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AVERAGING A SINGULARLY PERTURBED EQUATION WITH RAPIDLY OSCILLATING COEFFICIENTS IN A LAYER

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ABSTRACT. A complete asymptotic expansion is constructed for a second-order equation of elliptic type with a small parameter in the highest-order derivative and rapidly oscillating coefficients.

Bibliography: 11 titles.

In this paper in the layer $a < x_1 < b$ of the space $\mathbf{R}^n$ we study the problem

$$\begin{gathered} \varepsilon \frac{\partial}{\partial x_i} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} u^\varepsilon(x) + b_i\left(x, \frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_i} u^\varepsilon(x) + c\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x) = f(x), \\ u^\varepsilon|_{x_1=d} = \varphi_1(x), \qquad u^\varepsilon|_{x_1=b} = \varphi_2(x) \end{gathered} \tag{1}$$

with a small positive parameter ε. We assume that the coefficients $a_{ij}(x, y)$, $b_i(x, y)$, and $c(x, y)$ are smooth functions which are periodic in the second argument y. We further assume that the matrix (a_{ij}) satisfies the condition of uniform ellipticity

$$\lambda|\xi|^2 \leqslant a_{ij}(x, y)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \qquad 0 < \lambda < \Lambda.$$

Our aim is to find conditions on the coefficients of equation (1) under which the solution $u^\varepsilon(x)$ can be represented as an asymptotic series in powers of the small parameter ε and also to construct such an asymptotic expansion when these conditions are satisfied.

The first section is of independent interest. There we study the behavior of a solution of the equation

$$\begin{gathered} \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j} u(y) + b_i(y) \frac{\partial}{\partial y_i} u(y) = f(y), \\ u|_{y_1=0} = \varphi(y), \end{gathered} \tag{2}$$

defined in the cylinder $(y_1, \dots, y_n) \in (0, +\infty) \times T^{n-1}$ with toroidal cross-section for large y_1. Existence of a bounded solution of problem (2) and stabilization of any bounded solution to a constant as $y_1 \to \infty$ are proved under the assumption that the right side $f(y)$

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decays sufficiently fast. Necessary and sufficient conditions are obtained for uniqueness of a bounded solution of (2).

We note that for equations in divergence form a similar problem was solved in [1]. The behavior of solutions of various boundary value problems in domains close to a cylinder was investigated in [2] for equations of divergence type.

A method of constructing an asymptotic expansion of a solution of (1) is presented in §2. This method is applicable if an auxiliary vector field $\bar{b}_i(x)$ constructed on the basis of the coefficients of (1) satisfies certain conditions which are close to the condition of regular degeneracy of [3].

Near each of the boundary planes the asymptotic expansion of $u^\varepsilon(x)$ contains functions of boundary-layer type which are found by applying the results of §1. Here in a neighborhood of one of the boundary planes a boundary layer occurs already in the zeroth approximation, while in a neighborhood of the other it occurs only in first approximation.

In this same section the solvability of problem (1) is proved for sufficiently small ε, and the sign of the coefficient $c(x, y)$ can be arbitrary. Estimates of the solution in terms of the right side and the boundary functions are also obtained. These results are proved by probabilistic methods.

Equations of the form (1) with nonoscillating coefficients have been studied in [3], [4], and many other works. If the lower order terms $b_i(x, y)$ and $c(x, y)$ are not present in problem (1) an asymptotic expansion of the solution can be constructed by the methods of [5], but this expansion has a form different from that obtained by us.

§1

In an infinite cylinder which is the product of the half-line $(0, +\infty)$ and the $(n-1)$-dimensional torus T^{n-1} we consider the equation

$$\mathscr{A}_0 u \equiv \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j} u(y) + b_i(y) \frac{\partial}{\partial y_i} u(y) = f(y), \qquad u\big|_{y_1=0} = \varphi(y). \tag{2'}$$

Here the coefficients $a_{ij}(y)$ and $b_i(y)$ are bounded and measurable. Moreover, the matrix (a_{ij}) satisfies the condition

$$\lambda|\xi|^2 \leqslant a_{ij}(y)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \qquad 0 < \lambda < \Lambda.$$

We begin the study of problem (2′) with the simplest case where the coefficients $a_{ij}(y)$ and $b_i(y)$ are periodic in the variable y_1 and are twice continuously differentiable. We denote by $\mathscr{A}_0^*$ the operator formally adjoint to $\mathscr{A}_0$, and on the torus of periods T^n we consider the problem $\mathscr{A}_0^* p = 0$. A nontrivial solution $p(y)$ of this equation exists; it is unique up to a factor and vanishes nowhere (see [6]). In order to uniquely determine the choice of $p(y)$ we assume that its average over the torus T^n is equal to one. We now construct the auxiliary vector

$$\bar{b}_i = \int_{T^n} \left(\frac{\partial}{\partial y_j} a_{ij}(y) + b_i(y) \right) p(y)\, dy.$$

THEOREM 1. *Suppose that the operator $\mathscr{A}_0$ has coefficients which are twice continuously differentiable and periodic in the variable y_1, and suppose that the boundary condition in* (2′) *is given by a bounded function, while the right side $f(y)$ decays exponentially as $y_1 \to \infty$:*

$$\|\varphi\|_{L^\infty(T^{n-1})} < \infty, \qquad \|e^{\beta y_1} f\|_{L^\infty((0,+\infty)\times T^{n-1})} < \infty, \qquad \beta > 0.$$

Then any bounded solution of problem (2′) *stabilizes as* $y_1 \to \infty$ *to a constant at an exponential rate*:

$$|u(y) - \mu| \leqslant ce^{-\alpha y_1}, \qquad \alpha > 0.$$

The inequality $\bar{b}_1 \leqslant 0$ *holds if and only if a bounded solution of problem* (2′) *exists and is unique*; *the inequality* $\bar{b}_1 > 0$ *holds if and only if for any real* μ *problem* (2′) *has exactly one solution which tends to* μ *at infinity.*

PROOF. We first prove the assertion of the theorem for zero right side. To this end we consider the diffusion process corresponding to the operator $\mathscr{A}_0$:

$$d\xi_t^y = \sigma(\xi_t^y)\, dw_t + \tilde{b}_i(\xi_t^y)\, dt, \qquad \xi_0^y = y,$$

where

$$\sigma_{ik}(y)\sigma_{kj}(y) = 2a_{ij}(y), \qquad \tilde{b}_i(y) = \frac{\partial}{\partial y_j}c_{ij}(y) + b_i(y).$$

We denote by Γ_0 the lower base of the cylinder and by Γ_N the section of the cylinder by the hyperplane $\{y_1 = N\}$. Let $\tau(y)$ be the Markov time at which the process ξ_t^y first reaches the lower base Γ_0. If ξ_t^y does not reach Γ_0 we set $\tau(y) = \infty$.

ASSERTION 1. *Either the probability* $\mathbf{P}\{\tau(y) < \infty\}$ *of the process* ξ_t^y *reaching the lower base is equal to one for all* y, *or it is less than one for all* y *not lying on* Γ_0.

PROOF. We consider the process ξ_t^y in the finite cylinder $\{y\colon 0 < y_1 < N\}$, and we denote by $\tau_N(y)$ the Markov time of first passge of the process ξ_t^y onto the boundary $\Gamma_0 \cup \Gamma_N$ of this cylinder. According to [7], the probability $\mathbf{P}\{\xi_{\tau_N(y)}^y \in \Gamma_0\}$ satisfies the equation

$$\mathscr{A}_0 u_N = 0, \qquad u_N|_{\Gamma_0} = 1, \qquad u_N|_{\Gamma_N} = 0.$$

It is easy to verify by the maximum principle that the functions $u_N(y)$ are monotonically increasing in N and satisfy the inequality $0 \leqslant u_N(y) \leqslant 1$. Therefore a function $u(y) = \lim_{N\to\infty} u_N(y)$ is defined that, like all the $u_N(y)$, is a solution of the equation $\mathscr{A}_0 u = 0$ and satisfies the inequality $0 \leqslant u(y) \leqslant 1$.

On the other hand,

$$\lim_{N\to\infty} \mathbf{P}\left\{\xi_{\tau_N(y)}^y \in \Gamma_0\right\} = \mathbf{P}\{\tau(y) < \infty\}$$

for all y, and hence $u(y) = \mathbf{P}\{\tau(y) < \infty\}$. Assertion 1 now follows from the strong maximum principle.

ASSERTION 2. *Suppose that* $\mathbf{P}\{\tau(y) < \infty\} < 1$ *for all* y *not lying on* Γ_0. *Then there exists* $\theta > 0$ *such that for all* y

$$\mathbf{P}\{\tau(y) < \infty\} \leqslant ce^{-\theta y_1}. \tag{3}$$

PROOF. We use the strict Markov property of the diffusion process ξ_t^y. We may assume with no loss of generality that the period of the coefficients of $\mathscr{A}_0$ in the variable y_1 is equal to one. From the hypotheses of the assertion and the continuity of $\mathbf{P}\{\tau(y) < \infty\}$ it follows that

$$\max_{y\in\Gamma_1} \mathbf{P}\{\tau(y) < \infty\} = \theta_1 < 1.$$

Suppose now that $y \in \Gamma_2$. We denote the Markov time of the process ξ_t^y reaching the section Γ_1 by $\eta(y)$. By the strict Markov property of ξ_t^y we have (see [7])

$$\begin{aligned}\mathbf{P}\{\tau(y) < \infty\} &= \mathbf{P}\{(\eta(y) < \infty) \cap (\tau(y) - \eta(y) < \infty)\} \\ &= \mathsf{E}\big[\mathbf{P}\{(\eta(y) < \infty) \cap (\tau(y) - \eta(y) < \infty) | \mathscr{F}_{\eta(y)}\}\big] \\ &= \mathsf{E}\big(\chi_{\{\eta(y) < \infty\}} \mathbf{P}\{\tau(y) - \eta(y) < \infty | \mathscr{F}_{\eta(y)}\}\big) \\ &= \mathsf{E}\big(\chi_{\{\eta(y) < \infty\}} \mathbf{P}\{t(\eta(y)) < \infty\}\big) \\ &\leqslant \theta_1 \mathsf{E}(\chi_{\{\eta(y) < \infty\}}) = \theta_1 \mathbf{P}\{\eta(y) < \infty\} \leqslant \theta_1^2;\end{aligned}$$

here E denotes the expectation corresponding to the probability $\mathbf{P}$, and $\mathscr{F}_{\eta(y)}$ is the σ-algebra generated by the Markov time $\eta(y)$; here and henceforth χ denotes the characteristic function of a set.

In exactly the same way, for y lying on Γ_N we obtain by induction the inequality

$$\mathbf{P}\{\tau(y) < \infty\} \leqslant \theta_1^N.$$

Further, again using the strict Markov property, it is easy to obtain (3).

LEMMA 1. *Suppose that the probability* $\mathbf{P}\{\tau(y) < \infty\}$ *is equal to one for all* y. *Then* (2′) *has a unique bounded solution, and this solution stabilizes to a constant at an exponential rate.*

PROOF. *Existence.* In the bounded cylinders $\{0 < y_1 < N\}$ we consider the sequence of problems

$$\mathscr{A}_0 u_N = 0, \qquad u_N|_{\Gamma_0} = \varphi(y), \qquad u_N|_{\Gamma_N} = 0.$$

The solution of each of these is given by

$$u_N(y) = \mathsf{E}\Big(\chi_{\{\xi^y_{\tau_N(y)} \in \Gamma_0\}} \varphi\big(\xi^y_{\tau_N(y)}\big)\Big).$$

Since by hypothesis $\mathbf{P}\{\tau(y) < \infty\} = 1$ for any y, as $N \to \infty$ the probability $\mathbf{P}\{\tau_N(y) \neq \tau(y)\}$ converges to zero for all y. From this and Lebesgue's theorem it follows that for each y there exists the limit

$$u(y) = \lim_{N \to \infty} u_N(y) = \mathsf{E}\big(\varphi\big(\xi^y_{\tau(y)}\big)\big). \tag{4}$$

The limit function $u(y)$ is defined in the cylinder $\{0 \leqslant y_1 < \infty\}$, is bounded, and satisfies the equation $\mathscr{A}_0 u = 0$, since all the $u_N(y)$ satisfy this equation. Moreover, the function $\varphi(y)$ serves as boundary condition for $u(y)$ on Γ_0.

Uniqueness. We suppose that there is another bounded solution $v(y)$ of (2′) distinct from $u(y)$. We fix an arbitrary point y and note that in the cylinder $\{0 < y_1 < N\}$ we have

$$v(y) = \mathsf{E}\Big(v\big(\xi^y_{\tau_N(y)}\big)\Big). \tag{5}$$

It follows from the hypotheses of the lemma that

$$\lim_{N \to \infty} \mathbf{P}\Big\{\xi^y_{\tau_N(y)} \in \Gamma_0\Big\} = 1.$$

By hypothesis $v|_{\Gamma_0} = \varphi(y)$. Therefore, passing to the limit as $N \to \infty$ in (5) and using the boundedness of $v(y)$ we obtain $v(y) = \mathsf{E}(\varphi(\xi^y_{\tau(y)}))$. From this and (4) it follows readily that $v(y) = u(y)$.

The semigroup property. We shall prove the following property of the operator $\mathscr{A}_0$ which is important for our purposes. Let $z(y, \nu)$ be a bounded solution of the problem

$$\mathscr{A}_0 z = 0, \qquad z|_{\Gamma_\nu} = u|_{\Gamma_\nu}. \tag{6}$$

Then $z(y, \nu) = u(y)$ for $y_1 \geqslant \nu$.

Indeed, $u(y)|_{y_1 \geqslant \nu}$ is a bounded solution of (6). It remains to use the uniqueness of a bounded solution.

Stablization to a constant. In the cylinder $y_1 \geqslant k + 1/2$ we consider the problem

$$\mathscr{A}_0 z = 0, \qquad z|_{\Gamma_{k+1/2}} = \psi_\rho^x(y).$$

Here k is a natural number, and $\psi_\rho^x(y)$ is a smooth, nonnegative function with support in a ball of radius $\rho > 0$ with center at the point x which is equal to one in a ball of radius $\rho/2$. Then for any y in the section Γ_{k+1}

$$z(y) \geqslant \gamma(\rho) > 0, \tag{7}$$

where $\gamma(\rho)$ does not depend either on y or on x. Indeed, for fixed x inequality (7) follows from the maximum principle and the compactness of Γ_{k+1}. Further, using the probabilistic representation for $z(y)$ or the maximum principle, it is easy to verify the continuous dependence of $z(y)$ on x, after which the uniformity of the estimate (7) in x follows from the compactness of the torus $\Gamma_{k+1/2}$.

We denote by $\overline{m}(t)$ the maximum of $u(y)$ on the set Γ_t and by $\underline{m}(t)$ the minimum of $u(y)$ on Γ_t; we set $\omega(t) = \overline{m}(t) - \underline{m}(t)$. From the representation (4) for $u(y)$ and the semigroup property it follows that $\overline{m}(t)$ decreases monotonically, while $\underline{m}(t)$ increases monotonically. Our problem is to prove that $\omega(t) \leqslant ce^{-\alpha t}$, where the positive number α does not depend on φ but only on $\mathscr{A}_0$. To this end we consider the solution $u(y)$ over a single period ($k \leqslant y_1 \leqslant k + 1$), and we prove that $\omega(k + 1) \leqslant \beta_1\omega(k)$, where $\beta_1 < 1$ does not depend on k or $\varphi(y)$.

Because of the linearity of the problem, we may assume with no loss of generality that $\overline{m}(k) = 1$ and $\underline{m}(k) = -1$. If $\overline{m}(k + 1) \leqslant 1/2$ or $\underline{m}(k + 1) \geqslant -1/2$, then for β_1 it is possible to take $3/4$. Otherwise, on $\Gamma_{k+1/2}$ we have

$$\overline{m}(k + 1/2) > 1/2, \qquad \underline{m}(k + 1/2) < -1/2. \tag{8}$$

We denote by y' and y'' respectively the points of the maximum and minimum of $u(y)$. Noting that the quantity $|u(y)|$ is bounded by one, from Schauder's inequality we obtain

$$\|u\|_{C^1(k+1/4 \leqslant y_1 \leqslant k+3/4)} \leqslant \theta_2(\mathscr{A}_0). \tag{9}$$

The uniformity of this estimate in k follows from the periodicity of the coefficients of $\mathscr{A}_0$.

It follows from (8) and (9) that for some $\rho > 0$ in a ball of radius ρ with center at y' we have $u(y) > 1/4$. Similarly, in a neighborhood of the minimum point we have $u(y) < -1/4$. We now represent $u(y)$ as the sum $u_1(y) + u_2(y)$, where $u_1(y)$ is the solution of the problem

$$\mathscr{A}_0 u_1 = 0, \qquad u_1|_{\Gamma_{k+1/2}} = \tfrac{1}{4}\psi_\rho^{y'}(y),$$

and $u_2(y)$ is the solution of the same equation with the initial condition

$$u_2|_{\Gamma_{k+1/2}} = u|_{\Gamma_{k+1/2}} - \tfrac{1}{4}\psi_\rho^{y'}(y).$$

It is easy to see that $u_2|_{\Gamma_{k+1/2}} > -1$ on $\Gamma_{k+1/2}$; this same inequality therefore holds on Γ_{k+1} as well. From (7) we have

$$u_1|_{\Gamma_{k+1}} \geqslant \tfrac{1}{4}\gamma(\rho).$$

Hence,

$$u|_{\Gamma_{k+1}} = (u_1 + u_2)|_{\Gamma_{k+1}} > -1 + \tfrac{1}{4}\gamma(\rho).$$

In precisely the same way,

$$u|_{\Gamma_{k+1}} < 1 - \tfrac{1}{4}\gamma(\rho),$$

and so

$$\omega(k+1) \leqslant \left(1 - \tfrac{1}{4}\gamma(\rho)\right)\omega(k).$$

To complete the proof of the lemma it remains to choose for β_1 the quantity

$$\min(3/4, 1 - \gamma(\rho)/4).$$

REMARK. It is evident from the proof of Lemma 1 that in the estimate

$$|u(y) - \mu| \leqslant ce^{-\alpha y_1}$$

in our case the constant c can be chosen equal to $2e^{\alpha}\|\varphi\|_{L^\infty(\Gamma_0)}$.

LEMMA 2. *Suppose that* $\mathbf{P}\{\tau(y) < \infty\} < 1$ *for all* y *not lying on* Γ_0. *Then for any real* μ *there exists a unique solution of problem* (2′) *which converges to* μ *at infinity. The stabilization occurs exponentially. Problem* (2′) *has no other bounded solutions except those which stabilize to a constant.*

PROOF. As in the preceding lemma, in the bounded cylinders $\{0 < y_1 < N\}$ we consider the sequence of problems

$$\mathscr{A}_0 u_N = 0, \qquad u_N|_{\Gamma_0} = \varphi(y), \qquad u_N|_{\Gamma_N} = \mu.$$

Each of the solutions $u_N(y)$ is given by

$$u_N(y) = \mathsf{E}\Big(\mu\chi_{\{\xi^y_{\tau_N(y)}\in\Gamma_N\}} + \varphi\big(\xi^y_{\tau_N(y)}\big)\chi_{\{\xi^y_{\tau_N(y)}\in\Gamma_0\}}\Big).$$

It is clear that for any y the sequence $u_N(y)$ converges as $N \to \infty$ to the quantity

$$u(y) = \lim_{N\to\infty} u_N(y) = \mu\mathbf{P}\{\tau(y) = \infty\} + \mathsf{E}\big(\varphi\big(\xi^y_{\tau(y)}\big)\chi_{\{\tau(y)<\infty\}}\big).$$

Since all the functions $u_N(y)$ are solutions of the equation $\mathscr{A}_0 u_N = 0$, it follows that $\mathscr{A}_0 u = 0$. Further, it is easy to verify that $u|_{\Gamma_0} = \varphi(y)$.

We shall show that our solution $u(y)$ tends to μ exponentially as $y_1 \to \infty$; to this end we estimate the difference

$$\begin{aligned}|u(y) - \mu| &\leqslant |\mu|\mathbf{P}\{\tau(y) < \infty\} + \|\varphi\|_{L^\infty(\Gamma_0)}\mathbf{P}\{\tau(y) < \infty\}\\ &\leqslant c\Big(|\mu| + \|\varphi\|_{L^\infty(\Gamma_0)}\Big)e^{-\alpha y_1};\end{aligned}$$

the second inequality here follows from Assertion 2. It follows from the maximum principle that any solution of (2′) which tends to μ at infinity coincides with $u(y)$.

We shall verify that there are no other bounded solutions. We suppose that there is a bounded solution of (2′) which does not stabilize to a constant. We denote it by $v(y)$. The lack of stabilization implies that for some $\varepsilon_0 > 0$ for any k in the cylinder $\{y_1 > k\}$ there are points y' and y'' such that

$$|v(y') - v(y'')| > \varepsilon_0. \tag{10}$$

We now extend the coefficients of $\mathscr{A}_0$ to the cylinder $\{-\infty < y_1 < +\infty\}$ in periodic fashion, and we consider the problem

$$\mathscr{A}_0 z_N = 0, \qquad z_N|_{\Gamma_N} = v|_{\Gamma_N}, \tag{11}$$

which can be solved "below", i.e., on the set $(-\infty, N) \times T^{n-1}$. We observe that for any y, $-\infty < y_1 < N$, the probability that the process ξ_t^y defined in the cylinder $\{-\infty < y_1 < +\infty\}$ reaches the section Γ_N is equal to one. Indeed, for each finite ν the probability that the process ξ_t^y reaches the boundary $\Gamma_\nu \cup \Gamma_N$ of the cylinder $\{\nu < y_1 < N\}$ is equal to one (see [7]), and by the hypotheses of the lemma the probability of reaching the lower base Γ_ν for fixed y tends to zero as $\nu \to -\infty$. From what has been said it follows that Lemma 1 is applicable to problem (11); therefore $Z_N(y)$ stabilizes exponentially to a constant as $(y_1 - N) \to -\infty$. Here

$$z_N(y) = \mathsf{E}\Big(v\big(\xi^y_{\hat\tau_N(y)}\big)\Big),$$

where $\hat\tau_N(y)$ is the Markov time of first passage of ξ_t^y onto Γ_N.

In the cylinder $\{0 < y_1 < N\}$ the function $v(y)$ can be defined by

$$v(y) = \mathsf{E}\Big(v\big(\xi^y_{\tau_N(y)}\big)\Big).$$

Further, by Assertion 2

$$\mathbf{P}\{\hat\tau_N(y) = \tau_N(y)\} = \mathbf{P}\Big\{\xi^y_{\tau_N(y)} \in \Gamma_0\Big\} \leqslant \mathbf{P}\{\tau(y) < \infty\} \leqslant ce^{-\theta y_1},$$

and hence for any y' and y'' in the cylinder $\{k < y_1 < N/2\}$ on the basis of the formulas for the solutions $z_N(y)$ and $v(y)$ we have

$$\begin{aligned}|v(y') - v(y'')| &\leqslant |v(y') - z_N(y')| + |z_N(y') - z_N(y'')| + |z_N(y'') - v(y'')| \\ &\leqslant c\|\varphi\|_{L^\infty(\Gamma_0)}e^{-\theta k} + c\|v\|_{L^\infty(\Gamma_N)}e^{-\alpha N/2} + c\|\varphi\|_{L^\infty(\Gamma_0)}e^{-\theta k}.\end{aligned}$$

Choosing first k and then N sufficiently large, we obtain a contradiction to (10). The proof of the lemma is complete.

LEMMA 3. *The condition $\bar b_1 > 0$ is necessary for the inequality $\mathbf{P}\{\tau(y) < \infty\} < 1$ to be satisfied for all y not lying on Γ_0.*

PROOF. Suppose that $\mathbf{P}\{\tau(y) < \infty\} < 1$ for all y not lying on Γ_0. In this case Lemma 2 guarantees the existence of a solution of the problem

$$\mathscr{A}_0 u = 0, \qquad u|_{\Gamma_0} = 0, \qquad u|_{y_1 = \infty} = 1.$$

We multiply by $p(y)u(y)$ and integrate over the cylinder $0 \leqslant y_1 \leqslant N$:

$$I(N) = \int_0^N dy_1 \int_{\Gamma_{y_1}} \left(\frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j} u(y) + b_i(y) \frac{\partial}{\partial y_i} u(y)\right) p(y) u(y)\, dy'.$$

We now integrate $I(N)$ by parts:

$$
\begin{aligned}
I(N) = &-\int_0^N dy_1 \int_{\Gamma_{y_1}} \left(a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} u(y) \right) p(y)\, dy' \\
&-\int_0^N dy_1 \int_{\Gamma_{y_1}} \left(a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} p(y) \right) u(y)\, dy' \\
&-\frac{1}{2}\int_0^N dy_1 \int_{\Gamma_{y_1}} \frac{\partial}{\partial y_i}(b_i(y)p(y))u^2(y)\, dy' + \int_{\Gamma_N} u(y)p(y)a_{1j}(y)\frac{\partial}{\partial y_j}u(y)\, dy' \\
&+\frac{1}{2}\int_{\Gamma_N} b_1(y)p(y)u^2(y)\, dy' \\
= &-\int_0^N dy_1 \int_{\Gamma_{y_1}} p(y)a_{ij}(y)\frac{\partial}{\partial y_i}u(y)\frac{\partial}{\partial y_j}u(y)\, dy' \\
&+\frac{1}{2}\int_0^N dy_1 \int_{\Gamma_{y_1}} u^2(y)\left[\frac{\partial}{\partial y_j}\left(a_{ij}(y)\frac{\partial}{\partial y_i}p(y)\right) - \frac{\partial}{\partial y_i}(b_i(y)p(y))\right] dy' \\
&-\frac{1}{2}\int_{\Gamma_N} u^2(y)a_{1j}(y)\frac{\partial}{\partial y_j}p(y)\, dy' + \int_{\Gamma_N} u(y)p(y)a_{1j}(y)\frac{\partial}{\partial y_j}u(y)\, dy' \\
&+\frac{1}{2}\int_{\Gamma_N} b_1(y)p(y)u^2(y)\, dy' \\
= &\ 0.
\end{aligned}
$$

From the definition of $p(y)$ we obtain

$$
\int_0^N dy_1 \int_{\Gamma_{y_1}} u^2(y)\left[\frac{\partial}{\partial y_j}a_{ij}(y)\frac{\partial}{\partial y_i}p(y) - \frac{\partial}{\partial y_i}(b_i(y)p(y))\right] dy' = 0.
$$

We now integrate $I(N)$ with respect to N over the interval from η to $\eta + 1$:

$$
\begin{aligned}
&\int_\eta^{\eta+1} dN \int_0^N dy_1 \int_{\Gamma_{y_1}} p(y)a_{ij}(y)\frac{\partial}{\partial y_i}u(y)\frac{\partial}{\partial y_j}u(y)\, dy' \\
&\qquad -\frac{1}{2}\int_\eta^{\eta+1} dy_1 \int_{\Gamma_{y_1}} u^2(y)\left[a_{1j}(y)\frac{\partial}{\partial y_j}p(y) - b_1(y)p(y)\right] dy' \\
&\qquad +\int_\eta^{\eta+1} dy_1 \int_{\Gamma_{y_1}} u(y)p(y)a_{1j}(y)\frac{\partial}{\partial y_j}u(y)\, dy' = 0.
\end{aligned}
$$

In this equality we pass to the limit as $\eta \to \infty$. Since the function $u(y)$ converges to one as $y_1 \to \infty$ as its first derivatives tend to zero, the third integral on the left side of the last equality drops out, and we arrive at the relation

$$
\begin{aligned}
&\int_0^\infty dy_1 \int_{\Gamma_{y_1}} p(y)a_{ij}(y)\frac{\partial}{\partial y_i}u(y)\frac{\partial}{\partial y_j}u(y)\, dy' \\
&\qquad = \frac{1}{2}\int_{T^n} p(y)\left(\frac{\partial}{\partial y_j}a_{1j}(y) + b_1(y)\right) dy.
\end{aligned}
$$

The first integral here is positive, while the second is $\bar{b}_1$ by definition; therefore, $\bar{b}_1 > 0$.

LEMMA 4. *The condition* $\bar{b}_1 \leqslant 0$ *is necessary in order that* $\mathbf{P}\{\tau(y) < \infty\} = 1$ *for all* y.

PROOF. Suppose that $\mathbf{P}\{\tau(y) < \infty\} = 1$ for all y. In the cylinders $\{0 < y_1 < N\}$ we consider the sequence of problems

$$\mathscr{A}_0 u_N = 0, \qquad u_N|_{\Gamma_0} = 1, \qquad u_N|_{\Gamma_N} = 0. \tag{12}$$

It follows from the hypotheses of the lemma that on the set $\{0 < y_1 < 3\}$ the functions $u_N(y)$ tend to one as $N \to \infty$. The derivatives of these functions tend to zero on the set $\{1 \leqslant y_1 \leqslant 2\}$ as $N \to \infty$; this follows from Schauder's inequality [8]. As in the proof of Lemma 3, we multiply (12) by $p(y)u_N(y)$ and integrate over the cylinder $\{\eta < y_1 < N\}$. Integrating by parts, just as in the preceding lemma, we obtain

$$\begin{aligned} -\int_\eta^N dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u_N(y) \frac{\partial}{\partial y_j} u_N(y)\, dy' \\ + \frac{1}{2}\int_{\Gamma_\eta} u_N^2(y)\left(a_{1j}(y)\frac{\partial}{\partial y_j} p(y) - b_1(y)p(y)\right) dy' \\ - \int_{\Gamma_\eta} p(y) u_N(y) a_{j1}(y) \frac{\partial}{\partial y_j} u_N(y)\, dy' = 0. \end{aligned}$$

We integrate this equality with respect to η from 1 to 2. Passing to the limit as $N \to \infty$, we obtain

$$\begin{aligned} \lim_{N\to\infty} -\int_1^2 d\eta \int_\eta^N dy_1 \int_{\Gamma_{y_1}} \left(a_{ij}(y)\frac{\partial}{\partial y_i}u_N(y)\frac{\partial}{\partial y_j}u_N(y)\right) p(y)\, dy' \\ = \frac{1}{2}\int_{T^n}\left(\frac{\partial}{\partial y_j}a_{1j}(y) + b_1(y)\right) p(y)\, dy. \end{aligned}$$

Since the limit on the left is nonpositive, it follows that $\bar{b}_1 \leqslant 0$.

We now observe that Assertion 1 implies that the necessary conditions formulated in Lemmas 3 and 4 are also sufficient. The assertion of Theorem 1 has thus been proved in the special case $f(y) \equiv 0$.

We continue the proof of Theorem 1 in the general case; for this we consider the problem

$$\mathscr{A}_0 u = f(y), \qquad u|_{\Gamma_0} = 0.$$

As previously, we consider two cases independently: $\bar{b}_1 > 0$ and $\bar{b}_1 \leqslant 0$. Suppose first that $\bar{b}_1 \leqslant 0$. We denote by $u_N(y)$ the solution of the problem

$$\mathscr{A}_0 u_N = f(y), \qquad u_N|_{\Gamma_0} = 0, \qquad u_N|_{\Gamma_N} = 0.$$

We multiply both sides of this equation by $p(y)u_N(y)$ and integrate it over the cylinder $0 \leqslant y_1 \leqslant N$. Integrating by parts, we obtain

$$\begin{aligned} -\int_0^N dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u_N(y) \frac{\partial}{\partial y_j} u_N(y)\, dy' \\ = \int_0^N dy_1 \int_{\Gamma_{y_1}} f(y) p(y) u_N(y)\, dy'. \end{aligned} \tag{13}$$

For brevity we denote the function $f(y)p(y)$ by $\tilde{f}(y)$. Just as $f(y)$, it decays exponentially as $y_1 \to \infty$. We shall prove that $\tilde{f}(y)$ can be represented in the form

$$\tilde{f}(y) = \frac{\partial}{\partial y_i} g_i(y),$$

where all the $g_i(y)$, $i = 1, \ldots, n$, decay exponentially as $y_1 \to \infty$. Indeed, let $\tilde{f}_1(y_1)$ be the average of $\tilde{f}(y)$ over the section Γ_{y_1}:

$$\tilde{f}_1(y_1) = \int_{\Gamma_{y_1}} \tilde{f}(y)\, dy'.$$

As $g_1(y)$ we choose the function

$$g_1(y) = -\int_{y_1}^{\infty} \tilde{f}_1(z)\, dz.$$

It is clear that $g_1(y)$ decays exponentially as $y_1 \to \infty$. The difference $\tilde{f}_2(y) = \tilde{f}(y) - \tilde{f}_1(y)$ has zero mean over Γ_{y_1} for almost all y_1. We define the function $g_2(y)$ as follows:

$$g_2(y) = \int_0^{y_2} dz \int_{T^{n-2}} \tilde{f}_2(y_1, z, y_3, \ldots, y_n)\, dy_3 \cdots dy_n.$$

Continuing this process, we find the desired representation for $\tilde{f}(y)$.

Equality (13) can now be rewritten as follows:

$$\int_0^N dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u_N(y) \frac{\partial}{\partial y_j} u_N(y)\, dy' = \int_0^N dy_1 \int_{\Gamma_{y_1}} g_i(y) \frac{\partial}{\partial y_i} u_N(y)\, dy'.$$

From this by the exponential decay of the functions $g_i(y)$ we obtain

$$\|\nabla u_N\|_{L^2\{0<y_1<N\}} \leqslant c(f), \tag{14}$$

where the constant $c(f)$ does not depend on N. Noting that all the $u_N(y)$ vanish on Γ_0, from (14) we easily find that for any k, $1 < k < N-1$,

$$\|u_N\|_{L^2\{k-1<y_1<k+1\}} \leqslant cc(f)k. \tag{15}$$

From this by known estimates [9] we obtain

$$\|u_N\|_{C^0(\Gamma_k)} \leqslant c_2 c(f)k. \tag{16}$$

Estimates of the Hölder norms of $u_N(y)$ which are uniform in N (see [9]) enable us to choose from the sequence $u_N(y)$ a subsequence $u_{N'}(y)$ which converges uniformly on each compact set to some function $u(y)$. It is clear that $u(y)$ is a solution of the equation $\mathscr{A}_0 u = f(y)$ and satisfies the boundary conditions $u|_{\Gamma_0} = 0$. The estimates (15) and (16) remain valid for $u(y)$.

We now define the function $v_N(y, k)$ as the solution of the problem

$$\mathscr{A}_0 v_N = f(y), \qquad v_N|_{\Gamma_k} = 0, \qquad v_N|_{\Gamma_N} = 0.$$

For $y_1 > k$ the function $z_N(y, k) = v_N(y, k) - u_N(y)$ satisfies the equation

$$\mathscr{A}_0 z_N = 0, \qquad z_N|_{\Gamma_k} = -u_N|_{\Gamma_k}, \qquad z_N|_{\Gamma_N} = 0.$$

From the sequence $v_N(y, k)$ for each fixed k it is possible to select a subsequence converging to some function $v(y, k)$. In this case it is easy to verify that the function

$z(y, k) = v(y, k) - u(y)$ is a bounded solution of the problem

$$\mathscr{A}_0 z = 0, \qquad z|_{\Gamma_k} = -u|_{\Gamma_k}; \tag{17}$$

and hence from the part of the theorem already proved it follows that the oscillation of $z(y, k)$ on the set $2k \leqslant y_1 \leqslant 2k + 1$ does not exceed $c_3 k e^{-\alpha k}$. From the exponential decay of the right side $f(y)$ and estimates of the type (16) for $v(y, k)$ we obtain

$$\|v(\cdot, k)\|_{C^0\{2k \leqslant y_1 \leqslant 2k+1\}} \leqslant c_4 k e^{-\beta k};$$

here β is the coefficient in the argument of the exponential function in the estimate of the rate of decay of $f(y)$. For $y_1 > k$ the function $u(y)$ can be represented as the sum $u(y) = v(y, k) + z(y, k)$, and hence the oscillation of $u(y)$ on $\{2k \leqslant y_1 \leqslant 2k + 1\}$ does not exceed $c_5 e^{-\alpha_1 k}$, $\alpha_1 > 0$. From this it follows immediately that $u(y)$ converges exponentially to a constant.

The case $\bar{b}_1 > 0$ is considered similarly; the only difference is that in problem (17) there is the additional condition at infinity $z(y, k)|_{y_1 = \infty} = 0$. The proof of the theorem is complete.

REMARK 1. By analyzing the proof of the theorem it is easy to verify that in the estimate

$$|u(y) - \mu| \leqslant c e^{-\alpha y_1}$$

the quantity α depends only on the operator $\mathscr{A}_0$ and the number β present in the formulation of the theorem, while the constant c further depends also on $\|\varphi\|_{L^\infty(\Gamma_0)}$, $\|e^{\beta y_1} f\|_{L^\infty((0,+\infty)\times T^{n-1})}$ and μ.

REMARK 2. If in the hypotheses of Theorem 1 the coefficients of $\mathscr{A}_0$ are continuously differentiable l times, $l \geqslant 2$, while the right side $f(y)$ decays exponentially together with its derivatives to order l, then all the derivatives of the bounded solution $u(y)$ of order no higher than $l + 1$ also decays exponentially. This follows immediately from Schauder's inequalities.

Periodicity of the coefficients of problem (2′) in the variable y_1 played a fundamental role in the theorem just proved. The condition of periodicity of the coefficients of $\mathscr{A}_0$ in y_1 can be replaced by a considerably weaker condition. Namely, we have the following result.

THEOREM 2. *Suppose that the coefficients $a_{ij}(y)$ and $b_i(y)$ of the operator $\mathscr{A}_0$ are bounded and measurable, and that in the cylinder $\{0 < y_1 < \infty\}$ there exists a solution $p(y)$ of the homogeneous, formally adjoint equation $\mathscr{A}_0^* p = 0$ which for all y satisfies the inequality*

$$0 < \kappa_0 \leqslant p(y) \leqslant \kappa_1 < \infty.$$

Then for any $k > 0$ there exists the limit

$$\Lambda^k = \lim_{N\to\infty} \int_N^{N+k} dy_1 \int_{\Gamma_{y_1}} \left(b_1(y) p(y) - a_{j1}(y) \frac{\partial}{\partial y_j} p(y) \right) dy',$$

the quantities Λ^k are connected by the relation $\Lambda^k = k\Lambda^1$, and all the assertions of Theorem 1 continue to hold if in the hypotheses the quantity $\bar{b}_1$ is replaced by Λ^1 and the additional condition $\|\varphi\|_{H^{1/2}(\Gamma_0)} < \infty$ is imposed on the boundary condition $\varphi(y)$.

PROOF. As in the proof of Theorem 1, we first consider problem (2′) with zero right side. The existence of a bounded solution in this case can be obtained just as in Theorem 1. We observe only that the existence of a solution of the problem

$$\mathscr{A}_0 u_N = 0, \qquad u_N|_{\Gamma_0} = \varphi(y), \qquad u_N|_{\Gamma_N} = 0$$

for each N is ensured by the finiteness of the $H^{1/2}$-norm of $\varphi(y)$. We shall prove that any bounded solution of (2′) stabilizes to a constant as $y_1 \to \infty$. For this we multiply (2′) by the function $p(y)u(y)$ and integrate this equation as in Lemma 3. As a result we obtain

$$\begin{aligned} -\int_k^{k+s} d\eta \int_\eta^{N+\eta} dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} u(y)\, dy' \\ + \left(\int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} u(y) p(y) a_{1j}(y) \frac{\partial}{\partial y_j} u(y)\, dy' \right. \\ \left. - \frac{1}{2} \int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} \left(a_{i1}(y) \frac{\partial}{\partial y_i} p(y) - b_1(y) p(y) \right) u^2(y)\, dy' \right) \Bigg|_{t=k}^{k+N} = 0. \end{aligned} \tag{18}$$

Since $u(y)$ and $p(y)$ are solutions of $\mathscr{A}_0 u = 0$ and $\mathscr{A}_0^* p = 0$ respectively, the boundedness of these functions implies the boundedness of their H^1-norms in the cylinders $\{N < y_1 < N + s\}$ uniformly with respect to N in $(0, +\infty)$. Therefore, from (18) we find that

$$\int_0^\infty dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} u(y)\, dy' < \infty.$$

From the convergence of the last integral we obtain

$$\lim_{N \to \infty} \|\nabla u\|_{L^2\{N < y_1 < N+s\}} = 0.$$

Using Poincaré's inequality, it can be deduced from this that for some numerical function $\mu(N)$ the difference $u(y) - \mu(N)$ tends to zero in the norm of $L^2\{N < y_1 < N + s\}$ as $N \to \infty$. Therefore, according to [9], the maximum of $|u(y) - \mu(N)|$ in the cylinder $\{N + s/3 \leqslant y_1 \leqslant N + 2s/3\}$ also tends to zero as $N \to \infty$.

Further, using the methods of [8], it is possible to prove that the maximum principle holds for the equation $\mathscr{A}_0 u = 0$ in each cylinder $\{N < y_1 < N + s\}$.

From what has been said and from the boundedness of the solution $u(y)$ it now follows that the maximum and minimum of $u(y)$ on the section Γ_N have finite, equal limits as $N \to \infty$. Of course, this is equivalent to the stabilization of $u(y)$ to a constant.

Using the methods of Lemma 4, it is possible to prove that the uniqueness of a bounded solution of (2′) implies the existence of the limit Λ^k, $\Lambda^k = k\Lambda^1$, where $\Lambda^1 \leqslant 0$.

As in the proof of Lemma 3, it is possible to show that if for any real μ there exists a solution of (2′) converging to μ to infinity, then the limit Λ^k exists, $\Lambda^k = k\Lambda^1$, and $\Lambda^1 > 0$.

Thus, to complete the proof of Theorem 2 in the present special case it remains to verify that $u(y)$ stabilizes to a constant at an exponential rate.

Suppose first that $\Lambda' \leqslant 0$. It can be shown that in this case the maximum of $u(y)$ on the section Γ_N decays monotonically with respect to N, while the analogous minimum increases monotonically. By subtracting a constant from $u(y)$ if necessary, we may assume with no loss of generality that $u|_{y_1=\infty} = 0$. This solution $u(y)$ for any N satisfies the inequality

$$\|u\|_{L^2\{N<y_1<N+s\}} \leqslant c(s) \|\nabla u\|_{L^2\{N-s/2 \leqslant y_1 < N+s\}}. \tag{19}$$

We shall not consider the proof of this inequality in detail; we note only that here estimates of the maximum modulus of $u(y)$ in terms of its L^2-norm in a broader domain are used. In (18) we now put N equal to ∞. From the easily verified relation

$$\int_t^{t+s} d\eta \int_\eta^\infty dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} u(y)\, dy' = \int_t^\infty d\eta \int_\eta^{\eta+s} dy_1 \int_{\Gamma_{y_1}} p(y) a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} u(y)\, dy'$$

and (18) with $N = \infty$ we obtain

$$\int_t^\infty d\eta \int_\eta^{\eta+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' \leqslant c\left(\int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' + \max_{t \leqslant y_1 \leqslant t+s} u^2(y)\right),$$

which in turn together with (19) implies the following estimate (see [9]):

$$\int_t^\infty d\eta \int_\eta^{\eta+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' \leqslant c \int_{t-s}^{t+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy';$$

here we have used the fact that the maximum of $u^2(y)$ over Γ_N does not increase. For the positive, continuous function

$$I_1(t) = \int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy'$$

we have obtained

$$\int_t^\infty I_1(\eta)\, d\eta \leqslant c(I_1(t) + I_1(t-s)).$$

It can be shown that this inequality implies

$$\int_t^\infty I_1(\eta)\, d\eta \leqslant ce^{-\alpha t}, \qquad \alpha > 0.$$

From this it is easy to obtain the exponential rate of stabilization of $u(y)$.

In the case $\Lambda^1 > 0$ we define the function

$$I_2(t) = \frac{1}{2s} \int_{t-s/2}^{t+3s/2} dy_1 \int_{\Gamma_{y_1}} u(y)\, dy'.$$

As previously, we here assume that $u|_{y_1=\infty} = 0$; therefore, from (18) it is not hard to obtain

$$\begin{aligned}
&\int_t^\infty d\eta \int_\eta^{\eta+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' \\
&\quad + I_2^2(t) \int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} \left(b_1(y)p(y) - a_{i1}(y)\frac{\partial}{\partial y_i} p(y)\right) dy' \leqslant \delta I_2^2(t) \\
&\quad + \left(c + \frac{c_1}{\delta}\right)\left(\int_t^{t+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' + \max_{t \leqslant y_1 \leqslant t+s} (u(y) - I_2(t))^2\right);
\end{aligned}$$

here δ is an arbitrary positive number. Choosing it equal to $\Lambda^1/2$ and considering the estimate

$$\max_{t \leqslant y_1 \leqslant t+s} (u(y) + I_2(t))^2 \leqslant c\|\nabla u\|_{L^2\{t-s/2<y_1<t+3s/2\}},$$

we obtain for $u(y)$ the inequality

$$\int_t^\infty d\eta \int_\eta^{\eta+s} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy' \leqslant c \int_{t-s/2}^{t+3s/2} dy_1 \int_{\Gamma_{y_1}} |\nabla u(y)|^2\, dy',$$

from which, as above, we deduce an exponential estimate for the rate of stabilization of $u(y)$ to a constant.

The transition to a nonzero right side is accomplished as in Theorem 1.

REMARK 1. The condition of boundedness of the right side $f(y)$ was used in the proof of Theorems 1 and 2 only to obtain estimates of the type $\|u\|_{C^0(Q')} \leqslant c\|u\|_{L^2(Q)}$. However, according to [9], these estimates are valid under much less stringent conditions on $f(y)$. Therefore, in the formulations of both theorems the conditions on the right side $f(y)$ can be substantially relaxed.

REMARK 2. The assertions of both theorems remain valid also for right sides $f(y)$ in the space $H^{-1}\{0 < y_1 < \infty\}$, but in this case stabilization of the solution occurs not in the norm of C^0 but in the norm of L^2 over a section.

§2

In this section we construct an asymptotic expansion of the solution of problem (1) in powers of the small parameter ε under the assumption that the coefficients of the operator $\mathscr{A}^\varepsilon$ satisfy certain conditions which we call regularity conditions. The results of §1 are used to study the asymptotics near the boundary planes.

We henceforth assume that the coefficients $a_{ij}(x, y)$, $b_i(x, y)$ and $c(x, y)$ of the operator $\mathscr{A}^\varepsilon$ are smooth functions periodic in the second argument y and the derivatives of any order of the coefficients are bounded uniformly with respect to x and y.

Before formulating the main regularity condition, we introduce the following notation. We denote by $\mathscr{A}_0$ the operator

$$\mathscr{A}_0 = \frac{\partial}{\partial y_i} a_{ij}(x, y) \frac{\partial}{\partial y_j} + b_i(x, y) \frac{\partial}{\partial y_i},$$

which depends on x as a parameter, and by $p(x, y)$ a solution of the equation $\mathscr{A}_0^* p = 0$ which is periodic in y and is normalized by the condition

$$\int_{T^n} p(x, y)\, dy = 1;$$

under the assumptions made above regarding the coefficients of $\mathscr{A}^\varepsilon$ the function $p(x, y)$ is a smooth function of both arguments. Finally, we define the auxiliary vector field $\bar{b}_i(x)$:

$$\bar{b}_i(x) = \int_{T^n} \left(\frac{\partial}{\partial y_j} a_{ij}(x, y) + b_i(x, y) \right) p(x, y)\, dy.$$

DEFINITION. We say that problem (1) is *regular* if $|\bar{b}_1(x)| \geqslant E > 0$ everywhere in the layer $a \leqslant x_1 \leqslant b$.

To be specific, we henceforth assume that $\bar{b}_1(x) > 0$.

We shall begin with the formal construction of an asymptotic expansion without having for the time being either the existence of a solution $u^\varepsilon(x)$ or any estimates. Within the layer we seek a solution in the form of a series

$$u^\varepsilon(x) \sim u_0(x) + \sum_{k=1}^\infty \varepsilon^k \left(u_k\left(x, \frac{x}{\varepsilon}\right) + \mu_k(x) \right); \tag{20}$$

here we seek functions $u_k(x, y)$ which are all periodic in the second argument y. We represent $\mathscr{A}^\varepsilon$ in the following form:

$$\mathscr{A}^\varepsilon = \Bigg(\varepsilon^{-1}\left[\frac{\partial}{\partial y_i}a_{ij}(x, y)\frac{\partial}{\partial y_j} + b_i(x, y)\frac{\partial}{\partial y_i}\right]$$
$$+\varepsilon^0\left[\frac{\partial}{\partial x_i}a_{ij}(x, y)\frac{\partial}{\partial y_j} + \frac{\partial}{\partial y_i}a_{ij}(x, y)\frac{\partial}{\partial x_j} + b_i(x, y)\frac{\partial}{\partial x_i} + c(x, y)\right] \tag{21}$$
$$+ \varepsilon^1\left[\frac{\partial}{\partial x_i}a_{ij}(x, y)\frac{\partial}{x_j}\right]\Bigg)\Bigg|_{y=x/\varepsilon}$$
$$= \varepsilon^{-1}\mathscr{A}_0 + \varepsilon^0\mathscr{A}_1 + \varepsilon^1\mathscr{A}_2.$$

In the equation $\mathscr{A}^\varepsilon u^\varepsilon = f$ we represent $\mathscr{A}^\varepsilon$ in the form (21), and in place of u^ε we substitute the series (20). Equating the coefficients of like powers of ε, we obtain the following infinite system of equations:

$$\begin{aligned}
&\varepsilon^0: \mathscr{A}_0 u_1 + \mathscr{A}_1 u_0 = f,\\
&\varepsilon^1: \mathscr{A}_0 u_2 + \mathscr{A}_1 u_1 + \mathscr{A}_2 u_0 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\varepsilon^k: \mathscr{A}_0 u_{k+1} + \mathscr{A}_1 u_k + \mathscr{A}_2 u_{k-1} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

We begin the investigation of this system with the equation for ε^0. The condition for its solvability in the space of functions periodic in the variable y is the orthogonality of the functions $f(x) - \mathscr{A}_1 u_0$ and $p(x, y)$ in the space $L^2(T^n)$ for each x. This condition can be rewritten in the form of an equation for $u_0(x)$:

$$\bar{b}_i(x)\frac{\partial}{\partial x_i}u_0(x) + \bar{c}(x)u_0(x) = f(x); \tag{22}$$

here

$$\bar{c}(x) = \int_{T^n} c(x, y)p(x, y)\,dy.$$

The averaged equation (22), which the leading term of the asymptotic expansion $u_0(x)$ satisfies, is an equation of first order; therefore, of the two boundary conditions in problem (1) we can impose only one. For this condition we choose

$$u_0|_{x_1=b} = \varphi_2(x);$$

this choice is explained by our assumption that $\bar{b}_1 > 0$. Indeed, the boundary condition must be imposed on that boundary plane onto which there pass trajectories beginning within the layer of the equation generated by the field $\bar{b}_i(x)$: $\dot{x} = \bar{b}(x)$. After the function $u_0(x)$ has been found, from the same equation for ε^0 we find the function $u_1(x, y)$; the solution $u_1(x, y)$ is determined up to a function depending only on x. We choose this solution so that it depends smoothly on x. This can be achieved, for example, by requiring that $u_1(x, y)$ have zero mean value over a period for each x.

So far we have not kept track of the boundary conditions. We note that on the boundary $\{x_1 = a\}$ the difference between $\varphi(x)$ and the first two terms of the asymptotic series (20) has order $O(1)$, while on the boundary $\{x_1 = b\}$ it has order $O(\varepsilon)$. In order to deal with the errors in the boundary conditions, in a neighborhood of the boundary $\{x_1 = a\}$ we add to the series (20) an asymptotic series

$$z_0\left(x, \zeta, \frac{x'}{\varepsilon}\right) + \varepsilon z_1\left(x, \zeta, \frac{x'}{\varepsilon}\right) + \cdots, \qquad \zeta = \frac{x_1 - a}{\varepsilon}, \tag{23}$$

and in a neighborhood of $\{x_1 = b\}$ a series

$$\varepsilon v_1\left(x, \theta, \frac{x'}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \theta, \frac{x'}{\varepsilon}\right) + \cdots, \qquad \theta = \frac{x_1 - b}{\varepsilon}. \tag{24}$$

We seek all functions $z_k(x, \zeta, y')$ and $v_k(x, \theta, y')$ periodic in the variables $y' = (y_2, \ldots, y_n)$ which decay exponentially in ζ and θ toward the interior of the layer.

Before seeking these corrections, in a neighborhood of each of the boundary hyperplanes we expand the coefficients $a_{ij}(x, y)$, $b_i(x, y)$, and $c(x, y)$ in Taylor series in the variable x_1:

$$\begin{aligned} a_{ij}(x, y) &\sim a_{ij}(a, x', y) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial x_1^k} a_{ij}(a, x', y)(x_1 - a)^k, \\ b_i(x, y) &\sim b_i(a, x', y) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial x_1^k} b_i(a, x', y)(x_1 - a)^k, \\ c(x, y) &\sim c(a, x', y) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial x_1^k} c(a, x', y)(x_1 - a)^k \end{aligned} \tag{25}$$

and we form analogous expansions near the hyperplanes $\{x_1 = b\}$.

Above we represented $\mathscr{A}^\varepsilon$ as the sum (21) of operators acting on functions of the $2n$ independent variables x, y. In the same way, here we write $\mathscr{A}^\varepsilon$ in the form

$$\mathscr{A}^\varepsilon = \varepsilon^{-1}\tilde{\mathscr{A}}_0 + \varepsilon^0\tilde{\mathscr{A}}_1 + \varepsilon\tilde{\mathscr{A}}_2,$$

where $\tilde{\mathscr{A}}_0$, $\tilde{\mathscr{A}}_1$, and $\tilde{\mathscr{A}}_2$ act on functions of the variables x, ζ and y' and have a form entirely analogous to $\mathscr{A}_0$, $\mathscr{A}_1$, and $\mathscr{A}_2$. We shall not write them out in detail.

In each of the operators $\tilde{\mathscr{A}}_0$, $\tilde{\mathscr{A}}_1$, and $\tilde{\mathscr{A}}_2$ we now substitute in place of the coefficients $\tilde{a}_{ij}(x, \zeta, y')$, $\tilde{b}_i(x, \zeta, y')$, and $\tilde{c}(x, \zeta, y')$ the series (25), replacing the quantities $(x_1 - a)^k$ and $(x_1 - b)^k$ by $(\varepsilon\zeta)^k$ and $(\varepsilon\theta)^k$ respectively. Applying the operator $\mathscr{A}^\varepsilon$ written in this form to the asymptotic series (23) and (24) and equating coefficients of like powers of ε, we obtain two systems of equations one of which must be satisfied by the functions $z_k(x, \zeta, y')$ and the other by the functions $v_k(x, \theta, y')$.

To construct $z_0(x, \zeta, y')$ we use the equation for ε^{-1}:

$$\begin{aligned} &\frac{\partial}{\partial \zeta}\tilde{a}_{11}(a, x', \zeta, y')\frac{\partial}{\partial \zeta}z_0(x, \zeta, y') + \frac{\partial}{\partial \zeta}\tilde{a}_{1j}(a, x', \zeta, y')\frac{\partial}{\partial y_j}z_0(x, \zeta, y') \\ &\quad + \frac{\partial}{\partial y_i}\tilde{a}_{i1}(a, x', \zeta, y')\frac{\partial}{\partial \zeta}z_0(x, \zeta, y') + \frac{\partial}{\partial y_i}\tilde{a}_{ij}(a, x', \zeta, y')\frac{\partial}{\partial y_j}z_0(x, \zeta, y') \\ &\quad + b_1(a, x', \zeta, y')\frac{\partial}{\partial \zeta}z_0(x, \zeta, y') + b_i(a, x', \zeta, y')\frac{\partial}{\partial y_i}z_0(x, \zeta, y') = 0, \\ &\qquad\qquad z_0|_{\zeta=0} = \varphi_1(x) - u_0(a, x'); \end{aligned} \tag{26}$$

the indices i and j in this equation vary from 2 to n; the boundary conditions are chosen so as to eliminate the error of zeroth order on the boundary $\{x_1 = a\}$. The regularity condition enables us to apply Theorem 1 to problem (26); hence there exists a solution $z_0(x, \zeta, y')$ which decays exponentially together with its derivatives as $\zeta \to \infty$. In a neighborhood of $\{x_1 = b\}$ we now find $v_1(x, \theta, y')$ from the equation for ε^0:

$$\frac{\partial}{\partial\theta} a_{11}(b, x', \theta, y') \frac{\partial}{\partial\theta} v_1(x, \theta, y') + \frac{\partial}{\partial\theta} a_{1j}(b, x', \theta, y') \frac{\partial}{\partial y_j} v_1(x, \theta, y')$$

$$+ \frac{\partial}{\partial y_i} a_{i1}(b, x', \theta, y') \frac{\partial}{\partial\theta} v_1(x, \theta, y') + \frac{\partial}{\partial y_i} a_{ij}(b, x', \theta, y') \frac{\partial}{\partial y_j} v_1(x, \theta, y')$$

$$+ b_1(b, x', \theta, y') \frac{\partial}{\partial\theta} v_1(x, \theta, y') + b_i(b, x', \theta, y') \frac{\partial}{\partial y_i} v_1(x, \theta, y') = 0,$$

$$v_1|_{\theta=0} = -u_1\left(b, x', \frac{b}{\varepsilon}, y'\right).$$

From Theorem 1 it follows that there exists a unique bounded solution of this problem. This solution stabilizes at an exponential rate as $\theta \to -\infty$ to a function $\nu(x')$ depending only on x'.

We can now constuct $\mu_1(x)$. For this we write out the solvability condition of the equation corresponding to ε' in the system of equations for the functions $u_k(x, y)$:

$$\bar{b}_i(x) \frac{\partial}{\partial x_i} \mu_1(x) + \bar{c}(x)\mu_1(x) = g(x);$$

here $g(x)$ is a known smooth function. To this equation we add the boundary condition $\mu_1|_{x_1=b} = -\nu(x)$.

In the same way at the kth step the equation for ε^{k-1} in the first system of equations enables us to find $u_k(x, y)$. From the corresponding equations in neighborhoods of the boundary hyperplanes $\{x_1 = a\}$ and $\{x_1 = b\}$ we then find the functions $z_{k-1}(x, \zeta, y')$ and $v_k(x, \theta, y')$. Theorem 1 ensures the existence of such functions of boundary-layer type. Finally, the solvability condition for the equation corresponding to ε^k gives an equation for determining $\mu_k(x)$.

Thus, corresponding to the original equation (1) we have formed the asymptotic series

$$u^\varepsilon(x) \sim u_0(x) + \left(\dot{z}_0(x, \zeta, y') + \sum_{k=1}^{\infty} \varepsilon^k \big(u_k(x, y) + \mu_k(x) + z_k(x, \zeta, y') + v_k(x, \theta, y')\big) \right)\Bigg|_{y=x/\varepsilon, \zeta=(x_1-a)/\varepsilon, \theta=(x_1-b)/\varepsilon}.$$

We denote by $\sigma_s^\varepsilon(x)$ the sum of the first s terms of this series.

PROPOSITION. *Suppose that in problem* (1) *the right side $f(x)$ and the boundary functions $\varphi_1(x)$ and $\varphi_2(x)$ are bounded together with all derivatives uniformly with respect to x. Then*

$$\mathscr{A}^\varepsilon \sigma_s^\varepsilon = f(x) + \varepsilon^{s-1} g_s^\varepsilon(x),$$

$$\sigma_s^\varepsilon|_{x_1=a} = \varphi_1(x) + \varepsilon^{s+1} h_{1,s}^\varepsilon(x), \qquad \sigma_s^\varepsilon|_{x_1=b} = \varphi_2(x) + \varepsilon^{s+1} h_{2,s}^\varepsilon(x),$$

where the functions $\sigma_s^\varepsilon(x)$, $g_s^\varepsilon(x)$, and $h_{i,s}^\varepsilon(x)$ satisfy the inequalities

$$\|\sigma_s^\varepsilon\|_{C^l\{(a,b)\times\mathbf{R}^{n-1}\}} + \|g_s^\varepsilon\|_{C^l\{(a,b)\times\mathbf{R}^{n-1}\}} + \sum_{i=1}^{2} \|h_{i,s}^\varepsilon\|_{C^l(\mathbf{R}^{n-1})}$$
$$\leqslant c(s, l)\varepsilon^{-l}\left(\|f\|_{C^{l+s}\{(a,b)\times\mathbf{R}^{n-1}\}} + \sum_{i=1}^{2} \|\varphi_i\|_{C^{l+s}(\mathbf{R}^{n-1})} \right), \qquad l \geqslant 0, s \geqslant 0. \tag{27}$$

We omit the proof, since it is obvious.

We shall now prove the existence of a solution of problem (1) for sufficiently small ε and obtain estimates of this solution in terms of the boundary functions and the right side.

THEOREM 3. *Suppose that problem* (1) *is regular and its coefficients* $a_{ij}(x, y)$, $b_i(x, y)$, *and* $c(x, y)$ *are bounded uniformly with respect to* x *and* y *together with their derivatives of the first two orders. Then there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *and for any sufficiently smooth* $f(x)$ *and* $\varphi(x)$ *which are uniformly bounded in the layer problem* (1) *has a unique bounded solution. Moreover,*

$$\|u^\varepsilon\|_{L^\infty\{(a,b)\times\mathbf{R}^{n-1}\}} \leqslant c\left(\|f\|_{L^\infty\{(a,b)\times\mathbf{R}^{n-1}\}} + \sum_{i=1}^{2}\|\varphi_i\|_{L^\infty(\mathbf{R}^{n-1})}\right). \tag{28}$$

PROOF. We first prove the theorem under the assumption that $c(x, y) \leqslant 0$. This restriction will be removed below. We consider the operator

$$\mathscr{B}^\varepsilon = \varepsilon\frac{\partial}{\partial x_i}a_{ij}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j} + b_i\left(x, \frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}$$

and the diffusion process $\xi_t^{\varepsilon,x}$ corresponding to it, which for brevity we denote by ξ_t^ε.

LEMMA 5. *Let* $\tau^\varepsilon(x)$ *be the Markov time at which the process* ξ_t^ε *first reaches the boundary of the layer* $(a, b) \times \mathbf{R}^{n-1}$. *Then there exists a constant* c *not depending on* ε *or* x *such that* $\mathsf{E}\tau^\varepsilon(x) < c$.

PROOF. Let $v_0(x)$ be the solution of the auxiliary problem

$$\bar{b}_i(x)\frac{\partial}{\partial x_i}v_0(x) = -1, \qquad v_0|_{x_1=b} = 0.$$

It follows from the regularity condition that $v_0(x)$ is twice continuously differentiable in the layer. We find a function $v_1(x, y)$ periodic in y from the equation

$$\begin{aligned}\frac{\partial}{\partial y_i}a_{ij}(x, y)\frac{\partial}{\partial y_j}v_1(x, y) + b_i(x, y)\frac{\partial}{\partial y_i}v_1(x, y)\\ = -\frac{\partial}{\partial y_i}a_{ij}(x, y)\frac{\partial}{\partial x_j}v_0(x) - b_i(x, y)\frac{\partial}{\partial x_i}v_0(x) - 1.\end{aligned}$$

In order to uniquely determine the choice of solution $v_1(x, y)$ we assume that its mean on y for each x is equal to zero. From the construction of $v_0(x)$ and $v_1(x, y)$ we obtain

$$\mathscr{B}^\varepsilon\left(v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right)\right) = -1 + \varepsilon\gamma\left(x, \frac{x}{\varepsilon}\right),$$

in which $\gamma(x, y)$ is a function bounded in the layer. We now choose ε_0 so that $\varepsilon_0\gamma(x, y) < 1/2$ for all x and y. For suitable choice of the constant c_1 the sum $v_0(x) + \varepsilon v_1(x, x/\varepsilon) + c_1$ for all $\varepsilon < \varepsilon_0$ satisfies the following relations:

$$v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + c_1 > 0, \qquad \mathscr{B}^\varepsilon\left(v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + c_1\right) < -1/2.$$

Hence, according to [7],

$$\mathsf{E}\tau^\varepsilon(x) \leqslant 2\max\left(v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + c_1\right) < c_2.$$

Continuing the proof of the theorem, in the bounded cylinders $\{x\colon a < x_1 < b, |x'| < k\}$ we consider the sequence of problems

$$\mathscr{A}^\varepsilon u_k^\varepsilon = f, \qquad u_k^\varepsilon|_{|x'|=k} = 0, \qquad u_k^\varepsilon|_{x_1=a} = \varphi_1(x), \qquad u_k^\varepsilon|_{x_1=b} = \varphi_2(x). \tag{29}$$

For each k this problem has a unique solution for which there is the probabilistic representation

$$u_{(k)}^\varepsilon(x) = \mathsf{E}\left(\int_0^{\tau_k^\varepsilon(x)} f(\xi_t^\varepsilon)\exp\int_0^t c\left(\xi_s^\varepsilon, \frac{\xi_s^\varepsilon}{\varepsilon}\right) ds\,dt + \varphi^k\left(\xi_{\tau_k^\varepsilon(x)}^\varepsilon\right)\exp\int_0^{\tau_k^\varepsilon(x)} c\left(\xi_t^\varepsilon, \frac{\xi_t^\varepsilon}{\varepsilon}\right) dt\right);$$

here $t_k^\varepsilon(x)$ is the Markov time at which the process ξ_t^ε first reaches the boundary of the cylinder $\{x\colon a < x_1 < b, |x'| < k\}$; $\varphi^k(x)$ denotes a function which coincides with $\varphi(x)$ on the upper and lower bases of the cylinder and is equal to zero on the lateral surface.

It is easy to verify that each fixed ε the sequence $\tau_k^\varphi(x)$ converges as $k \to \infty$ to $\tau^\varepsilon(x)$; therefore, on the basis of Lemma 5 and Lebesgue's theorem we can pass to the limit in the preceding equality and find the bounded solution of (1):

$$u^\varepsilon(x) = \mathsf{E}\left(\int_0^{\tau^\varepsilon(x)} f(\xi_t^\varepsilon)\exp\int_0^t c\left(\xi_s^\varepsilon, \frac{\xi_s^\varepsilon}{\varepsilon}\right) ds\,dt + \varphi\left(\xi_{\tau^\varepsilon(x)}^\varepsilon\right)\exp\int_0^{\tau^\varepsilon(x)} c\left(\xi_t^\varepsilon, \frac{\xi_t^\varepsilon}{\varepsilon}\right) dt\right).$$

The estimate (28) is derived from this representation in elementary fashion by means of Lemma 5.

We shall prove uniqueness of the bounded solution. Using the strict Markov property of the process ξ_t^ε, it is possible to obtain the estimate

$$\mathbf{P}\left\{\left|\xi_{\tau_k^\varepsilon(x)}^\varepsilon - x\right| > N\right\} < ce^{-\delta N}, \qquad \delta > 0. \tag{30}$$

in which the constants c and δ do not depend on k, ε or x.

We now suppose that there exists a nonzero bounded solution of the equation

$$\mathscr{A}^\varepsilon v = 0, \qquad v|_{x_1=a} = 0, \qquad v|_{x_1=b} = 0,$$

and we consider a sequence of problems of the form

$$\mathscr{A}^\varepsilon v_k = 0, \qquad v_k|_{|x'|=k} = v|_{|x'|=k}, \qquad v_k|_{x_1=a} = 0, \qquad v_k|_{x_1=b} = 0.$$

The functions $v_k(x)$ so defined coincide with $v(x)$ inside the cylinder $\{x\colon a < x_1 < b, |x'| < k\}$. On the other hand, using the probabilistic representation of $v_k(x)$ and (30), we find that for each fixed x the quantity $v_k(x)$ tends to zero as $k \to \infty$. This contradiction proves the uniqueness of a bounded solution.

We now show how to eliminate the restriction $c(x, y) \leqslant 0$. To this end we prove the following lemma.

LEMMA 6. *There exists a number N_0 such that for any $\theta > 0$ the following inequality holds uniformly with respect to x and $\varepsilon < \varepsilon_0(\theta)$*:

$$\mathbf{P}\{\tau^\varepsilon(x) > N\} \leqslant ce^{-\theta(N-N_0)}. \tag{31}$$

PROOF. Since the process ξ_t^ε possesses the Markov property, it suffices to prove the existence of an N_0 such that for each $\beta > 0$ for all x and $\varepsilon < \varepsilon_0(\beta)$ the following inequality holds:

$$\mathbf{P}\{\tau^\varepsilon(x) > N_0\} < \beta. \tag{32}$$

We shall show that for N_0 it is possible to choose a constant bounded above by $2\mathsf{E}\tau^\varepsilon(x)$.

According to [11], $\mathsf{E}\tau^\varepsilon(x)$ and $\mathsf{E}(\tau^\varepsilon(x))^2$ satisfy the equations

$$\mathscr{B}^\varepsilon(\mathsf{E}\tau^\varepsilon(x)) = -1 \tag{33}$$

and

$$\mathscr{B}^\varepsilon\big(\mathsf{E}(\tau^\varepsilon(x))^2\big) = -2\mathsf{E}\tau^\varepsilon(x)$$

respectively with zero boundary conditions. Using these equations, we see by direct verification that the square of the variance $\mathsf{E}(\tau^\varepsilon(x))^2 - (\mathsf{E}\tau^\varepsilon(x))^2$ of the Markov time $\tau^\varepsilon(x)$ is a solution of the problem

$$\begin{gathered}\mathscr{B}^\varepsilon(\operatorname{Var}\tau^\varepsilon(x))^2 = 2\varepsilon a_{ij}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}\mathsf{E}\tau^\varepsilon(x)\frac{\partial}{\partial x_j}\mathsf{E}\tau^\varepsilon(x),\\ \operatorname{Var}\tau^\varepsilon(x)|_{x_1=a} = 0, \qquad \operatorname{Var}\tau^\varepsilon(x)|_{x_1=b} = 0.\end{gathered} \tag{34}$$

Since for nonpositive $c(x, y)$ the theorem has already been proved, we can estimate the variance of $\tau^\varepsilon(x)$ if we construct an approximate solution of the last problem.

A procedure for constructing the formal asymptotics of a solution of (33) was described in detail above. This asymptotic expression has the form

$$\mathsf{E}\tau^\varepsilon(x) \sim m_0(x) + \varepsilon m_1\left(x, \frac{x}{\varepsilon}\right) + q_0\left(x', \frac{x_1 - a}{\varepsilon}, \frac{x'}{\varepsilon}\right) + \cdots. \tag{35}$$

Substituting now the first several terms of (35) into the right side of (34) in place of $\mathsf{E}\tau^\varepsilon(x)$, we obtain an asymptotic expression for this right side:

$$\varepsilon\tilde{m}_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^{-1}\tilde{q}_0\left(x', \frac{x_1 - a}{\varepsilon}, \frac{x'}{\varepsilon}\right) + \cdots;$$

here $\tilde{q}_0(x', \zeta, y')$ is the boundary layer near the hyperplane $\{x_1 = a\}$, and the function $\tilde{m}_1(x, y)$ is periodic in y. We shall seek a solution of (34) in the form of a series

$$(\operatorname{Var}\tau^\varepsilon(x))^2 \sim \varepsilon\tilde{\tilde{m}}_1\left(x, \frac{x}{\varepsilon}\right) + \tilde{\tilde{q}}_0\left(x', \frac{x_1 - a}{\varepsilon}, \frac{x'}{\varepsilon}\right) + \varepsilon\tilde{\tilde{q}}_1\left(x' \frac{x_1 - a}{\varepsilon}, \frac{x'}{\varepsilon}\right) + \cdots.$$

Applying the same methods as above, we construct a function $\tilde{\tilde{q}}_0(x', \zeta, y')$ of boundary-layer type near the hyperplane $\{x_1 = a\}$ and a function $\tilde{\tilde{m}}_1(x, y)$ periodic in y, etc.

In our expansion of $(\operatorname{Var}\tau^\varepsilon(x))^2$ all terms except for $\tilde{\tilde{q}}_0(x', \zeta, y')$ have order $O(\varepsilon)$. Since $q_0(x, \zeta, y')$ is exponentially small inside the layer, for all x, $a - d < x_1 < b$, we obtain the estimate $\operatorname{Var}\tau^\varepsilon(x) \leqslant c\sqrt{\varepsilon}$. This enables us to deduce (32) from the Tchebycheff inequality.

In order to prove (32) for all x in the layer, we extend the coefficients $a_{ij}(x, y)$ and $b_i(x, y)$ of the operator $\mathscr{B}^\varepsilon$ to a broader layer $(a - d, b) \times \mathbf{R}^{n-1}$ with preservation of smoothness and periodicity in y. Carrying out constructions analogous to the above in this broader layer, we prove (32) for all x in the original layer.

The estimate of Lemma 6 now implies the finiteness of $(\mathsf{E}\exp(\tilde{c}\tau^\varepsilon(x)))$ for all sufficiently small ε; here $\tilde{c}$ is chosen so that $|c(x, y)| < \tilde{c}$ for all x and y. According to [4], from this it follows that for all k a solution of problem (29) exists, is uniformly bounded with respect to k, and can be represented in probabilistic form. Passing to the limit as $k \to \infty$ in these equations, we find a bounded solution of problem (1).

The estimate (28) and the uniqueness of a bounded solution also follow from Lemma 6. The proof of the theorem is complete.

COROLLARY. *Suppose that in the hypotheses of Theorem* 3 *the coefficients and the data of problem* (1) *are uniformly bounded together with their first l derivatives, $l \geqslant 2$. Then for all $k \leqslant l$*

$$\|u^\varepsilon\|_{C^k\{(a,b)\times\mathbf{R}^{n-1}\}} \leqslant c(k)\varepsilon^{-k}\left(\|f\|_{C^k\{(a,b)\times\mathbf{R}^{n-1}\}} + \sum_{i=1}^{2}\|\varphi_i\|_{C^k(\mathbf{R}^{n-1})}\right). \tag{36}$$

For the proof it is necessary to use (28) and Schauder's inequality.

We are now in a position to justify the asymptotic expression for the solution of problem (1) constructed above.

THEOREM 4. *Suppose that problem* (1) *is regular and its coefficients $a_{ij}(x, y)$, $b_i(x, y)$, and $c(x, y)$ are bounded uniformly with respect to x and y together with their first l derivatives. Suppose further that the data of problem* (1) *are also uniformly bounded together with their derivatives to order l. Then the first l terms of the asymptotic expansion for the solution of problem* (1) *are well defined, and there are the following error estimates*:

$$\begin{aligned}&\|u^\varepsilon - \sigma_s^\varepsilon\|_{C^k\{(a,b)\times\mathbf{R}^{n-1}\}}\\ &\qquad\leqslant c(s,k)\varepsilon^{s+1-2k}\left(\|f\|_{C^{s+k}\{(a,b)\times\mathbf{R}^{n-1}\}} + \sum_{i=1}^{2}\|\varphi_i\|_{C^{s+k}(\mathbf{R}^{n-1})}\right);\\ &\qquad\qquad s>0,\ k\geqslant 0,\ s+k\leqslant l.\end{aligned}$$

The proof reduces to substituting (27) into (28) and (36).

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