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AVERAGING ON A BACKGROUND OF VANISHING VISCOSITY

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ABSTRACT. Elliptic equations of the form

$$\left(\mu a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\varepsilon^{-1}v_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}\right)u^{\mu,\varepsilon}(x)=0,$$
$$u^{\mu,\varepsilon}|_{\partial\Omega}=\varphi(x)$$

with periodic coefficients are considered; μ and ε are small parameters. For potential fields $v(y)$ and constants $a_{ij}=\delta_{ij}$, the asymptotic behavior as $\mu\to 0$ of the coefficients of the averaged operator (which is customarily also called the effective diffusion) is studied. It is shown that as $\mu\to 0$ the effective diffusion $\sigma(\mu)=\sigma_{ij}(\mu)$ decays exponentially, and the limit $\lim_{\mu\to 0}\mu\ln\sigma(\mu)$ is found.

Sufficient conditions are found for the existence of a limit operator as μ and ε tend to 0 simultaneously. The structure of this operator depends on the symmetry reserve of the coefficients $a_{ij}(y)$ and $v_i(y)$; in particular, it may decompose into independent operators in subspaces of lower dimension.

This paper is devoted to problems of averaging elliptic equations with small diffusion. Problems of this kind are encountered, for example, in investigating the diffusion of charged particles in the field of an attractive lattice.

The corresponding stationary equation has the form

$$\left(\mu\Delta+\varepsilon^{-1}v_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}\right)u^{\mu,\varepsilon}(x)=0; \tag{1}$$

here $v(y)$ is a periodic vector field which can be interpreted as the attractive force of a crystal lattice, and Δ is the Laplace operator.

On the boundary of a domain Ω we prescribe the Dirichlet condition

$$u^{\mu,\varepsilon}|_{\partial\Omega}=\varphi(x) \tag{2}$$

and we shall study the solution of problem (1), (2) for small μ and ε. The picture of the behavior of this solution may be completely different depending on the relation between the parameters μ and ε.

In the case where $\mu\sim 1$ the averaging problem for equation (1), (2) was solved in [1] and [2]. The case $1<\mu<\varepsilon^{-1}$ submits to averaging in a similar way. For constant $\mu\varepsilon$ we arrive at the classical averaging problem [3], [4].

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In this paper we consider the opposite case where $\mu \to 0$ in equation (1), (2) or in the more general equation

$$\mu a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i \partial x_j}u^{\mu,\varepsilon}(x)+\varepsilon^{-1}v_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}u^{\mu,\varepsilon}(x)=0, \qquad u^{\mu,\varepsilon}|_{\partial\Omega}=\varphi(x). \quad (3)$$

With this relation between the parameters the behavior of the solution $u^{\mu,\varepsilon}(x)$ becomes very sensitive to the structure of the vector field $v(y)$.

In §1 we assume that μ is constant. The matrix of the averaged coefficients of problem (3), also called for short the effective diffusion, depends on μ. It is shown that in the potential field $v(y)$ the effective diffusion is exponentially small, and its logarithmic asymptotics is found (for comparison we recall that in a solenoidal field v the effective diffusion is not less than μ [5]). We mention the asymptotics of the effective diffusion was found by another method in [6], but our approach is very graphic and admits generalization to nonpotential fields $v(y)$.

In the subsequent sections we study problem (1), (2) and (3) under the assumption that the parameters μ and ε are connected by the relation $\mu=\varepsilon^{\alpha}$, $\alpha>0$. In this case the solution u^{ε} may not have a limit as $\varepsilon\to 0$ even for smooth coefficients $v_i(y)$ and $a_{ij}(y)$, since in investigating (1)–(3) different additional conditions are imposed on their coefficients. In §§2–5 we consider equations in which the vector field $v(y)$ has a family of periodically distributed, asymptotically stable, attractive fixed points. For small ε the diffusion defined by such an equation is approximately described by a random walk over the periodic lattice formed by the singular points of the field. The transition probabilities of the random walk depend on the symmetry reserve of the coefficients of the equation.

In §§2 and 3 we investigate problems whose coefficients are invariant under any motion of $\mathbf{R}^n$ taking the period cube into itself. §2 is devoted to the proof of a tentative theorem formulated in terms of transition probabilities of the diffusion process. In §3 various sufficient conditions are then given for the applicability of this theorem. The limit operator here is always the Laplace operator.

The object of §4 is the averaging of problems with fewer symmetries. Absence of symmetry in part of the variables leads to degeneration of the limit diffusion in certain directions; the averaged problem therefore splits into independent problems in planes of lower dimension. In the absence of symmetry the limit operator is, as a rule, an operator of first order. The problem on the contact of two periodic media is solved here. We mention that for constant $\mu\varepsilon$ similar problems were considered in [7].

Finally, §5 contains the proofs of various technical results.

§1. Estimates of the effective diffusion

In this section we study the behavior of the effective diffusion in problem (1), (2) as $\mu\to 0$, i.e., the asymptotic behavior of the coefficients of the averaged operator of this problem.

We recall the concept of effective diffusion. Suppose that on $T^n=\mathbf{R}^n/\mathbf{Z}^n$, where $\mathbf{Z}^n$ is the integral lattice, we are given a process $\xi^{\mu}_{t,x}$ governed by the operator

$$A_\mu=\mu\Delta+v(x)\nabla,$$

where $v(x)$ is a potential field, $v(x)=-\nabla U(x)$, with $U(x)\in C^2(T^n)$. The effective diffusion (see [5]) is defined from the relation

$$\frac{1}{\sqrt{t}}\int_{(\xi^\mu_{0,x},\,\xi^\mu_{t,x})} d\overline{x} \sim N(0, \sigma(\mu)), \qquad t \to \infty, \tag{4}$$

where $N(0, \sigma(\mu))$ is a normal random variable with covariance matrix $\sigma(\mu)$ and mean 0, while the integral of the form $d\overline{x} = (dx_1, \dots, dx_n)$ is taken over a smooth path homotopic to the path $(\xi^\mu_{0,x}, \xi^\mu_{t,x}) = (\xi^\mu_{s,x}, 0 \le s \le t)$ with fixed initial and end points. Relation (4) is satisfied uniformly with respect to $x \in T^n$ (see [5]), and hence this relation holds also for the corresponding stationary process ξ^μ_t. Since $A^*_\mu e^{-U(x)/\mu} = 0$, it follows that $p_0(x) = c_\mu e^{-U(x)/\mu}$ is an invariant measure of the Markov process $\xi^\mu_{t,x}$ $(\int_{T^n} p_0\,dx = 1)$. The probability density of transition of the process $\xi^\mu_{t,x}$ is denoted by $p(t, x, y)$.

Together with $\xi^\mu_{t,x}$ we consider the process $\eta^\mu_{t,x}$ corresponding to the operator A_μ in $\mathbf{R}^n$. For it there is also a central limit theorem (see [5])

$$(\eta^\mu_{t,x} - x)/\sqrt{t} \sim N(0, \sigma(\mu)), \qquad t \to \infty,$$

with the same $\sigma(\mu)$; the convergence is uniform with respect to x. For equation (1) this implies that $\sigma(\mu)/2$ is the coefficient matrix of the averaged operator.

Suppose on T^n the potential $U(y)$ has only a finite number of singular points, only one of which is a minimum point. We assume with no loss of generality that $\min U(y) = 0$ and that the minimum point coincides with the origin. We introduce the quantity

$$U_0 = \inf_{\{x(t)\}} \sup_t U(x(t));$$

the infimum here is taken over all continuous curves joining the origin and the end of the coordinate unit vector e_i, and we assume that U_0 does not depend on i, $i = 1, \dots, n$.

We note that the number U_0 has the meaning of a level of "veined percolation" of the potential $U(y)$ (see [8]): this is the minimum value of the constant h for which the set $\{x \in \mathbf{R}^n : U(x) \le h\}$ has an unbounded connected component. The condition of the independence of U_0 and i implies coincidence of the percolation levels in all directions.

THEOREM 1. *Under the above assumptions,*

$$\lim_{\mu\to 0} \mu \ln \sigma(\mu) = -U_0. \tag{5}$$

PROOF. We shall show that the process $\xi^\mu_{t,x}$ possesses uniform mixing properties with respect to μ. We formulate this assertion in the following form. Let

$$\partial f(t, x)/\partial t = A_\mu f(t, x), \qquad f|_{t=0} = f_0(x);$$

$$|f| = \max_{x\in T^n} |f(t, x)|, \qquad \|f\| = \left(\int_{T^n} f^2\,dx\right)^{1/2}.$$

LEMMA 1. *For any $\delta > 0$ and $s > 0$ there exists $c(\delta, s) > 0$ such that, for all $t > T_0 = c(\delta, s)e^{\delta/\mu}$,*

$$|f(t, \cdot) - \overline{f}| \le e^{-s/\mu}\|f_0\|; \tag{6}$$

here $f_0(x)$ *is an arbitrary function of* $L^2(T^n)$, *and*

$$\overline{f} = \int_{T^n} f_0(x) p_0(x)\, dx. \tag{7}$$

The proof will be given in §5.

COROLLARY. *For all* $t > kT_0$, k *a natural number,*

$$|f(t, \cdot) - \overline{f}| \le e^{-ks/\mu} \|f_0\|. \tag{8}$$

For the proof it suffices to use Lemma 1 and the semigroup property of the solution $f(t, x)$.

Without yet specifying the choice of s, we suppose that it is arbitrary but such that $e^{-s/\mu} < 1/2$ for $\mu < 1$; we omit the dependence of $c(\delta, s)$ on s.

We set $t_k = kT_0 = kc(\delta)e^{\delta/\mu}$ and $f_k(x) = f(t_k, x)$. From (8) we then get

$$|f_k - \overline{f}| \le 2^{-k} \|f_0\|. \tag{9}$$

Let $T = NT_0 = Nc(\delta)e^{\delta/\mu}$ where N is an integer. We decompose the trajectories of the stationary process $\{\xi_t^\mu, 0 \le t \le T\}$ into N parts $(\xi_0^\mu, \xi_{T_0}^\mu)$, $(\xi_{T_0}^\mu, \xi_{2T_0}^\mu)$, $\dots$, $(\xi_{(N-1)T_0}^\mu, \xi_{NT_0}^\mu)$. Since the potential $U(x)$ has a unique local minimum on T^n, any set of the form $\{x \in T^n | U(x) < h\}$ is connected. Further, by the choice of U_0 the vector-form $d\overline{x} = (dx_1, \dots, dx_n)$ is exact on the set $\mathscr{O}_0 = \{x \in T^n | U(x) < U_0\}$.

We join the points $\xi_{kT_0}^\mu$ in the set $\mathscr{O}_0$ with the point 0 by a smooth curve lying in $\mathscr{O}_0$; the remaining points $\xi_{kT_0}^\mu$ we join with 0 by the shortest geodesic. We prescribe the order of passing about the closed path $l_k = (0, \xi_{kT_0}^\mu, \xi_{(k+1)T_0}^\mu, 0)$ and set $\eta_k = \int_{l_k} d\overline{x} \in \mathbf{Z}^n$. Then $\int_{(\xi_0^\mu, \xi_T^\mu)} d\overline{x} = \sum_{k=0}^{N-1} \eta_k + \zeta_N$, where $|\zeta_N| < c$, uniformly with respect to N and μ.

The sequence η_k is stationary, since the process ξ_t^μ is stationary. In this case

$$\mathsf{E}\sum_{k=0}^{N-1} \eta_k + \mathsf{E}\zeta_N = N\mathsf{E}\eta_0 + \mathsf{E}\zeta_N = \mathsf{E}\int_{(\xi_0^\mu, \xi_T^\mu)} d\overline{x} = 0$$

(see [5]), whence $\mathsf{E}\eta_k = 0$. We verify that the limit theorem of [9] (Theorem 20.1) is applicable to this sequence:

Suppose the sequence η_k *is stationary* $\mathsf{E}\eta_k = 0$, $\mathsf{E}\eta_k^2 < \infty$, *and there is a nonnegative nonincreasing function* $\varphi(k)$ *such that, for any* k, s, $E_1 \in F(\eta_0, \dots, \eta_s)$, *and* $E_2 \in F(\eta_{s+k}, \eta_{s+k+1}, \dots)$,

$$|\mathbf{P}(E_1 \cap E_2) - \mathbf{P}(E_1)\mathbf{P}(E_2)| \le \varphi(k)\mathbf{P}(E_1), \qquad \sum_{k=1}^{\infty} \varphi^{1/2}(k) < \infty. \tag{10}$$

Then the family of processes $(\sqrt{\varepsilon}\sum_{k=1}^{[t/\varepsilon]} \eta_k)$ *converges weakly as* $\varepsilon \to 0$ *to a Wiener process with covariance*

$$\sigma(\mu) = \mathsf{E}\eta_0 \otimes \eta_0 + \sum_{k=1}^{\infty} \mathsf{E}(\eta_0 \otimes \eta_k + \eta_k \otimes \eta_0), \qquad x \otimes y = \{(x_i y_i)\}.$$

We shall show that relation (10) with mixing coefficient $\varphi(k)$ equal to $1/2^{k-1}$ is satisfied for the sequence η_k. We denote by $F_{a,b}$ the σ-algebra generated by the random variables $(\xi_t^\mu, a \le t \le b)$. Then η_k is measurable relative to $F_{kT_0, (k+1)T_0}$. If χ_A is the characteristic function of the set A, then

$$\begin{aligned}|\mathbf{P}(E_1 \cap E_2) - \mathbf{P}(E_1)\mathbf{P}(E_2)| &= |\mathsf{E}\chi_{E_1}\chi_{E_2} - \mathsf{E}\chi_{E_1}\mathsf{E}\chi_{E_2}| \\ &= |\mathsf{E}(\chi_{E_1}\mathsf{E}(\chi_{E_2} - \mathsf{E}\chi_{E_2}|F_{0,(s+1)T_0}))|.\end{aligned}$$

We set $f_0(\xi^\mu_{(s+k)T_0}) = \mathsf{E}(\chi_{E_2} - \mathsf{E}\chi_{E_2}|\xi^\mu_{(s+k)T_0})$. Then

$$\int_{T^n} f_0(x)p_0(x)\,dx = \mathsf{E}f_0(\xi^\mu_{(s+k)T_0}) = 0$$

and, by (9),

$$\begin{aligned}|\mathbf{P}(E_1 \cap E_2) - \mathbf{P}(E_1)\mathbf{P}(E_2)| &= |\mathsf{E}(\chi_{E_1}\mathsf{E}(f_0(\xi^\mu_{(k+s)T_0})|F_{0,(s+1)T_0}))| \\ &\le \mathsf{E}(\chi_{E_1}|\mathsf{E}(f_0(\xi^\mu_{(s+k)T_0})|F_{0,(s+1)T_0})|) \\ &\le \mathbf{P}(E_1)|f_{k-1}| \le 2^{1-k}\mathbf{P}(E_1)\|f_0\| \le 2^{1-k}\mathbf{P}(E_1),\end{aligned}$$

where we have used the fact that $|f_0| \le 1$. We shall verify the finiteness of the covariance matrix $\mathsf{E}\eta_0 \otimes \eta_0$ and find its logarithmic asymptotics as $\mu \to 0$. We consider the process $\eta_{t,x}$ in $\mathbf{R}^n$ corresponding to the operator A_μ. For each $z \in \mathbf{Z}^n$ we construct the connected component of the set $\{x \in \mathbf{R}^n | U(x) < U_0\}$ containing z, which we denote by $\mathscr{O}_0^z$. From the conditions of the theorem it follows that for any basis of integral vectors $z_1, \dots, z_n$ we have the relations $\overline{\mathscr{O}_0^0} \cap \overline{\mathscr{O}_0^{z_i}} \ne \varnothing$, $i = 1, \dots, n$. From this, the uniqueness of the local minimum, and the finiteness of the number of singular points on a period it follows (see [10], Chapter 4, Theorem 3.1 and 1.2) that for any $\delta_1 > 0$ there exists $T > 0$ such that

$$\mathbf{P}\{|\eta_{T,x} - z_i| < \kappa\} \ge e^{-(U_0+\delta_1)/\mu}, \qquad i = 1, \dots, n, \tag{11}$$

for all $x \in \mathscr{O}_0^0$; κ is an arbitrary fixed constant. It will be proved below that, with a suitable choice of κ, for all $t \le T_0$ uniformly with respect to x in $\{x \in \mathbf{R}^n | |x-z| < \kappa\}$ the quantity $\mathbf{P}\{\eta_{t,x} \in \mathscr{O}_0^z\} \to 1$ as $\mu \to 0$. By the Markov property of $\eta_{t,x}$ it follows from the last relation and (11) that

$$\mathbf{P}\{\eta_0 = z_i\} \ge ce^{-(U_0+\delta_1)/\mu}, \qquad i = 1, \dots, n. \tag{12}$$

To prove a similar upper bound in the region $\mathscr{O}_\delta$, which is the connected component of the set $\{x \in \mathbf{R}^n | U(x) < U_0 - \delta\}$ containing 0, we consider all singular points of the potential: $0 = x_0, x_1, \dots, x_k$ such that $0 = U(x_0) < U(x_1) \le \dots \le U(x_k) < U_0 - \delta$. This collection of points is finite because of the compactness of $\mathscr{O}_\delta$ in $\mathbf{R}^n$ and the assumption on the finiteness of the number of singular points on a period. Let

$$B_\delta^j = \{x \in \mathscr{O}_\delta | |x - x_j| \le \delta\}, \qquad B_\delta = \bigcup_{j=0}^k B_\delta^j.$$

For sufficiently small δ the sets $B_{4\delta}$ and $\mathscr{O}_\delta$ do not intersect. We introduce the

Markov times

$$
\begin{gathered}
\tau_1 = \inf\{t \mid \eta_{t,x} \notin \mathscr{O}_\delta \backslash B_{\delta_2}\}, \qquad \delta_2 < \delta/2, \\
\nu_1 = \inf\{t > \tau_1 \mid \eta_{t,x} \notin B_\delta\}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\tau_k = \inf\{t > \nu_{k-1} \mid \eta_{t,x} \notin \mathscr{O}_\delta \backslash B_{\delta_2}\}, \\
\nu_k = \inf\{t > \tau_k \mid \eta_{t,x} \notin B_\delta\}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{gathered}
$$

From standard estimates of the fundamental solution [11] it follows that for any $s > 0$ we can find $c(\delta)$, not depending on μ, such that for all x in ∂B_δ

$$
\mathbf{P}\{\tau_1 < c(\delta)\} < e^{-s/\mu}. \tag{13}
$$

We shall prove the estimate

$$
\mathbf{P}\{\eta_{T_0,x} \notin \mathscr{O}_\delta\} \le c \exp\left(-\frac{U_0 - U(x) - \delta_1}{\mu}\right),
$$

where $\delta_1 \to 0$ as $\delta \to 0$ and $x \in \mathscr{O}_\delta$. We denote $\bigcup_{i=0}^{j} B_\delta^i$ by H_δ^j. According to [10], Chapter 6, Lemma 2.1, uniformly with respect to $x \in \partial H_\delta^j$ we have

$$
\mathbf{P}\{\eta_{\tau_1,x} \in \partial B_{\delta_2}^l\} \le c \exp\left(-\frac{U(x_l) - U(x_j) - \delta_1}{\mu}\right), \tag{14}
$$

$$
\mathbf{P}\{\eta_{\tau_1,x} \in \partial \mathscr{O}_\delta\} \le c \exp\left(-\frac{U_0 - U(x_j) - \delta_1}{\mu}\right), \tag{15}
$$

where $l > j$ and $\delta_1 \to 0$ and $\delta \to 0$. For $l > j$ and $x \in \partial H_\delta^j$ we define the quantities

$$
\begin{gathered}
p^{j,l}(x) = \mathbf{P}(\{\tau_j^0 < T_0\} \cap \{\eta_{\tau_j^0,x} \in \partial B_{\delta_2}^l\}), \\
p^{j,k+1}(x) = \mathbf{P}(\{\tau_j^0 < T_0\} \cap \{\eta_{\tau_j^0,x} \in \partial \mathscr{O}_\delta\});
\end{gathered}
$$

here τ_j^0 is the Markov time when the process $\eta_{t,x}$ first reaches the set $\partial\mathscr{O}_\delta \cup \bigcup_{i=j+1}^{k} \partial B_{\delta_2}^i$. From (13) for $s = 2U_0$ and the strong Markov property of $\eta_{t,x}$ we have

$$
\begin{aligned}
p^{j,l}(x) \le \sum_{i=1}^{T_0/c(\delta)} &\left(\mathbf{P}(\{\eta_{\tau_i,x} \in B_{\delta_2}^l\}) \cap \left\{\bigcap_{m=1}^{i-1} (\eta_{\tau_m,x} \in \partial H_{\delta_2}^j)\right\}\right) \\
&+ \mathbf{P}\left(\left\{\bigcap_{m=1}^{i-1} (\eta_{\tau_m,x} \in \partial H_{\delta_2}^j)\right\} \cap \{\tau_i - \nu_{i-1} < c(\delta)\}\right) \\
\le \sum_{i=1}^{T_0/c(\delta)} &\mathbf{P}\left(\{\eta_{\tau_1,\eta_{\nu_{i-1},x}} \in B_{\delta_2}^l\} \cap \left\{\bigcap_{m=1}^{i-1} (\eta_{\tau_m,x} \in \partial H_{\delta_2}^j)\right\}\right) + \frac{T_0}{c(\delta)} e^{-2U_0/\mu}.
\end{aligned}
$$

From this with the help of (14) we obtain

$$p^{j,l}(x) \leq \frac{T_0}{c(\delta)} \exp\left(-\frac{U(x_l) - U(x_j) - \delta_1}{\mu}\right) + e^{-U_0/\mu}$$
$$\leq c \exp\left(-\frac{U(x_l) - U(x_j) - \delta_1 - \delta}{\mu}\right).$$

Similarly,

$$p^{j,k+1}(x) \leq c \exp\left(-\frac{U_0 - U(x_j) - \delta_1 - \delta}{\mu}\right).$$

Further, by the strong Markov property of $\eta_{t,x}$ for $x \in B_\delta^j$ we have

$$\mathbf{P}\{\eta_{T_0,x} \in \mathscr{O}_\delta\} \leq \sum_{j=i_0<i_1<\cdots<i_{s+1}=k+1} \prod_{m=0}^{s} p^{i_m, i_{m+1}}(x)$$
$$\leq c \exp\left(-\frac{U_0 - U(x_j) - (k+1)\delta - (k+1)\delta_1}{\mu}\right).$$

In exactly the same way for an arbitrary $x \in \mathscr{O}_\delta$

$$\mathbf{P}\{\eta_{T_0,x} \notin \mathscr{O}_\delta\} \leq c \exp\left(-\frac{U_0 - U(x) - (k+1)\delta - (k+1)\delta_1}{\mu}\right). \qquad (16)$$

Considering now that ξ_t^μ is distributed over a period with density $c_\mu e^{-U(x)/\mu}$ and integrating (16) with this density, we obtain

$$\mathbf{P}\{\eta_0 \neq 0\} \leq \int_{\mathscr{O}_\delta} p_0(x)\mathbf{P}\{\eta_{T_0,x} \notin \mathscr{O}_\delta\}\, dx + ce^{-(U_0-\delta)/\mu} \leq ce^{-(U_0-\delta_1)/\mu}, \qquad (17)$$

where $\delta_1 \to 0$ as $\delta \to 0$.

We now prove the matrix inequality

$$c_1 e^{-(U_0+\delta_1)/\mu} \leq \mathsf{E}\eta_0 \otimes \eta_0 \leq c_2 e^{-(U_0-\delta_1)/\mu}. \qquad (18)$$

The lower bound obviously follows from (12) and the fact that η_0 is an integer. Further, according to [10], there exists $L > 0$ such that for any $x \in [-1/2, 1/2]^n$

$$\mathbf{P}\{|\eta_{1,x}| \geq kL\} < e^{-2kU_0/\mu}. \qquad (19)$$

With the help of the Markov property of $\eta_{t,x}$, from this we obtain

$$\mathbf{P}\{|\eta_{T_0,x}| \geq kT_0L\} \leq T_0 e^{-2kU_0/\mu} \leq c(e^{-(2U_0-\delta_1)/\mu})^k,$$

which together with (17) gives

$$\mathsf{E}\eta_0 \otimes \eta_0 \leq T_0^2 L^2 e^{-(U_0-\delta_1)/\mu} + \sum_{m=1}^{\infty} cm^2 T_0^2 L^2 (e^{-(2U_0-\delta_1)/\mu})^m \leq ce^{-(U_0-\delta_1)/\mu}.$$

Finally, the estimate $\sigma(\mu) < ce^{-(U_0-\delta_1)/\mu}$ follows from (18) and the inequality ([9], §20, Lemma 1)

$$\mathsf{E}\eta_0 \otimes \eta_k \leq 2(\varphi(k))^{1/2}\mathsf{E}\eta_0 \otimes \eta_0.$$

The lower bound for $\sigma(\mu)$ requires more delicate arguments. In the conditions of Lemma 1 we choose $s \geq 2U_0$. For the sequence η_k constructed on the basis of the step size $T_0 = c(\delta)e^{\delta/\mu}$ we then have φ-mixing with $\varphi = (e^{-2U_0/\mu})^{k-1}$. We now specify (12): for this choice of T_0

$$\mathbf{P}\{\eta_0 = z_i\} \geq c \exp\left(-\frac{U_0 + \delta_1 - \delta}{\mu}\right), \qquad i = 1, \ldots, n, \tag{20}$$

where δ_1 may be chosen arbitrarily small. Indeed,

$$\begin{aligned}\mathbf{P}\{\eta_0 = z_i\} &\geq \sum_{j=1}^{T_0/T} \mathbf{P}\left(\bigcup_{l=1}^{j-1}\{\eta_{lT,\cdot} \in \mathscr{O}_0^0\} \cap \{|\eta_{jT,\cdot} - z_i| < \kappa\} \cap \bigcap_{l=j+1}^{T_0/T}\{\eta_{lT,\cdot} \in \mathscr{O}_0^{z_i}\}\right) \\ &\geq \sum_{j=1}^{T_0/T} c e^{-(U_0+\delta_1)/\mu} \geq c e^{\delta/\mu} e^{-(U_0+\delta_1)/\mu} = c \exp\left(-\frac{U_0 + \delta_1 - \delta}{\mu}\right);\end{aligned}$$

here T and κ are taken from (11), and (17) and the Markov property of $\eta_{t,x}$ are also used.

In place of η_i we now consider the quantities $\overline{\eta}_i$ corresponding to the time $\overline{T}_0 = c(\delta)e^{\Xi\delta/\mu}$, $\Xi > 1$. By construction $\overline{\eta}_0 = \sum_{i=0}^{\overline{T}_0/T_0-1} \eta_i$. As noted above, φ-mixing with the same $\varphi(k)$ remains in force for $\overline{\eta}_i$, while the coefficients of the limit distributions of the sums η_i and $\overline{\eta}_i$ are obviously connected by the relation $\overline{\sigma}(\mu) = \sqrt{\overline{T}_0/T_0}\sigma(\mu)$; it therefore suffices to obtain a lower bound for $\overline{\eta}_i$. Formula (20) for $\overline{\eta}_0$ takes the following form:

$$\mathbf{P}\{\overline{\eta}_0 = z_i\} \geq c \exp\left(-\frac{U_0 + \delta_1 - \Xi\delta}{\mu}\right), \qquad i = 1, \ldots, n,$$

and hence for an appropriate choice of Ξ

$$\mathsf{E}(\overline{\eta}_0 \otimes \overline{\eta}_0) \geq 8\mathsf{E}(\eta_0 \otimes \eta_0). \tag{21}$$

Further,

$$\begin{aligned}\mathsf{E}(\overline{\eta}_0 \otimes \overline{\eta}_1) &= \mathsf{E}\left(\sum_{i=0}^{\overline{T}_0/T_0-1} \eta_i \otimes \sum_{i=0}^{\overline{T}_0/T_0-1} \eta_{i+\overline{T}_0/T_0}\right) \\ &= \mathsf{E}(\eta_{\overline{T}_0/T_0} \otimes \eta_{\overline{T}_0/T_0-1}) + \sum_{j\geq i+2} \mathsf{E}(\eta_i \otimes \eta_j).\end{aligned}$$

On the right side of the last equality each term of the second sum can be estimated by the quantity $2\varphi^{1/2}(2)\mathsf{E}(\eta_0 \otimes \eta_0)$, while the first term satisfies $\mathsf{E}(\eta_{\overline{T}_0/T_0} \otimes \eta_{\overline{T}_0/T_0-1}) \leq \mathsf{E}(\eta_0 \otimes \eta_0)$, which by virtue of the choice of φ gives $\mathsf{E}(\overline{\eta}_0 \otimes \overline{\eta}_1) \leq 2\mathsf{E}(\eta_0 \otimes \eta_0)$. Combining this obtained with (21) and considering the φ-mixing of $\overline{\eta}_i$, we obtain

$$\mathsf{E}(\overline{\eta}_0 \otimes \overline{\eta}_0) + \sum_{k=1}^{\infty} \mathsf{E}(\overline{\eta}_0 \otimes \overline{\eta}_k + \overline{\eta}_k \otimes \overline{\eta}_0) \geq \mathsf{E}\eta_0 \otimes \eta_0,$$

which completes the proof.

To conclude the section we present an answer in the nonisotropic case, i.e., we suppose that under the conditions of Theorem 1 the quantity $\inf_{\{x(t)\}} \sup_t U(x(t))$

depends on the point of $\mathbf{Z}^n$ to which the path $x(t)$ joins the origin. Let

$$U_0^1 = \min_{i\in\mathbf{Z}^n\setminus\{0\}} \inf_{\{x(t),\, x(0)=0,\, x(1)=i\}} \sup_t U(x(t)) \tag{22}$$

and let z_1 be a unit vector collinear with the vector i_1 delivering the minimum of (22). We denote by $\{z_1, \dots, z_k\}$ the linear hull of $z_1, \dots, z_k$ and construct

$$U_0^2 = \min_{i\in\mathbf{Z}^n\setminus\{0\},\, i\notin\{z_1\}} \inf_{\{x(t),\, x(0)=0,\, x(1)=i\}} \sup_t U(x(t)) \tag{23}$$

and we denote by z_2 a unit vector orthogonal to z_1 and lying in the plane $\{z_1, i_2\}$ formed by z_1 and the vector i_2 realizing the minimum (23). Continuing this procedure, at the kth step we find

$$U_0^k = \min_{i\in\mathbf{Z}^n\setminus\{0\},\, i\notin\{z_1,\dots,z_{k-1}\}} \inf_{\{x(t),\, x(0)=0,\, x(1)=i\}} \sup_t U(x(t))$$

and we construct a unit vector z_k orthogonal to the subspace $\{z_1, \dots, z_{k-1}\}$ and lying in $\{z_1, \dots, z_{k-1}, i_k\}$

Let $A(\overline{U}_0)$ be a matrix having diagonal form in the coordinates $z_1, \dots, z_n$ with coefficients $U_0^1, \dots, U_0^n$. In this case a minor modification of the proof of Theorem 1 enables us to obtain the following assertion.

THEOREM 2. *For the covariance matrix $\sigma(\mu)$ the following asymptotics holds*:

$$\lim_{\mu\to 0} \mu \ln \sigma(\mu) = -A(\overline{U}_0).$$

§2. Averaging of problems with symmetric coefficients

In this and the following sections we shall assume that the coefficients of equation (3) are infinitely differentiable and that the matrix $a_{ij}(y)$ is uniformly elliptic:

$$\lambda_0|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \lambda_1|\xi|^2, \qquad 0<\lambda_0<\lambda_1,\ \xi\in\mathbf{R}^n.$$

The conditions most used below we formulate as individual conditions.

I. The coefficients $a_{ij}(y)$ and $v_i(y)$ are periodic in all variables with unit period.

II. The vector field $v(y)$ has asymptotically stable, attractive singular points at the vertices of the integral lattice.

III. For any isometry S ($SS^* = I$, $\det S = 1$) of the space $\mathbf{R}^n$ taking the cube of periods $I^n = (-1/2, 1/2)^n$ into itself

$$a_{ij}(Sy) = Sa_{ij}(y)S^*, \qquad v(Sy) = Sv(y).$$

Condition III implies symmetry of the equation relative to motions preserving the cube of periods.

In this section we shall consider equations of the form (3), assuming that $\mu = \varepsilon^\alpha$ with $\alpha > 0$, and that conditions I–III hold. In order to formulate still another condition we construct the diffusion process ξ_t^ε corresponding to the operator

$$B^\varepsilon = \varepsilon^\alpha a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + v_i(y)\frac{\partial}{\partial y_i}.$$

We denote by K_r^R the ball of radius R with center at the origin from which all balls of radius r with centers at the remaining points of $\mathbf{Z}^n$ contained entirely within it have been removed. Let $p_r^R(y, \varepsilon)$ be the probability that the process issuing from y first reaches the boundary of K_r^R at a point not lying on the sphere $\{y\in\mathbf{R}^n \mid |y| = R\}$. We set $\overline{p}_r^R(\varepsilon) = \min_{y\in I^n} p_r^R(y, \varepsilon)$.

THEOREM 3. *Suppose conditions* I–III *are satisfied, and suppose there exist positive constants,* r, R, *and* β *such that the closed ball of radius* r *is attracted by the field* $v(y)$ *toward the origin, and* $\overline{p}_r^R(\varepsilon) > \beta$ *for all* $\varepsilon > 0$. *Then* $u^\varepsilon(x)$ *converges as* $\varepsilon \to 0$ *to a function* $u^0(x)$ *harmonic in* Ω *with the boundary condition* $\varphi(x)$.

REMARK. The condition $\overline{p}_r^R(\varepsilon) > \beta$ has conditional character. Essentially, this condition forbids mixing of a diffusion particle at large distances without seizure by one of the centers of attraction. Thus, it cannot be satisfied if $v(y)$ has unbounded integral curves on $\mathbf{R}^n$. For a potential periodic field this is usually impossible for topological reasons, and it will be shown below that in the case of a potential field $v(y)$ the conditions of the theorem are satisfied for any potential having on a period a finite number of singular points of which only one is a minimum point.

In the proof of the theorem an important role is played by the following fact.

LEMMA 2. *Suppose that in the domain* $\Omega \subset \mathbf{R}^n$

$$\varepsilon a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} z^\varepsilon(y) + v_i(y)\frac{\partial}{\partial y_i} z^\varepsilon(y) = 0, \tag{24}$$

where $c|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le c^{-1}|\xi|^2$, *and suppose the vector field* $v(y)$ *has the asymptotically stable singular point* 0. *Then for any compact set* $K \subset \Omega$ *which is attracted by the field* $v(y)$ *toward the point* 0 *there exist constants* c_1 *and* c_2, *not depending on* ε, *such that*

$$\max_{y\in K} z^\varepsilon(y) - \min_{y\in K} z^\varepsilon(y) \le c_1 \sup_{\Omega} |z^\varepsilon(y)| e^{-c_2/\varepsilon}. \tag{25}$$

The proof will be given in §5.

Using the strong Markov property of $\xi^\varepsilon_{t,y}$, we can easily verify

PROPOSITION 1. *Under the conditions of Theorem* 3,

$$1 - \overline{p}_r^R(\varepsilon) \le c_0 e^{-cR}, \qquad c > 0.$$

We introduce the following notation: $Q_r^j = \{x \in \mathbf{R}^n \mid |x - j| \le r\}$, $j \in \mathbf{Z}^n$;

$$L_r^j = \bigcup_{i\ne j} Q_r^t, \qquad \tau_0 = \inf(t|\xi^\varepsilon_{t,x} \in L_r^0), \qquad p^\varepsilon(y, j) = \mathbf{P}(\xi^\varepsilon_{\tau_0, y} \in Q_r^j\}.$$

PROPOSITION 2. *On the set* $\mathbf{R}^n \backslash L_r^0$ *the functions* $p^\varepsilon(y, j)$ *satisfy the equation*

$$B^\varepsilon p^\varepsilon(y, j) = 0 \tag{26}$$

and the following collection of boundary conditions:

$$p^\varepsilon(y, j)\big|_{\partial Q_r^i} = 0, \quad i \ne j; \qquad p^\varepsilon(y, j)\big|_{\partial Q_r^j} = 1. \tag{27}$$

Moreover,

$$\sum_j p^\varepsilon(y, j) \equiv 1. \tag{28}$$

PROOF. Equality (28) follows immediately from Proposition 1. To prove (26), on the expanding family of sets K_r^{NR} we consider the following sequence of problems:

$$B^\varepsilon u^\varepsilon_{N,j} = 0, \qquad u^\varepsilon_{N,j}\big|_{\partial Q_r^i} = 0, \quad i \ne j;$$

$$u^\varepsilon_{N,j}\big|_{\partial Q_r^j} = 1, \qquad u^\varepsilon_{N,j}\big|_{|y|=NR} = 0.$$

The solution of each of them determines the probability that the point of first exit of the process $\xi^{\varepsilon}_{t,y}$ onto the boundary of K^{NR}_r lies on Q^j_r. It is obvious that the sequence $u^{\varepsilon}_{N,j}$ is monotone increasing in N and is bounded by one; therefore the limit $u^{\varepsilon}_j = \lim_{N\to\infty} u^{\varepsilon}_{N,j}$ exists. The functions $u^{\varepsilon}_j(y)$ are generalized solutions of problem (26), (27), and so by virtue of the smoothness of the coefficients they are classical solutions of it. On the other hand, the event $\{\xi^{\varepsilon}_{\tau_0,y} \in Q^j_r\}$ can be represented in the form $\bigcup_{N=1}^{\infty}\{\xi^{\varepsilon}_{\tau_N,y} \in Q^j_r\}$; here τ_N denotes the Markov time when the process $\xi^{\varepsilon}_{t,y}$ first reaches the boundary of K^{NR}_r. Therefore, $\lim_{N\to\infty} u^{\varepsilon}_{N,j}(y) = p^{\varepsilon}(y, j)$ for each y. Hence $p^{\varepsilon}(y, j) = u^{\varepsilon}_j(y)$, and the proposition is proved.

From Lemma 2 and the last two propositions we obtain

PROPOSITION 3. *In each ball εQ^j_r lying entirely in the region Ω the solution $u^{\varepsilon}(x)$ of problem* (3) *satisfies*

$$\max_{\varepsilon Q^j_r} u^{\varepsilon}(x) - \min_{\varepsilon Q^j_r} u^{\varepsilon}(x) \le \|\varphi\|_{L^{\infty}(\partial\Omega)} e^{-c/\varepsilon^{\alpha}}.$$

PROPOSITION 4. *The following inequalities hold uniformly with respect to $\varepsilon > 0$ in Q^0_r:*

$$\max_{Q^0_r} p^{\varepsilon}(y, j) - \min_{Q^0_r} p^{\varepsilon}(y, j) \le \exp\left(-c_1|j| - \frac{c}{\varepsilon^{\alpha}}\right).$$

Our next step is to pass from equations (3) to a family of difference schemes. For this we note that the functions $u^{\varepsilon}(\varepsilon y)$ in the region $\varepsilon^{-1}\Omega$ satisfy the equation

$$B^{\varepsilon} u^{\varepsilon}(\varepsilon y) = 0, \qquad u^{\varepsilon}(\varepsilon y)\Big|_{\partial(\varepsilon^{-1}\Omega)} = \varphi(\varepsilon y). \tag{29}$$

We consider an arbitrary integral vector j in $\varepsilon^{-1}\Omega$ at a distance of no more than $1/\sqrt{\varepsilon}$ from the boundary $\partial(\varepsilon^{-1}\Omega)$. According to [12], for $u^{\varepsilon}(\varepsilon y)$ there is the representation

$$u^{\varepsilon}(\varepsilon j) = \mathsf{E}(u^{\varepsilon}(\varepsilon \xi^{\varepsilon}_{\eta_j, j})); \tag{30}$$

here η_j is the Markov time that the process $\xi^{\varepsilon}_{t,y}$ first reaches the boundary of the set $\varepsilon^{-1}\Omega\backslash L^j_r$. With the help of Propositions 1, 3, and 4 we can transform (30) as follows:

$$u^{\varepsilon}(\varepsilon j) = \sum_{|i-j|<1\sqrt{\varepsilon}} p^{\varepsilon}(0, i-j) u^{\varepsilon}(\varepsilon i) + O(e^{-c/\varepsilon^{\alpha_1}}), \tag{31}$$

where $\alpha_1 = \min(\alpha, 1/2)$. We now introduce the quantities

$$q^{\varepsilon}(i) = p^{\varepsilon}(0, i)\left(\sum_{|j|<1/\sqrt{\varepsilon}} p^{\varepsilon}(0, j)\right)^{-1}, \quad |i| < \frac{1}{\sqrt{\varepsilon}}; \qquad q^{\varepsilon}(i) = 0, \quad i \ge \frac{1}{\sqrt{\varepsilon}}.$$

Since by Propositions 1 and 2

$$1 - \sum_{|i|<1/\sqrt{\varepsilon}} p^{\varepsilon}(0, i) = O(e^{-c/\sqrt{\varepsilon}}),$$

equality (31) is unchanged if in it we replace $p^{\varepsilon}(0, i)$ by $q^{\varepsilon}(i)$:

$$\mu^{\varepsilon}(\varepsilon j) = \sum_i q^{\varepsilon}(i-j) u^{\varepsilon}(\varepsilon i) + O(e^{-c/\varepsilon^{\alpha_1}}). \tag{32}$$

We note the following properties of the numbers $q^\varepsilon(i)$:

1) $q^\varepsilon(i) \geq 0$, $\sum_i q^\varepsilon(i) = 1$, and $q^\varepsilon(0) = 0$.

2) $q^\varepsilon(i) \leq c_1 e^{-c|i|}$, where c and c_1 do not depend on ε.

3) They are symmetric relative to any isometry preserving the cube of periods.

In particular, from 1) and 2) it follows that, uniformly with respect to $\varepsilon > 0$,

$$0 < \sigma \leq \sum_{i \in \mathbf{Z}^n} |i|^2 q^\varepsilon(i) \leq \sigma^{-1} < \infty. \tag{33}$$

We have thus arrived at a family of difference schemes connecting the quantities $u^\varepsilon(\varepsilon j)$ on the set $\{j \in \varepsilon^{-1}\Omega | \rho(j, \partial(\varepsilon^{-1}\Omega)) \geq 1/\sqrt{\varepsilon}\}$. However, so far we know nothing about the behavior of $u^\varepsilon(\varepsilon j)$ near $\partial(\varepsilon^{-1}\Omega)$. Correct definition of the boundary conditions affords the following assertion, in which $p^\varepsilon(\delta, \Omega, j)$ denotes the probability that ξ_t^ε reaches the boundary $\partial(\varepsilon^{-1}\Omega)$ without leaving a δ/ε-neighborhood of the point j.

PROPOSITION 5. *For any* $\delta > 0$

$$\lim_{\varepsilon \to 0} \min_{\{j \in \mathbf{Z}^n | \rho(j, \partial(\varepsilon^{-1}\Omega)) \leq 1/\sqrt{\varepsilon}\}} p^\varepsilon(\delta, \Omega, j) = 1.$$

For the proof we consider a family of random walks in $\mathbf{R}^n$ depending on ε with independent increments and transition probabilities $p^\varepsilon(0, i)$. From 1)–3) it follows that for this family of random walks normalized in a suitable manner the Lindeberg condition is satisfied, and therefore ([13], Russian p. 525, English p. 446) it converges weakly as $\varepsilon \to 0$ to a Wiener process in $\mathbf{R}^n$. From this property of weak convergence it follows that the limit relation of Proposition 5 is satisfied with the diffusion $\xi_{t,j}^\varepsilon$ in the definition of $p^\varepsilon(\delta, \Omega, j)$ replaced by the random walk.

We further construct the Markov times: $\mu_1^\varepsilon(j)$ is the time when $\xi_{t,j}^\varepsilon$ first reaches any of the balls $Q_r^{i_1}$, $i_1 \neq j$; $\mu_2^\varepsilon(j)$ is the first time after $\mu_1^\varepsilon(j)$ of reaching $Q_r^{i_2}$, $i_2 \neq i_1$; etc. According to Proposition 4, in the Markov chain $\xi^\varepsilon_{\mu_k^\varepsilon(j), j}$ the transition probabilities from Q_r^i to Q_r^l differ from $p^\varepsilon(0, l-i)$ by a quantity of order $O(\exp(-c_1|l-i| - c/\varepsilon^{\alpha_1})$, and hence the proposition follows from the analogous assertion for the random walk constructed above. The proposition is proved.

In a neighborhood of $\partial\Omega$ we now introduce coordinates, the first of which is directed along the normal to $\partial\Omega$, while the remainder are coordinates on the boundary. The image of the point εj in an $\sqrt{\varepsilon}$-neighborhood of $\partial\Omega$ is carried onto $\partial\Omega$ along the first of the coordinates introduced; we denote it by $\overline{x}(\varepsilon j)$.

PROPOSITION 6. *In an* $\sqrt{\varepsilon}$-*neighborhood of* $\partial\Omega$

$$\lim_{\varepsilon \to 0} \max_{\rho(\varepsilon j, \partial\Omega) \leq 1/\sqrt{\varepsilon}} |u^\varepsilon(\varepsilon j) - \varphi(\overline{x}(\varepsilon j))| = 0.$$

PROOF. It suffices to write the solution $u^\varepsilon(x)$ of problem (3) in probabilistic form and then use Proposition 5 and the continuity of the function $\varphi(x)$.

We further consider the auxiliary family of difference schemes

$$\begin{aligned} v_j^\varepsilon &= \sum_i q^\varepsilon(i-j) v_i^\varepsilon, \qquad \rho(\varepsilon j, \partial\Omega) > \sqrt{\varepsilon}; \\ v_j^\varepsilon &= \varphi(\overline{x}(\varepsilon j)), \qquad \rho(\varepsilon j, \partial\Omega) \leq \sqrt{\varepsilon}. \end{aligned} \tag{34}$$

From [14] it follows that a family of difference schemes of the form (34) whose coefficients are subject to conditions 1)–3) is compact, and any limit point of this family is a solution of an elliptic equation of the form $\Delta u^0(x) = 0$, $u^0|_{\partial\Omega} = \varphi(x)$. Therefore, the entire family of grid functions converges as $\varepsilon \to 0$ to a function harmonic in Ω with the boundary condition $\varphi(x)$.

To complete the proof of the theorem it remains to note that the maximum principle is applicable to the difference schemes (34), and to use Proposition 6 and standard estimates for elliptic difference schemes, according to which an exponentially right side in (32) gives an exponentially small correction to the solution of the corresponding difference equations. The theorem is proved.

REMARK. The results of Theorem 3 remain in force for equations of the form (3) in which the vector field $v(y)$ on a period has an attracting limit cycle or an attracting region satisfying certain additional conditions. Work with such equations requires involving the method developed in [10].

§3. Averaging symmetric equations. Realization of the tentative theorem

Verifying the conditions of the theorem of §3 may be very difficult. We shall give some sufficient conditions for its applicability.

Our first result pertains to equations for which the attractive zones of the field $v(y)$ do not abut one another.

THEOREM 4. *Suppose the coefficients of problem* (3) *satisfy conditions* I–III, *and suppose one of the following conditions is satisfied*:

P1. *On I^n the field $v(y)$ has compact support and forms a nonpositive inner product with the radius vector.*

P2. *In I^n the support of $v(y)$ coincides with the closure of some region G with a smooth boundary, all points of G are attracted by the field $v(y)$ to the origin, and in a sufficiently small neighborhood of ∂G the angle between $v(y)$ and the direction of the inner normal does not exceed $\pi/2$.*

Then $u^\varepsilon(x)$ converge as $\varepsilon \to 0$ to a function harmonic in Ω with the boundary condition $\varphi(x)$.

PROOF. We shall verify the conditions of Theorem 3. We choose $r > 0$ such that the closed ball of radius r with center at a point with integral coordinates is attracted by the field $v(y)$ to this point, and we shall verify that $\overline{p}_r^R(\varepsilon)$ can be bounded below, uniformly with respect to $\varepsilon > 0$, by a positive constant if $R > 2\sqrt{n}$.

Since the process $\xi^\varepsilon_{t,y}$ possesses the strong Markov property, in the case P1 it suffices to use the following assertions. First of all, with probability one $\xi^\varepsilon_{t,y}$ reaches ∂I^n for all $y \in I^n$. Suppose, further, that $s = \rho(G, \partial I^n)$. We construct the set $\omega^1 = (-1/2+s, 1/2-s)^n$, $\omega^2 = (-1/2-s, 1/2+s)^n$, $\omega^3 = \{y \in \mathbf{R}^n \,|\, |y| < 1/2 - 2s/3\}$, and $\omega = \omega^2 \backslash \{\omega^1 \cup \omega^3\}$. Let π_1 be the Markov time when the process $\xi^\varepsilon_{t,y}$ first reaches the boundary $\partial\omega$. Then, uniformly with respect to $y \in \partial I^n$ and $\varepsilon > 0$,

$$\mathbf{P}\{\xi^\varepsilon_{\pi_1,y} \in \partial\omega^3\} \geq \nu_1 > 0, \tag{35}$$

since in ω this probability is delivered by an operator which does not depend on ε.

We now consider $\xi^\varepsilon_{t,y}$ in the spherical layer $\{y | r < |y| < 1/2 - s/3\}$. Let π_2 be the Markov time when $\xi^\varepsilon_{t,y}$ first reaches the boundary of the spherical layer. The probability $\mathbf{P}\{|\xi^\varepsilon_{\pi_2,y}| = r\}$ satisfies $B^\varepsilon u = 0$, $u|_{|y|=r} = 1$, and $u|_{|y|=1/2-s/3} = 0$. To estimate the solution we choose a barrier function of the form

$$z_0 = c(e^{-\gamma|y|} - e^{-\gamma(1/2-s/3)}).$$

Since by hypothesis $(v(y), y) \leq 0$, for sufficiently large γ in the indicated spherical layer for all $\varepsilon > 0$ we have $B^\varepsilon z_0 \geq 0$, whence by the maximum principle $\mathbf{P}\{|\xi^\varepsilon_{\pi_2, y}| = r\} \geq \nu_2 > 0$ for all $y \in \{y | |y| = 1/2 - 2s/3\}$, which together with (35) and the strong Markov property of $\xi^\varepsilon_{t,y}$ gives the required assertion.

We now suppose that P2 holds. We define $s = \rho(G, \partial I^n)$,

$$\omega^4 = (-1/2 - s, 1/2 + s)^n \backslash G$$

and the Markov time $\pi_3 = \inf\{t > 0 | \xi^\varepsilon_{t,y} \notin \omega^4\}$. Since by hypothesis $v(y) = 0$ in ω^4, it follows that $\mathbf{P}\{\xi^\varepsilon_{\pi_3, y} \in \partial G\}$ does not depend on ε for all $y \in \omega^4$, and hence, uniformly with respect to $y \in \partial I^n$ and $\varepsilon > 0$,

$$\mathbf{P}(\xi^\varepsilon_{\pi_3, y} \in \partial G\} \geq \nu_3 > 0. \tag{36}$$

In a neighborhood of ∂G we introduce coordinates $z_1, \dots, z_n$, where z_1 is the inner normal to ∂G, while $z_2, \dots, z_n$ are coordinates on the boundary, so that $\partial G = \{z | z_1 = 0\}$. In the new coordinates B^ε takes the form

$$B^\varepsilon_1 = \varepsilon a^1_{ij}(z) \frac{\partial^2}{\partial z_i \partial z_j} + v^1_i(z, \varepsilon) \frac{\partial}{\partial z_i},$$

where for some sufficiently small $\delta > 0$ in a neighborhood $\{z | |z_1| < \delta\}$ of the boundary ∂G we have the estimate $v^1_1(z, \varepsilon) \geq -c\varepsilon$, which follows directly from P2. Suppose π_4 is the Markov time at which the process $\xi^\varepsilon_{t,y}$ first reaches the boundary of the set $\{z | |z_1| < \delta\}$. The probability $\mathbf{P}\{(\xi^\varepsilon_{\pi_4, y})_1 = +\delta\}$ is a solution of the problem $B^\varepsilon_1 u = 0$, $u|_{z_1=-\delta} = 0$, $u|_{z_1=\delta} = 1$. A lower bound for it is given by a barrier function of the form $(e^{\gamma z_1} - e^{-\gamma\delta})(e^{\gamma\delta} - e^{-\gamma\delta})$ for sufficiently large γ. From this, uniformly with respect to $y \in \partial G$ and $\varepsilon > 0$,

$$\mathbf{P}\{(\xi^\varepsilon_{\pi_4, y})_1 = +\delta\} \geq \nu_4 > 0. \tag{37}$$

Finally, since all the points of G are attracted by the field $v(y)$ toward the origin, uniformly with respect to $y \in \{y \in G | z_1(y) = \delta\}$ the probability $\mathbf{P}\{|\xi^\varepsilon_{\pi_5, y}| = r\}$ converges to one as $\varepsilon \to 0$ (see [10]); here π_5 is the Markov time when the process $\xi^\varepsilon_{t,y}$ first reaches the boundary of the set $G \backslash \{y | |y| \leq r\}$. The required estimate follows from the last relations via (36), (37), and the strong Markov property of $\xi^\varepsilon_{t,y}$.

Another sufficient condition for convergence is given by

THEOREM 5. *Suppose equation* (3) *satisfies conditions* I–III, *and suppose that each point of I^n is attracted by the field $v(y)$ toward the origin. Suppose also that for each $z \in \partial I^n$ and $y \in I^n$ such that $|z - y|$ is sufficiently small, the angle between the vector of the inner normal at the point z to ∂I^n and $v(y)$ does not exceed $\pi/2$. Then the assertion of Theorem* 3 *holds.*

The proof, which consists in verifying the conditions of Theorem 3, is altogether identical to the proof of the preceding theorem.

REMARK. From the conditions of Theorem 5 and the periodicity of the field $v(y)$ it follows that $v(y)$ must be tangent to the boundary of the cube ∂I^n.

We consider the potential case. The next result is a direct corollary of Theorem 3 and the estimates of §1.

THEOREM 6. *Suppose conditions* I–III *are satisfied, where* $v(y) = -\nabla U(y)$ *and the potential* $U(y)$ *has on the torus of periods only a finite number of singular points* ($\nabla U(y) = 0$), *among which only one is a minimum point. Then as* $\varepsilon \to 0$ *solutions* $u^\varepsilon(x)$ *of problem* (3) *converge to a function harmonic in* Ω *with the boundary condition* $\varphi(x)$.

§4. Further examples

In this section we give further examples of the application of the properties obtained above of the diffusion process ξ_t^ε.

It is frequently necessary to investigate equations of the form (3) in which there is symmetry in not all but only a part of the variables $y_1, \dots, y_k$, $0 \le k < n$ (for $k = 0$ there is no symmetry).

As in [10], suppose that

$$W(y) = \inf_{T>0} \inf_{\{x(t),\, x(0)=0,\, x(T)=y\}} \int_0^T |\sigma^{-1}(x(t))(\dot{x}(t) - v(x(t)))|^2 \, dt,$$

where the matrix $\sigma(y)$ is such that $\sigma(y)\sigma^*(y) = (a_{ij}(y))$.

THEOREM 7. *Suppose the coefficients of problem* (3) *satisfy conditions* I *and* II, *and condition* III *in part of the variables* $y_1, \dots, y_k$, $k \ge 1$, *and suppose that each point of the cube* I^n *is attracted by the field* $v(y)$ *toward the origin. Suppose further that the restriction of* $W(y)$ *to* ∂I^n *achieves a minimum only at points situated within those* $2k$ *faces of the cube which are parallel to the subspace* $\{y \in \mathbf{R}^n | y_1 = \cdots = y_k = 0\}$. *Suppose, finally, that in a neighborhood of* ∂I^n *the field* $v(y)$ *forms an acute angle with the vector of the inner normal to* ∂I^n. *Then as* $\varepsilon \to 0$ *solutions* $u^\varepsilon(x)$ *of problem* (3) *converge to a function* $u_0(x)$ *which in each section of the region* Ω *by the plane* $\{x \in \mathbf{R}^n | x_{k+1} = c_{k+1}, \dots, x_n = c_n\}$ *satisfies the equation*

$$\sum_{i=1}^k \frac{\partial^2}{\partial x_i^2} u_0(x) = 0, \qquad u_0|_{\partial\Omega} = \varphi(x).$$

This assertion, as in the proof of Theorem 3, is obtained by passing to a family of random walks on the discrete lattice. A typical transition time is characterized by the quantity W, while the random walk itself decomposes into independent random walks over planes of lower dimension.

For $k = 0$, i.e. in the absence of symmetry, the limit operator may be of first order, in contrast to the potential case.

Suppose conditions I and II are satisfied, and suppose all points of I^n are attracted by the field $v(y)$ toward the origin, where in a neighborhood of ∂I^n the field $v(y)$ forms an acute angle with the vector of the inner normal. We suppose that the restriction of $W(y)$ to ∂I^n achieves a minimum only on one face of the cube I^n— say on the face where $y_1 = 1/2$. Then as $\varepsilon \to 0$ the solutions $u^\varepsilon(x)$ of problem (3) converge to a solution of the equation $\partial u_0(x)/\partial x_1 = 0$ with boundary condition $\varphi(x)$ given on that portion of the boundary where the first coordinate axis is directed to the exterior of the region Ω.

In this case the random walk corresponding to the process $\xi^{\varepsilon}_{t,y}$ reduces to a sequence of jumps in the direction of the first coordinate vector, which leads to degeneration of the limit operator to an operator of first order.

In the remainder of the section we consider the problem of the contact of two media. We consider equation (1), (2) with $v(y) = -\nabla U(y)$.

Suppose there are two symmetric periodic potentials $U_1(y)$ and $U_2(y)$ possessing the following properties:

A. $U_1(y)$ and $U_2(y)$ have strict minima at points of $\mathbf{Z}^n$.

B. All points of I^n are attracted by each of the potentials toward the origin.

C. The restriction of $U_i(y)$, $i = 1, 2$, to ∂I^n achieves a minimum only at interior points of the $(n-1)$-dimensional faces of ∂I^n.

D. $\min_{\partial I^n} U_1(y) < \min_{\partial I^n} U_2(y)$.

We call problem (1), (2) the problem of contact of two media if the potential $U(y)$ of the field $v(y)$ coincides with $U_1(y)$ for $y_1 < -1/2 + \delta$, $\delta > 0$, with $U_2(y)$ for $y_1 > 1/2 - \delta$, and is periodic and symmetric in the variables $y_2, \dots, y_n$.

The conditions arising in the limit equation on the separation boundary of the two media, i.e., on the hyperplane $\{x \in \mathbf{R}^n | x_1 = 0\}$, depend on the structure of the potential in the layer $\{y \in \mathbf{R}^n | -1/2 + \delta < y_1 < 1/2 - \delta\}$. Apparently, the most natural is the following behavior of $U(y)$ for $-1/2 < y_1 < 1/2$:

a) $U(y)$ has a minimum point at zero which attracts all trajectories beginning in I^n.

b) The restriction of $U(y)$ to ∂I^n has minimum points only on that face of the cube where $y_1 = -1/2$.

If these conditions are satisfied, $u^{\varepsilon}(x)$ converge as $\varepsilon \to 0$ to a solution of the problem

$$u_0(x) = \begin{cases} u_0^+(x), & x_1 \geq 0, \\ u_0^-(x), & x_1 \leq 0; \end{cases}$$

$$\Delta u_0^+(x) = 0, \quad x \in \{x \in \Omega | x_1 > 0\}; \qquad \Delta u_0^-(x) = 0, \quad x \in \{x \in \Omega | x_1 < 0\};$$

$$u_0|_{\partial\Omega} = \varphi(x), \qquad u_0^+\big|_{x_1=0} = u_0^-\big|_{x_1=0}, \qquad \partial u_0^- / \partial x_1\big|_{x_1=0} = 0.$$

To illustrate the fact that the limit problem may depend on the properties of the potential in the layer we consider two examples of the contact of two media. In the first of them we replace condition b) by the following condition:

b′) The restriction of $U(y)$ to ∂I^n has minimum points only on those faces of the cube where $-1/2 < y_1 < 1/2$. The limit problem then has the form

$$u_0(x) = \begin{cases} u_0^+(x), & x_1 \geq 0, \\ u_0^-(x), & x_1 \leq 0; \end{cases}$$

$$\Delta u_0^+(x) = 0, \quad x \in \{x \in \Omega | x_1 > 0\}; \qquad \Delta u_0^-(x) = 0, \quad x \in \{x \in \Omega | x_1 < 0\};$$

$$u_0|_{\partial\Omega} = \varphi(x), \qquad u_0|_{x_1=0} = \overset{0}{u}_0(x);$$

$$\sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2} \overset{0}{u}_0(x) = 0, \qquad \overset{0}{u}_0\big|_{x \in \partial\Omega \cap \{x_1=0\}} = \varphi|_{x_1=0}.$$

In the second example we consider a potential for which the minimum on each of the faces $\{y \in \partial I^n | y_1 = -1/2\}$ and $\{y \in \partial I^n | y_1 = 1/2\}$ of the cube I^n is strictly less than the minimum on the union of the hyperplane $\{y \in \mathbf{R}^n | y_1 = 0\}$ with the remaining faces of the cube. In this case the limit function is a solution of the problem

$$u_0(x) = \begin{cases} u_0^+(x), & x_1 > 0, \\ u_0^-(x), & x_1 < 0; \end{cases} \qquad \Delta u_0^+(x) = 0, \; x \in \{x \in \Omega | x_1 > 0\};$$

$$\Delta u_0^-(x) = 0, \qquad x \in \{x \in \Omega | x_1 < 0\};$$

$$u_0|_{\partial\Omega} = \varphi(x), \qquad \partial U_0^\pm / \partial x_1\Big|_{x_1=0} = 0,$$

and, generally speaking, it is discontinuous on the hyperplane $\{x \in \mathbf{R}^n | x_1 = 0\}$.

§5. Proofs of technical lemmas

PROOF OF LEMMA 2. We assume with no loss of generality that the point of attraction is situated at 0. We choose $R > 0$ so that the ball $B_R = \{x \in \mathbf{R}^n | \, |x| \le R\}$ is attracted by the field $v(x)$ toward the origin. By hypothesis, there exists $\delta > 0$ such that the trajectories of the equation $\dot{x} = v(x)$ issuing from points of the sphere $S_{2\delta} = \{x \in \mathbf{R}^n | \, |x| = 2\delta\}$ for all $t > 0$ lie in $B_{r/2}$ and $x(t) \to 0$ as $t \to \infty$. By the continuity of the dependence of the solution on the initial conditions and the compactness of the sphere, from this it follows that there exists $t_0 > 0$ such that $x(t) \in B_\delta$ for all $t > t_0$. Suppose $\xi^\mu_{t,x}$ is the diffusion process corresponding to the operator

$$A_\mu = \mu a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + v_i(x) \frac{\partial}{\partial x_i},$$

and let τ^0 be the Markov time when the process $\xi^\mu_{t,x}$ first reaches the boundary of the spherical layer

$$Q_{e^{-\kappa/\mu}, 2\delta} = \{x \in \mathbf{R}^n | e^{-\kappa/\mu} < |x| < 2\delta\}.$$

We shall prove that, uniformly with respect to $x \in S_\delta$,

$$\mathbf{P}\{\xi^\mu_{\tau^0, x} \in S_{e^{-\kappa/\mu}}\} \ge c e^{-(q\kappa + \lambda\delta)/\mu}, \tag{38}$$

where q and λ do not depend on μ. For this in the spherical layer $Q_{\varepsilon^{-\kappa/\mu}, 2\delta}$ we consider the problem $A_\mu u = 0$, $u|_{|x| = e^{-\kappa/\mu}} = 0$, $u|_{|x| = 2\delta} = 1$, whose solution determines the probability

$$\mathbf{P}\{\xi^\mu_{\tau^0, x} \in B_{2\delta}\} = 1 - \mathbf{P}\{\xi^\mu_{\tau^0, x} \in B_{e^{-\kappa/\mu}}\}.$$

To construct a barrier function we consider the auxiliary equation

$$\left(\mu\Delta + \mu(q - 2n)\frac{1}{r}\frac{\partial}{\partial r} + \lambda\frac{\partial}{\partial r}\right)\overline{u} = 0, \qquad \overline{u}|_{|x| = e^{-\kappa/\mu}} = 0, \qquad \overline{u}|_{|x| = 2\delta} = 1.$$

Since the solution $\overline{u}$ depends only on $r = |x|$, the last equation can be rewritten as follows:

$$\left(\mu\frac{d^2}{dr^2} + \mu\frac{q}{r}\frac{d}{dr} + \lambda\frac{d}{dr}\right)\overline{u} = 0, \qquad \overline{u}(e^{-\kappa/\mu}) = 0, \qquad \overline{u}(2\delta) = 1.$$

After this it can be integrated explicitly:

$$\overline{u}(r) = \left(\int_{e^{-\kappa/\mu}}^{2\delta} \tau^{-q} e^{-\lambda\tau/\mu}\, d\tau\right)^{-1} \int_{e^{-\kappa/\mu}}^{r} \tau^{-q} e^{-\lambda\tau/\mu}\, d\tau.$$

We estimate $1-\overline{u}(\delta)$:

$$1-\overline{u}(\delta)=\int_{\delta}^{2\delta}\tau^{-q}e^{-\lambda\tau/\mu}\,d\tau\left(\int_{e^{-\kappa/\mu}}^{2\delta}\tau^{-q}e^{-\lambda\tau/\mu}\,d\tau\right)^{-1}$$
$$\geq \mu^{-1}\delta^{1-q}e^{-\lambda\delta/\mu}(e^{\kappa q/\mu})^{-1}\geq e^{-(\lambda\delta+\kappa q)/\mu}. \tag{39}$$

We also note that $\overline{u}(r)$ is monotone increasing with r. To prove the estimate $u(x)\leq \overline{u}(|x|)$ we act with the operator A_μ on $(u-\overline{u})$:

$$\begin{aligned}A_\mu(u-\overline{u}) &= -A_\mu\overline{u}=\left(-\mu a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}-v_i(x)\frac{\partial}{\partial x_i}\right)\overline{u}\\ &= -\left(\mu\tilde{a}(x)\frac{\partial^2}{\partial r^2}+\mu\tilde{b}(x)\frac{1}{r}\frac{\partial}{\partial r}+\tilde{v}(x)\frac{\partial}{\partial r}\right)\overline{u}\\ &= -\tilde{a}(x)\left(\mu\frac{\partial^2}{\partial r^2}+\mu\frac{\tilde{b}(x)}{\tilde{a}(x)}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\tilde{v}(x)}{\tilde{a}(x)}\frac{\partial}{\partial r}\right)\overline{u}.\end{aligned}$$

In the last equality the function $\tilde{a}(x)$ can be bounded below by the minimum eigenvalue of the matrix $(a_{ij}(x))$, and $\tilde{b}(x)$ can be bounded above by the quantity $n\Lambda$, where Λ is the maximum eigenvalue of $(a_{ij}(x))$, $|\tilde{v}(x)|\leq|v(x)|$. Further,

$$\begin{aligned}\left(\mu\frac{\partial^2}{\partial r^2}+\mu\frac{\tilde{b}(x)}{\tilde{a}(x)}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\tilde{v}(x)}{\tilde{a}(x)}\frac{\partial}{\partial r}\right)\overline{u} &= \left(\mu\frac{\partial^2}{\partial r^2}+\mu\frac{q}{r}\frac{\partial}{\partial r}+\lambda\frac{\partial}{\partial r}\right)\overline{u}\\ &\quad+\mu\left(\frac{\tilde{b}(x)}{\tilde{a}(x)}-q\right)\frac{1}{r}\frac{\partial}{\partial r}\overline{u}+\left(\frac{\tilde{v}(x)}{\tilde{a}(x)}-\lambda\right)\frac{\partial}{\partial r}\overline{u}\\ &= -\left(q-\frac{\tilde{b}(x)}{\tilde{a}(x)}\right)\frac{\mu}{r}\frac{\partial}{\partial r}\overline{u}-\left(\lambda-\frac{\tilde{v}(x)}{\tilde{a}(x)}\right)\frac{\partial}{\partial r}\overline{u}.\end{aligned}$$

For a suitable choice of q and λ we have $q-\tilde{b}(x)/\tilde{a}(x)>0$ and $\lambda-\tilde{v}(x)/\tilde{a}(x)>0$, and hence

$$A_\mu(u-\overline{u})=-\tilde{a}(x)\left(q-\frac{\tilde{b}(x)}{\tilde{a}(x)}\right)\frac{\mu}{r}\frac{\partial}{\partial r}\overline{u}-\tilde{a}(x)\left(\lambda-\frac{\tilde{v}(x)}{\tilde{a}(x)}\right)\frac{\partial}{\partial r}\overline{u}>0.$$

From this by the maximum principle $u\leq\overline{u}$, and hence

$$1-u\geq 1-\overline{u}\geq e^{-(\lambda\delta+\kappa q)/\mu}.$$

We now construct the sequence of Markov times

$$\begin{aligned}\tau_1 &= \inf\{t>0\,|\,\xi^\mu_{t,x}\notin B_R\backslash B_\delta\},\\ \nu_1 &= \inf\{t>\tau_1\,|\,\xi^\mu_{t,x}\notin B_{2\delta}\backslash B_{e^{-\kappa/\mu}}\},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \tau_k &= \inf\{t>\nu_{k-1}\,|\,\xi^\mu_{t,x}\notin B_R\backslash B_\delta\},\\ \nu_k &= \inf\{t>\tau_k\,|\,\xi^\mu_{t,x}\notin B_{2\delta}\backslash B_{e^{-\kappa/\mu}}\},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

According to [10], for $x \in S_{2\delta}$ we have $\mathbf{P}\{\xi^{\mu}_{\tau_1, x} \in S_\delta\} \le e^{-\theta/\mu}$, $\theta > 0$, and so, by (38),

$$\mathbf{P}\{\xi^{\mu}_{\bar{\tau}, x} \in B_R\} = \sum_{i=1}^{\infty} \mathbf{P}\left(\bigcap_{j=1}^{i-1}\{\xi^{\mu}_{\nu_j, x} \in S_{2\delta}\} \cap \bigcap_{j=1}^{i-1}\{\xi^{\mu}_{\tau_j, x} \in S_\delta\} \cap \{\xi^{\mu}_{\tau_i, x} \in S_R\}\right)$$
$$\le \sum_{i=0}^{\infty} e^{-\theta/\mu}(1 - e^{-(q\kappa+\lambda\delta)/\mu})^i = \exp\left(-\frac{\theta - (q\kappa + \lambda\delta)}{\mu}\right);$$

here $\bar{\tau}$ is the Markov time when the process $\xi^{\mu}_{t,x}$ first reaches the boundary of $B_R \backslash B_{e^{-\kappa/\mu}}$. For a suitable choice of δ and κ the preceding estimate gives

$$\mathbf{P}\{\xi^{\mu}_{\bar{\tau}, x} \in B_R\} \le e^{-\theta_1/\mu}, \qquad \theta_1 > 0. \tag{40}$$

In the ball $B_{e^{-\kappa/\mu}}$, by standard estimates of the derivatives of a solution of an elliptic equation in terms of the maximum modulus in a broader domain we have

$$\max_{B_{e^{-\kappa/\mu}}} u(x) - \min_{B_{e^{-\kappa/\mu}}} u(x) \le c\frac{1}{\mu}\sup_{\Omega}|u(x)|e^{-\kappa/\mu}. \tag{41}$$

Representing $u(x)$ in probabilistic form and using (40) and (41), we now get

$$\max_{B_{2\delta}} u(x) - \min_{B_{2\delta}} u(x) \le c\sup_{\Omega}|u(x)|e^{-\theta/\mu}, \qquad \theta_2 > 0. \tag{42}$$

We now consider an arbitrary compact set $K \subset \Omega$ which is attracted by the field $v(x)$ toward the origin. Let $\tilde{\tau} = \inf\{t > 0 | \xi^{\mu}_{t,x} \notin \Omega\backslash B_{2\delta}\}$. According to [10], for x in K

$$\mathbf{P}\{\xi^{\mu}_{\tilde{\tau}, x} \notin B_{2\delta}\} \le e^{-\theta_3/\mu}, \qquad \theta_3 > 0. \tag{43}$$

To complete the proof it remains to estimate $u(x)$ in probabilistic form, and use (42) and (43).

Proof of Lemma 1. Let $\xi^{\mu}_{t,y}$ be the diffusion process corresponding to the operator A_μ on the torus of periods, and let $p(t, x, y)$ be the density of the transition probability of this process ($p(t, x, y)$ is also a fundamental solution of the operator $\partial/\partial t - A_\mu$ on T^n). In the proof of the lemma the decisive step is the following inequality: for each $\delta > 0$ there exist $t_0(\delta)$ and $\delta_1 > 0$ such that for all $x \in T^n$ and $y \in Q_{\delta_1} = \{y \in T^n \,|\, |y - \mathcal{O}| \le \delta_1\}$ we have

$$p(t_0, (\delta), x, y) \ge e^{-\delta/2\mu}; \tag{44}$$

here $\mathcal{O}$ is the attracting singular point of the field $v(y)$. To prove (44) it suffices to use a result of [16] and [17]: for any $t > 0$, x, and y

$$\lim_{\mu\to 0} -\mu \ln p(t, x, y) = \inf_{\{\varphi(t),\, \varphi(0)=x,\, \varphi(t)=y\}} \int_0^t |\dot{\varphi} - v(\varphi(t))|^2\, dt,$$

we then consider the structure of the vector field $v(y)$ on T^n.

We denote by $\omega(t)$ the oscillation of a solution of the problem

$$(\partial/\partial t - A_\mu)u = 0, \qquad u|_{t=0} = f_0,$$

at time t:

$$\omega(t) = \max_{x\in T^n} u(t, x) - \min_{x\in T^n} u(t, x),$$

and we estimate $\omega(t_0(\delta)+t)$ in terms of $\omega(t)$. We assume with no loss of generality that $\max_x u(t, x) = -\min_x u(t, x) = \frac{1}{2}\omega(t)$. We denote by $Q^+_{\delta_1}$ and $Q^-_{\delta_1}$ the subsets of Q_{δ_1} where $u(t, x) \geq 0$, and $u(t, x) < 0$ respectively. Then, by (44),

$$\begin{aligned} u(t_0(\delta)+t, x) &= \int_{T^n} p(t_0(\delta), x, y)u(t, y)\,dy \\ &\leq (1 - e^{-\delta/2\mu}\operatorname{meas} Q^-_{\delta_1})\max_x u(t, x) = \tfrac{1}{2}\omega(t)(1 - e^{-\delta/2\mu}\operatorname{meas} Q^-_{\delta_1}) \end{aligned}$$

and, similarly,

$$u(t_0(\delta)+t, x) \geq -\tfrac{1}{2}\omega(t)(1 - e^{-\delta/2\mu}\operatorname{meas} Q^+_{\delta_1}).$$

Hence

$$\begin{aligned} \omega(t_0(\delta)+t) &\leq \omega(t)(1 - \tfrac{1}{2}c^{-\delta/2\mu}(\operatorname{meas} Q^-_{\delta_1} + \operatorname{meas} Q^+_{\delta_1})) \\ &= \omega(t)(1 - \tfrac{1}{2}e^{-\delta/2\mu}\operatorname{meas} Q_{\delta_1}) = \omega(t)(1 - \kappa(\delta)e^{-\delta/2\mu}). \end{aligned}$$

Choosing now for T_0 the quantity $c(\delta)t_0(\delta)e^{\delta/\mu}/\kappa(\delta)$, we find that

$$\omega(T_0 + t) \leq \omega(t)(1 - \kappa(\delta)e^{-\delta/2\mu})^{c(\delta)e^{\delta/\mu}/\kappa(\delta)} \leq \omega(t)(\tfrac{1}{2})^{c(\delta)e^{\delta/2\mu}},$$

and for a suitable choice of $c(\delta)$ we obtain

$$\omega(T_0 + t) \leq \omega(t)(\tfrac{1}{2})^{c(\delta)e^{\delta/2\mu}} \leq \omega(t)e^{-c/\mu}$$

for any prescribed c.

Inequality (6) can be derived from this by means of standard estimates of the maximum modulus of a solution of a second-order parabolic equation in terms of its L^2-norm in a broader domain (see [11]).

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