describe the effective matrix $g^0$, we need to consider auxiliary problems on the periodicity cell. Suppose that a vector-valued function $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), \dots, v_d(\mathbf{x}))^t$, $\mathbf{x} \in \mathbb{R}^d$, is a $\mathbb{Z}^d$-periodic solution of the problem

$$\int_{\mathbb{R}^d} a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))\,d\mathbf{y} = \int_{\mathbb{R}^d} a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})\,d\mathbf{y}, \quad \int_\Omega \mathbf{v}(\mathbf{y})\,d\mathbf{y} = 0; \tag{2.1}$$

the symbol $\Omega$ stands for the periodicity cell $[0,1)^d$. Note that problem (2.1) has a unique $\mathbb{Z}^d$-periodic solution. It turns out that this solution is bounded.

**Lemma 2.1.** *Let conditions* (1.2)–(1.5) *be fulfilled, and assume that* $M_1(a) < \infty$ *and* $\mathbf{v}(\mathbf{x})$ *is the* $\mathbb{Z}^d$ *-periodic solution of problem* (2.1)*. Then* $v_j \in L_\infty(\mathbb{R}^d)$ *and* $\|v_j\|_{L_\infty(\mathbb{R}^d)} \leqslant C(a,\mu)$, $j = 1, \dots, d$.

Let $g^0$ be a $(d\times d)$-matrix with the entries $\frac{1}{2}g_{kl}$, $k,l = 1,\dots,d$, given by

$$g_{kl} = \int_\Omega d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{y}\Big((x_k-y_k)(x_l-y_l) - v_l(\mathbf{x})(x_k-y_k) - v_k(\mathbf{x})(x_l-y_l)\Big)a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y}), \quad k,l = 1,\dots,d. \tag{2.2}$$

The matrix $g^0$ is positive definite (see Section 6). The effective operator $\mathbb{A}^0 = -\operatorname{div} g^0\nabla$ is defined on the Sobolev space $H^2(\mathbb{R}^d)$.

The following result was obtained in [8].

**Theorem 2.2** ([8])**.** *Assume that conditions* (1.2)–(1.5) *are satisfied, and let* $M_3(a) < \infty$*. Then the resolvent* $(\mathbb{A}_\varepsilon + I)^{-1}$ *converges to the resolvent* $(\mathbb{A}^0 + I)^{-1}$ *of the effective operator in the operator norm on* $L_2(\mathbb{R}^d)$*. Moreover, the following order-sharp error estimate holds*:

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathrm{C}_1(a,\mu)\varepsilon, \quad \varepsilon > 0. \tag{2.3}$$

In this note, we obtain a more accurate approximation of the resolvent $(\mathbb{A}_\varepsilon + I)^{-1}$. Our main result is

**Theorem 2.3.** *Assume that conditions* (1.2)–(1.5) *are satisfied, and let* $M_4(a) < \infty$*. Then*

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1} - \varepsilon K_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathrm{C}_2(a,\mu)\varepsilon^2, \quad \varepsilon > 0. \tag{2.4}$$

*Here the operator* $K_\varepsilon$ *is given by*

$$K_\varepsilon = -\sum_{j=1}^d [v_j^\varepsilon]\partial_j(\mathbb{A}^0 + I)^{-1} + \sum_{j=1}^d (\mathbb{A}^0 + I)^{-1}\partial_j[v_j^\varepsilon],$$

*where* $[v_j^\varepsilon]$ *is the operator of multiplication by the function* $v_j(\mathbf{x}/\varepsilon)$.

*Remark* 1. If in the assumptions of Theorem 2.2, condition $M_3(a) < \infty$ is replaced with

$$\int_{\mathbb{R}^d} |\mathbf{x}|^k a(\mathbf{x})\,d\mathbf{x} < \infty, \tag{2.5}$$

where $2 < k < 3$, then

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon^{k-2}, \quad \varepsilon > 0.$$

If in the assumptions of Theorem 2.3, condition $M_4(a) < \infty$ is replaced by (2.5) with $3 < k < 4$, then

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1} - \varepsilon K_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon^{k-2}, \quad \varepsilon > 0.$$

## 3. OPERATOR-THEORETIC APPROACH

In order to find an approximation for the resolvent of the operator (1.1), we modify the operator-theoretic approach, that was suggested and developed by Birman and Suslina in the works [1–3], which focused on self-adjoint second-order elliptic differential operators. This approach is based on the scaling transformation, the direct integral decomposition for periodic operators, and calculation of an approximation for the resolvent in terms of the spectral characteristics of the operator at the bottom of the spectrum (threshold characteristics). For differential operators, at the third step, the methods of the analytic perturbation theory applied.

The first two steps, namely, the scaling transformation and the decomposition of $\mathbb{A}$ into the direct integral of the operators $\mathbb{A}(\boldsymbol{\xi})$ with the help of the unitary Gelfand transform, remain the same. Here and in what follows, $\mathbb{A} = \mathbb{A}_{\varepsilon_0},\ \varepsilon_0 = 1$. The operators $\mathbb{A}(\boldsymbol{\xi})$ act in the space $L_2(\Omega)$ and depend on the parameter $\boldsymbol{\xi} \in \widetilde{\Omega}$. Here $\Omega := [0,1)^d$ is the cell of $\mathbb{Z}^d$ and $\widetilde{\Omega} := [-\pi,\pi)^d$ is the cell of the dual lattice $(2\pi\mathbb{Z})^d$. The problem is reduced to studying the asymptotics of the resolvent $(\mathbb{A}(\boldsymbol{\xi}) + \varepsilon^2 I)^{-1}$ for small $\varepsilon > 0$. However, the methods of the analytic perturbation theory are no longer applicable to the family of operators $\mathbb{A}(\boldsymbol{\xi})$. In contrast with the case of differential operators, the operator family studied here is not analytic. Instead, we have only the finite smoothness of the family $\mathbb{A}(\boldsymbol{\xi})$, which is granted by the assumption on the finiteness of the first few moments of the coefficient $a(\mathbf{x})$.

According to the operator-theoretic approach, to obtain an approximation for the resolvent $(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}$ for small $\varepsilon$, it suffices to find the asymptotics of the operator-valued functions $F(\boldsymbol{\xi})$ and $\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi})$, $\boldsymbol{\xi} \to 0$; here $F(\boldsymbol{\xi})$ is the spectral projection of the operator $\mathbb{A}(\boldsymbol{\xi})$ corresponding to some neighborhood of zero. Traditionally, these asymptotics were calculated via the asymptotics of the first eigenvalue $\lambda_1(\boldsymbol{\xi})$ of $\mathbb{A}(\boldsymbol{\xi})$. We consistently apply an alternative approach, which relies on integrating the resolvent $(\mathbb{A}(\boldsymbol{\xi}) - \zeta I)^{-1}$ along a suitable contour on the complex plane. In [8, Sec. 4.2], this approach was called the “third method”.

## 4. THE OPERATOR $\mathbb{A}$: DIRECT INTEGRAL DECOMPOSITION AND ESTIMATES

We recall that $\mathbb{A} = \mathbb{A}_{\varepsilon_0}$, where $\varepsilon_0 = 1$. Then $\mathbb{A} = \mathbb{A}(a,\mu)$ is given by

$$\mathbb{A}u(\mathbf{x}) := \int_{\mathbb{R}^d} a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))\,d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d.$$

The operator $\mathbb{A}$ can be represented as $\mathbb{A} = p(\mathbf{x}) - \mathbb{B}$, where $p(\mathbf{x}) := \int_{\mathbb{R}^d} a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})\,d\mathbf{y}$, $\mathbf{x} \in \mathbb{R}^d$, and

$$\mathbb{B}u(\mathbf{x}) := \int_{\mathbb{R}^d} a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})u(\mathbf{y})\,d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d.$$

By the Schur lemma, the operator $\mathbb{B}$ is bounded and $\|\mathbb{B}\|_{L_2\to L_2} \leqslant \|\mu\|_{L_\infty}\|a\|_{L_1}$. Moreover, the potential $p(\mathbf{x})$ satisfies $\|p\|_{L_\infty} \leqslant \|\mu\|_{L_\infty}\|a\|_{L_1}$. Hence, the operator $\mathbb{A} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is bounded. Obviously, under assumptions (1.2) and (1.3), the potential $p(\mathbf{x})$ is real-valued and the operator $\mathbb{B}$ is self-adjoint. Consequently, the operator $\mathbb{A}$ is also self-adjoint. Clearly, the potential $p(\mathbf{x})$ satisfies the estimates

$$\mu_-\|a\|_{L_1(\mathbb{R}^d)} \leqslant p(\mathbf{x}) \leqslant \mu_+\|a\|_{L_1(\mathbb{R}^d)}, \quad \mathbf{x} \in \mathbb{R}^d. \tag{4.1}$$

Under assumptions (1.2) and (1.3), the quadratic form of the operator $\mathbb{A}$ admits the following representation (see, e.g., [5] or [8, Sec. 1.2]):

$$(\mathbb{A}u,u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mathbf{x}\,d\mathbf{y}\,a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})|u(\mathbf{x}) - u(\mathbf{y})|^2, \quad u \in L_2(\mathbb{R}^d). \tag{4.2}$$

Therefore, the operator $\mathbb{A}$ is nonnegative and

$$\mu_-(\mathbb{A}_0 u,u) \leqslant (\mathbb{A}u,u) \leqslant \mu_+(\mathbb{A}_0 u,u), \quad u \in L_2(\mathbb{R}^d); \tag{4.3}$$

here $\mathbb{A}_0 = \mathbb{A}(a,\mu_0)$ with $\mu_0 \equiv 1$. Then $\mathbb{A}_0 = p_0 - \mathbb{B}_0$, where $p_0 = \int_{\mathbb{R}^d} a(\mathbf{y})\,d\mathbf{y}$ and $\mathbb{B}_0$ is the convolution operator with the kernel $a$. The Fourier transform translates $\mathbb{B}_0$ into the operator of multiplication by the function $\widehat{a}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} e^{-i\langle\boldsymbol{\xi},\mathbf{z}\rangle}a(\mathbf{z})\,d\mathbf{z}$, $\boldsymbol{\xi} \in \mathbb{R}^d$. It follows that the operator $\mathbb{A}_0$ is unitarily equivalent to the operator of multiplication by the function $\widehat{a}(\mathbf{0}) - \widehat{a}(\boldsymbol{\xi})$. Hence, $\lambda_0 = 0$ belongs to the spectrum of $\mathbb{A}_0$. Since $\mathbb{A}_0$ is a nonnegative operator, $\lambda_0$ is the spectral edge. In view of estimates (4.3), the point $\lambda_0 = 0$ is also the lower edge of the spectrum of $\mathbb{A}$.

Due to conditions (1.2)–(1.5), the operator $\mathbb{A}$ commutes with the shift operators $S_{\mathbf{n}}$ defined by $S_{\mathbf{n}}u(\mathbf{x}) = u(\mathbf{x}+\mathbf{n})$, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{n} \in \mathbb{Z}^d$. This means that $\mathbb{A}$ and $\mathbb{B}$ are periodic operators with a periodicity lattice $\mathbb{Z}^d$. Recall the definition of the Gelfand transform $\mathcal{G}$. First, $\mathcal{G}$ is defined on the Schwarz class $\mathcal{S}(\mathbb{R}^d)$ as follows:

$$\mathcal{G}u(\boldsymbol{\xi},\mathbf{x}) := (2\pi)^{-d/2} \sum_{\mathbf{n}\in\mathbb{Z}^d} u(\mathbf{x}+\mathbf{n})e^{-i\langle\boldsymbol{\xi},\mathbf{x}+\mathbf{n}\rangle}, \quad \boldsymbol{\xi} \in \widetilde{\Omega}, \quad \mathbf{x} \in \Omega, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Then $\mathcal{G}$ is extended by continuity up to the unitary mapping $\mathcal{G} : L_2(\mathbb{R}^d) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega)\, d\boldsymbol{\xi} = L_2(\widetilde{\Omega}\times\Omega)$.

Like all periodic operators, $\mathbb{A}$ and $\mathbb{B}$ are decomposed into the direct integrals:

$$\mathbb{A} = \mathcal{G}^*\Big(\int_{\widetilde{\Omega}} \oplus \mathbb{A}(\boldsymbol{\xi})\, d\boldsymbol{\xi}\Big)\mathcal{G}, \quad \mathbb{B} = \mathcal{G}^*\Big(\int_{\widetilde{\Omega}} \oplus \mathbb{B}(\boldsymbol{\xi})\, d\boldsymbol{\xi}\Big)\mathcal{G}. \tag{4.4}$$

Here $\mathbb{A}(\boldsymbol{\xi})$ and $\mathbb{B}(\boldsymbol{\xi})$ are self-adjoint bounded operators in $L_2(\Omega)$ defined by

$$\mathbb{A}(\boldsymbol{\xi})u(\mathbf{x}) = p(\mathbf{x})u(\mathbf{x}) - \mathbb{B}(\boldsymbol{\xi})u(\mathbf{x}), \quad u \in L_2(\Omega),$$
$$\mathbb{B}(\boldsymbol{\xi})u(\mathbf{x}) = \int_{\Omega} \widetilde{a}(\boldsymbol{\xi},\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})u(\mathbf{y})\, d\mathbf{y}, \quad u \in L_2(\Omega),$$

where

$$\widetilde{a}(\boldsymbol{\xi},\mathbf{z}) := \sum_{\mathbf{n}\in\mathbb{Z}^d} a(\mathbf{z}+\mathbf{n})e^{-i\langle\boldsymbol{\xi},\mathbf{z}+\mathbf{n}\rangle}, \quad \boldsymbol{\xi} \in \widetilde{\Omega}, \quad \mathbf{z} \in \mathbb{R}^d.$$

The first relation in (4.4) is understood in the following sense: for any $u \in L_2(\mathbb{R}^d)$ and $v = \mathbb{A}u$, we have $\mathcal{G}v(\boldsymbol{\xi},\cdot) = \mathbb{A}(\boldsymbol{\xi})\mathcal{G}u(\boldsymbol{\xi},\cdot)$, $\boldsymbol{\xi} \in \widetilde{\Omega}$. The second one has a similar meaning.

The operator $\mathbb{B}(\boldsymbol{\xi})$ is compact (see [8]); by the Schur lemma, its norm satisfies the estimate $\|\mathbb{B}(\boldsymbol{\xi})\| \leqslant \mu_+\|a\|_{L_1}$, $\boldsymbol{\xi} \in \widetilde{\Omega}$. We conclude that the essential spectrum of $\mathbb{A}(\boldsymbol{\xi})$ coincides with the essential range of $p(\cdot)$. Due to the compactness of $\mathbb{B}(\boldsymbol{\xi})$ and lower bound (4.1), for any $\boldsymbol{\xi} \in \widetilde{\Omega}$, the spectrum of $\mathbb{A}(\boldsymbol{\xi})$ in the interval $(-\infty, \mu_-\|a\|_{L_1})$ is discrete.

In [8, Lemma 1.1], the following representation for the quadratic form of $\mathbb{A}(\boldsymbol{\xi}) = \mathbb{A}(\boldsymbol{\xi}; a, \mu)$ was obtained:

$$(\mathbb{A}(\boldsymbol{\xi})u,u) = \frac{1}{2}\int_{\Omega} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{y}\, a(\mathbf{x}-\mathbf{y})\mu(\mathbf{x},\mathbf{y})\big|e^{i\langle\boldsymbol{\xi},\mathbf{x}\rangle}u(\mathbf{x}) - e^{i\langle\boldsymbol{\xi},\mathbf{y}\rangle}u(\mathbf{y})\big|^2, \quad u \in L_2(\Omega), \quad \boldsymbol{\xi} \in \widetilde{\Omega}. \tag{4.5}$$

It is assumed here that the function $u \in L_2(\Omega)$ is extended periodically to the whole $\mathbb{R}^d$.

Let $\mathbb{A}_0(\boldsymbol{\xi}) = \mathbb{A}(\boldsymbol{\xi}; a, \mu_0)$, where $\mu_0 \equiv 1$. Relations (1.3) and (4.5) imply the estimates

$$\mu_-(\mathbb{A}_0(\boldsymbol{\xi})u,u) \leqslant (\mathbb{A}(\boldsymbol{\xi})u,u) \leqslant \mu_+(\mathbb{A}_0(\boldsymbol{\xi})u,u), \quad u \in L_2(\Omega), \quad \boldsymbol{\xi} \in \widetilde{\Omega}. \tag{4.6}$$

The operators $\mathbb{A}_0(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \widetilde{\Omega}$, are diagonalized by means of the unitary discrete Fourier transform. Namely, $\mathbb{A}_0(\boldsymbol{\xi})$ is unitarily equivalent to the operator $\mathfrak{A}_0(\boldsymbol{\xi})$ acting in $\ell_2(\mathbb{Z}^d)$ as multiplication by the symbol

$$\widehat{a}(\mathbf{0}) - \widehat{a}(2\pi\mathbf{n}+\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \big(1-\cos(\langle\mathbf{z},\boldsymbol{\xi}+2\pi\mathbf{n}\rangle)\big)a(\mathbf{z})\, d\mathbf{z}, \quad \mathbf{n} \in \mathbb{Z}^d. \tag{4.7}$$

Under conditions (1.2)–(1.5), using the described diagonalization for $\mathbb{A}_0(\boldsymbol{\xi})$ and analyzing the symbol (4.7), we obtain the following: $\operatorname{Ker}\mathbb{A}_0(\mathbf{0}) = \mathcal{L}\{\mathbf{1}_\Omega\}$;

$$(\mathbb{A}_0(\boldsymbol{\xi})u,u) \geqslant \mathcal{C}_1(a)\|u\|^2_{L_2(\Omega)}, \quad u \in L_2(\Omega) \ominus \mathcal{L}\{\mathbf{1}_\Omega\}, \quad \boldsymbol{\xi} \in \widetilde{\Omega},$$
$$(\mathbb{A}_0(\boldsymbol{\xi})u,u) \geqslant \mathcal{C}_2(a)|\boldsymbol{\xi}|^2\|u\|^2_{L_2(\Omega)}, \quad u \in L_2(\Omega), \quad \boldsymbol{\xi} \in \widetilde{\Omega},$$

with some positive constants $\mathcal{C}_1(a)$, $\mathcal{C}_2(a)$ depending on $a$. Combining these relations with (4.6), we obtain $\operatorname{Ker}\mathbb{A}(\mathbf{0}) = \mathcal{L}\{\mathbf{1}_\Omega\}$;

$$(\mathbb{A}(\boldsymbol{\xi})u,u) \geqslant \mu_-\mathcal{C}_1(a)\|u\|^2_{L_2(\Omega)}, \quad u \in L_2(\Omega) \ominus \mathcal{L}\{\mathbf{1}_\Omega\}, \quad \boldsymbol{\xi} \in \widetilde{\Omega}, \tag{4.8}$$
$$(\mathbb{A}(\boldsymbol{\xi})u,u) \geqslant \mu_-\mathcal{C}_2(a)|\boldsymbol{\xi}|^2\|u\|^2_{L_2(\Omega)}, \quad u \in L_2(\Omega), \quad \boldsymbol{\xi} \in \widetilde{\Omega}. \tag{4.9}$$

We also need the following estimate which is proved by the Schur lemma under conditions (1.2)–(1.5) and condition $M_1(a) < \infty$:

$$\|\mathbb{A}(\boldsymbol{\xi}) - \mathbb{A}(\boldsymbol{\eta})\| \leqslant \mu_+ M_1(a)|\boldsymbol{\xi}-\boldsymbol{\eta}|, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \widetilde{\Omega}. \tag{4.10}$$

## 5. THRESHOLD APPROXIMATIONS FOR THE OPERATOR $\mathbb{A}(\boldsymbol{\xi})$

Under conditions (1.2)–(1.5), the lower edge of the spectrum of the operator $\mathbb{A}(\mathbf{0})$ is an isolated simple eigenvalue $\lambda_0 = 0$. Let $d_0 := d_0(a,\mu)$ be the distance from the point $\lambda_0$ to the rest of the spectrum of $\mathbb{A}(\mathbf{0})$. Estimate (4.8) implies that $d_0(a,\mu) \geqslant \mu_- \mathcal{C}_1(a)$. Let

$$\delta_0 = \delta_0(a,\mu) := \mu_- \mathcal{C}_1(a) \left(3M_1(a)\mu_+\right)^{-1}.$$

Under conditions (1.2)–(1.5) and $M_1(a) < \infty$, applying the perturbation theory arguments, we deduce from estimate (4.10) that for all $\boldsymbol{\xi}$ such that $|\boldsymbol{\xi}| \leqslant \delta_0(a,\mu)$ the spectrum of the operator $\mathbb{A}(\boldsymbol{\xi})$ on the interval $[0, d_0/3]$ consists of just one simple eigenvalue; while the interval $(d_0/3, 2d_0/3)$ does not contain points of the spectrum of $\mathbb{A}(\boldsymbol{\xi})$.

Under conditions (1.2)–(1.5) and $M_k(a) < \infty$ the operator-valued function $\mathbb{A}(\cdot)$ is $k$ times continuously differentiable in the operator norm. We have

$$\partial^\alpha \mathbb{A}(\boldsymbol{\xi}; a,\mu) u(\mathbf{x}) = \int_\Omega \widetilde{a}_\alpha(\boldsymbol{\xi}, \mathbf{x}-\mathbf{y}) \mu(\mathbf{x},\mathbf{y}) u(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \Omega, \quad u \in L_2(\Omega);$$
$$\widetilde{a}_\alpha(\boldsymbol{\xi}, \mathbf{z}) := (-1)(-i)^{|\alpha|} \sum_{\mathbf{n} \in \mathbb{Z}^d} (\mathbf{z}+\mathbf{n})^\alpha a(\mathbf{z}+\mathbf{n}) e^{-i\langle \boldsymbol{\xi}, \mathbf{z}+\mathbf{n}\rangle}, \quad \alpha \in \mathbb{Z}_+^d, \quad |\alpha| \leqslant k.$$

Then we can apply the Hadamard formula. For instance, if $M_4(a) < \infty$, then we have

$$\begin{gathered}\mathbb{A}(\boldsymbol{\xi}) = \mathbb{A}(\mathbf{0}) + [\Delta_1 \mathbb{A}](\boldsymbol{\xi}) + [\Delta_2 \mathbb{A}](\boldsymbol{\xi}) + [\Delta_3 \mathbb{A}](\boldsymbol{\xi}) + \mathbb{K}_3(\boldsymbol{\xi}),\\ [\Delta_1 \mathbb{A}](\boldsymbol{\xi}) := \sum_{j=1}^d \partial_j \mathbb{A}(\mathbf{0}) \xi_j, \quad [\Delta_2 \mathbb{A}](\boldsymbol{\xi}) := \frac{1}{2} \sum_{k,l=1}^d \partial_k \partial_l \mathbb{A}(\mathbf{0}) \xi_k \xi_l, \quad [\Delta_3 \mathbb{A}](\boldsymbol{\xi}) := \frac{1}{6} \sum_{j,k,l=1}^d \partial_j \partial_k \partial_l \mathbb{A}(\mathbf{0}) \xi_j \xi_k \xi_l,\end{gathered} \tag{5.1}$$

$$\|\mathbb{K}_3(\boldsymbol{\xi})\| \leqslant \frac{1}{24} \mu_+ M_4(a) |\boldsymbol{\xi}|^4, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu). \tag{5.2}$$

Let $F(\boldsymbol{\xi})$ be the spectral projection of $\mathbb{A}(\boldsymbol{\xi})$ that corresponds to the interval $[0, d_0/3]$. The symbol $\mathfrak{N}$ stands for the kernel $\operatorname{Ker} \mathbb{A}(\mathbf{0}) = \mathcal{L}\{\mathbf{1}_\Omega\}$; by $P$ we denote the orthogonal projection onto $\mathfrak{N}$; then $P = (\cdot, \mathbf{1}_\Omega)\mathbf{1}_\Omega$. Let $\Gamma$ be a contour on the complex plane that is equidistant to the interval $[0, d_0/3]$ and passes through the middle point of the interval $(d_0/3, 2d_0/3)$. By the Riesz formula, the following representations are valid:

$$F(\boldsymbol{\xi}) = -\frac{1}{2\pi i} \oint_\Gamma (\mathbb{A}(\boldsymbol{\xi}) - \zeta I)^{-1}\, d\zeta, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu), \tag{5.3}$$

$$\mathbb{A}(\boldsymbol{\xi}) F(\boldsymbol{\xi}) = -\frac{1}{2\pi i} \oint_\Gamma (\mathbb{A}(\boldsymbol{\xi}) - \zeta I)^{-1} \zeta\, d\zeta, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu); \tag{5.4}$$

here we integrate along the contour $\Gamma$ counterclockwise.

We obtain an approximation for the operator $F(\boldsymbol{\xi})$ with an error $O(|\boldsymbol{\xi}|^2)$ and an approximation for the operator $\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi})$ with an error $O(|\boldsymbol{\xi}|^4)$. Earlier less accurate approximations for $F(\boldsymbol{\xi})$ and $\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi})$ were obtained in [8] with errors $O(|\boldsymbol{\xi}|)$ and $O(|\boldsymbol{\xi}|^3)$, respectively; see Propositions 5.1(1°) and 5.3.

For $|\boldsymbol{\xi}| \leqslant \delta_0(a,\mu)$ and $\zeta \in \Gamma$, we write

$$R(\boldsymbol{\xi}, \zeta) := (\mathbb{A}(\boldsymbol{\xi}) - \zeta I)^{-1}, \quad R_0(\zeta) := R(\mathbf{0}, \zeta), \quad \Delta\mathbb{A}(\boldsymbol{\xi}) := \mathbb{A}(\boldsymbol{\xi}) - \mathbb{A}(\mathbf{0}).$$

We apply the resolvent identity

$$R(\boldsymbol{\xi}, \zeta) = R_0(\zeta) - R(\boldsymbol{\xi}, \zeta) \Delta\mathbb{A}(\boldsymbol{\xi}) R_0(\zeta), \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu), \quad \zeta \in \Gamma. \tag{5.5}$$

The length of the contour $\Gamma$ is equal to $\frac{\pi+2}{3} d_0$, and both resolvents on the contour $\Gamma$ satisfy the estimates

$$\|R(\boldsymbol{\xi}, \zeta)\| \leqslant 6 d_0^{-1}, \quad \|R_0(\zeta)\| \leqslant 6 d_0^{-1}, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu), \quad \zeta \in \Gamma. \tag{5.6}$$

Iterating the resolvent identity (5.5) and using representations (5.3), (5.4), estimates (5.6), and an appropriate version of the Hadamard formula for $\mathbb{A}(\boldsymbol{\xi})$ (cf. (5.1), (5.2)), we deduce the required approximations. When calculating the contour integrals, we also use the following representation of the resolvent of $\mathbb{A}(\mathbf{0})$:

$$R_0(\zeta) = R_0(\zeta) P + R_0(\zeta) P^\perp = -\frac{1}{\zeta} P + R_0(\zeta) P^\perp, \quad \zeta \in \Gamma,$$

and take into account the fact that the operator-valued function $R_0^\perp(\zeta) := R_0(\zeta) P^\perp$ is holomorphic inside the contour $\Gamma$. This way we obtain the following results.

**Proposition 5.1.** *Suppose that conditions* (1.2)–(1.5) *are satisfied.*
$1^\circ$. *If* $M_1(a) < \infty$, *then*

$$\|F(\boldsymbol{\xi}) - P\| \leqslant C_1(a,\mu)|\boldsymbol{\xi}|, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu). \tag{5.7}$$

$2^\circ$. *If* $M_2(a) < \infty$, *then*

$$F(\boldsymbol{\xi}) = P + [F]_1(\boldsymbol{\xi}) + \Phi(\boldsymbol{\xi}), \quad [F]_1(\boldsymbol{\xi}) := \sum_{j=1}^{d} F_j \xi_j,$$
$$\|\Phi(\boldsymbol{\xi})\| \leqslant C_2(a,\mu)|\boldsymbol{\xi}|^2, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu).$$

*The operators* $F_j$ *are given by*

$$F_j = -P\partial_j \mathbb{A}(\mathbf{0}) P^\perp \mathbb{A}(\mathbf{0})^{-1} P^\perp - P^\perp \mathbb{A}(\mathbf{0})^{-1} P^\perp \partial_j \mathbb{A}(\mathbf{0}) P, \quad j = 1, \dots, d. \tag{5.8}$$

*Here* $\mathbb{A}(\mathbf{0})^{-1}$ *is understood as the inverse operator to* $\mathbb{A}(\mathbf{0})|_{\mathfrak{N}^\perp} : \mathfrak{N}^\perp \to \mathfrak{N}^\perp$.

Representation (5.8) can be “deciphered” in terms of solutions of auxiliary problems.

**Proposition 5.2.** *Under the assumptions of Proposition* 5.1($2^\circ$), *the operators* $F_j$ *admit the following representations*:

$$F_j = i(\cdot, v_j)\mathbf{1}_\Omega - i(\cdot, \mathbf{1}_\Omega) v_j, \quad j = 1, \dots, d, \tag{5.9}$$

*where* $v_j(\mathbf{x})$ *are components of the periodic solution* $\mathbf{v}(\mathbf{x})$ *of problem* (2.1).

**Proposition 5.3** ([8])**.** *Suppose that conditions* (1.2)–(1.5) *are satisfied and* $M_3(a) < \infty$. *Then*

$$\mathbb{A}(\boldsymbol{\xi}) F(\boldsymbol{\xi}) = [G]_2(\boldsymbol{\xi}) + \Psi(\boldsymbol{\xi}), \quad [G]_2(\boldsymbol{\xi}) := \frac{1}{2} \sum_{k,l=1}^{d} G_{kl} \xi_k \xi_l, \tag{5.10}$$

$$\|\Psi(\boldsymbol{\xi})\| \leqslant C_3(a,\mu)|\boldsymbol{\xi}|^3, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu). \tag{5.11}$$

*The operators* $G_{kl}$ *are given by*

$$G_{kl} = P\partial_k \partial_l \mathbb{A}(\mathbf{0}) P - P\partial_k \mathbb{A}(\mathbf{0}) P^\perp \mathbb{A}(\mathbf{0})^{-1} P^\perp \partial_l \mathbb{A}(\mathbf{0}) P - P\partial_l \mathbb{A}(\mathbf{0}) P^\perp \mathbb{A}(\mathbf{0})^{-1} P^\perp \partial_k \mathbb{A}(\mathbf{0}) P, \quad k, l = 1, \dots, d. \tag{5.12}$$

Representation (5.12) can be deciphered in terms of solutions of auxiliary problems.

**Proposition 5.4** ([8])**.** *Under the assumptions of Proposition* 5.3, *the operators* $G_{kl}$ *admit the representations* $G_{kl} = g_{kl} P$, $k, l = 1, \dots, d$, *where* $g_{kl}$ *are given by* (2.2). *Thus,*

$$[G]_2(\boldsymbol{\xi}) = \frac{1}{2} \sum_{k,l=1}^{d} g_{kl} \xi_k \xi_l P = \langle g^0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle P, \quad \boldsymbol{\xi} \in \mathbb{R}^d,$$

*where* $g^0$ *is the effective matrix.*

**Proposition 5.5.** *Assume that conditions* (1.2)–(1.5) *hold and* $M_4(a) < \infty$. *Then*

$$\mathbb{A}(\boldsymbol{\xi}) F(\boldsymbol{\xi}) = [G]_2(\boldsymbol{\xi}) + [G]_3(\boldsymbol{\xi}) + \Upsilon(\boldsymbol{\xi}),$$
$$\|\Upsilon(\boldsymbol{\xi})\| \leqslant C_4(a,\mu)|\boldsymbol{\xi}|^4, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu).$$

*Here* $[G]_2(\boldsymbol{\xi})$ *is defined by* (5.10), (5.12), *and* $[G]_3(\boldsymbol{\xi}) = \frac{1}{6} \sum_{j,k,l=1}^{d} G_{jkl} \xi_j \xi_k \xi_l$ *satisfies the relations*

$$\|[G]_3(\boldsymbol{\xi})\| \leqslant C_5(a,\mu)|\boldsymbol{\xi}|^3, \quad \boldsymbol{\xi} \in \mathbb{R}^d,$$
$$P[G]_3(\boldsymbol{\xi}) P = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

## 6. APPROXIMATION FOR THE RESOLVENT $(\mathbb{A}+\varepsilon^2 I)^{-1}$

From (4.9), it follows that

$$(\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi})u,u) \geqslant \mu_- \mathcal{C}_2(a)|\boldsymbol{\xi}|^2 (F(\boldsymbol{\xi})u,u), \quad u\in L_2(\Omega), \quad |\boldsymbol{\xi}|\leqslant \delta_0(a,\mu). \tag{6.1}$$

According to Propositions 5.1, 5.3, 5.4 we have $\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi}) = \langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle P + O(|\boldsymbol{\xi}|^3)$ and $F(\boldsymbol{\xi}) = P + O(|\boldsymbol{\xi}|)$ as $|\xi|\to 0$. Substituting these expansions in (6.1) and letting $u=\mathbf{1}_\Omega$, we obtain $\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle \geqslant \mu_-\mathcal{C}_2(a)|\boldsymbol{\xi}|^2 + O(|\boldsymbol{\xi}|^3)$. This implies that the matrix $g^0$ is positive definite: $g^0\geqslant \mu_-\mathcal{C}_2(a)\mathbf{1}$.

Denote

$$\Xi(\boldsymbol{\xi},\varepsilon) := (\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}F(\boldsymbol{\xi}) - \left(\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2\right)^{-1}P, \quad \boldsymbol{\xi}\in\widetilde{\Omega}, \quad \varepsilon>0. \tag{6.2}$$

Obviously,

$$\Xi(\boldsymbol{\xi},\varepsilon) = F(\boldsymbol{\xi})(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}(F(\boldsymbol{\xi})-P) + (F(\boldsymbol{\xi})-P)(\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P \\ - F(\boldsymbol{\xi})(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}\left(\mathbb{A}(\boldsymbol{\xi})F(\boldsymbol{\xi}) - \langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle P\right)(\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P. \tag{6.3}$$

Using this identity, the estimate $g^0\geqslant \mu_-\mathcal{C}_2(a)\mathbf{1}$, and relations (4.9), (5.7), (5.10), (5.11), we deduce the following statement.

**Proposition 6.1** ([8]). *Let conditions* (1.2)–(1.5) *be fulfilled, and assume that* $M_3(a)<\infty$. *Then the operator* (6.2) *satisfies the estimate*

$$\|\Xi(\boldsymbol{\xi},\varepsilon)\| \leqslant \frac{2C_1(a,\mu)|\boldsymbol{\xi}|}{\mu_-\mathcal{C}_2(a)|\boldsymbol{\xi}|^2+\varepsilon^2} + \frac{C_3(a,\mu)|\boldsymbol{\xi}|^3}{(\mu_-\mathcal{C}_2(a)|\boldsymbol{\xi}|^2+\varepsilon^2)^2}, \quad |\boldsymbol{\xi}|\leqslant\delta_0(a,\mu),\ \varepsilon>0.$$

Taking into account the obvious estimate

$$\|(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}(I-F(\boldsymbol{\xi}))\| \leqslant \frac{3}{d_0}, \quad |\boldsymbol{\xi}|\leqslant\delta_0(a,\mu), \quad \varepsilon>0, \tag{6.4}$$

from Proposition 6.1, we obtain the following result.

**Theorem 6.2** ([8]). *Assume that conditions* (1.2)–(1.5) *are satisfied and* $M_3(a)<\infty$. *Then*

$$\left\|(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1} - (\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P\right\| \leqslant C_6(a,\mu)\varepsilon^{-1}, \quad \varepsilon>0, \quad |\boldsymbol{\xi}|\leqslant\delta_0(a,\mu). \tag{6.5}$$

Obviously, for $\boldsymbol{\xi}\in\widetilde{\Omega}$ such that $|\boldsymbol{\xi}|>\delta_0(a,\mu)$, the left-hand side of (6.5) does not exceed $2(\mu_-\mathcal{C}_2(a)\delta_0^2+\varepsilon^2)^{-1}$. Hence, estimate (6.5) holds for any $\boldsymbol{\xi}\in\widetilde{\Omega}$, perhaps with a different constant $C_6(a,\mu)$.

Let $\mathbb{A}^0$ be the effective operator defined in Section 2. Using the unitary Gelfand transform, we decompose $\mathbb{A}^0$ into the direct integral:

$$\mathbb{A}^0 = \mathcal{G}^*\Big(\int_{\widetilde{\Omega}} \oplus \mathbb{A}^0(\boldsymbol{\xi})\,d\boldsymbol{\xi}\Big)\mathcal{G}. \tag{6.6}$$

Here $\mathbb{A}^0(\boldsymbol{\xi})$ is the self-adjoint operator in $L_2(\Omega)$ given by $\mathbb{A}^0(\boldsymbol{\xi}) = (\mathbf{D}+\boldsymbol{\xi})^* g^0(\mathbf{D}+\boldsymbol{\xi})$, $\mathbf{D}=-i\nabla$, $\operatorname{Dom}\mathbb{A}^0(\boldsymbol{\xi}) = \widetilde{H}^2(\Omega)$. The space $\widetilde{H}^2(\Omega)$ is defined as a subspace of $H^2(\Omega)$ consisting of functions whose $\mathbb{Z}^d$-periodic extension to $\mathbb{R}^d$ belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Note that $\mathbb{A}^0(\boldsymbol{\xi})P = \langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle P$. Hence, $\left(\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2\right)^{-1}P = (\mathbb{A}^0(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}P$. By the discrete Fourier transform, we deduce that

$$\left\|(\mathbb{A}^0(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}(I-P)\right\| = \sup_{0\neq\mathbf{n}\in\mathbb{Z}^d}(\langle g^0(2\pi\mathbf{n}+\boldsymbol{\xi}),2\pi\mathbf{n}+\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1} \leqslant (\mu_-\mathcal{C}_2(a)\pi^2+\varepsilon^2)^{-1}, \quad \boldsymbol{\xi}\in\widetilde{\Omega}. \tag{6.7}$$

Combining this inequality and Theorem 6.2 yields the following theorem.

**Theorem 6.3** ([8]). *Suppose that conditions* (1.2)–(1.5) *are satisfied and* $M_3(a)<\infty$. *Then*

$$\left\|(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1} - (\mathbb{A}^0(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}\right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant \mathrm{C}_1(a,\mu)\varepsilon^{-1}, \quad \varepsilon>0, \quad \boldsymbol{\xi}\in\widetilde{\Omega}.$$

Now we proceed to a more accurate approximation for the resolvent $(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1}$. Using Propositions 5.1, 5.3, 5.5, and 6.1 together with representation (6.3) and estimates (6.4), (6.7), we obtain the following statement.

**Theorem 6.4.** *Suppose that conditions* (1.2)–(1.5) *are satisfied and* $M_4(a) < \infty$. *Then*

$$(\mathbb{A}(\boldsymbol{\xi})+\varepsilon^2 I)^{-1} = (\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P + [F]_1(\boldsymbol{\xi})(\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P + (\langle g^0\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\varepsilon^2)^{-1}P[F]_1(\boldsymbol{\xi}) + Y(\boldsymbol{\xi},\varepsilon),$$
$$\|Y(\boldsymbol{\xi},\varepsilon)\| \leqslant C_7(a,\mu), \quad \varepsilon > 0, \quad |\boldsymbol{\xi}| \leqslant \delta_0(a,\mu).$$

*Here* $[F]_1(\boldsymbol{\xi}) = \sum_{j=1}^d F_j\xi_j$, *and the operators* $F_j$ *are given by* (5.9).

From Theorem 6.4, considering Lemma 2.1, it is easy to deduce the following result.

**Theorem 6.5.** *Suppose that conditions* (1.2)–(1.5) *are satisfied and* $M_4(a) < \infty$. *Then*

$$\big\|(\mathbb{A}(\boldsymbol{\xi}) + \varepsilon^2 I)^{-1} - (\mathbb{A}^0(\boldsymbol{\xi}) + \varepsilon^2 I)^{-1} - K(\boldsymbol{\xi},\varepsilon)\big\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant \mathrm{C}_2(a,\mu)$$

*for* $\varepsilon > 0$ *and* $\boldsymbol{\xi} \in \widetilde{\Omega}$. *Here*

$$K(\boldsymbol{\xi},\varepsilon) := -i\sum_{j=1}^d [v_j](D_j+\xi_j)(\mathbb{A}^0(\boldsymbol{\xi}) + \varepsilon^2 I)^{-1} + i\sum_{j=1}^d (\mathbb{A}^0(\boldsymbol{\xi}) + \varepsilon^2 I)^{-1}(D_j+\xi_j)[v_j].$$

Using the direct integral decompositions (4.4) and (6.6), we deduce the following two statements from Theorems 6.3 and 6.5.

**Theorem 6.6** ([8])**.** *Let conditions* (1.2)–(1.5) *be fulfilled, and assume that* $M_3(a) < \infty$. *Then*

$$\big\|(\mathbb{A} + \varepsilon^2 I)^{-1} - (\mathbb{A}^0 + \varepsilon^2 I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathrm{C}_1(a,\mu)\varepsilon^{-1}, \quad \varepsilon > 0.$$

**Theorem 6.7.** *Let conditions* (1.2)–(1.5) *be fulfilled, and assume that* $M_4(a) < \infty$. *Then*

$$\big\|(\mathbb{A} + \varepsilon^2 I)^{-1} - (\mathbb{A}^0 + \varepsilon^2 I)^{-1} - K(\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathrm{C}_2(a,\mu), \quad \varepsilon > 0.$$

*Here*

$$K(\varepsilon) := -\sum_{j=1}^d [v_j]\partial_j(\mathbb{A}^0 + \varepsilon^2 I)^{-1} + \sum_{j=1}^d (\mathbb{A}^0 + \varepsilon^2 I)^{-1}\partial_j[v_j]. \tag{6.8}$$

## 7. PROOF OF MAIN RESULTS

Let us introduce the scaling transformation (a family of unitary operators $T_\varepsilon$): $T_\varepsilon u(\mathbf{x}) := \varepsilon^{d/2}u(\varepsilon\mathbf{x})$, $u \in L_2(\mathbb{R}^d)$, $\varepsilon > 0$. It is easily seen that $\mathbb{A}_\varepsilon = \varepsilon^{-2}T_\varepsilon^*\mathbb{A}T_\varepsilon$, $\varepsilon > 0$. Hence, $(\mathbb{A}_\varepsilon + I)^{-1} = T_\varepsilon^*\varepsilon^2(\mathbb{A} + \varepsilon^2 I)^{-1}T_\varepsilon$, $\varepsilon > 0$. Similarly, $(\mathbb{A}^0 + I)^{-1} = T_\varepsilon^*\varepsilon^2(\mathbb{A}^0 + \varepsilon^2 I)^{-1}T_\varepsilon$, $\varepsilon > 0$. Since $T_\varepsilon$ is unitary, then

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \varepsilon^2\|(\mathbb{A} + \varepsilon^2 I)^{-1} - (\mathbb{A}^0 + \varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.$$

Combining this with Theorem 6.6, we obtain the required estimate (2.3). Theorem 2.2 is proved.

In a similar fashion, taking into account the relation $K_\varepsilon = T_\varepsilon^*\varepsilon K(\varepsilon)T_\varepsilon$, $\varepsilon > 0$, where $K(\varepsilon)$ is the operator (6.8), we obtain

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1} - \varepsilon K_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \varepsilon^2\|(\mathbb{A} + \varepsilon^2 I)^{-1} - (\mathbb{A}^0 + \varepsilon^2 I)^{-1} - K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.$$

Together with Theorem 6.7, this implies estimate (2.4) and completes the proof of Theorem 2.3.

The detailed presentation of the above results can be found in preprint [9].

## FUNDING

The research of A. Piatnitski and E. Zhizhina was partially supported by the project “Pure Mathematics in Norway” and the UiT Aurora project MASCOT. The research of A. Piatnitski was supported by the megagrant of Ministry of Science and Higher Education of the Russian Federation, project 075-15-2022-1115.

The research of V. Sloushch and T. Suslina was supported by Russian Science Foundation, project no. 22-11-00092.

## CONFLICTS OF INTEREST

The authors of this work declare that they have no conflicts of interest.

## REFERENCES

[1] M. Sh. Birman and T. A. Suslina, "Second Order Periodic Differential Operators. Threshold Properties and Homogenization", *Algebra i Analiz*, **15**:5 (2003), 1–108; English transl. *St. Petersburg Math. J.*, **15**:5 (2004), 639–714.

[2] M. Sh. Birman and T. A. Suslina, "Homogenization with Corrector Term for Periodic Elliptic Differential Operators", *Algebra i Analiz*, **17**:6 (2005), 1–104; English transl. *St. Petersburg Math. J.*, **17**:6 (2006), 897–973.

[3] M. Sh. Birman and T. A. Suslina, "Homogenization with Corrector for Periodic Differential Operators. Approximation of Solutions in the Sobolev Class $H^1(\mathbb{R}^d)$", *Algebra i Analiz*, **18**:6 (2006), 1–130; English transl. *St. Petersburg Math. J.*, **18**:6 (2007), 857–955.

[4] A. Braides and A. Piatnitski, "Homogenization of Quadratic Convolution Energies in Periodically Perforated Domains", *Adv. Calc. Var.*, 2019, DOI 10.1515/acv-2019-0083.

[5] Yu. Kondratiev, S. Molchanov, S. Pirogov, and E. Zhizhina, "On Ground State of Some Non Local Schrödinger Operators", *Appl. Anal.*, **96**:8 (2017), 1390–1400.

[6] A. Piatnitski and E. Zhizhina, "Periodic Homogenization of Nonlocal Operators with a Convolution-Type Kernel", *SIAM J. Math. Anal.*, **49**:1 (2017), 64–81.

[7] A. Piatnitski and E. Zhizhina, "Homogenization of Biased Convolution Type Operators", *Asymptotic Anal.*, **115**:3-3 (2019), 241–262.

[8] A. Piatnitski, V. Sloushch, T. Suslina, and E. Zhizhina, "On Operator Estimates in Homogenization of Nonlocal Operators of Convolution Type", *J. Diff. Equ.*, **352** (2023), 153–188.

[9] A. Piatnitski, V. Sloushch, T. Suslina, and E. Zhizhina, "Homogenization of Nonlocal Convolution Type Operators: Approximation for the Resolvent with Corrector", 2023.