ON THE LIMIT BEHAVIOR OF THE DOMAIN OF DEPENDENCE OF A HYPERBOLIC EQUATION WITH RAPIDLY OSCILLATING COEFFICIENTS

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ABSTRACT. In this paper, the behavior of the support of the solution to the Cauchy problem for a hyperbolic equation of the form

$$\frac{\partial^2}{\partial t^2} u^\varepsilon(x, t) - \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j} u^\varepsilon + b_i\left(x, \frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} u^\varepsilon + c\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon = 0$$

with periodic, rapidly oscillating coefficients $a_{ij}(y)$ and small parameter ε, is studied. It is proved that, for small ε, the domain of dependence of this equation is close to some convex cone with rectilinear generators.

In the case when the coefficients a_{ij} depend essentially on only one argument, e.g. y_1, this limit cone can be found explicitly. Its construction uses the Hamiltonian, which does not depend on ε and does not correspond to any differential operator.

Bibliography: 8 titles.

Consider the Cauchy problem for a hyperbolic equation of the form

$$\frac{\partial^2}{\partial t^2} u^\varepsilon(x, t) - \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j} u^\varepsilon(x, t) = 0,$$

$$u^\varepsilon(x, t)\big|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t} u^\varepsilon(x, t)\big|_{t=0} = \psi(x). \tag{1}$$

The matrix of coefficients $a_{ij}(y)$ is periodic on $\mathbf{R}^n$ with cube of periods $[0, 1]^n$, and satisfies the uniform ellipticity condition

$$\lambda_0 |\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \Lambda_0 |\xi|^2, \quad 0 < \lambda_0 < \Lambda_0,$$

for any $\xi \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$. The summation sign with respect to repeated indices will be omitted. Assume, also, that the coefficients $a_{ij}(y)$ are sufficiently smooth, for example $C^\infty(\mathbf{R}^n)$. We shall consider problem (1) only with initial data $\varphi(x)$ and $\psi(x)$ of compact support. Under the assumptions made above, problem (1) has a unique solution for each $\varepsilon > 0$. If the initial functions $\varphi(x)$ and $\psi(x)$ are sufficiently smooth, then, according to [1] and [2], as $\varepsilon \to 0$ the solutions $u^\varepsilon(x, t)$ of problem (1) converge in $L^2(\mathbf{R}^{n+1})$ to a function

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$u^0(x, t)$, which is the solution of the equation with constant coefficients q_{ij} (see [1] and [2]):

$$\frac{\partial^2}{\partial t^2} u^0(x, t) - q_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u^0(x, t) = 0,$$

$$u^0(x, t)|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t} u^0(x, t)|_{t=0} = \psi(x).$$

(2)

From the explicit formulas for the solution of equation (2) (see [3]), it follows that the support of the solution $u^0(x, t)$ of problem (2) lies in the set

$$K^0 = \bigcup_{x_0 \in \operatorname{supp} \varphi \cup \operatorname{supp} \psi} Q^0_{x_0},$$

where $Q^0_{x_0}$ is the closed cone

$$t^2 - q^{ij}(x_i - x_{0i})(x_j - x_{0j}) \geqslant 0.$$

Here (q^{ij}) is the matrix inverse of (q_{ij}).

For each fixed ε, a set K^ε can be singled out for problem (1), outside which the solution $u^\varepsilon(x, t)$ vanishes. This set has the form

$$K^\varepsilon = \bigcup_{x_0 \in \operatorname{supp} \varphi \cup \operatorname{supp} \psi} Q^\varepsilon_{x_0},$$

but the domains $Q^\varepsilon_{x_0}$ have a more complicated structure than $Q^0_{x_0}$.

The goal of this paper is the study of the domains of dependence $Q^\varepsilon_{x_0}$ for equation (1) as $\varepsilon \to 0$, and their connection with the domains $Q^0_{x_0}$. It will be proved that the domain $Q^\varepsilon_{x_0}$ tends, as $\varepsilon \to 0$, to some limit domain Q_{x_0} in the following sense:

$$\lim_{\varepsilon \to 0} \sup_{\substack{(x,t) \in \partial Q^\varepsilon_{x_0} \\ t < T}} \rho\big((x, t), \partial Q_{x_0}\big) = 0.$$

(3)

In contrast to the domains $Q^\varepsilon_{x_0}$, which have a rather complicated structure, the domain Q_{x_0} is a cone with rectilinear generators, whose form does not depend on the point x_0. A cross section of this cone by the plane $\{t = \text{const}\}$ contains a convex set. If the coefficients of (1) depend only on the one variable y_1, explicit formulas can be derived for the boundary ∂Q_{x_0}, where the limit (3) in this case exists uniformly in T and the following estimate is valid:

$$\sup_{(x,t) \in \partial Q^\varepsilon_{x_0}} \rho\big((x, t), \partial Q_{x_0}\big) < C\sqrt{\varepsilon}.$$

The domain Q_{x_0} turns out to be wider than $Q^0_{x_0}$.

If the coefficients of (1) depend on all the variables, then explicit formulas for the limit domain of dependence can be derived only in isolated special cases. The obstacle here is the extremely nonregular behavior of the bicharacteristics of problem (1) (see [5] and [6]). At the conclusion, some examples will be given.

§1. Existence of the limit domain of dependence

First of all, let us point out two procedures for constructing the set $Q^\varepsilon_{x_0}$. The first of these consists of the following. Extend out from the point x_0 all possible bicharacteristics of problem (1) lying on the zero level of the Hamiltonian. According to [4], there exists a domain in $\mathbf{R}^{n+1}_+$ whose boundary is formed by the projections of these bicharacteristics

onto (x, t)-space, outside of which these projections no longer exist. This domain is $Q^\varepsilon_{x_0}$. In order to construct the domain of dependence by the second procedure, at each point $(\tilde{x}, \tilde{t})$ of $\mathbf{R}^{n+1}_+$, consider the cone

$$(t - \tilde{t})^2 - a^{ij}\left(\frac{\tilde{x}}{\varepsilon}\right)(x_i - \tilde{x}_i)(x_j - \tilde{x}_j) \geqslant 0, \tag{4}$$

aligned in the positive direction along the t-axis. Further, from the point x_0, draw all possible curves whose tangents at all of their points lie inside the corresponding cones (4). The set which comprises these curves is $Q^\varepsilon_{x_0}$.

THEOREM 1. *There exists a convex cone Q_{x_0}, whose form does not depend on the point x_0, such that for all $T < \infty$*

$$\lim_{\varepsilon \to 0} \sup_{\substack{(x,t) \in \partial Q^\varepsilon_{x_0} \\ t < T}} \rho\left((x, t), \partial Q_{x_0}\right) = 0.$$

The convergence is uniform with respect to x_0.

PROOF. Denote by $\mathcal{P}^\varepsilon_{x_0}(\tilde{t})$ the cross section of the domain $Q^\varepsilon_{x_0}$ by the plane $\{t = \tilde{t}\}$. Let ε_0 be an arbitrary number less than one, say $\varepsilon_0 = \frac{1}{2}$. Set $\varepsilon_{k+1} = \varepsilon_k^2$. Now fix some direction $p = (p_1, \ldots, p_n) \neq 0$ and let $x_\varepsilon(p)$ be the point of intersection of the boundary $\partial \mathcal{P}^\varepsilon_{x_0}(1)$ with the ray emanating from x_0 in the direction p, which is farthest removed from x_0. We get a sequence of points $x_{\varepsilon_0}(p)$, $x_{\varepsilon_1}(p)$, $x_{\varepsilon_k}(p)$ on the ray p emanating from x_0. For convenience, let us introduce a coordinate on this ray—distance from x_0. Using the second procedure for constructing the limit domain of dependence, it is easy to show that there exists a constant C_1, not depending on the number k, such that

$$x_{\varepsilon_{k+1}}(p) > x_{\varepsilon_k}(p) - C_1 \varepsilon_k \tag{5}$$

for all k. Moreover, the segment $[x_0, x_{\varepsilon_k}(p) - C_1 \varepsilon_k]$ lies entirely in $\mathcal{P}^{\varepsilon_{k+1}}_{x_0}(1)$. In fact, since $x_{\varepsilon_k}(p)$ is contained in $\mathcal{P}^{\varepsilon_k}_{x_0}(1)$, then a curve passes through it, whose tangents at all of its points lie inside the corresponding cones (4).

Let us construct a curve which passes through the point $\bar{x}$ along the ray $x_0 + \tau p$, where

$$\bar{x} > x_{\varepsilon_k}(p) - C_1 \varepsilon_k.$$

For that, consider the set of points which differ from x_0 by an integral number of periods. Shrink the set obtained this way $(\varepsilon_k)^{-1}$ times and select from the resulting set the point x' closest to x_0. Since $|x_0 - x'| < \sqrt{n}\, \varepsilon_{k+1}$, x_0 and x' can be joined by a curve whose length does not exceed $\sqrt{n/\lambda}\,\varepsilon_{k+1}$. From the point x' we now extend a curve shrunk $(\varepsilon_k)^{-1}$ times which joins x_0 and $x_{\varepsilon_k}(p)$. Repeating this procedure up to the moment of time $t = 1 - C\varepsilon_k$, we will move away from the ray $x_0 + \tau p$ no more than $C\varepsilon_k$. The remaining time is expended in reaching this ray. It is clear that one can also get any point on the segment $[x_0, x_{\varepsilon_k}(p) - C_1\varepsilon_k]$.

From (5) and the boundedness above of the sequence $x_{\varepsilon_0}(p), \ldots, x_{\varepsilon_k}(p), \ldots$ we deduce the existence of the limit

$$\lim_{k \to \infty} x_{\varepsilon_k}(p) = \tilde{x}(p).$$

Upon singling out the quantity $\tilde{x}(p)$ on each ray emanating from x_0, we get a set which we shall denote by $\mathcal{P}_{x_0}(1)$. With the help of arguments analogous to those given above, one

can prove that the set $\mathscr{P}_{x_0}(1)$ is convex, and obtain the existence of the limit

$$\lim_{\varepsilon \to 0} \sup_{x \in \partial \mathscr{P}_{x_0}^\varepsilon(1)} \rho\left(x, \partial \mathscr{P}_{x_0}(1)\right) = 0.$$

It is now easy to check that one must take as Q_{x_0} the cone with vertex at $(x_0, 0)$ and with cross section $\mathscr{P}_{x_0}(1)$ by the plane $\{t = 1\}$. Actually, in order to construct $\mathscr{P}_{x_0}(t)$, it suffices to shrink all the curves and the period $(t)^{-1}$ times.

§2. Construction of the limit domain of dependence for an equation whose coefficients depend on one argument

In this section, we shall suppose that the coefficients a_{ij} depend essentially only on the argument y_1. For ease of presentation, consider first the case $n = 2$ and $a_{12} = a_{21} \equiv 0$. Problem (1) assumes the following form:

$$\frac{\partial^2}{\partial t^2} u^\varepsilon(x, t) - \frac{\partial}{\partial x_1} a_{11}\left(\frac{x_1}{\varepsilon}\right) \frac{\partial}{\partial x_1} u^\varepsilon(x, t) - \frac{\partial}{\partial x_2} a_{22}\left(\frac{x_1}{\varepsilon}\right) \frac{\partial}{\partial x_2} u^\varepsilon(x, t) = 0,$$

$$\tag{6}$$

$$u^\varepsilon(x, t)|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t} u^\varepsilon(x, t)|_{t=0} = \psi(x).$$

We shall investigate the behavior of the bicharacteristics of this equation; therefore, let us find the corresponding Hamiltonian:

$$H^\varepsilon(x, p, t, E) = E^2 - a_{11}\left(\frac{x_1}{\varepsilon}\right) p_1^2 - a_{22}\left(\frac{x_1}{\varepsilon}\right) p_2^2.$$

We are only interested in those bicharacteristics for which

$$H^\varepsilon(x, p, t, E) = 0. \tag{7}$$

From now on, we shall use the following version of (7):

$$p_1 = \pm \left(\sqrt{a_{11}\left(\frac{x_1}{\varepsilon}\right)}\right)^{-1} \sqrt{E^2 - a_{22}\left(\frac{x_1}{\varepsilon}\right) p_2^2}. \tag{8}$$

Note that p_2 is constant along the bicharacteristics, and that, by homogeneity of the Hamiltonian, it suffices to consider only the bicharacteristics on which $E = 1$.

From now on, we shall study separately two types of trajectories:

1. The quantity $(E^2 - a_{22}(x_1/\varepsilon) p_2^2)$ is positive for all x_1.

2. This quantity changes sign.

It is easy to see that trajectories of the second type stay inside the layer $\{|x_1 - x_{01}| < \varepsilon\}$ for all τ.

Let us consider the first case. It will be clear later on that trajectories of the first type carry basic information concerning the domain of dependence. From the continuity of $a_{22}(y)$ it follows that there exist points $x_1' \leqslant x_{01}$ and $x_1'' \geqslant x_{01}$ close to x_{01} such that

$$a_{22}\left(\frac{x_1'}{\varepsilon}\right) = a_{22}\left(\frac{x_1''}{\varepsilon}\right) = a_{22\max}.$$

We construct four bicharacteristics with the following initial conditions:

$$x_1|_{\tau=0} = x_{01}, \quad p_1|_{\tau=0} = \pm \left(\sqrt{a_{11}\left(\frac{x_{01}}{\varepsilon}\right)}\right)^{-1} \sqrt{1 - a_{22}\left(\frac{x_{01}}{\varepsilon}\right) p_2^2},$$

$$x_2|_{\tau=0} = x_{02}, \quad p_2|_{\tau=0} = \left(\sqrt{a_{22\max}}\right)^{-1}.$$

The projections of these bicharacteristics onto the (x_1, x_2)-plane divide this plane into four regions. Let us denote by W_1 the one which contains the set $\{(x_1, x_2) \mid x_1 > x_1''\}$. Analogously, W_2 is the region containing $\{(x_1, x_2) \mid x_1 < x_1'\}$. To each p_2 in the interval $(-(\sqrt{a_{22\max}})^{-1}, (\sqrt{a_{22\max}})^{-1})$, which we shall denote by $\mathcal{L}$ for brevity, we can associate two trajectories: on one of them $p_1 > 0$, and on the other $p_1 < 0$. If these trajectories are now projected onto (x, t)-space, then two mappings $(p_2, t) \to (x_1(p_2, t), x_2(p_2, t))$ arise, which we denote by R_1^ε and R_2^ε. The following three assertions are valid; we omit their proofs.

ASSERTION 1. R_1^ε and R_2^ε are diffeomorphisms of the region $\mathcal{L} \times (0, +\infty)$ onto W_1 and W_2, respectively.

ASSERTION 2. The following estimate is valid for any trajectory of the second type:

$$|x_2^\varepsilon(t) - x_{02}| \leqslant \sqrt{a_{22\max}}\, t.$$

ASSERTION 3. Consider the portion of the boundary of the domain $Q_{x_0}^\varepsilon$ included in the strip $x_1' \leqslant x_1 \leqslant x_1''$. There exists a function $\rho(\varepsilon)$, $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for any boundary point

$$\sqrt{a_{22\max}}\, t - \rho(\varepsilon) \leqslant |x_2^\varepsilon(t) - x_{02}|.$$

For now we shall not study in greater detail the behavior of the trajectories, but we will find the Hamiltonian which corresponds to the limit domain of dependence Q_{x_0}. This is naturally called the *mean Hamiltonian*. In order to construct this Hamiltonian, let us recall formula (8):

$$p_1 = \pm \sqrt{\frac{E^2 - a_{22}(x_1/\varepsilon)p_2^2}{a_{11}(x_1/\varepsilon)}}.$$

For bicharacteristics of the first type, in this formula there is either a plus sign for all τ, or a minus sign for all τ. It will be proved below that the mean Hamiltonian has the form

$$H(x, p, t, E) = \pm p_1 - M\left\{ \sqrt{\frac{E^2 - a_{22}(y)p_2^2}{a_{11}(y)}} \right\}, \qquad (9)$$

where $M\{f\}$ is the average of a periodic function f over its period. Before proving this, let us study the properties of the equations associated to the Hamiltonian (9):

$$\dot{x}_1 = \pm 1, \quad \dot{x}_2 = M\left\{ \frac{a_{22}(y)p_2}{D(y)} \right\},$$

$$\dot{p}_1 = 0, \quad \dot{p}_2 = 0, \quad \dot{t} = M\left\{ \frac{1}{D(y)} \right\}. \qquad (10)$$

Here $D(y)$ denotes the radical $\sqrt{a_{11}(y)(1 - a_{22}(y)p_2^2)}$. Note that the system (10) is defined only for p_2 in the interval $\mathcal{L}$. The projections of the solutions of the system (10) onto (x, t)-space are rays emanating from the point $(x_0, 0)$. By symmetry it suffices to consider just one sign in (9), e.g. the plus sign.

ASSERTION 4. *The mapping* $(p_2, t) \rightarrow (x_1(p_2, t), x_2(p_2, t))$ *given by* (10) *is a diffeomorphism of the region* $\mathcal{L} \times (0, +\infty)$ *onto* $\{(x_1, x_2) \mid x_1 > x_{01}\}$.

This assertion is a special case of Lemma 2, although it is easy to give a direct proof, which we omit here.

Thus, the two families of rays corresponding to the plus and minus signs in (10) bound some cone Q_{x_0} in $\mathbf{R}_+^{n+1}$. This is the desired limit domain of dependence.

THEOREM 2. *Let* $x^\varepsilon(t)$ *and* $\bar{x}(t)$ *be solutions of the systems of equations corresponding to the Hamiltonians* $H^\varepsilon(x, p, t, E)$ *and* $H(x, p, t, E)$, *respectively, and to the same initial condition*

$$x \mid_{\tau=0} = x_0, \quad p_2 \mid_{\tau=0} = p_2, \quad t \mid_{\tau=0} = 0, \quad E \mid_{\tau=0} = 1.$$

Here p_2 *is an arbitrary number in the interval* $\mathcal{L}$. *Then*

$$|x^\varepsilon(t) - \bar{x}(t)| \leqslant C\varepsilon \tag{11}$$

uniformly with respect to p_2, t *and* x_0.

PROOF. Consider both systems of equations

$$\dot{x}_1 = 2a_{11}\left(\frac{x_1}{\varepsilon}\right)p_1, \qquad\qquad \dot{x}_1 = 1,$$

$$\dot{x}_2 = 2a_{22}\left(\frac{x_1}{\varepsilon}\right)p_2, \qquad\qquad \dot{x}_2 = M\left\{\frac{a_{22}(y)p_2}{D(y)}\right\},$$

$$\dot{p}_1 = \frac{\partial}{\partial x_1}\left(a_{11}\left(\frac{x_1}{\varepsilon}\right)p_1^2 + a_{22}\left(\frac{x_1}{\varepsilon}\right)p_2^2\right), \quad \dot{p}_1 = 0,$$

$$\dot{p}_2 = 0, \qquad\qquad \dot{p}_2 = 0,$$

$$\dot{t} = 2, \qquad\qquad \dot{t} = M\left\{\frac{1}{D(y)}\right\}.$$

Let us introduce the new independent variable x_1 into the first of these. This change is permissible since, for trajectories of the first type, $\dot{x}_1 = 2a_{11}(x_1/\varepsilon)p_1 > 0$. After changing the variable, the system assumes the following form:

$$\dot{x}_1 = 1, \quad \dot{x}_2 = \frac{a_{22}(x_1/\varepsilon)p_2}{a_{11}(x_1/\varepsilon)p_1} = \frac{a_{22}(x_1/\varepsilon)p_2}{D(x_1/\varepsilon)},$$

$$\dot{p}_2 = 0, \quad \dot{t} = (D(x_1/\varepsilon))^{-1}.$$

On comparing the mean system of equations with the resulting system, it is easy to obtain the required estimate (11). The uniformity of this estimate with respect to t and x_0 is obvious. Let us prove its uniformity with respect to p_2. For that, observe that for x_1 satisfying the condition $x_1 - x_{01} = \varepsilon N$, where N is any integer, the trajectories pass through the same point. Now consider both trajectories at the same moment of time t: $t = t^\varepsilon(x_1^\varepsilon) = \bar{t}(\bar{x}_1)$. This equality is equivalent to the following relation between x_1^ε and $\bar{x}_1$:

$$\int_{\varepsilon N}^{x_1^\varepsilon} \frac{dz}{D(z/\varepsilon)} = (\bar{x}_1 - \varepsilon N)M\left\{\frac{1}{D(y)}\right\}. \tag{12}$$

By the inequalities $\varepsilon N \leqslant x_1^\varepsilon \leqslant \varepsilon(N+1)$ and $\varepsilon N \leqslant \bar{x}_1 \leqslant \varepsilon(N+1)$, we get immediately that $|x_1^\varepsilon - \bar{x}_1| < \varepsilon$. It remains to estimate the expression

$$x_2^\varepsilon(x_1^\varepsilon) - \bar{x}_2(\bar{x}_1) = \int_{\varepsilon N}^{x_1^\varepsilon} \frac{a_{22}(z/\varepsilon)p_2 \, dz}{D(z/\varepsilon)} - (\bar{x}_1 - \varepsilon N)M\left\{ \frac{a_{22}(y)p_2}{D(y)} \right\}$$

$$= \int_{\varepsilon N}^{x_1^\varepsilon} \frac{a_{22\max}p_2 \, dz}{D(z/\varepsilon)} - (\bar{x}_1 - \varepsilon N)M\left\{ \frac{a_{22\max}p_2}{D(y)} \right\}$$

$$+ \int_{\varepsilon N}^{x_1^\varepsilon} \frac{(a_{22}(z/\varepsilon) - a_{22\max})p_2 \, dz}{D(z/\varepsilon)} - (\bar{x}_1 - \varepsilon N)M\left\{ \frac{(a_{22}(y) - a_{22\max})p_2}{D(y)} \right\}.$$

By (12), the first terms add up to zero, and therefore
$$|x_2^\varepsilon(x_1^\varepsilon) - \bar{x}_2(\bar{x}_1)|$$

$$\leqslant \left| \int_{\varepsilon N}^{x_1^\varepsilon} \frac{\sqrt{a_{22\max} - a_{22}(y_1/\varepsilon)}}{\sqrt{a_{11}(y_1/\varepsilon)}} \, dy_1 \right| + \left| M\left\{ \frac{\sqrt{a_{22\max} - a_{22}(y_1)}}{\sqrt{a_{11}(y_1)}} \right\}(\bar{x}_1 - \varepsilon N) \right| \leqslant C\varepsilon,$$

where the constant C depends only on λ_0 and Λ_0.

So, the limit domain of dependence Q_{x_0} is a cone whose form does not depend on x_0. The boundary of $\mathscr{P}_{x_0}(1)$ is given parametrically by the following equations:

$$x_1 - x_{01} = \pm \left(M\left\{ \frac{1}{D(y)} \right\} \right)^{-1},$$

$$x_2 - x_{02} = M\left\{ \frac{a_{22}(y)p_2}{D(y)} \right\}\left(M\left\{ \frac{1}{D(y)} \right\} \right)^{-1}. \tag{13}$$

The parameter p_2 in this formula ranges over the set $|p_2| < (\sqrt{a_{22\max}})^{-1}$. Let us find the points of intersection of the boundary of $\mathscr{P}_{x_0}(1)$ with the lines passing through x_0 parallel to the coordinate axes. These four points have the following coordinates:

$$\left(x_{01} + \left(M\left\{ \left(\sqrt{a_{11}(y_1)} \right)^{-1} \right\} \right)^{-1}, x_{02} \right), \quad \left(x_{01} - \left(M\left\{ \left(\sqrt{a_{11}(y_1)} \right)^{-1} \right\} \right)^{-1}, x_{02} \right),$$

$$\left(x_{01}, x_{02} + \sqrt{a_{22\max}} \right), \quad \left(x_{01}, x_{02} - \sqrt{a_{22\max}} \right).$$

The smoothness of $\partial \mathscr{P}_{x_0}(1)$, as we see from (13), can be disrupted only at the points of intersection with the line parallel to the x_2-axis. Let us find the one-sided derivatives at these points:

$$\lim_{p_2 \to (\sqrt{a_{22\max}})^{-1}} \frac{\sqrt{a_{22\max}} - \bar{x}_2(p_2)}{\bar{x}_1(p_2) - 0} = \int_0^1 \frac{\sqrt{a_{22\max} - a_{22}(y_1)}}{\sqrt{a_{11}(y_1)}} \sqrt{a_{22\max}} \, dy_1.$$

Hence we deduce that the constructed cone Q_{x_0} has a smooth boundary if and only if the function a_{22} does not depend on y_1, i.e., is constant. It is interesting now to compare the sets $\mathscr{P}_{x_0}(1)$ and $\mathscr{P}_{x_0}^0(1)$. The set $\mathscr{P}_{x_0}^0(1)$ is the ellipse (see [1] and [2])

$$a^2 x_1^2 + b^2 x_2^2 = 1,$$

where $a = (M\{(\sqrt{a_{11}(y_1)})^{-1}\})^{1/2}$ and $b = (M\{a_{22}(y_1)\})^{-1/2}$. If both coefficients $a_{11}(y_1)$ and $a_{22}(y_1)$ are not constant, then $\mathscr{P}_{x_0}^0(1)$ is strictly contained in $\mathscr{P}_{x_0}(1)$.

Let us proceed to the equation of the general form (1). The coefficients a_{ij}, as before, depend only on y_1. To this equation there corresponds the Hamiltonian

$$H^\varepsilon(x, p, t, E) = E^2 - a_{ij}\left(\frac{x_1}{\varepsilon}\right)p_i p_j.$$

We proceed right to the construction of the mean Hamiltonian. For that, we solve the equation

$$E^2 - a_{ij}\left(\frac{x_1}{\varepsilon}\right)p_i p_j = 0$$

with respect to the variable p_1, and then take the mean of both sides with respect to y_1:

$$\overline{H}(x, p, t, E) = p_1 - M\left\{\frac{a_{1l}(y_1)}{a_{11}(y_1)}p_l \pm \sqrt{\frac{(a_{1l}(y_1)p_l)^2 + a_{11}(y_1) - a_{11}(y_1)a_{ij}(y_1)p_ip_j}{a_{11}(y_1)}}\right\}.$$

LEMMA 1. *Let (a_{ij}) be a positive definite matrix. Then the matrix*

$$b_{sr} = \left(a_{sr} - \frac{a_{1s}a_{r1}}{a_{11}}\right)\bigg|_{s,r=2}^{n}$$

is also positive definite.

PROOF. Consider an arbitrary nonzero vector $(\xi_2,\ldots,\xi_n)$ in $\mathbf{R}^{n-1}$. Since (a_{ij}) is positive definite, we have

$$a_{11}\tau^2 + 2a_{1l}\tau\xi_l + \sum_{i,j=2}^{n} a_{ij}\xi_i\xi_j > 0$$

for all τ. Thus, the discriminant of this trinomial is less than zero:

$$\left(\sum_{l=2}^{n} a_{1l}\xi_l\right)^2 - a_{11}\sum_{i,j=2}^{n} a_{ij}\xi_i\xi_j < 0.$$

Hence the assertion of the lemma quickly follows.

Using Lemma 1, we may write down the mean Hamiltonian in the following form:

$$\overline{H}(x, p, t, E) = p_1 - M\left\{\frac{a_{1l}(y_1)}{a_{11}(y_1)}\right\}p_l \pm M\left\{\frac{\sqrt{1 - b_{ij}(y_1)p_ip_j}}{\sqrt{a_{11}(y_1)}}\right\}.$$

Here $b_{ij}(y_1)$ is a smooth, positive definite matrix, and

$$\lambda_0'|\xi|^2 \leqslant b_{ij}(y_1)\xi_i\xi_j \leqslant \Lambda_0'|\xi|^2.$$

THEOREM 3. *Consider the region in the space of $(p_2,\ldots,p_n) \in \mathbf{R}^{n-1}$ given by*

$$\mathcal{L} = \{p_2,\ldots,p_n \,|\, b_{ij}(y_1)p_ip_j < 1\}.$$

The following inequality is satisfied for trajectories of the equations with Hamiltonians $H^\varepsilon(x, p, t, E)$ and $\overline{H}(x, p, t, E)$ coming from the same point x_0 with the same $p' = (p_2,\ldots,p_n)$ in $\mathcal{L}$:

$$|x^\varepsilon(t) - \overline{x}(t)| < C\varepsilon. \tag{14}$$

The estimate is uniform with respect to x_0, t and p′ satisfying the condition

$$b_{ij}(y_1)p_ip_j < 1 - \delta,$$

where δ is an arbitrary positive number.

PROOF. First, $\mathcal{C}$ is actually an open set. This follows from the fact that the functions $b_{ij}(y_1)$ are continuous and defined on a compact set. The rest of the proof repeats the first part of the proof of Theorem 2.

Note that the set $\mathcal{C}$ is convex.

REMARK. In contrast with the two-dimensional case considered above, already in the three-dimensional case the uniformity of estimate (14) breaks down on approaching the boundary of $\mathcal{C}$. For any positive ε one can find trajectories which, after a time t of order one, separate from each other by a quantity of order one. Nevertheless, the limit domain of dependence propagates linearly with respect to t, a fact which follows from Theorem 1. This will be proved below by another method which enables Q_{x_0} to be found, gives an estimate of the proximity of the sets Q_{x_0} and $Q_{x_0}^\varepsilon$, and proves the uniformity of this proximity for all t.

Let us show that the boundary of Q_{x_0} coincides with the set of projections of the bicharacteristics of the condition of Theorem 3 onto (x, t)-space.

LEMMA 2. *The mapping $(p_2, \ldots, p_n, t) \to (\bar{x}_1(t), \ldots, \bar{x}_n(t))$ is a diffeomorphism from the region $\mathcal{C} \times (0, +\infty)$ onto the half-space $\{(x_1, \ldots, x_n) \mid x_1 > x_{01}\}$.*

PROOF. The mapping which puts the vector $(\bar{x}_1(t), \ldots, \bar{x}_n(t))$ in correspondence with the vector $(p_2, \ldots, p_n, t)$ is linear in t. Therefore, it suffices to prove that the mapping

$$x_2 = -M\left\{\frac{a_{12}(y)}{a_{11}(y)}\right\} + M\left\{\frac{b_{2i}(y)p_i}{G(y)}\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n = -M\left\{\frac{a_{1n}(y)}{a_{11}(y)}\right\} + M\left\{\frac{b_{ni}(y)p_i}{G(y)}\right\},$$

which we shall denote by Q, is a diffeomorphism from $\mathcal{C}$ to $\mathbf{R}^{n-1}$. Here, $G(y)$ denotes the radical $\sqrt{a_{11}(y)(1 - b_{ij}(y)p_ip_j)}$. For the proof, consider the auxiliary mapping F^{-1} from $\mathcal{C}$ onto the unit ball:

$$F^{-1}: p′ \to (\max|p′|)^{-1}p′,$$

where $\max|p′|$ is taken in the direction of the vector $p′$ in $\mathcal{C}$. The mapping F^{-1} is a homeomorphism. Let us show that QF maps the unit ball in $\mathbf{R}^{n-1}$ onto all of $\mathbf{R}^{n-1}$. For that, consider an arbitrary vector ξ in $\mathbf{R}^{n-1}$. It is clear that, for $p′$ in $\mathcal{C}$ sufficiently close to the boundary $\partial\mathcal{C}$,

$$(Qp′ - \xi, p′) > 0.$$

Therefore, there exists $\rho < 1$ such that for any y, $|y| = \rho$,

$$(QFy - \xi, y) > 0.$$

By the continuity of QF ([7], Lemma 4.3), it follows that for some y, $|y| < \rho$,

$$QFy = \xi.$$

This is equivalent to

$$Qp' = \xi$$

for some $p' \in \mathcal{L}$.

Let us check that Q is a local diffeomorphism. We have

$$\frac{\partial x_l}{\partial p_k} = M\left\{\frac{b_{kl}(y)}{G(y)}\right\} + M\left\{\frac{b_{li}(y)p_i b_{kj}(y)p_j}{(G(y))^3}\right\}. \qquad (15)$$

The matrix of the first terms is positive definite, since $(b_{ij}(y_1))$ is positive definite for all y_1. Regarding the second terms on the right-hand side of (15), we note that the matrix $c_{ij} = \lambda_i \lambda_j$, for an arbitrary vector $(\lambda_1, \ldots, \lambda_n)$, is nonnegative:

$$c_{ij}\xi_i\xi_j = (\lambda_i\xi_i)^2 \geqslant 0.$$

It remains to prove that, as a local diffeomorphism, the mapping Q is a global diffeomorphism. Indeed, we shall prove that the equation $Qp' = x'$ has a unique solution for each x'. Since the constant vector in the definition of the mapping Q does not affect uniqueness, we shall neglect it. Assume that there are two solutions for some x'. It is easy to check that in a small neighborhood of the origin in the space $(x_2, \ldots, x_n) \in \mathbf{R}^{n-1}$, the solution is unique and the set of such solutions is a neighborhood of the origin in $(p_2, \ldots, p_n) \in \mathbf{R}^{n-1}$. Let ρ_0 be the precise upper bound of ρ such that for all x' in the open ball of radius ρ the solution is unique. By hypothesis, $0 < \rho_0 < \infty$. We claim that there exists $\tilde{x}$, $|\tilde{x}| = \rho_0$, whose inverse image consists of at least two distinct elements $\tilde{p}$ and $\tilde{\tilde{p}}$. Assume the contrary. Then there exist a sequence $\tilde{x}_1, \tilde{x}_2, \ldots$ and two sequences $\tilde{p}_1, \tilde{p}_2, \ldots$ and $\tilde{\tilde{p}}_1, \tilde{\tilde{p}}_2, \ldots$ such that $Q(\tilde{p}_i) = Q(\tilde{\tilde{p}}_i) = \tilde{x}_i$ and $|\tilde{x}_i| \to \rho_0$ as $i \to \infty$. It is clear that all the $\tilde{p}_i$ and $\tilde{\tilde{p}}_i$ belong to some compact set in $\mathcal{L}$. Since Q is a local diffeomorphism, it follows that $|\tilde{p}_i - \tilde{\tilde{p}}_i| \geqslant \delta_0 > 0$. Passing to a subsequence, we find that $\tilde{p}_i \to \tilde{p}$ and $\tilde{\tilde{p}}_i \to \tilde{\tilde{p}}$, and so $|\tilde{p} - \tilde{\tilde{p}}| \geqslant \delta_0$. Passing to the limit, we have, by continuity of Q,

$$Q(\tilde{p}) = Q(\tilde{\tilde{p}}) = \tilde{x}.$$

Since $|\tilde{x}_i| \to \rho_0$, $|\tilde{x}| = \rho_0$. Thus, $\tilde{x}$ exists. But then, since Q is a local diffeomorphism, two solutions exist for x' in a whole neighborhood of $\tilde{x}$. This contradicts the fact that ρ_0 is the precise upper bound. The lemma is proved.

The cross section of the constructed cone by the plane $\{t = 1\}$ has the form

$$x_1 - x_{01} = \pm\left(M\left\{\frac{1}{G(y)}\right\}\right)^{-1},$$

$$x_2 - x_{02} = \frac{\mp M\left\{\dfrac{a_{12}(y)}{a_{11}(y)}\right\} + M\left\{\dfrac{b_{2i}(y)p_i}{G(y)}\right\}}{M\left\{\dfrac{1}{G(y)}\right\}},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \qquad (16)$$

$$x_n - x_{0n} = \frac{\mp M\left\{\dfrac{a_{1n}(y)}{a_{11}(y)}\right\} + M\left\{\dfrac{b_{ni}(y)p_i}{G(y)}\right\}}{M\left\{\dfrac{1}{G(y)}\right\}}.$$

Here the parameters ($p_2, \ldots, p_n$) vary in the region $\mathcal{L}$. From Theorem 1, Theorem 3 and Lemma 2, it follows that the surface (16) is the boundary of the set $\mathcal{P}_{x_0}(1)$. Let us clarify what happens to the cross section of $\mathcal{P}_{x_0}(1)$ by the plane $\{x_1 = \delta\}$ as $\delta \to 0$. For that, we shall prove several auxiliary statements.

Consider the ellipsoid in $\mathbf{R}^d$

$$c_{ij}\xi_i\xi_j = 1. \tag{17}$$

For any point ξ of this ellipsoid, we define the conjugate point in the following manner: $\eta_i = c_{ij}\xi_j$. The resulting set of points is again an ellipse, whose equation is $c^{ij}\eta_i\eta_j = 1$.

LEMMA 3. *Let the ellipsoids* $\mathfrak{I}_1, \ldots, \mathfrak{I}_N$ *of the form* (17) *have a common point* ξ_0. *Then all its conjugate points lie in one hyperplane* $\{\eta \mid (\eta, \xi_0) = 1\}$, *where the conjugate ellipsoids* $\mathfrak{I}_1^*, \ldots, \mathfrak{I}_N^*$ *are tangent to this hyperplane at the points conjugate to* ξ_0. *The converse is also true: if the ellipsoids of the form* (17) *are tangent to the same hyperplane, then the points conjugate to the tangency points coincide. Moreover,* $(\mathfrak{I}^*)^* = \mathfrak{I}$.

The proof is carried out by direct verification.

LEMMA 4. *Consider a family of ellipsoids* $\mathfrak{I}_\alpha$ *satisfying the condition* $\lambda \mid \xi \mid^2 \leqslant c_{ij}(\alpha)\xi_i\xi_j \leqslant \Lambda \mid \xi \mid^2$. *Let* $\mathcal{L}$ *be the intersection of the ellipsoids* $\mathfrak{I}_\alpha$, *and let* S *be the convex hull of the set of conjugate ellipsoids* $\mathfrak{I}_\alpha^*$. *Let the ellipsoid* $\mathfrak{I}_0$ *contain* $\mathcal{L}$. *Then the conjugate ellipsoid* $\mathfrak{I}_0^*$ *is contained in* S.

PROOF. Without loss of generality, we can assume that the family of matrices $(c_{ij}(\alpha))$ is closed in the space of $d \times d$ matrices. Using the conditions of the ellipsoids $\mathfrak{I}_\alpha$, we can show that the set S has boundary of class C^1. Suppose that, for some point ξ_0 of $\mathfrak{I}_0$, the conjugate point η_0 turns out to be outside S. Shrink the conjugate ellipsoid so that it does not have points outside S, but could only have points on the boundary ∂S. Such points are tangency points. On the other hand, during this procedure the ellipsoid $\mathfrak{I}_0$ expands and, as before, will contain $\mathcal{L}$. Consider one of the tangency points thus formed, call it $\bar{\eta}$. Let us first prove that ∂S consists of two types of points: points which are conjugate to points lying in $\partial\mathcal{L}$, and points which arise on squeezing the convex hull and which are not conjugate to any of the ellipsoids $\mathfrak{I}_\alpha$. For that, choose an arbitrary point $\hat{\xi}$ on the ellipsoid $\mathfrak{I}_{\alpha_0}$ lying outside $\partial\mathcal{L}$, and construct the point $\overset{\star}{\xi}$ lying on the intersection of the ray $\hat{\xi}$ with $\partial\mathcal{L}$. From the definition of $\mathcal{L}$, it follows that $\overset{\star}{\xi}$ lies on the surface of some ellipsoid $\mathfrak{I}_{\alpha_1}$. The ellipsoids conjugate to $\mathfrak{I}_{\alpha_0}$ and $\mathfrak{I}_{\alpha_1}$ are tangent to parallel planes at the points $\hat{\eta}$ and $\overset{\star}{\eta}$ conjugate to $\hat{\xi}$ and $\overset{\star}{\xi}$, where the ellipsoid $\mathfrak{I}_{\alpha_0}^*$ is tangent to the plane lying closer to the origin. From this it is clear that $\hat{\eta}$ cannot lie on ∂S. Now let Γ be the plane passing through $\bar{\eta}$ and tangent to $\mathfrak{I}_0^*$ and ∂S. From the above comments it follows that on $\partial\mathcal{L}$ a point $\tilde{\xi}$ can be found lying on the surface of an ellipsoid $\mathfrak{I}_{\alpha_2}$, whose conjugate point lies on Γ, while the conjugate ellipsoid $\mathfrak{I}_{\alpha_2}^*$ is tangent to Γ at this point. Consequently, $\tilde{\xi}$ lies simultaneously on $\mathfrak{I}_0$ and $\partial\mathcal{L}$. We have arrived at a contradiction.

Now consider the family of ellipsoids in $\mathbf{R}^{n-1}$ given by the matrices $b_{ij}(y)$,

$$b_{ij}(y)p_ip_j = 1.$$

Construct the sets $\mathcal{L}$ and S corresponding to this family. We shall show that, as $\delta \to 0$, the cross section of the set $\mathcal{P}_{x_0}(1)$ by the plane $\{x_1 - x_{01} = \delta\}$ converges to S:

$$\sup_{x' \in \partial\mathcal{P}_{x_0}(1) \cap \{(x_1 - x_{01}) = \delta\}} \rho(x', \partial S) \to 0.$$

This results immediately from the following lemma.

LEMMA 5. *Consider the surface given by the equations*

$$x_1 = \left(M\left\{ \frac{1}{G(y)} \right\} \right)^{-1} = \delta,$$

$$x_2 = \left(M\left\{ \frac{1}{G(y)} \right\} \right)^{-1} M\left\{ \frac{b_{2j}(y)p_j}{G(y)} \right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (18)$$

$$x_n = \left(M\left\{ \frac{1}{G(y)} \right\} \right)^{-1} M\left\{ \frac{b_{nj}(y)p_j}{G(y)} \right\}.$$

Project an arbitrary point x on this surface onto $(x_2,\ldots,x_n)$-space. Denote the set of points obtained in this fashion by $I(\delta)$. Now define the function $r(\delta)$ by the equality

$$r(\delta) = \sup_{x' \in I(\delta)} \rho(x', \partial S).$$

It is asserted that $r(\delta)$ tends to zero as $\delta \to 0$, where

$$r(\delta) \leqslant c\sqrt{\delta}. \qquad (19)$$

PROOF. Consider an arbitrary vector $(p_2,\ldots,p_n)$ in the region $\mathfrak{L}$ for which $x_1(1) = \delta$. All the points y in a period of $b_{ij}(y)$ divide up into two groups. To one group belong points where

$$1 - b_{ij}(y)p_i p_j > \delta,$$

and all the remaining points belong to the other. In accordance with this, let us split the integrals in (18) into the sum of two integrals over each of the indicated sets of y:

$$(x_2,\ldots,x_n) = (\tilde{x}_2,\ldots,\tilde{x}_n) + (\tilde{\tilde{x}}_2,\ldots,\tilde{\tilde{x}}_n).$$

It is easy to see that the integrals over the first set have order $\sqrt{\delta}$:

$$|(\tilde{x}_2,\ldots,\tilde{x}_n)| < c\sqrt{\delta};$$

therefore, it suffices to prove the proximity of the point $(\tilde{\tilde{x}}_2,\ldots,\tilde{\tilde{x}}_n)$ to ∂S. Taking the convexity of S into account, it is easy to derive from (18) that $(x_2,\ldots,x_n) \in S$. Further, let $(\tilde{p}_2,\ldots,\tilde{p}_n)$ be the point of intersection of the ray $(p_2,\ldots,p_n)$ with $\partial\mathfrak{L}$. To the point $(\tilde{p}_2,\ldots,\tilde{p}_n)$ there corresponds the plane Γ_1 to which are tangent the ellipsoids conjugate to the ellipsoids passing through $(\tilde{p}_2,\ldots,\tilde{p}_n)$. The set ∂S is also tangent to Γ_1, where the whole set S lies on one side of Γ_1. Now note that, for all points y in the second set, the vector $(b_{2i}p_i,\ldots,b_{ni}p_i)$ stays away from the plane Γ_1 a distance of order $O(\delta)$. Hence, it is easy to derive that the distance from $(x_2,\ldots,x_n)$ to Γ_1 does not exceed $c\sqrt{\delta}$. Since $(x_2,\ldots,x_n) \in S$ in this case, it follows that $\rho((x_2,\ldots,x_n), \partial S) < c\sqrt{\delta}$. The lemma is proved.

We come now to the proof of the uniform proximity of the domains $\mathfrak{P}^\varepsilon_{x_0}(t)$ and $\mathfrak{P}_{x_0}(t)$ for all t.

LEMMA 6. *Consider the trajectories $x^{\varepsilon}(t)$ and $\bar{x}(t)$ corresponding to the Hamiltonians $H^{\varepsilon}(x, p, t, E)$ and $\bar{H}(x, p, t, E)$, respectively, and identical initial conditions, where $(p_2, \ldots, p_n) \in \mathfrak{L}$. Then the estimate for the projections of these trajectories in the direction $(p_0, \ldots, p_n)$*

$$| ((x_2^{\varepsilon}(t), \ldots, x_n^{\varepsilon}(t))(p_2, \ldots, p_n)) - ((\bar{x}_2(t), \ldots, \bar{x}_n(t))(p_2, \ldots, p_n)) | \leqslant c\varepsilon$$

is valid uniformly with respect to all x_0, t and $p' \in \mathfrak{L}$.

The proof is the same as that of Theorem 2.

Now consider again the set $\mathscr{P}_{x_0}(1)$. It can be proved that for $(p_2, \ldots, p_n)$ close to $\partial \mathfrak{L}$, the quantities $x_1(p')$ and $\rho(x'(p'), \partial S)$ have the same order of smallness

$$\rho(x'(p'), \partial S) < cx_1(p'). \tag{20}$$

These calculations are not complicated, but they are extremely tedious, and so we omit them. This means that in (19) one can replace $\sqrt{\delta}$ by δ.

Now let $(x_1, \ldots, x_n)$ belong to $\partial \mathscr{P}_{x_0}(t) \cap \{0 \leqslant x_1 - x_{01} \leqslant \varepsilon\}$. From the linear dependence of $\bar{x}$ on t and (20) it follows that

$$\rho((x_2, \ldots, x_n), t \cdot \partial S) < c\varepsilon.$$

From Lemma 6 it is now easy to derive that, in the layer $\{0 \leqslant x_1 - x_{01} \leqslant \varepsilon\}$,

$$\sup_{x \in \partial \mathscr{P}_{x_0}^{\varepsilon}(t) \cap \{0 \leqslant x_1 - x_{01} \leqslant \varepsilon\}} \rho((x_1, \ldots, x_n), \partial \mathscr{P}_{x_0}(t)) < c\varepsilon. \tag{21}$$

We now use the fact that, for $x_1 - x_{01} = \varepsilon$, the sets $\partial \mathscr{P}_{x_0}^{\varepsilon} \cap \{x_1 - x_{01} = \varepsilon\}$ and $\partial \mathscr{P}_{x_0} \cap \{x_1 - x_{01} = \varepsilon\}$ coincide. Taking the point $(x_1, \ldots, x_n)$ in $\partial \mathscr{P}_{x_0} \cap \{x_1 - x_{01} = \varepsilon\}$, we can repeat the same procedure in the layer $\{\varepsilon \leqslant x_1 - x_{01} \leqslant 2\varepsilon\}$, with insignificant changes, as in the layer $\{0 \leqslant x_1 - x_{01} \leqslant \varepsilon\}$, thanks to which we get the estimate

$$\sup_{x \in \partial \mathscr{P}_{x_0}^{\varepsilon}(t) \cap \{\varepsilon \leqslant x_1 - x_{01} \leqslant 2\varepsilon\}} \rho(x, \partial \mathscr{P}_{x_0}(t)) < c\varepsilon.$$

Continuing this process, we get

$$\sup_{x \in \partial \mathscr{P}_{x_0}^{\varepsilon}(t)} \rho(x, \partial \mathscr{P}_{x_0}(t)) < c\varepsilon$$

uniformly with respect to $t \in (0, \infty)$.

THEOREM 4. *Let the coefficients of equation (1) depend only on the coordinate y_1. Let Q_{x_0} be the family of cones constructed above. Then the following estimate for the domain of dependence is satisfied uniformly with respect to x_0 and $t \in (0, \infty)$:*

$$\sup_{x \in \partial \mathscr{P}_{x_0}^{\varepsilon}(t)} \rho(x, \partial \mathscr{P}_{x_0}(t)) < c\varepsilon. \tag{22}$$

PROOF. The major portion of the proof has already been dealt with. For its completion it remains to study the bicharacteristics of the second type entering the layer. But for them the estimate (22) inside the layer follows from the analogous estimate outside the layer.

§3. Some examples and generalizations

We first present several problems to which the foregoing construction carries over without any substantial modifications. For one thing, the domain of dependence $Q_{x_0}^{\varepsilon}$, and

hence the limit domain of dependence Q_{x_0}, is unchanged if, instead of (1), we consider the equation

$$u^\varepsilon(x, t) - a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^\varepsilon(x, t) - b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}u^\varepsilon(x, t) - c\left(\frac{x}{\varepsilon}\right)u^\varepsilon(x, t) = 0,$$

$$u^\varepsilon(x, t)|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t}u^\varepsilon(x, t)|_{t=0} = \psi(x).$$

The divergence form of notation in problem (1) was chosen because, for this kind of problem, the behavior of the solution for small ε has been most fully studied; here the limit operator has constant coefficients which are easy to find in certain special cases.

For the limit domain of dependence Q_{x_0}, it is easy to get a distinctive "estimate from above". For that, it is necessary to consider the family of ellipsoids in $\mathbf{R}^n$

$$a_{ij}(y)p_i p_j = 1$$

and to construct for it the regions $\mathcal{L}$ and S. Then $\mathcal{P}_{x_0}(1) \subset S$.

Further, in place of the coefficients $a_{ij}(x/\varepsilon)$, one can consider coefficients $a_{ij}(x, x/\varepsilon)$, where the functions $a_{ij}(x, y)$ are smooth with respect to both variables, satisfy the uniform ellipticity condition, and are periodic in y. In this case, at each point for a fixed first argument, one must construct the corresponding cone depending on x, and further, having such a cone at each x, construct the limit domain of dependence in the usual way.

Let us consider the last example, in which the domain of dependence has been found explicitly. In fact, let the coefficients a_{ij} be such that $a_{ij} \equiv 0$ for $i \neq j$, and the functions a_{ii} depend essentially only on the argument y_i:

$$\frac{\partial^2}{\partial t^2}u^\varepsilon(x, t) - \frac{\partial}{\partial x_i}a_{ii}\left(\frac{x_i}{\varepsilon}\right)\frac{\partial}{\partial x_i}u^\varepsilon(x, t) = 0,$$

$$u^\varepsilon(x, t)|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t}u^\varepsilon(x, t)|_{t=0} = \psi(x).$$

In this case, in the corresponding system of ordinary differential equations, the variables separate, and we get

$$a_{ii}\left(\frac{x_i}{\varepsilon}\right)p_i^2 = c_i = \text{const}.$$

As a result, the limit domain of dependence has the form

$$\mathcal{P}_{x_0}(1) = \left\{A_{ii}(x_i - x_{0i})^2 = 1\right\},$$

where

$$A_{ii} = \left(M\left\{\frac{1}{\sqrt{a_{ii}(y)}}\right\}\right)^2.$$

Now let $(x, t) \in Q_{x_0}^\varepsilon$. Let $U(x, t)$ be an arbitrary, small neighborhood of (x, t). According to [8], no matter how small the neighborhood of x_0, one can choose a function $\varphi_\varepsilon(x)$ on it such that the solution $u^\varepsilon(x, t)$ of problem (1) with initial condition $\varphi_\varepsilon(x)$ does not vanish identically in $U(x, t)$. However, in this case the functions $\varphi_\varepsilon(x)$ are generally different for different ε. It is natural to ask whether a function $\varphi(x)$ could be constructed which would not depend on ε and, for small ε, would satisfy the aforementioned property.

The answer to this question is affirmative. Let us present a plane for constructing such a function $\varphi(x)$. Choose a sequence ε_n which converges sufficiently rapidly to zero, e.g. $\varepsilon_n = 1/2^n$. As has already been mentioned above, for each ε_n we can construct a function $\varphi_{\varepsilon_n}(x)$ satisfying the required conditions. We take for $\varphi(x)$ the sum of the series

$$\varphi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi_{\varepsilon_n}(x),$$

where the coefficients α_n decay so quickly that the series converges together with all its derivatives, and the solution $u^{\varepsilon}(x, t)$ with initial function $\varphi(x)$ differs from zero in an arbitrarily small neighborhood of the point $(x, t) \in \partial Q_{x_0}$, for sufficiently small ε.

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