A PARABOLIC EQUATION WITH RAPIDLY OSCILLATING COEFFICIENTS

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The aim of this paper is the construction of an asymptotic expansion for the solution of the Cauchy problem for the parabolic equation

$$\frac{\partial}{\partial t} u_\varepsilon (x, t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial}{\partial x_j} u_\varepsilon(x, t) = f(x, t), \qquad (1)$$
$$u_\varepsilon(x, t)|_{t=0} = f_1(x)$$

with coefficients periodic in R^{n+1} $a_{ij}(y, \tau)$, $y=\frac{x}{\varepsilon}$, $\tau=\frac{t}{\varepsilon^2}$.

We note that the formal expansion constructed by the method in [1] does not satisfy the initial conditions in this case; therefore for the construction of the asymptotic $u_\varepsilon(x, t)$ it is necessary to introduce a correction of boundary layer type. The asymptotic expansion takes the form

$$u_\varepsilon(x, t) \sim u_0(x, t) + \sum_{k=1}^{\infty} \varepsilon^k \left(\tilde{w}_k\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^2}\right) + \tilde{z}_k\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^2}\right)\right);$$

here the functions $\tilde{z}_k(x, y, t, \tau)$ decay exponentially as $\tau \to \infty$ and are periodic in y; the functions $\tilde{w}_k(x, y, t, \tau)$ are periodic in τ and y.

In the problem under consideration corrections of boundary layer cause undamped terms for $t \sim 1$ in the higher terms of the expansion. Therefore for $k > 1$ the functions $\tilde{w}_k$ differ from approximations obtained in [1,2].

The boundary-value problem in a layer for an elliptic equation was solved in [3].

1. Notation and Definitions. Denote by $\|\cdot\|$ and $\|\cdot\|_1$ the norms in $L^2(R^n)$ and $H^1(R^n)$, respectively. Later the spaces $L^2(0, T; H^1(R^n))$ and $L^\infty(0, T; L^2(R^n))$ and their norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are defined in the usual manner. Suppose $\Pi^n=[0, 1]^n$. The mean of the Π^n-periodic function over a period we denote by $M\{f\}$; $H^1(\Pi^n)$ is the space of Π^n-periodic functions lying in $H^1_{loc}(R^n)$. As usual, $S(R^n)$ is the Schwartz space; $S_T(R^n)$ is the space of smooth functions in $(R^n \times [0, T])$ lying in $S(R^n)$ uniformly with respect to $t \in [0, T]$.

2. Construction of the asymptotic expansion. Consider problem (1). The coefficients $a_{ij}(y,\tau)$ are Π^{n+1}-periodic and satisfy the condition of uniform ellipticity

$$c_1|\xi|^2 \leqslant a_{ij}(y,\tau)\xi_i\xi_j \leqslant c_2|\xi|^2,$$

where $0<c_1<c_2$, $\xi\in R^n$. Here and later we omit the summation sign for repeating indices.

For convenience we consider the case when the coefficients a_{ij} are symmetric and do not depend on t. We introduce a rapidly oscillating variable $y=\frac{x}{\varepsilon}$ and we seek the solution of the original problem in the form of a function $u_\varepsilon(x,y,t)$ of (2n + 1) variables. In this connection the relation

$$\frac{\partial}{\partial x}u\left(x,\frac{x}{\varepsilon}\right)=\left(\frac{\partial}{\partial x}+\varepsilon^{-1}\frac{\partial}{\partial y}\right)u(x,y)\Big|_{y=\frac{x}{\varepsilon}}$$

holds, Using this relation, we write A_ε in the following form:

$$A_\varepsilon\varphi\left(x,\frac{x}{\varepsilon},t\right)=(\varepsilon^{-2}A_2-\varepsilon^{-1}A_1-A_0)\varphi(x,y,t)\Big|_{y=\frac{x}{\varepsilon}},$$

where A_2, A_1, A_0 denote the operators

$$A_2=-\frac{\partial}{\partial y_i}a_{ij}(y)\frac{\partial}{\partial y_j},\quad A_1=-\frac{\partial}{\partial y_i}a_{ij}(y)\frac{\partial}{\partial x_j}-a_{ij}(y)\frac{\partial^2}{\partial x_i\partial y_j},$$

$$A_0=\frac{\partial}{\partial t}-a_{ij}(y)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}.$$

We seek the solution of (1) in the form of a series

$$u_\varepsilon(x,y,t)=u_0(x,t)+\sum_{k=1}^{\infty}\varepsilon^k(u_k(x,t)+v_k(x,y,t)).$$

Substituting this series into (1) and equating coefficients of like powers of ε we obtain the system of equations

$$\begin{array}{c|l} \varepsilon^{-2} & A_2u_0=0 \\ \varepsilon^{-1} & A_2w_1+A_1u_0=0 \\ \varepsilon^{0} & A_2w_2+A_1w_1+A_0u_0=f \end{array}\qquad \begin{array}{c|l} \varepsilon^{1} & A_2w_3+A_1w_2+A_0w_1=0 \\ \cdots & \cdots\cdots\cdots \\ \varepsilon^{k} & A_2w_{k+2}+A_1w_{k+1}+A_0w_k=0 \\ \cdots & \cdots\cdots\cdots \end{array}\qquad (2)$$

Here w_k is $(u_k + v_k)$. We seek the function $v_k(x,y,t)$ in the space $\mathring{H}^1(\Pi^n)$ with respect to y. We solve (2) with the following initial conditions:

$$u_0(x,t)|_{t=0}=f_1(x);\quad u_k(x,t)|_{t=0}=0,\qquad k\geqslant 1. \qquad (3)$$

Theorem 1. Suppose $f_1(x)\in S(R^n)$, $f(x,t)\in S_T(R^n)$. Then the system of equations (2), (3) is successively solvable. The function $u_0(x,t)$ satisfies the parabolic equation with constant coefficients

$$\frac{\partial}{\partial t}u_0(x,t)-Qu_0(x,t)=f(x,t),$$

$$u_0|_{t=0}=f_1(x),$$

where the operator Q has the form

$$Qu(x,t) = q_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u(x,t) = \left(M\{a_{ij}(y)\} - M\left\{a_{il}(y)\frac{\partial\chi_j(y)}{\partial y_l}\right\}\right)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j},$$

and the vector-function $\chi(y)$ of class $\mathring{H}^1(\Pi^n)$ is a solution of the equation

$$A_2(\chi(y) - y) = 0, \qquad y = (y_1, \dots, y_n).$$

Proof. The approximations w_k are constructed by standard methods [1,2].

Remark. The $v_k(x,y,t)$ obtained are of the following form:

$$v_1(x,y,t) = \frac{\partial}{\partial x_k}u_0(x,t)\chi_k(y), \quad v_r(x,y,t) = \sum_{k=1}^{N(r)} \chi_k^r(y)\mu_k^r(x,t).$$

Denote by $u_\varepsilon^k(x,t)$ and $r_\varepsilon^k(x,t)$ the expressions $\sum_{l=1}^{k}\varepsilon^l w_l$ and $(u_\varepsilon - u_\varepsilon^k)$ respectively. By Theorem 1, it is easy to prove the following theorem.

Theorem 2. The functions $u_\varepsilon^{k+2}(x,t)$ satisfy (1) with accuracy $O(\varepsilon^k)$:

$$A_\varepsilon r_\varepsilon^{k+2} = \varepsilon^k g_\varepsilon(x,t),$$

where $\|g_\varepsilon(x,t)\|_{L^2(0,T;H^{-1}(R^n))} = O(1)$.

Corollary. The error in the first approximation $r_\varepsilon^1(x,t)$ satisfies the estimate

$$\|r_\varepsilon^1\|_\infty < c\varepsilon, \quad \|r_\varepsilon^1\|_1 < c\varepsilon.$$

The formal expansion constructed above does not satisfy the initial conditions, therefore we add a correction of boundary layer type to all terms of the asymptotic $u_\varepsilon(x,t)$, which decays exponentially as $\tau \to \infty$ and corrects the initial condition:

$$\bar{u}_\varepsilon(x,y,t,\tau) = u_0(x,t) + \varepsilon(w_1(x,y,t) + z_1(x,y,t,\tau)) + \dots$$

Substituting this series into (1), with regard to (2) we find equations for $z_k(x,y,t,\tau)$

$$\begin{array}{l|l} \varepsilon^{-1} & A_3 z_1 = 0 \\ \varepsilon^0 & A_3 z_2 + A_1 z_1 = 0 \end{array} \qquad \begin{array}{l|l} \varepsilon^1 & A_3 z_3 + A_1 z_2 + A_0 z_1 = 0 \\ \cdots & \cdots\cdots \\ \varepsilon^k & A_3 z_{k+2} + A_1 z_{k+1} + A_0 z_k = 0 \\ \cdots & \cdots\cdots \end{array} \qquad (4)$$

$$z_k(x,y,t,\tau)|_{t,\tau=0} = -v_k(x,y,t)|_{t=0},$$

where A_3 denotes the operator $\left(\frac{\partial}{\partial\tau} + A_2\right)$. We will seek the solution of the system (4) in the following form:

$$z_k(x,y,t,\tau) = \sum_{m=1}^{N(k)} \alpha_m^k(y,\tau)\varphi_m^k(x,t).$$

Theorem 3. The system of equations (4) is successively solvable.

Proof. Consider equation (4) for k = -1 (4.-1):

$$\frac{\partial}{\partial \tau} z_1(x, y, t, \tau) - \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j} z_1(x, y, t, \tau) = 0,$$

$$z_1(x, y, t, \tau)|_{t,\tau=0} = -v_1(x, y, 0).$$

Having written $z_1(x, y, t, \tau)$ in the form $\left(\sum_{k=1}^{n} \alpha_k^1(y, \tau) \frac{\partial}{\partial x_k} \varphi^1(x, t)\right)$, we obtain a system of equations for $\alpha_k^1(y, \tau)$:

$$\frac{\partial}{\partial \tau} \alpha_k^1(y, \tau) - \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j} \alpha_k^1(y, \tau) = 0,$$

$$\alpha_k^1|_{\tau=0} = -\chi_k^1(y).$$

The solutions $\alpha_k^1(y, \tau)$ are continuous as functions $[0, \infty) \to \mathring{H}^1(\Pi^n)$ and satisfy the estimate

$$\|\alpha_k^1\|_{\mathring{H}^1(\Pi^n)} \leqslant ce^{-\lambda_1 \tau}, \qquad (5)$$

where λ_1 is the first nonzero eigenvalue of A_2. For $\varphi_k^1(x, t)$ we set the initial condition

$$\varphi_k^1(x, t)|_{t=0} = \frac{\partial}{\partial x_k} f_1(x).$$

In equation (4.0) we take the mean of both sides with respect to y:

$$\frac{d}{d\tau} M\{z_2\} = \left(M\left\{a_{ij}(y) \frac{\partial}{\partial y_j} \alpha_k^1(y, \tau)\right\}\right) \frac{\partial}{\partial x_i} \varphi_k^1(x, t). \qquad (6)$$

From (6) we determine $M\{z_2\}$. In view of (5)

$$M\{z_2\} = (\beta_{km}^2 + \gamma_{km}^2(\tau)) \frac{\partial}{\partial x_m} \varphi_k^1(x, t),$$

where β_{km}^2 are constants, $\gamma_{km}^2(\tau)$ decay exponentially:

$$|\gamma_{km}^2(\tau)| < ce^{-\lambda_1 \tau}.$$

In order to find the oscillating part of z_2 we examine the equation

$$A_2\alpha_{km}^2(y, \tau) = \left(a_{mj}(y) \frac{\partial}{\partial y_j} \alpha_k^1(y, \tau) - M\left\{a_{mj}(y) \frac{\partial}{\partial y_j} \alpha_k^1(y, \tau)\right\}\right) + \frac{\partial}{\partial y_j} a_{mj}(y) \alpha_k^1(y, \tau), \qquad (7)$$

$$\alpha_{km}^2(y, \tau)|_{\tau=0} = -\chi_{km}^2(y).$$

Lemma 1. The solution of (7) is continuous as a function $[0, \infty) \to L^2(\Pi^n)$ and satisfies the estimate

$$\|\alpha_{km}^2(y, \tau)\|_{L^2(\Pi^n)} \leqslant ce^{-\frac{\lambda_1}{2}\tau}, \quad \|\alpha_{km}^2(y, \tau)\|_{L^2(\tau, \infty; \mathring{H}^1(\Pi^n))} \leqslant ce^{-\frac{\lambda_1}{2}\tau}.$$

Proof. Since the operator A_2 does not depend on τ, there exists a basis in $H^1(\Pi^n)$ of eigenfunctions of A_2. Expanding the solution in a series in this basis we obtain the needed estimate from the exponential decay of the right side of (7).

We proceed to equation (4.1):

$$A_2 z_3(x, y, t, \tau) = -A_1 z_2(x, y, t, \tau) - A_0 z_1(x, y, t, \tau).$$

The term

$$\frac{\partial}{\partial y_i} a_{ij}(y) \beta^2_{km} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_m} \varphi^1_k(x, t)$$

on the right side of this equation does not depend on τ, and the remaining terms decay exponentially; therefore

$$z_3 = (\theta^3_{krm}(y) + \beta^3_{krm} + \gamma^3_{krm}(\tau) + \alpha^3_{krm}(y, \tau)) \frac{\partial}{\partial x_r} \frac{\partial}{\partial x_m} \varphi^1_k(x, t) + g^3(x, t),$$

where $\theta^3_{krm}(y) = \beta^2_{kr}\chi_m(y)$, and $\alpha^3_{krm}(y, \tau)$, β^3_{krm} and $\gamma^3_{krm}(\tau)$ we seek as in (4.0). The function $g^3(x,t)$ of class $S_T(R^n)$ will be found later.

In (4.2) we select the terms on the right side which do not decay exponentially in τ and which have nonzero mean with respect to y:

$$-\beta^2_{kr} \frac{\partial}{\partial x_r} \left(\frac{\partial}{\partial t} \varphi^1_k(x, t) - a_{ij}(y) \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_j} \varphi^1_k(x, t) + \right.$$
$$\left. + a_{ij}(y) \frac{\partial}{\partial y_j} \chi_m(y) \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_m} \varphi^1_k(x, t) \right).$$

For the solvability of (4.2) it is necessary that the mean with repsect to y of this expression be equal to zero. Taking into account that $\varphi^1_k(x; t)|_{|x|=\infty} = 0$, we obtain

$$\frac{\partial}{\partial t} \varphi^1_k(x, t) - Q\varphi^1_k(x, t) = 0,$$
$$\varphi^1_k(x, t)|_{t=0} = \frac{\partial}{\partial x_k} f_1(x).$$

In the sum $z_4(x, y, t, \tau)$ may be written in the same form as z_3.

On a k-dimensional sphere the mean with respect to y in the right side of (4.k) is

$$\frac{\partial}{\partial t} g^{k-2}(x, t) - Qg^{k-2}(x, t) + v^k(x, t),$$

where $v^k(x, t)$ is a known function of class $S_T(R^n)$. The condition of solvability of (4.k) gives an equation for $g^{k-2}(x, t)$:

$$\frac{\partial}{\partial t} g^{k-2}(x, t) - Qg^{k-2}(x, t) = -v^k(x, t),$$
$$g^{k-2}(x, t)|_{t=0} = 0.$$

Continuing the process we obtain a solution to the system (4). The theorem is shown.

From this theorem it is easy to obtain the estimates

$$\|\bar{r}_\varepsilon^k(x, t)\|_\infty \leqslant c\varepsilon^k, \quad \|\bar{r}_\varepsilon^k\|_2 \leqslant c\varepsilon^k.$$

3. The General Case. In this section the coefficients a_{ij} depend on t. We construct an asymptotic expansion as in the preceding section.

Lemma 2. The equation

$$\frac{\partial}{\partial\tau}\chi_k(y, \tau) - \frac{\partial}{\partial y_i} a_{ij}(y, \tau)\frac{\partial}{\partial y_j}\chi_k(y, \tau) = \frac{\partial}{\partial y_i} a_{ik}(y, \tau)$$

has a unique Π^{n+1}-periodic solution which lies in $L^2(0, 1; \mathring{H}^1(\Pi^n))$.

The proof of this lemma is given in [4, ch. 3].

Lemma 3. Consider the Cauchy problem

$$\frac{\partial}{\partial\tau}\alpha(y, \tau) - \frac{\partial}{\partial y_i} a_{ij}(y, \tau)\frac{\partial}{\partial y_j}\alpha(y, \tau) = g(y, \tau),$$
$$\alpha(y, \tau)|_{\tau=0} = g_1(y).$$

Suppose $g_1(y) \in \mathring{L}^2(\Pi^n)$, and suppose for $g(y, \tau)$ the estimate

$$\|g(y, \tau)\|_{L^2(\tau, \infty; \mathring{H}^{-1}(\Pi^n))} \leqslant ce^{-\lambda\tau}$$

is true. Then there exists a $\lambda_0 > 0$, such that

$$\|\alpha(y, \tau)\|_{L^2(\Pi^n)} \leqslant c_3e^{-\lambda_0\tau}, \quad \|\alpha(y, \tau)\|_{L^2(\tau, \infty; \mathring{H}^1(\Pi^n))} \leqslant c_3e^{-\lambda_0\tau}.$$

Proof. From [4] the solution $\alpha(y, \tau)$ is a continuous function $[0, \infty) \to L^2(\Pi^n)$. In view of the well-known relation $\|\alpha(y)\|^2_{L^2(\Pi^n)} \leqslant \delta_0 \|\alpha(y)\|^2_{\mathring{H}^1(\Pi^n)}$ it is sufficient to show the following statement: the solution $f^2(\tau)$ of the equation

$$\int_\tau^\infty \delta(t) f^2(t)\, dt - f^2(\tau) = \varphi(\tau) \tag{8}$$

decays exponentially as $\tau \to \infty$, if

$$\delta(\tau) \geqslant \delta_0 > 0; \ |\varphi(\tau)| < ce^{-\lambda\tau}, \qquad \lambda > 0;$$

$f^2(\tau)$ and $\varphi(\tau)$ are continuous.

We shall show that for λ_0 it is possible to take the number

$$\lambda_0 = \min\left(\frac{\lambda}{2}, \ \ln\left(1 + \frac{\delta_0}{2}\right)\right).$$

For this we shall prove the inequality $f^2(\tau) \leqslant c_4e^{-\lambda_0\tau}$. Pick τ_0 such that for every $\tau > \tau_0 - 1$ the inequality

$$|\varphi(\tau)| \leqslant ce^{-\lambda\tau} \leqslant \frac{1}{10} e^{-\lambda} e^{-\lambda_0\tau} \tag{9}$$

holds. By the continuity of $c_4 > 1$, there exists a $f^2(\tau)$ such that $|f^2(\tau)| < c_4e^{-\lambda_0\tau}$ for $\tau < \tau_0$. We shall show that this inequality is true for all τ. Assume that there exists a $\tau_1 > \tau_0$, for which $f^2(\tau_1) = c_4e^{-\lambda_0\tau_1}$; $f^2(\tau) < c_4e^{-\lambda_0\tau}$ for $\tau < \tau_1$. From (8) it follows that

$$f^2(\tau_1-1)-f^2(\tau_1)=\int\limits_{\tau_1-1}^{\tau_1}\delta(t)f^2(t)\,dt+\varphi(\tau_1)-\varphi(\tau_1-1).$$

But $f^2(\tau)$ is the sum of monotonically decreasing functions and of functions not greater than $ce^{-\lambda\tau}$; therefore

$$f^2(\tau_1-1)-f^2(\tau_1)\geqslant\delta_0 f^2(\tau_1)-2(\delta_0+1)\,e^{-\lambda(\tau_1-1)}.$$

Taking advantage of (9) we obtain

$$f^2(\tau_1-1)-f^2(\tau_1)\geqslant\delta_0 f^2(\tau_1)-2(\delta_0+1)\frac{1}{10}e^{-\lambda_0\tau_1}.$$

In view of the choice of τ_1,

$$f^2(\tau_1-1)>(1+\delta_0)f^2(\tau_1)-\frac{1}{5}(1+\delta_0)f^2(\tau_1).$$

Assuming $\delta_0\geqslant 1$, we have

$$f^2(\tau_1-1)>\left(1+\frac{3}{5}\delta_0\right)f^2(\tau_1)>\left(1+\frac{\delta_0}{2}\right)f^2(\tau_1),$$

which contradicts the choice of τ_1. The lemma is shown.

Theorem 4. For the remaining terms of the expansion the following estimates holds:

$$\|r_\varepsilon^k(x,t)\|_2<c\varepsilon^k,\quad \|\bar{r}_\varepsilon^k(x,t)\|_\infty<c\varepsilon^k.$$

If $a_{ij}(y,\tau)$ are smooth functions, then

$$\|\bar{r}_\varepsilon^k\|_{C^m}\leqslant c\varepsilon^{k-2m}.$$

Remark. Analogous results may be obtained for the equation

$$\frac{\partial}{\partial t}u_\varepsilon(x,t)-\frac{\partial}{\partial x_i}a_{ij}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\frac{\partial}{\partial x_j}u_\varepsilon(x,t)+c\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)u_\varepsilon(x,t)=f(x,t),$$

$$u_\varepsilon|_{t=0}=f_1(x),$$

where the functions $c(y,\tau)$ is bounded and Π^{n+1}-periodic, and the a_{ij} are the same as in (1).

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