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An existence result for nonisothermal immiscible incompressible 2-phase flow in heterogeneous porous media

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In the present article, we study the temperature effects on two-phase immiscible incompressible flow through a porous medium. The mathematical model is given by a coupled system of 2-phase flow equations and an energy balance equation. The model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy-Muskat and the capillary pressure laws. The problem is written in terms of the phase formulation; ie, the saturation of one phase, the pressure of the second phase, and the temperature are primary unknowns. The major difficulties related to this model are in the nonlinear degenerate structure of the equations, as well as in the coupling in the system. Under some realistic assumptions on the data, we show the existence of weak solutions with the help of an appropriate regularization and a time discretization. We use suitable test functions to obtain a priori estimates. We prove a new compactness result to pass to the limit in nonlinear terms.

KEYWORDS
immiscible incompressible, nonisothermal two–phase flow, nonlinear degenerate system, existence, porous media

1 | INTRODUCTION

Modeling 2-phase flow through porous media is an important topic that spans a broad spectrum of engineering disciplines. Examples include geothermal systems, oil reservoir engineering, ground-water hydrology, and thermal energy storage. More recently, modeling multiphase flow received an increasing attention in connection with gas migration in a nuclear waste repository and sequestration of CO_2.

This work aims to incorporate the temperature effects into immiscible incompressible 2-phase flow in heterogeneous porous media. The basic equations for nonisothermal 2-phase flow in a porous medium involve mass conservation, Darcy's law, energy conservation, saturation, and capillary pressure constraint equations. The governing fluid and heat transport equations used to model thermal recovery processes are highly nonlinear. As fluid properties are defined as a function of temperature and pressure, there is a strong coupling between the mass balance and energy balance equations.

During recent decades, mathematical analysis and numerical simulation of multiphase flows in porous media have been the subject of investigation of many researchers owing to important applications in reservoir simulation. There is an extensive literature on this subject. We will not attempt a literature review here but will merely mention a few references. Here we restrict ourselves to the mathematical analysis of such models. We refer, for instance, to the books[1-7] and the references therein. The mathematical analysis of the system describing isothermal flow of 2 incompressible immiscible fluids in porous media is quite understood. Existence, uniqueness of weak solutions to these equations, and their regularity have been been shown under various assumptions on physical data; see for instance some related studies[1,2,4,8-14] and the references therein. A recent review of the mathematical analysis developed for immiscible 2-phase flow in porous media and compressible miscible flow in porous media can be viewed in Amaziane et al.[15]

Let us note that all the aforementioned works are restricted to the case where flows are under isothermal conditions, contrarily to the present work. This assumption is too restrictive for some realistic problems, such as thermally enhanced oil recovery, geothermal energy production, and high-level radioactive waste repositories. The present work was motivated by a need to incorporate the thermal behavior for such problems. In this work, a coupled reservoir 2-phase flow model is described, which accounts for varying reservoir temperature to capture flow physics accurately. Although considerable progress has been made in the computational simulation of 2-phase problems under nonisothermal conditions (see, eg, some related works[16-27] and the refrences therein), to the best knowledge of the authors, the mathematical analysis of such coupled models under nonisothermal conditions is still incomplete. The previous results on the existence of a weak solution of a simplified system describing nonisothermal 2-phase flow in porous media were obtained in Bocharov and Monakhov[28-31] and then revisited in Monakhov.[32] The model studied in these works considers a simplified form of the conservation of energy expression, where there is no energy exchange between the phases, which reduces to the conventional advective-diffusive scalar equation for the temperature, which is too restrictive for realistic problems. More precisely, the corresponding system consists of 3 equations: The first 2 equations describe the evolution of the phase pressures and the saturation, and the last equation the evolution of the temperature function, all 3 equations being coupled. However, in contrast to our model, the above mentioned works deal with a simplified version of the temperature equation; in particular, the evolution term does not involve a coefficient that depends on the saturation function and the porosity as in our model. This fact allows one to obtain the compactness result for the temperature directly from the uniform estimates for the spatial derivatives and the estimate for the time derivative of the temperature in the space H^{-1}. This is not the case in our situation, we have to prove the compactness of the saturation function and of the temperature simultaneously. Closer to the present problem, in Li and Sun,[33] the authors proved the global existence of a weak solution for the 1D multicomponent heat-air-vapor transport model in porous textile materials. Recently, existence of a global weak solution for Richard's model arising from the heat and moisture flow through a partially saturated porous medium was obtained in Beneš and Pažanin.[34]

Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling, the strong nonlinearity, and the degeneracy of the diffusion term in the saturation equation. In particular, the degeneracy of the relative permeability implies that we have no uniform estimates for the gradients of the phase pressures. This is the reason why we have to pass to the formulation of our problem in terms of the global pressure and saturation. But even in this formulation, we have no uniform estimates for the gradient of the saturation. This creates the difficulties in the proof of the compactness results. Also, because of the degeneracy and strong coupling, the solutions do not have much regularity. We follow the strategy used in Amaziane et al[35] and Galusinski and Saad[36]; that is, we first regularize the phase pressures in our model by introducing a small parameter η, and then we use the time discretization to get a sequence of elliptic problems, which introduces second small parameter h. To apply the fixed point argument to the regularized problem, we need 2 further regularizations, which are used in construction of the fixed point mapping.

The presence of the temperature brings additional difficulties in obtaining a priori estimates and passage to the limit and makes the proof essentially more involved. Our approach also relies on the proof of a new compactness result. Thus, we extend our previous results to the case of nonisothermal 2-phase flow in porous media. This study was intended as a first step to the homogenization of nonisothermal immiscible incompressible 2-phase flow through heterogeneous reservoirs.

The rest of the paper is organized as follows. In the next section, the basic equations for 2-phase flow through porous media and heat transport are presented. First, in Section 2.1, we give a short description of the physical model and formulate the corresponding mathematical problem. In Section 2.2, we provide the assumptions on the data. Then in Section 2.3, we introduce the notion of the nonisothermal global pressure generalizing the well-known notion of the global pressure introduced earlier in the analysis of the incompressible and compressible 2-phase flow. In Section 3, we present our main result and a short description of the scheme of its proof. In Section 4, we introduce the regularized problem with a regularization parameter $\eta > 0$, for all the equations of the model including that corresponding to the temperature function and its time discretization with a small parameter $h > 0$, and, using the Leray-Schauder fixed point theorem, we establish, as in Amaziane et al,[35] the existence of a weak solution to this problem. Moreover, we prove the maximum principle for the saturation and temperature functions. Section 5 is devoted to the study of the non degenerate system. In Section 5.1, we use suitable test functions to get uniform estimates with respect to h. In Section 5.2, a new approach is proposed to prove a compactness result adapted to our model. The proof is essentially based on Simon's compactness theorem for the spaces of functions depending on the space and time variables (see Simon[37]). These estimates allow us to pass to the limit, as h tends to zero, and to justify the existence of weak solutions of the regularized problem with continuous time. In section 6 we complete the proof of the main result. To this end, we perform the limit as η tends to 0 and obtain a solution of the degenerate system. This part of the proof relies on the compactness results established in Section 5.2. Lastly, some concluding remarks are forwarded.

2 | FORMULATION OF THE PROBLEM

The section is organized as follows. First, in Section 2.1, we introduce the physical-mathematical model, which will be studied in the paper. In Section 2.2, we formulate the main assumptions on the data. In Section 2.3, we introduce the notion of the nonisothermal global pressure and obtain some important relations, which will be used below in the derivation of the a priori estimates.

2.1 | Governing equations

We consider a nonisothermal immiscible incompressible 2-phase flow process in a porous reservoir $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$, which is a bounded Lipschitz domain. The time interval of interest is $(0, \mathcal{T})$ and $Q = \Omega \times (0, \mathcal{T})$. Let $\Phi = \Phi(x)$ be the porosity of Ω; $K = K(x)$ be the absolute permeability tensor of Ω; ϱ_w, ϱ_o and ϱ_s are the densities of the wetting and nonwetting phases, and the skeleton, respectively; $S_w = S_w(x, t), S_o = S_o(x, t)$ are the saturations of the wetting and nonwetting phases; $k_{r,w} = k_{r,w}(S_w), k_{r,o} = k_{r,o}(S_w)$ are the relative permeabilities of the wetting and nonwetting phases; $p_w = p_w(x, t), p_o = p_o(x, t)$ are the pressures of the wetting and nonwetting phases; $T = T(x, t)$ is the temperature; $h_w = h_w(T), h_o = h_o(T), h_s = h_s(T)$ are specific enthalpies of the wetting and nonwetting phases and the solid part; $\mu_w = \mu_w(T)$ and $\mu_o = \mu_o(T)$ are the viscosities of the wetting and nonwetting phases, respectively; and $k_T = k_T(x)$ is the effective thermal conductivity of the combined 3-phase system.

In this paper, we focus our attention on a model where both fluids are assumed incompressible; that is, the densities of the wetting and nonwetting phases are strictly positive constants, and the skeleton density is also assumed to be a strictly positive constant. It is assumed that no exchange of mass between the 2 phases can take place and each phase remains homogeneous. We assume that the phase enthalpies are linear functions of the temperature and are given by

$$h_w(T) = C_w T, \quad h_o(T) = C_o T, \quad h_s(T) = C_s T, \quad c_w = \varrho_w C_w, \quad c_o = \varrho_o C_o, \quad c_s = \varrho_s C_s, \tag{2.1}$$

where C_w, C_o, C_s are the specific heat capacities of the wetting, the nonwetting, and the solid part, respectively, and c_w, c_o, c_s are the specific heat capacities per unit volume. They are assumed to be positive constants.

The basic equations for nonisothermal 2-phase flow in a porous medium involve mass conservation, Darcy-Muskat's law, energy conservation, saturation, and capillary pressure constraint equations. For nonisothermal 2-phase flows, one more governing equation is required to determine the temperature field. This can be obtained from the total energy conservation equation for a combined solid matrix-multiphase mixture system and by invoking the assumption that local thermodynamic equilibrium prevails among the solid matrix and the phases.

In what follows, for the sake of presentation simplicity, we neglect the source terms. Let

$$\mathbf{\Psi}(S, T) \overset{\text{def}}{=} \psi(S)\ T \quad \text{where} \quad \psi(S) \overset{\text{def}}{=} \big(c_w S + c_o[1 - S]\big)\Phi(x) + c_s\ [1 - \Phi(x)\big]. \tag{2.2}$$

Then the conservation of mass in each phase and conservation of energy can be written as (see, eg, Chen et al,[3] Helmig,[5] Kaviany,[38] and Wu[39]):

$$\begin{cases} 0 \leqslant S \leqslant 1 \quad \text{in } \ Q; \\ \Phi(x)\dfrac{\partial S}{\partial t} - \operatorname{div}\left\{ K(x)\lambda_w(S,T)\big(\nabla p_w - \varrho_w\,\vec{g}\big)\right\} = 0 \quad \text{in } \ Q; \\ -\Phi(x)\dfrac{\partial S}{\partial t} - \operatorname{div}\left\{ K(x)\lambda_o(S,T)\big[\nabla p_o - \varrho_o\,\vec{g}\big]\right\} = 0 \quad \text{in } \ Q; \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial t} - \ \operatorname{div}\left\{ TK(x)\Big[c_w\lambda_w(S,T)\big(\nabla p_w - \varrho_w\vec{g}\big) + c_o\lambda_o(S,T)\big(\nabla p_o - \varrho_o\vec{g}\big)\Big]\right\} - \\ \qquad - \operatorname{div}\,(k_T\nabla T) = 0 \quad \text{in } \ Q; \\ P_c(S) = p_o - p_w \quad \text{in } \ Q, \end{cases} \tag{2.3}$$

where here and in the rest of the paper, we set $S \stackrel{\text{def}}{=} S_w$; the velocities of the phases $\vec{q}_w, \vec{q}_o$ are defined by the Darcy-Muskat law:

$$\vec{q}_w \stackrel{\text{def}}{=} -K(x)\ \lambda_w(S,T)\left[\nabla p_w - \varrho_w\,\vec{g}\right], \quad \text{with } \ \lambda_w(S,T) \stackrel{\text{def}}{=} \frac{k_{r,w}(S)}{\mu_w(T)}; \tag{2.4}$$

$$\vec{q}_o \stackrel{\text{def}}{=} -K(x)\ \lambda_o(S,T)\left[\nabla p_o - \varrho_o\,\vec{g}\right], \quad \text{with } \ \lambda_o(S,T) \stackrel{\text{def}}{=} \frac{k_{r,o}(S)}{\mu_o(T)}; \tag{2.5}$$

with $\vec{g}$ is the gravity vector.

The system 2.3 has to be completed by appropriate boundary and initial conditions. We assume that the boundary $\partial\Omega$ consists of 2 parts, Γ_1 and Γ_2, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $|\Gamma_1| > 0$. The boundary conditions read:

$$\begin{cases} p_o(x,t) = p_w(x,t) = T(x,t) = 0 \quad \text{on } \ \Gamma_1 \times (0,\mathcal{T}); \\ \vec{q}_w \cdot \vec{\nu} = \vec{q}_o \cdot \vec{\nu} = k_T\nabla T \cdot \vec{\nu} = 0 \quad \text{on } \ \Gamma_2 \times (0,\mathcal{T}); \end{cases} \tag{2.6}$$

where the velocities $\vec{q}_w, \vec{q}_o$ are defined in 2.4 and 2.5.

The initial conditions read:

$$p_w(x,0) = p_w^0(x), \quad p_o(x,0) = p_o^0(x), \quad \text{and} \quad T(x,0) = T^0(x) \quad \text{in } \ \Omega. \tag{2.7}$$

2.2 | Main assumptions

The main assumptions on the data are as follows:

(A.1) The porosity $\Phi \in L^\infty(\Omega)$, and there are positive constants ϕ_-, ϕ^+ such that $0 < \phi_- < \phi^+$ and

$$0 < \phi_- \leqslant \Phi(x) \leqslant \phi^+ < 1 \quad \text{a.e. in } \ \Omega. \tag{2.8}$$

(A.2) The tensors $K, k_T \in (L^\infty(\Omega))^{d\times d}$ and there exist constants K_-, K^+ such that $0 < K_- < K^+$ and

$$K_-|\xi|^2 \leqslant K(x)\ \xi\ \cdot \xi,\ k_T(x)\ \xi\ \cdot \xi \leqslant K^+|\xi|^2 \ \text{ for all } \xi \in \mathbb{R}^d, \ \text{ a.e. in } \ \Omega. \tag{2.9}$$

(A.3) The capillary pressure function $P_c \in C^1([0,1];\mathbb{R}^+)$. Moreover, it is a decreasing function of the saturation, ie, $P_c'(S) < 0$ in $[0,1]$ and $P_c(1) = 0$.

(A.4) The functions $k_{r,w}, k_{r,o}$ belong to the space $C(\mathbb{R})$ and satisfy the following properties: (1) $0 \leqslant k_{r,w}, k_{r,o} \leqslant 1$ on $\mathbb{R}$; (2) $k_{r,w}(S) = 0$ for $S \leqslant 0$ and $k_{r,o}(S) = 0$ for $S \geqslant 1$; $k_{r,w}(S) = 1$ for $S \geqslant 1$ and $k_{r,o}(S) = 1$ for $S \leqslant 0$; (3) there is a positive constant k_0 such that $k_{r,w}(S) + k_{r,o}(S) \geqslant k_0 > 0$ for all $S \in \mathbb{R}$.

(A.5) The viscosities $\mu_w, \mu_o \in C^1(\mathbb{R})$ are functions of the temperature T. Moreover, these functions, for any $T \in \mathbb{R}$, satisfy the following bounds:

$$0 < \mathsf{m}_w \leqslant \mu_w(T) \leqslant \mathsf{M}_w, \quad 0 \leqslant |\mu_w'(T)| \leqslant \mathsf{M}_w < +\infty; \tag{2.10}$$

$$0 < \mathsf{m}_o \leqslant \mu_o(T) \leqslant \mathsf{M}_o, \quad 0 \leqslant |\mu_o'(T)| \leqslant \mathsf{M}_o < +\infty. \tag{2.11}$$

(A.6) The function α defined in 2.18 is such that $\alpha \in C^1([0,1];\mathbb{R}^+)$. Moreover, $\alpha(0)=\alpha(1)=0$ and $\alpha>0$ in $(0,1)$.

(A.7) The function β^{-1}, inverse of β defined in 2.18, is a Hölder function of order θ with $\theta \in (0,1)$ on the interval $[0,\beta(1)]$. Namely, there exists a positive constant C_β such that for all $u_1, u_2 \in [0,\beta(1)]$ the following inequality holds:

$$\left|\beta^{-1}(u_1)-\beta^{-1}(u_2)\right| \leqslant C_\beta\ |u_1-u_2|^\theta.$$

(A.8) The initial data for the phase pressures are such that $p_o^0, p_w^0 \in L^2(\Omega)$ and $0 \leqslant p_o^0 - p_w^0 \leqslant P_c(0)$. The initial data for the saturation $0 \leqslant S^0 \leqslant 1$ are defined by the capillary pressure law: $p_o^0 - p_w^0 = P_c(S^0)$. The initial temperature $T^0 \in L^\infty(\Omega)$ satisfies $T_m \leqslant T^0(x) \leqslant T_M$ for some constants T_m and T_M, such that $T_m \leqslant 0 \leqslant T_M$.

The assumptions (A.1) to (A.8) are classical for 2-phase flow in porous media.

Remark 1.

- Let us note that it follows from (A.4) and (A.5) that the relative mobility functions λ_w, λ_o, defined in 2.4, 2.5, belong to the space $C([0,1]\times\mathbb{R};\mathbb{R}^+)$ and satisfy the following properties: (i) $\lambda_w(0,T)=0$ and $\lambda_o(1,T)=0$; (ii) there is a positive constant L_0 such that $\lambda(S,T) \stackrel{\text{def}}{=} \lambda_w(S,T)+\lambda_o(S,T) \geqslant L_0 > 0$ for $S,T \in \mathbb{R}$.
- Condition (A.5) is in good accordance, for example, with the Reynolds model for shear viscosity (see, eg, Seeton[40]). Namely, $\mu(T)=\mu_0\exp(-bT)$, where μ_0 and b are constants.

2.3 | Global pressure and useful relations

In the sequel, we will use a formulation obtained after transformation using the concept of the so-called global pressure. For isothermal incompressible immiscible 2-phase flow, this concept was introduced for the first time in Antontsev et al[1] and Chavent and Jaffré.[2] Then it was generalized to the nonisothermal case in Bocharov and Monakhov.[28-31] This concept plays a crucial mathematical role for a priori estimates and compactness results. Following Bocharov and Monakhov,[28] we define the nonisothermal global pressure P as follows:

$$p_o = \mathsf{P} + \int_1^S \frac{\lambda_w}{\lambda}(\xi,T)\ P_c'(\xi)\ d\xi \stackrel{\text{def}}{=} \mathsf{P} + \mathsf{G}_o(S,T), \tag{2.12}$$

where

$$\lambda(S,T) \stackrel{\text{def}}{=} \lambda_w(S,T)+\lambda_o(S,T). \tag{2.13}$$

Then using the capillary pressure relation in 2.3, one can easily obtain that

$$p_w = \mathsf{P} - \int_1^S \frac{\lambda_o}{\lambda}(\xi,T)\ P_c'(\xi)\ d\xi \stackrel{\text{def}}{=} \mathsf{P} + \mathsf{G}_w(S,T). \tag{2.14}$$

It is easy to see that

$$\nabla p_o = \nabla\mathsf{P} + \frac{\lambda_w}{\lambda}(S,T)\ \nabla P_c(S) + \mathsf{B}_o\ \nabla T \quad \text{and} \quad \nabla p_w = \nabla\mathsf{P} - \frac{\lambda_o}{\lambda}(S,T)\ \nabla P_c(S) - \mathsf{B}_w\ \nabla T, \tag{2.14}$$

where

$$\mathsf{B}_o = \mathsf{B}_o(S,T) \stackrel{\text{def}}{=} \int_1^S \frac{\partial}{\partial T}\left[\frac{\lambda_w}{\lambda}(\xi,T)\right]\ P_c'(\xi)\ d\xi; \quad \mathsf{B}_w = \mathsf{B}_w(S,T) \stackrel{\text{def}}{=} \int_1^S \frac{\partial}{\partial T}\left[\frac{\lambda_o}{\lambda}(\xi,T)\right]\ P_c'(\xi)\ d\xi. \tag{2.15}$$

Taking into account that $\mathsf{B}_w = -\mathsf{B}_o$, after some calculations, we obtain

$$\lambda_o\ |\nabla p_o|^2 + \lambda_w\ |\nabla p_w|^2 = \lambda\ |\nabla\mathsf{P}|^2 + \frac{\lambda_w\lambda_o}{\lambda}\ |\nabla P_c|^2 + \lambda\ \mathsf{B}_o^2\ |\nabla T|^2 + 2\ \lambda\ \mathsf{B}_o\ \nabla\mathsf{P}\cdot\nabla T. \tag{2.16}$$

Remark 2. The equality 2.16 can be interpreted in the following way: The total energy of the fluid is separated into the convection energy, capillary energy, and thermal energy. However, there is no complete separation between convective and thermal energies because the heat is transported by the convection, and therefore, a nonquadratic term $2\ \lambda\ \mathsf{B}_o\ \nabla\mathsf{P}\cdot\nabla T$ is present in the equation.

We observe that in contrast to the isothermal case the second term on the right-hand side of 2.16 depends both on the saturation and the temperature. However, in our analysis, we will need a function depending on the saturation only and having a bounded gradient. We introduce this function as follows:

$$\beta(S) \stackrel{\text{def}}{=} \int_0^S \alpha(\xi)|P_c'(\xi)|\,d\xi \quad \text{with} \quad \alpha(\xi) \stackrel{\text{def}}{=} \sqrt{\frac{\frac{k_{r,w}(\xi)}{\mathsf{M}_w}\cdot\frac{k_{r,o}(\xi)}{\mathsf{M}_o}}{\frac{k_{r,w}(\xi)}{\mathsf{m}_w}+\frac{k_{r,o}(\xi)}{\mathsf{m}_o}}}\,, \tag{2.18}$$

where the constants $\mathsf{M}_w, \mathsf{M}_o, \mathsf{m}_w, \mathsf{m}_o$ are defined in condition (A.5).

Furthermore, we introduce the functions

$$\Lambda_0(S,T) \stackrel{\text{def}}{=} \frac{\mathsf{M}_o\mathsf{M}_w}{\mathsf{m}_o\mathsf{m}_w}\,\frac{k_{r,o}(S)\mathsf{m}_w + k_{r,w}(S)\mathsf{m}_o}{k_{r,o}(S)\mu_w(T) + k_{r,w}(S)\mu_o(T)}; \tag{2.19}$$

$$\Lambda_1(S,T) \stackrel{\text{def}}{=} \sqrt{\Lambda_0(S,T)}\sqrt{\frac{\lambda_w(S,T)\lambda_o(S,T)}{\lambda(S,T)}}. \tag{2.20}$$

Note that the function Λ_0, due to (A.4) and (A.5), satisfies

$$0 < \Lambda_{0,min} \leqslant \Lambda_0(S,T) \leqslant \Lambda_{0,max} < +\infty, \tag{2.21}$$

with some constants $\Lambda_{0,min}$ and $\Lambda_{0,max}$. The function Λ_1 preserves degeneration because it is zero for $S = 0$ and $S = 1$. With these new functions, we can write:

$$\lambda_o \nabla p_o = \lambda_o \nabla \mathsf{P} + \Lambda_1\ \nabla\beta(S) + \lambda_o \mathsf{B}_o\ \nabla T, \tag{2.22}$$

$$\lambda_w \nabla p_w = \lambda_w \nabla \mathsf{P} - \Lambda_1\ \nabla\beta(S) + \lambda_w \mathsf{B}_o\ \nabla T, \tag{2.23}$$

$$\lambda_o\ |\nabla p_o|^2 + \lambda_w\ |\nabla p_w|^2 = \lambda\ |\nabla \mathsf{P}|^2 + \Lambda_0|\nabla\beta(S)|^2 + \lambda\ \mathsf{B}_o^2\ |\nabla T|^2 + 2\ \lambda\ \mathsf{B}_o\ \nabla \mathsf{P}\cdot\nabla T. \tag{2.24}$$

By simple calculation, we can prove the following estimate.

Lemma 2.1. *Assume (A.1) to (A.8) hold. Then there exists a constant C such that*

$$|B_o(S,T)| \leqslant C, \tag{2.25}$$

for all $T \in \mathbb{R}$ and all $S \in [0,1]$. If the functions $k_{r,w}$ and $k_{r,o}$ are extended on $\mathbb{R}$ as bounded continuous functions and P_c is extended on $\mathbb{R}$ as bounded smooth function, then 2.25 holds for all $T, S \in \mathbb{R}$.

3 | FORMULATION OF THE MAIN RESULT

Before presenting our main result, we introduce the following Sobolev space:

$$H^1_{\Gamma_1}(\Omega) \stackrel{\text{def}}{=} \left\{u \in H^1(\Omega)\ :\ u = 0 \ \text{ on } \ \Gamma_1\right\}.$$

The space $H^1_{\Gamma_1}(\Omega)$ is a Hilbert space. The norm in this space is given by $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

Now we give the definition of weak solutions to the system 2.3 to 2.7 and then state our main result.

Theorem 3.1. *Let assumptions (A.1) to (A.8) be fulfilled. Then there exists a set of functions $\{p_o, p_w, S, T\}$ such that*

(I) The functions p_o, p_w, S, T have the following regularity properties:

$$p_w, p_o \in L^2(Q) \quad \textit{and} \quad \sqrt{\lambda_w(S,T)}\ \nabla p_w, \sqrt{\lambda_o(S,T)}\ \nabla p_o \in L^2(Q); \tag{3.1}$$

$$\beta(S) \in L^2(0,\mathcal{T};H^1(\Omega)) \quad \textit{and} \quad \mathsf{P} \in L^2(0,\mathcal{T};H^1(\Omega)); \tag{3.2}$$

$$\frac{\partial}{\partial t}(\Phi S) \in L^2(0,\mathcal{T};H^{-1}(\Omega)); \tag{3.3}$$

$$T \in L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{3.4}$$

$$\frac{\partial \boldsymbol{\Psi}}{\partial t} \in L^2(0, \mathcal{T}; H^{-1}(\Omega)), \tag{3.5}$$

where the function $\boldsymbol{\Psi}$ is defined in 2.2.

(II) The maximum principle for the saturation holds:

$$0 \leqslant S \leqslant 1 \quad a.e.\ in \quad Q. \tag{3.6}$$

(III) The maximum principle for the temperature holds:

$$T_m \leqslant T \leqslant T_M \quad a.e.\ in \quad Q. \tag{3.7}$$

(IV) For any $\varphi_w, \varphi_o, \varphi_T \in C^1([0, \mathcal{T}]; H^1(\Omega))$ satisfying $\varphi_w = \varphi_o = \varphi_T = 0$ on $\Gamma_1 \times (0, \mathcal{T})$ and $\varphi_w(x, \mathcal{T}) = \varphi_o(x, \mathcal{T}) = \varphi_T(x, \mathcal{T}) = 0$, we have

$$-\int_Q \Phi(x) S \frac{\partial \varphi_w}{\partial t}\, dx\, dt - \int_\Omega \Phi(x) S^0(x) \varphi_w(x, 0)\, dx + \int_Q K(x) \lambda_w(S, T) \left[\nabla p_w - \varrho_w \vec{g}\right] \cdot \nabla \varphi_w\, dx\, dt = 0; \tag{3.8}$$

$$\int_Q \Phi(x) S \frac{\partial \varphi_o}{\partial t}\, dx\, dt + \int_\Omega \Phi(x) S^0 \varphi_o(x, 0)\, dx + \int_Q K(x) \lambda_o(S, T) \left[\nabla p_o - \varrho_o\, \vec{g}\right] \cdot \nabla \varphi_o\, dx\, dt = 0; \tag{3.9}$$

$$\begin{aligned} &-\int_Q \boldsymbol{\Psi} \frac{\partial \varphi_T}{\partial t}\, dx\, dt - \int_\Omega \boldsymbol{\Psi}^0 \varphi_T(x, 0)\, dx \\ &+ \int_Q \left\{ TK(x) \left[c_w \lambda_w(S, T) \left(\nabla p_w - \vec{g}\right) + c_o \lambda_o(S, T) \left(\nabla p_o - \varrho_o \vec{g}\right) \right] + k_T \nabla T \right\} \cdot \nabla \varphi_T\, dx\, dt = 0, \end{aligned} \tag{3.10}$$

where $\boldsymbol{\Psi^0} \stackrel{def}{=} \psi(S^0) T^0$.

Remark 3. Let us note that the initial conditions in Theorem 3.1 are satisfied in a weak sense as follows:

For any $\chi \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(x) S(x, t) \chi(x)\, dx, \int_\Omega \boldsymbol{\Psi}(x, t) \chi(x)\, dx \in C\left([0, \mathcal{T}]\right), \tag{3.11}$$

and it holds

$$\left(\int_\Omega \Phi(x) S\ \chi\ \ dx \right)(0) = \int_\Omega \Phi(x) S^0\ \chi\ \ dx \quad \text{and} \quad \left(\int_\Omega \boldsymbol{\Psi}\ \chi\ \ dx \right)(0) = \int_\Omega \boldsymbol{\Psi^0}\ \chi\ \ dx. \tag{3.12}$$

Consider the first relation in 3.12. It is easy to see that if we set $\varphi_w = v(t)\omega(x)$, where $v \in \mathcal{D}(0, T)$ and $\omega \in H^1_{\Gamma_1}(\Omega)$, in 3.8, we get

$$\frac{d}{d\,t} \int_\Omega \Phi(x) S(x, t)\ \omega(x)\ \ dx + \int_\Omega K(x) \lambda_w(S, T)(\nabla p_w - \vec{g}) \cdot \nabla \omega\ \ dx = 0 \tag{3.13}$$

in the sense of distributions. From the regularity properties of the solutions, we deduce that the integral $t \mapsto \int_\Omega \Phi(x) S(x, t)\ \omega(x)\ dx$ belongs to space $W^{1,1}(0, T)$ and, consequently, this function is continuous. Multiplying 3.13 by $v \in C^\infty([0, T])$ such that $v(0) = 1$ and $v(T) = 0$ and integrating by parts, we get

$$-\left(\int_\Omega \Phi(x) S\ \omega(x)\ \ dx \right)(0) = -\int_Q K(x) \lambda_w(S, T)(\nabla p_w - \vec{g}) \cdot \nabla \varphi_w\ \ dx\, dt + \int_Q \Phi(x) S\ \omega(x) \frac{d\,v}{d\,t}\ \ dx\, dt.$$

Comparing this equation and 3.8, where $\varphi_w(x, t) = v(t)\omega(x)$, we observe that

$$\left(\int_\Omega \Phi(x) S\ \omega(x)\ \ dx \right)(0) = \int_\Omega \Phi(x) S^0(x)\ \omega(x)\ \ dx, \tag{3.14}$$

which makes the initial condition at $t = 0$ well defined. The second relation in 3.12 can be shown in the same way.

The proof of Theorem 3.1 relies on an appropriate regularization, a time discretization, a priori estimates based on the concept of nonisothermal global pressure and maximum principles for the saturation and the temperature. It will be done in 2 main steps. First, we consider the following regularized system:

$$\begin{cases} \Phi\frac{\partial S^\eta}{\partial t} - \operatorname{div}\left\{K\lambda_w(S^\eta, T^\eta)\left[\nabla p_w^\eta - \varrho_w\vec{g}\right] + \eta\,\nabla\left(p_w^\eta - p_o^\eta\right)\right\} = 0 \quad \text{in } Q; \\ -\Phi\frac{\partial S^\eta}{\partial t} - \operatorname{div}\left\{K\lambda_o(S^\eta, T^\eta)\left[\nabla p_o^\eta - \varrho_o\,\vec{g}\right] + \eta\,\nabla\left(p_o^\eta - p_w^\eta\right)\right\} = 0 \quad \text{in } Q; \\ \frac{\partial \boldsymbol{\Psi}^\eta}{\partial t} - \operatorname{div}\left\{K\,T^\eta\left[c_w\lambda_w(S^\eta, T^\eta)\left(\nabla p_w^\eta - \varrho_w\vec{g}\right) + c_o\lambda_o(S^\eta, T^\eta)\left(\nabla p_o^\eta - \varrho_o\vec{g}\right)\right]\right\} - \\ -c_w\,\eta\,\operatorname{div}\left\{T^\eta\nabla\left(p_w^\eta - p_o^\eta\right)\right\} - c_o\,\eta\,\operatorname{div}\left\{T^\eta\nabla\left(p_o^\eta - p_w^\eta\right)\right\} - \operatorname{div}\left(k_T\nabla T^\eta\right) = 0 \quad \text{in } Q; \\ P_c\left(S^\eta\right) = p_o^\eta - p_w^\eta \quad \text{in } Q, \end{cases} \tag{3.15}$$

where $\eta > 0$ is a small positive parameter.

The boundary conditions for the system 3.15 read:

$$\begin{cases} p_o^\eta = p_w^\eta = T^\eta = 0 \quad \text{on } \Gamma_1 \times (0, \mathcal{T}); \\ \vec{q}_w^{\,\eta}\cdot\vec{\nu} = \vec{q}_o^{\,\eta}\cdot\vec{\nu} = k_T\nabla T^\eta\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_2 \times (0, \mathcal{T}); \end{cases} \tag{3.16}$$

where

$$\vec{q}_w^{\,\eta} \stackrel{\text{def}}{=} -K(x)\,\lambda_w(S^\eta, T^\eta)\left[\nabla p_w^\eta - \varrho_w\vec{g}\right] - \eta\nabla\left(p_w^\eta - p_o^\eta\right);$$

$$\vec{q}_o^{\,\eta} \stackrel{\text{def}}{=} -K(x)\,\lambda_o(S^\eta, T^\eta)\left[\nabla p_o^\eta - \varrho_o\,\vec{g}\right] - \eta\,\nabla\left(p_o^\eta - p_w^\eta\right).$$

Finally, the initial conditions are the same as for the original problem:

$$p_w^\eta(x,0) = p_w^0(x), \quad p_o^\eta(x,0) = p_o^0(x), \quad \text{and} \quad T^\eta(x,0) = T^0(x) \quad \text{in } \Omega. \tag{3.17}$$

3.1 | Notational convention

In what follows, the upper index corresponds to the "working parameter," ie, to the parameter for which we study the limit behavior of the corresponding functions.

The existence result for the system 3.15 to 3.17 will be formulated and proved in Section 5. The proof of this result is based on the existence result for a system with a time discretization. Namely, we will consider the following nondegenerate elliptic problem:

$$\begin{cases} \Phi\Delta_h S_\eta^h - \operatorname{div}\left\{K\lambda_w(S_\eta^h, T_\eta^h)\left[\nabla p_{w,\eta}^h - \varrho_w\vec{g}\right] + \eta\nabla\left(p_{w,\eta}^h - p_{o,\eta}^h\right)\right\} = 0; \\ -\Phi\Delta_h S_\eta^h - \operatorname{div}\left\{K\lambda_o(S_\eta^h, T_\eta^h)\left[\nabla p_{o,\eta}^h - \varrho_o\,\vec{g}\right] + \eta\,\nabla\left(p_{o,\eta}^h - p_{w,\eta}^h\right)\right\} = 0; \\ \Delta_h\boldsymbol{\Psi}_\eta^h - \operatorname{div}\left\{K\,T_\eta^h\left[c_w\lambda_w(S_\eta^h, T_\eta^h)\left[\nabla p_{w,\eta}^h - \varrho_w\vec{g}\right] + c_o\lambda_o(S_\eta^h, T_\eta^h)\left[\nabla p_{o,\eta}^h - \varrho_o\,\vec{g}\right]\right]\right\} - \\ -c_w\,\eta\,\operatorname{div}\left\{T_\eta^h\nabla\left(p_{w,\eta}^h - p_{o,\eta}^h\right)\right\} - c_o\,\eta\,\operatorname{div}\left\{T_\eta^h\nabla\left(p_{o,\eta}^h - p_{w,\eta}^h\right)\right\} - \operatorname{div}\left(k_T\nabla T_\eta^h\right) = 0; \\ P_c(S_\eta^h) = p_{o,\eta}^h - p_{w,\eta}^h, \end{cases} \tag{3.18}$$

where

$$\Delta_h S_\eta^h \stackrel{\text{def}}{=} \frac{S_\eta^h - S_\eta^\star}{h}, \qquad \text{and} \quad \Delta_h\boldsymbol{\Psi}_\eta^h \stackrel{\text{def}}{=} \frac{\boldsymbol{\Psi}_\eta^h - \boldsymbol{\Psi}_\eta^\star}{h}, \tag{3.19}$$

$S_\eta^\star, \boldsymbol{\Psi}_\eta^\star$ are given functions, and the boundary conditions 3.16 are imposed.

The rest of the paper is organized as follows. In Section 4, we are dealing with the time discrete regularized model. The existence result is proved in 2 main steps. In the first step, we consider the system 3.18, 3.19, and 3.16 with nondegenerate relative permeabilities $k^{\epsilon}_{r,w} \stackrel{\text{def}}{=} k_{r,w}+\epsilon$, $k^{\epsilon}_{r,o} \stackrel{\text{def}}{=} k_{r,o}+\epsilon$ with $\epsilon > 0$, and then we apply the Leray-Schauder fixed point theorem. In the second step, we pass to the limit as $\epsilon \to 0$. In Section 5, we pass to the limit as $h \to 0$. This proves the existence result for the regularized system 3.15. Finally, in Section 6, we pass to the limit as $\eta \to 0$ to prove the main result of the paper.

Remark 4. In fact, the temperature equation does not need a regularization. However, in 3.15, we add the term

$$-c_w\, \eta \quad \text{div}\ \left\{T^{\eta}\nabla\left(p_w^{\eta}-p_o^{\eta}\right)\ \right\} - c_o\, \eta \quad \text{div}\ \left\{T^{\eta}\nabla\left(p_o^{\eta}-p_w^{\eta}\right)\right\}$$

with the idea to have the accordance between the 3 equations.

4 | EXISTENCE RESULT FOR THE ELLIPTIC SYSTEM 3.18

In this section, we deal with the time discrete regularized model 3.18, 3.19 with the boundary conditions 3.16, where the dependence on the parameter η will not be indicated explicitly for the sake of brevity.

The main result of the section is given by the following theorem.

Theorem 4.1. *Let assumptions (A.1) to (A.8) be fulfilled, η be a fixed positive parameter, and let* $0 \leqslant S^{\star} \leqslant 1$, $T_m \leqslant T^{\star} \leqslant T_M$ *and* $\mathbf{\Psi}^{\star} \stackrel{def}{=} c_w\ \Phi(x)\ S^{\star}T^{\star} + c_o\ \Phi(x)\ (1-S^{\star})\ T^{\star} + c_s\ [1-\Phi(x)]\ T^{\star}$ *be given functions. Then for all $h > 0$, there exists a set of functions* $\{p_o^h, p_w^h, S^h, T^h\}$ *such that*

(I) The functions p_o^h, p_w^h, S^h, and T^h have the following regularity properties:

$$p_w^h, p_o^h, T^h \in H^1_{\Gamma_1}(\Omega) \quad and \quad 1-S^h \in H^1_{\Gamma_1}(\Omega). \tag{4.1}$$

(II) The maximum principle for the saturation holds:

$$0 \leqslant S^h \leqslant 1 \quad a.\ e.\ in \quad \Omega. \tag{4.2}$$

(III) The maximum principle for the temperature holds:

$$T_m \leqslant T^h \leqslant T_M \quad a.e.\ in \quad \Omega. \tag{4.3}$$

(IV) For any $\varphi_w, \varphi_o, \varphi_T \in H^1_{\Gamma_1}(\Omega)$,

$$\int_{\Omega} \left\{\Phi(x)\Delta_h S^h\ \varphi_w + \left[K(x)\lambda_w(S^h,T^h)\left(\nabla p_w^h - \varrho_w\vec{g}\right) + \eta\nabla\left(p_w^h - p_o^h\right)\right]\cdot\nabla\varphi_w\right\} \quad dx = 0;$$

$$\int_{\Omega} \left\{-\Phi(x)\Delta_h S^h\ \varphi_o + \left[K(x)\lambda_o(S^h,T^h)\left(\nabla p_o^h - \varrho_o\ \vec{g}\right) - \eta\nabla\left(p_w^h - p_o^h\right)\right]\cdot\nabla\varphi_o\right\}\ dx = 0;$$

$$\begin{aligned}\int_{\Omega} \Big\{ \Delta_h\mathbf{\Psi}^h\varphi_T &+ \Big(K(x)\ T^h\ \Big[c_w\lambda_w(S^h,T^h)\big(\nabla p_w^h - \varrho_w\vec{g}\big) + c_o\lambda_o(S^h,T^h)\big(\nabla p_o^h - \varrho_o\vec{g}\big)\Big]\Big)\cdot\nabla\varphi_T + \\ &+ \left[c_w\ \eta\ T^h\ \nabla\big(p_w^h - p_o^h\big) + c_o\ \eta\ T^h\ \nabla\big(p_o^h - p_w^h\big) + k_T\nabla T^h\right]\cdot\nabla\varphi_T\Big\}\ dx = 0.\end{aligned}$$

4.1 | Regularized system of equations

First, we shortly describe the scheme of the proof of Theorem 4.1. We follow the steps developed in Khalil and Saad[41] and Amaziane et al.[35] Before establishing Theorem 4.1, which is the main goal of this section, we consider a regularized problem. Namely, we consider the system 3.18, 3.19, and 3.16 with nondegenerate relative permeabilities $k^{\epsilon}_{r,w}, k^{\epsilon}_{r,o}$ given by

$$k^{\epsilon}_{r,w} \stackrel{\text{def}}{=} k_{r,w} + \epsilon \quad \text{and} \quad k^{\epsilon}_{r,o} \stackrel{\text{def}}{=} k_{r,o} + \epsilon, \tag{4.4}$$

where $\epsilon > 0$ is a small parameter. In addition, we replace the regularization terms in 3.18 with their projections on finite-dimensional subspaces defined in terms of the eigenbasis of the Laplace operator in Ω with corresponding boundary conditions. This further regularization allows us to truncate high frequencies in the additional terms containing the parameter η and makes it possible to apply the Leray-Schauder fixed point theorem.

The passage to the nondegenerate mobilities leads to the loss of the maximum principle for the saturation S^h. In this connection, the functions $\lambda_w^\epsilon, \lambda_o^\epsilon$ are extended on $S \in (-\infty, 0]$ and $S \in [1, +\infty)$ as functions of the temperature in such a way that the extended functions are continuous and strictly positive (see (A.4)). It is clear that they are bounded in $\mathbb{R}\times\mathbb{R}$. For the same reason, we introduce the extension of the functions S^h and T^h. Namely,

$$\mathsf{Z}_S(S^h) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } S^h \leqslant 0; \\ S^h & \text{for } S^h \in [0,1]; \\ 1 & \text{for } S^h \geqslant 1. \end{cases}, \quad \mathsf{Z}_T(T^h) \stackrel{\text{def}}{=} \begin{cases} T_m & \text{for } T^h \leqslant T_m; \\ T^h & \text{for } T^h \in [T_m, T_M]; \\ T_M & \text{for } T^h \geqslant T_M. \end{cases} \tag{4.5}$$

Similarly, to write the saturations S^h as functions of the unknowns p_w^h and p_o^h, we construct an extension $\widehat{P}_c$ of the capillary pressure function P_c with the following properties:

$$\widehat{P}_c \in C^1(\mathbb{R}), \ \ \widehat{P}_c' \leqslant -m_0 < 0, \ \ \widehat{P}_c' \text{ is bounded on } \mathbb{R} \text{ and } \widehat{P}_c|_{[0,1]} = P_c. \tag{4.6}$$

This is possible because of the condition (A.3). Finally, we introduce an *extended capillary pressure law*; $S^h = \widehat{P}_c^{-1}\left(p_o^h - p_w^h\right)$.

Remark 5. The nondegenerate mobilities $\lambda_w^\epsilon, \lambda_o^\epsilon$ and the extended capillary pressure function $\widehat{P}_c$ can be used to define the global pressure by the formula 2.12. The global pressure defined in this way will be denoted P^ϵ. Consistently, the functions $\mathsf{B}_o, \alpha, \beta, \Lambda_0$, and Λ_1 defined in 2.15, 2.18, 2.19, and 2.20, when defined from nondegenerate mobilities and the extended capillary pressure function, will be denoted by $\mathsf{B}_o^\epsilon, \alpha^\epsilon, \beta^\epsilon, \Lambda_0^\epsilon$, and Λ_1^ϵ. With this notation, we preserve the equalities 2.22 to 2.24, which can be written as

$$\lambda_o^\epsilon \nabla p_o^\epsilon = \lambda_o^\epsilon \nabla \mathsf{P}^\epsilon + \Lambda_1^\epsilon \ \nabla\beta^\epsilon(S^\epsilon) + \lambda_o^\epsilon \mathsf{B}_o^\epsilon \ \nabla T^\epsilon; \tag{4.7}$$

$$\lambda_w^\epsilon \nabla p_w^\epsilon = \lambda_w^\epsilon \nabla \mathsf{P}^\epsilon - \Lambda_1^\epsilon \ \nabla\beta^\epsilon(S^\epsilon) + \lambda_w^\epsilon \mathsf{B}_o^\epsilon \ \nabla T^\epsilon; \tag{4.8}$$

$$\lambda_o^\epsilon \ |\nabla p_o^\epsilon|^2 + \lambda_w^\epsilon \ |\nabla p_w^\epsilon|^2 = \lambda^\epsilon \ |\nabla \mathsf{P}^\epsilon|^2 + \Lambda_0^\epsilon |\nabla\beta^\epsilon(S^\epsilon)|^2 + \lambda^\epsilon \ (\mathsf{B}_o^\epsilon)^2 \ |\nabla T^\epsilon|^2 + 2 \ \lambda^\epsilon \ \mathsf{B}_o^\epsilon \ \nabla \mathsf{P}^\epsilon \cdot \nabla T^\epsilon. \tag{4.9}$$

It is easy to verify that the perturbed functions $\mathsf{B}_o^\epsilon(S,P)$ and $\Lambda_i^\epsilon(S,P)$ are bounded functions and converge uniformly on $\mathbb{R}\times\mathbb{R}$ to $\mathsf{B}_o(S,P)$ and $\Lambda_i(S,P)$ when $\epsilon \to 0$. Also, $\alpha^\epsilon(S) \to \alpha(S)$ uniformly on $\mathbb{R}$ when $\epsilon \to 0$ (however, this is not true for β^ϵ) and the functions λ^ϵ and Λ_0^ϵ are bounded below by constants independent of ϵ (see (A.4) and (2.21)).

The existence of solutions to 3.18, 3.19, and 3.16 is proved in 3 steps. At the first step, assuming that the parameters $\epsilon, N, h, \eta > 0$ are fixed, we study the following regularized elliptic system (the dependence on the parameters $h, \eta > 0$ is omitted for brevity):

$$\begin{aligned} \int_\Omega \Phi(x)\Delta_h S^{\epsilon,N} \ \varphi_w \ dx + \int_\Omega K(x)\lambda_w^\epsilon(S^{\epsilon,N}, T^{\epsilon,N})(\nabla p_w^{\epsilon,N} - \varrho_w\vec{g})\cdot\nabla\varphi_w \ dx \\ + \eta\int_\Omega \nabla\left(\mathcal{P}_N\left[p_w^{\epsilon,N}\right] - \mathcal{P}_N\left[p_o^{\epsilon,N}\right]\right)\cdot\nabla\varphi_w \ dx = 0; \end{aligned} \tag{4.10}$$

$$\begin{aligned} -\int_\Omega \Phi(x)\Delta_h S^{\epsilon,N} \ \varphi_o \ dx + \int_\Omega K(x)\lambda_o^\epsilon(S^{\epsilon,N}, T^{\epsilon,N})(\nabla p_o^{\epsilon,N} - \varrho_o\vec{g})\cdot\nabla\varphi_o \ dx \\ + \eta\int_\Omega \nabla\left(\mathcal{P}_N\left[p_o^{\epsilon,N}\right] - \mathcal{P}_N\left[p_w^{\epsilon,N}\right]\right)\cdot\nabla\varphi_o \ dx = 0; \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
&\int\limits_\Omega \Delta_h \mathbf{\Psi}^{\epsilon,N} \varphi_T \, dx + \int\limits_\Omega K(x) \mathsf{Z}_T(T^{\epsilon,N}) \, c_w \lambda_w^\epsilon(S^{\,\epsilon,N}, T^{\,\epsilon,N}) (\nabla p_w^{\,\epsilon,N} - \varrho_w \vec{g}) \cdot \nabla \varphi_T dx \\
&\qquad + \int\limits_\Omega K(x) \mathsf{Z}_T(T^{\epsilon,N}) \, c_o \lambda_o^\epsilon(S^{\,\epsilon,N}, T^{\,\epsilon,N}) (\nabla p_o^{\,\epsilon,N} - \varrho_o \vec{g}) \cdot \nabla \varphi_T dx + \int\limits_\Omega k_T \nabla T^{\epsilon,N} \cdot \nabla \varphi_T \, dx \\
&\qquad + \int\limits_\Omega (c_o - c_w) \, \eta \, \mathsf{Z}_T(T^{\epsilon,N}) \, \nabla \left(\mathcal{P}_N \left[p_o^{\epsilon,N} \right] - \mathcal{P}_N \left[p_w^{\epsilon,N} \right] \right) \cdot \nabla \varphi_T \, dx = 0,
\end{aligned}
\tag{4.12}
$$

for all $\varphi_w, \varphi_o, \varphi_T \in H^1_{\Gamma_1}(\Omega)$. Here $\mathcal{P}_N$ is the orthogonal projector in $L^2(\Omega)$ on the first N eigenvectors of the eigenproblem

$$
\begin{aligned}
-\Delta p_i &= \lambda_i p_i \quad \text{in} \quad \Omega; \\
p_i &= 0 \quad \text{on} \quad \Gamma_1; \\
\nabla p_i \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_2.
\end{aligned}
\tag{4.13}
$$

We scale the functions p_i such that they form an orthonormal basis in $H^1_{\Gamma_1}(\Omega)$. It is then easy to see that there exists a constant C_N such that for all $p \in L^2(\Omega)$ it holds

$$
\|\nabla \mathcal{P}_N \left[p \right] \|_{L^2(\Omega)} \leqslant C_N \; \|p\|_{L^2(\Omega)}. \tag{4.14}
$$

We also set

$$
\Delta_h S^{\epsilon,N} \stackrel{\text{def}}{=} \frac{\mathsf{Z}_S(S^{\,\epsilon,N}) - S^\star}{h}, \qquad \Delta_h \mathbf{\Psi}^{\epsilon,N} \stackrel{\text{def}}{=} \frac{\mathbf{\Psi}^{\epsilon,N} - \mathbf{\Psi}^\star}{h} \tag{4.15}
$$

with

$$
S^{\epsilon,N} = \widehat{P}_c^{-1} \left(p_o^{\epsilon,N} - p_w^{\epsilon,N} \right) \tag{4.16}
$$

and

$$
\mathbf{\Psi}^{\epsilon,N} \stackrel{\text{def}}{=} \Big((c_w \mathsf{Z}_S(S^{\,\epsilon,N}) + c_o[1 - \mathsf{Z}_S(S^{\,\epsilon,N})]) \Phi(x) + c_s \left[1 - \Phi(x) \right] \Big) \, \mathsf{Z}_T(T^{\epsilon,N}). \tag{4.17}
$$

At the first step, we establish the existence of the solution to 4.10 to 4.12. The second step is concerned with the passage to the limit as $N \to \infty$, while the third step with the passage to the limit as $\epsilon \to 0$.

4.2 | Step 1: application of a fixed point theorem

In this section, for fixed $N > 0$ and $\epsilon > 0$, we prove the existence of solutions to system 4.10 to 4.12. For the sake of brevity, we omit here the dependence of the solutions on the parameters N, ϵ.

We apply the following version of the Leray-Schauder fixed point theorem (see, eg, Gilbarg and Trudinger[42]):

Theorem 4.2. (Leray-Schauder's fixed point theorem.)
Let $\mathcal{M}$ be a continuous and compact map of a Banach space $\mathcal{B}$ into itself. Suppose that the set of $x \in \mathcal{B}$ such that $x = \sigma \mathcal{M} x$ for some $\sigma \in [0, 1]$ is bounded. Then the map $\mathcal{M}$ has a fixed point.

The main result of Section 4.2 is the following proposition.

Proposition 4.1. *Assume that $S^\star, \mathbf{\Psi}^\star \in L^\infty(\Omega)$. Then there exists a triple of functions $\{p_w, p_o, T\} \in \left[H^1_{\Gamma_1}(\Omega) \right]^3$, solution to 4.10 to 4.12.*

Proof of Proposition 4.1. The proof is based on the Leray-Schauder fixed point theorem. Let $\mathcal{M}$ be a map from $\left[L^2(\Omega) \right]^3$ to $\left[L^2(\Omega) \right]^3$ defined by

$$
\mathcal{M} \left(\overline{p}_w, \overline{p}_o, \overline{T} \right) \stackrel{\text{def}}{=} \{ p_w, p_o, T \}, \tag{4.18}
$$

where the triple $\{p_w, p_o, T\}$ is the unique solution of the following system of equations:

$$\begin{aligned}\int_\Omega \Phi(x)\Delta_h\overline{S}\ \varphi_w\ dx + \int_\Omega K(x)\lambda_w^\epsilon(\overline{S},\overline{T})\nabla p_w\cdot\nabla\varphi_w\ \ dx \\ - \int_\Omega K(x)\lambda_w^\epsilon(\overline{S},\overline{T})\ \varrho_w\vec{g}\cdot\nabla\varphi_w\ \ dx + \eta\int_\Omega \nabla\left(\mathcal{P}_N\left[\overline{p}_w\right] - \mathcal{P}_N\left[\overline{p}_o\right]\right)\cdot\nabla\varphi_w\ \ dx = 0;\end{aligned} \tag{4.19}$$

$$\begin{aligned}-\int_\Omega \Phi(x)\Delta_h\overline{S}\ \varphi_o\ dx + \int_\Omega K(x)\lambda_o^\epsilon(\overline{S},\overline{T})\ \nabla p_o\cdot\nabla\varphi_o\ \ dx \\ - \int_\Omega K(x)\lambda_o^\epsilon(\overline{S},\overline{T})\ \varrho_o\ \vec{g}\cdot\nabla\varphi_o\ \ dx + \eta\int_\Omega \nabla\left(\mathcal{P}_N\left[\overline{p}_o\right] - \mathcal{P}_N\left[\overline{p}_w\right]\right)\cdot\nabla\varphi_o\ \ dx = 0;\end{aligned} \tag{4.20}$$

$$\begin{aligned}\int_\Omega \Delta_h\overline{\mathbf{\Psi}}\varphi_T\ dx + \int_\Omega k_T\nabla T\cdot\nabla\varphi_T\ dx + \\ + \int_\Omega K(x)\ \mathsf{Z}_T(\overline{T})\ c_w\lambda_w^\epsilon(\overline{S},\overline{T})\nabla p_w\cdot\nabla\varphi_T dx + \int_\Omega K(x)\ \mathsf{Z}_T(\overline{T})\ c_o\lambda_o^\epsilon(\overline{S},\overline{T})\nabla p_o\cdot\nabla\varphi_T dx - \\ - \int_\Omega K(x)\ \mathsf{Z}_T(\overline{T})\ c_w\varrho_w\lambda_w^\epsilon(\overline{S},\overline{T})\ \vec{g}\cdot\nabla\varphi_T\ dx - \int_\Omega K(x)\ \mathsf{Z}_T(\overline{T})\ c_o\varrho_o\lambda_o^\epsilon(\overline{S},\overline{T})\ \vec{g}\cdot\nabla\varphi_T\ dx + \\ + \int_\Omega (c_o - c_w)\eta\mathsf{Z}_T(\overline{T})\nabla\left(\mathcal{P}_N\left[\overline{p}_o\right] - \mathcal{P}_N\left[\overline{p}_w\right]\right)\cdot\nabla\varphi_T\ dx = 0,\end{aligned} \tag{4.21}$$

for all $\varphi_w, \varphi_o, \varphi_T \in H^1_{\Gamma_1}(\Omega)$. Here

$$\Delta_h\overline{S} \stackrel{\text{def}}{=} \frac{\mathsf{Z}_S(\overline{S}) - S^\star}{h}, \quad \Delta_h\overline{\mathbf{\Psi}} \stackrel{\text{def}}{=} \frac{\overline{\mathbf{\Psi}} - \mathbf{\Psi}^\star}{h}$$

with $\overline{S} \stackrel{\text{def}}{=} \widehat{P}_c^{-1}(\overline{p}_o - \overline{p}_w)$, and $\overline{\mathbf{\Psi}} \stackrel{\text{def}}{=} \left(c_w\mathsf{Z}_S(\overline{S}) + c_o[1 - \mathsf{Z}_S(\overline{S})] + c_s\ [1 - \Phi(x)]\right)\mathsf{Z}_T(\overline{T})$. Weobserve that the map $\mathcal{M}$ is constructed by evaluating all nonlinear terms in 4.10, 4.11, and 4.12 at the known arguments $\overline{p}_w, \overline{p}_o, \overline{T}$. The fixed point of the map $\mathcal{M}$ is then a solution to 4.10 to 4.12.

The system 4.19 to 4.21 can be rewritten in the following form:

$$\mathbf{W}_w(p_w, \varphi_w) = \mathbf{f}_w(\varphi_w); \quad \mathbf{W}_o(p_o, \varphi_o) = \mathbf{f}_o(\varphi_o); \quad \mathbf{W}_T(T, \varphi_T) = \mathbf{f}_T(\varphi_T; p_w, p_o), \tag{4.22}$$

where

$$\mathbf{W}_w(p_w, \varphi_w) \stackrel{\text{def}}{=} \int_\Omega K\lambda_w^\epsilon(\overline{S},\overline{T})\nabla p_w\cdot\nabla\varphi_w\ dx, \quad \mathbf{W}_o(p_o, \varphi_o) \stackrel{\text{def}}{=} \int_\Omega K\lambda_o^\epsilon(\overline{S},\overline{T})\nabla p_o\cdot\nabla\varphi_o\ dx,$$

and

$$\mathbf{W}_T(T, \varphi_T) \stackrel{\text{def}}{=} \int_\Omega k_T\nabla T\cdot\nabla\varphi_T\ dx,$$

with the terms $\mathbf{f}_w(\varphi_w)$, $\mathbf{f}_o(\varphi_o)$, and $\mathbf{f}_T(\varphi_T; p_w, p_o)$ given by the remaining terms in the corresponding equations 4.19 to 4.21.

We observe that $\mathbf{W}_w(\cdot,\cdot)$, $\mathbf{W}_o(\cdot,\cdot)$, and $\mathbf{W}_T(\cdot,\cdot)$ are bilinear, continuous, and coercive mappings in the space $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$, and $\mathbf{f}_w(\cdot), \mathbf{f}_o(\cdot), \mathbf{f}_T(\cdot)$ are linear continuous mapping in $H^1_{\Gamma_1}(\Omega)$ (p_w and p_o are known functions in $\mathbf{f}_T$). Then we can apply the Lax-Milgram theorem to get the existence of $\{p_w, p_o, T\} \in \left[H^1_{\Gamma_1}(\Omega)\right]^3$, which ensures that the map $\mathcal{M}$ is well defined in $[L^2(\Omega)]^3$.

Furthermore, using standard energy estimates for the problems 4.22 and estimate 4.14, we obtain immediately the following estimates for the solutions of 4.22 (see also Khalil and Saad[43]):

$$\|p_w\|^2_{H^1_{\Gamma_1}(\Omega)}, \ \|p_o\|^2_{H^1_{\Gamma_1}(\Omega)}, \ \|T\|^2_{H^1_{\Gamma_1}(\Omega)} \leqslant \mathsf{C}, \tag{4.23}$$

where $\mathsf{C} = \mathsf{C}(\Omega, \eta, h, \epsilon, N, \phi^+, K^+, K^-, \|S^\star\|_{L^2(\Omega)}, \|T^\star\|_{L^2(\Omega)}, \|\overline{p}_w\|_{L^2(\Omega)}, \|\overline{p}_o\|_{L^2(\Omega)})$. □

Now to apply the fixed point theorem, we have to establish the following result.

Lemma 4.1. *Let $\mathcal{M}$ be the map defined in 4.18. Then*

- *$\mathcal{M}$ is a continuous operator, which maps every bounded subset of $L^2(\Omega)$ into a relatively compact set;*
- *there exists $r > 0$ such that, if $\{p_w, p_o, T\} = \sigma\ \mathcal{M}(p_w, p_o, T)$ with $\sigma \in (0, 1)$, then*

$$\| \{p_w, p_o, T\} \|_{[L^2(\Omega)]^3} \leqslant r.$$

Proof of Lemma 4.1. Let us prove the first statement of the lemma. Claim that $\mathcal{M}$ maps every bounded subset of $L^2(\Omega)$ into a relatively compact set follows from 4.23 and compact embedding $H^1(\Omega) \subset L^2(\Omega)$. To show the continuity, we consider a sequence $\{\overline{p}_{w,n}, \overline{p}_{o,n}, \overline{T}_n\}_{(n\to\infty)}$, which converges to $\{\overline{p}_w, \overline{p}_o, \overline{T}\} \in \left[L^2(\Omega)\right]^3$, and let us prove that the sequence

$$\{p_{w,n}, p_{o,n}, T_n\} = \mathcal{M}\left(\overline{p}_{w,n}, \overline{p}_{o,n}, \overline{T}_n\right)$$

converges to $\{p_w, p_o, T\} = \mathcal{M}\left(\overline{p}_w, \overline{p}_o, \overline{T}\right)$.

The sequences $\{p_{w,n}\}_{(n\to\infty)}, \{p_{o,n}\}_{(n\to\infty)}, \{T_n\}_{(n\to\infty)}$ satisfy the Equations 4.19 to 4.21 and by 4.23 are bounded in $H^1_{\Gamma_1}(\Omega)$. Then, up to a subsequence, we have the following convergence results:

$$p_{w,n} \to p_w, \quad p_{o,n} \to p_o, \quad T_n \to T, \tag{4.24}$$

weakly in $H^1_{\Gamma_1}(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Therefore, we can pass to the limit $n \to \infty$ in 4.19 to 4.21 and show that the limit $\{p_w, p_o, T\}$ is a solution to 4.19 to 4.21. Because the solution of 4.19 to 4.21 is unique, this proves the continuity of the map $\mathcal{M}$ and achieves the proof of statement (1) of Lemma 4.1.

Now we turn to the proof of statement (2) of Lemma 4.1. Assume that there exists $\sigma \in (0, 1)$ such that

$$\{p_w, p_o, T\} = \sigma\ \mathcal{M}(p_w, p_o, T).$$

Then $\{p_w, p_o, T\}$ satisfies the following equations:

$$\begin{aligned} \int_\Omega K(x)\lambda_w^\epsilon(S,T)\nabla p_w \cdot \nabla\varphi_w\, dx &= -\sigma \int_\Omega \Phi(x)\Delta_h S\ \varphi_w\, dx \\ +\sigma \int_\Omega K(x)\lambda_w^\epsilon(S,T)\varrho_w\, \vec{g}\cdot\nabla\varphi_w\, dx &- \eta\, \sigma \int_\Omega \nabla\left(\mathcal{P}_N\left[p_w\right] - \mathcal{P}_N\left[p_o\right]\right)\cdot\nabla\varphi_w\, dx; \end{aligned} \tag{4.25}$$

$$\begin{aligned} \int_\Omega K(x)\lambda_o^\epsilon(S,T)\ \nabla p_o \cdot \nabla\varphi_o\, dx &= \sigma \int_\Omega \Phi(x)\ \Delta_h S\ \varphi_o\, dx \\ +\sigma \int_\Omega K(x)\ \lambda_o^\epsilon(S,T)\varrho_o\, \vec{g}\cdot\nabla\varphi_o\ \ dx &- \eta\, \sigma \int_\Omega \nabla\left(\mathcal{P}_N\left[p_o\right] - \mathcal{P}_N\left[p_w\right]\right)\cdot\nabla\varphi_o\, dx; \end{aligned} \tag{4.26}$$

$$\begin{aligned} \int_\Omega k_T\nabla T\cdot\nabla\varphi_T\, dx &= -\sigma \int_\Omega \Delta_h \boldsymbol{\Psi}\varphi_T\, dx- \\ -\int_\Omega K(x)\ \mathsf{Z}_T(T)c_w\lambda_w^\epsilon(S,T)\nabla p_w\cdot\nabla\varphi_T dx &- \int_\Omega K(x)\ \mathsf{Z}_T(T)c_o\lambda_o^\epsilon(S,T)\nabla p_o\cdot\nabla\varphi_T dx+ \\ +\sigma\int_\Omega K(x)\ \mathsf{Z}_T(T)\ c_w\varrho_w\lambda_w^\epsilon(S,T)\,\vec{g}\cdot\nabla\varphi_T\, dx &+ \sigma\int_\Omega K(x)\ \mathsf{Z}_T(T)\ c_o\varrho_o\lambda_o^\epsilon(S,T)\,\vec{g}\cdot\nabla\varphi_T\, dx- \\ &-\sigma\,\eta\int_\Omega (c_o - c_w)\mathsf{Z}_T(T)\nabla\left(\mathcal{P}_N\left[p_o\right] - \mathcal{P}_N\left[p_w\right]\right)\cdot\nabla\varphi_T\, dx. \end{aligned} \tag{4.27}$$

First, we set $\varphi_w = p_w$ in 4.25 and $\varphi_o = p_o$ in 4.26. Summing these equations, we get

$$\begin{aligned}&\int_\Omega K(x)\lambda_w^\epsilon(S,T)\nabla p_w\cdot\nabla p_w\,dx+\int_\Omega K(x)\lambda_o^\epsilon(S,T)\,\nabla p_o\cdot\nabla p_o\,dx\\&-\sigma\int_\Omega K(x)\lambda_w^\epsilon(S,T)\varrho_w\,\vec{g}\cdot\nabla p_w\,dx-\sigma\int_\Omega K(x)\,\lambda_o^\epsilon(S,T)\,\varrho_o\,\vec{g}\cdot\nabla p_o\,\,dx\\&-\sigma\int_\Omega\Phi(x)\frac{\mathsf{Z}_S(S)-S^\star}{h}\,\widehat{P}_c(S)\,dx+\eta\,\sigma\int_\Omega\nabla\left(\mathcal{P}_N\left[p_o\right]-\mathcal{P}_N\left[p_w\right]\right)\cdot(\nabla p_o-\nabla p_w)\,dx=0.\end{aligned}\tag{4.28}$$

Now taking into account the definitions of the functions $\lambda_w^\epsilon, \lambda_o^\epsilon$ given in 4.4, the definition of the function Z_S, Cauchy-Schwartz, and Friedrichs' inequalities, from 4.28, we obtain that

$$\epsilon\,\|\nabla p_w\|^2_{L^2(\Omega)}+\epsilon\,\|\nabla p_o\|^2_{L^2(\Omega)}+\eta\,\sigma\int_\Omega\left|\nabla\left(\mathcal{P}_N\left[p_o\right]-\mathcal{P}_N\left[p_w\right]\right)\right|^2dx\leqslant C_1\,\left[1+\|S^\star\|^2_{L^2(\Omega)}\right],\tag{4.29}$$

where C_1 is a constant, which depends on ϵ but does not depend on σ and N.

Now we turn to the estimate for the temperature. We set $\varphi_T = T$ in 4.27 to get

$$\begin{aligned}&\int_\Omega k_T\nabla T\cdot\nabla T\,dx+\sigma\int_\Omega\Delta_h\mathbf{\Psi}\,T\,dx+\\&+\int_\Omega K(x)\,\mathsf{Z}_T(T)c_w\lambda_w^\epsilon(S,T)\nabla p_w\cdot\nabla T\,dx+\int_\Omega K(x)\,\mathsf{Z}_T(T)c_o\lambda_o^\epsilon(S,T)\nabla p_o\cdot\nabla T\,dx-\\&-\sigma\int_\Omega K(x)\,\mathsf{Z}_T(T)\,\Big(c_w\varrho_w\lambda_w^\epsilon(S,T)+c_o\varrho_o\lambda_o^\epsilon(S,T)\Big)\,\vec{g}\cdot\nabla T\,dx\\&+\sigma\,\eta\,(c_o-c_w)\int_\Omega\,\mathsf{Z}_T(T)\nabla\left(\mathcal{P}_N\left[p_o\right]-\mathcal{P}_N\left[p_w\right]\right)\cdot\nabla T\,dx.\end{aligned}\tag{4.30}$$

Now taking into account the uniform bound 4.29, the Cauchy-Schwartz inequality, from 4.30, we obtain that

$$\|\nabla T\|^2_{L^2(\Omega)}\leqslant C_2,$$

where C_2 is a constant, which is independent of σ and N.

Thus, all the conditions of Leray-Schauder's theorem are fulfilled and Proposition 4.1 is proved.

4.3 | Step 2: passage to the limit as $N \to +\infty$

In this section, we pass to the limit as $N \to +\infty$ in 4.10 to 4.12. For the sake of simplicity, we omit the dependence on the parameter ϵ in the functions depending on N. It follows from the previous section that the triple of functions $\{\mathsf{p}_w^N, p_o^N, T^N\} \in \left[H^1_{\Gamma_1}(\Omega)\right]^3$ is the solution of the system 4.10 to 4.12. We get the following estimate:

$$\epsilon\,\|\nabla p_w^N\|^2_{L^2(\Omega)}+\epsilon\,\|\nabla p_o^N\|^2_{L^2(\Omega)}+\|\nabla T^N\|^2_{L^2(\Omega)}+\eta\int_\Omega\left|\nabla\left(\mathcal{P}_N\left[p_w^N\right]-\mathcal{P}_N\left[p_o^N\right]\right)\right|^2dx\leqslant C,\tag{4.31}$$

where C is a constant, which does not depend on N. Then (up to subsequence), we obtain the following convergence results:

$$p_w^N\to p_w^\epsilon\ \text{ weakly in }\ H^1_{\Gamma_1}(\Omega),\ \ \text{strongly in }\ L^2(\Omega),\ \ \text{and a.e. in }\ \Omega;\tag{4.32}$$

$$p_o^N\to p_o^\epsilon\ \text{ weakly in }\ H^1_{\Gamma_1}(\Omega),\ \ \text{strongly in }\ L^2(\Omega),\ \ \text{and a.e. in }\ \Omega;\tag{4.33}$$

$$T^N\to T^\epsilon\ \text{ weakly in }\ H^1_{\Gamma_1}(\Omega),\ \ \text{strongly in }\ L^2(\Omega),\ \ \text{and a.e. in }\ \Omega.\tag{4.34}$$

Taking into account that $S = \widehat{P}_c^{-1}(p_o - p_w)$, we also have

$$\mathsf{Z}_S(S^N)\to\mathsf{Z}_S(S^{\,\epsilon})\ \text{ strongly in }\ L^2(\Omega)\ \text{ and a.e. in }\ \Omega.\tag{4.35}$$

Finally, it is easy to show from properties of the spectral basis 4.13 that for $\alpha \in \{w, g\}$,

$$\mathcal{P}_N\left[p_\alpha\right] \to p_\alpha \quad \text{weakly in} \quad H^1_{\Gamma_1}(\Omega). \tag{4.36}$$

Now we pass to the limit in 4.10 to 4.12 as $N \to +\infty$ using the convergence results 4.32 to 4.36. The corresponding system of equations is as follows:

$$\int_\Omega \Phi(x)\Delta_h\, S^\epsilon \varphi_w\, dx + \int_\Omega K(x)\lambda_w^\epsilon(S^{\,\epsilon}, T^\epsilon)(\nabla p_w^\epsilon - \varrho_w\, \vec{g}) \cdot \nabla\varphi_w \;\; dx - \eta \int_\Omega \nabla\hat{P}_c(S^\epsilon) \cdot \nabla\varphi_w \;\; dx = 0; \tag{4.37}$$

$$-\int_\Omega \Phi(x)\Delta_h S^\epsilon \varphi_o\, dx + \int_\Omega K(x)\lambda_o^\epsilon(S^\epsilon, T^\epsilon)(\nabla p_o^\epsilon - \varrho_o\, \vec{g}) \cdot \nabla\varphi_o \;\; dx + \eta \int_\Omega \nabla\hat{P}_c(S^\epsilon) \cdot \nabla\varphi_o \;\; dx = 0; \tag{4.38}$$

$$\begin{aligned}&\int_\Omega \Delta_h \boldsymbol{\Psi}^\epsilon \varphi_T\, dx + \int_\Omega k_T \nabla T^\epsilon \cdot \nabla\varphi_T\, dx + \int_\Omega (c_o - c_w)\eta \mathsf{Z}_T(T^\epsilon)\nabla\hat{P}_c(S^\epsilon) \cdot \nabla\varphi_T\, dx \\ &+ \int_\Omega K(x)\, \mathsf{Z}_T(T^\epsilon)\;\left(c_w\lambda_w^\epsilon(S^\epsilon, T^\epsilon)\left(\nabla p_w^\epsilon - \varrho_w \vec{g}\right) + c_o\lambda_o^\epsilon(S^\epsilon, T^\epsilon)\left(\nabla p_o^\epsilon - \varrho_o \vec{g}\right)\right) \cdot \nabla\varphi_T dx = 0,\end{aligned} \tag{4.39}$$

for all $\varphi_w, \varphi_o, \varphi_T \in H^1_{\Gamma_1}(\Omega)$.

4.4 | Uniform estimates with respect to ϵ

First, we notice that as in the previous sections we omit the dependence of the corresponding functions on η, h and keep the dependence on the small parameter ϵ, only.

It follows from the results of Section 4.3 that for any $\epsilon > 0$, there is $\left\{p_w^\epsilon, p_o^\epsilon, T^\epsilon\right\} \in \left[H^1_{\Gamma_1}(\Omega)\right]^3$, which is the solution of 4.37 to 4.39. First, we obtain uniform estimates (with respect to ϵ) for the solutions to pass to the limit in 4.37 to 4.39 as $\epsilon \to 0$. These estimates are given by the following lemma.

Lemma 4.2. *Let $\left\{p_w^\epsilon, p_o^\epsilon, T^\epsilon\right\}$ be a solution to 4.37 to 4.39, and let P^ϵ (the nonisothermal global pressure) and β^ϵ be the functions defined in 2.12 and 2.18, respectively. Then we have*

$$\{\sqrt{\epsilon}\;\; \nabla p_w^\epsilon\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; L^2(\Omega); \tag{4.40}$$

$$\{\sqrt{\epsilon}\;\; \nabla p_o^\epsilon\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; L^2(\Omega); \tag{4.41}$$

$$\{\nabla\hat{P}_c(S^{\,\epsilon})\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; L^2(\Omega); \tag{4.42}$$

$$\{T^\epsilon\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; H^1(\Omega); \tag{4.43}$$

$$\{\mathsf{P}^\epsilon\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; H^1(\Omega); \tag{4.44}$$

$$\{\beta^\epsilon(S^{\,\epsilon})\}_{\epsilon>0} \;\; \textit{is uniformly bounded in} \;\; H^1(\Omega). \tag{4.45}$$

Proof of Lemma 4.2. We set $\varphi_w = p_w^\epsilon$ in 4.37 and $\varphi_w = p_o^\epsilon$ in 4.38. Then taking a sum of these equations, we get

$$\begin{aligned}&\epsilon \int_\Omega K(x)\nabla p_w^\epsilon \cdot \nabla p_w^\epsilon \;\; dx + \epsilon \int_\Omega K(x)\nabla p_o^\epsilon \cdot \nabla p_o^\epsilon \;\; dx+ \\ &+ \int_\Omega K(x)\lambda_w^\epsilon(S^{\,\epsilon}, T^\epsilon)\nabla p_w^\epsilon \cdot \nabla p_w^\epsilon \;\; dx + \int_\Omega K(x)\lambda_o^\epsilon(S^\epsilon, T^\epsilon)\; \nabla p_o^\epsilon \cdot \nabla p_o^\epsilon \;\; dx- \\ &- \int_\Omega K(x)\lambda_w^\epsilon(S^{\,\epsilon}, T^\epsilon)\; \varrho_w\vec{g} \cdot \nabla p_w^\epsilon \;\; dx - \int_\Omega K(x)\lambda_o^\epsilon(S^\epsilon, T^\epsilon)\; \varrho_o\; \vec{g} \cdot \nabla p_o^\epsilon \;\; dx+ \\ &+ \eta \int_\Omega |\nabla\hat{P}_c(S^{\,\epsilon})|^2 \;\; dx = \int_\Omega \frac{\Phi(x)}{h} \;\; \left[\mathsf{Z}_S(S^{\,\epsilon}) - S^\star\right] \hat{P}_c(S^{\,\epsilon})\; dx.\end{aligned}$$

Using the Cauchy-Schwartz inequality and taking into account the condition (A.2), we find that there exists a constant C that does not depend on ϵ and η, such that

$$\epsilon \int_\Omega |\nabla p_w^\epsilon|^2\, dx + \epsilon \int_\Omega |\nabla p_o^\epsilon|^2\, dx + \int_\Omega \left\{ \lambda_w^\epsilon(S^\epsilon, T^\epsilon)\, |\nabla p_w^\epsilon|^2 + \lambda_o^\epsilon(S^\epsilon, T^\epsilon)\, |\nabla p_o^\epsilon|^2 + \eta |\nabla \hat{P}_c(S^\epsilon)|^2 \right\} dx \leqslant C. \tag{4.46}$$

The uniform estimates 4.40 to 4.42 follow immediately from 4.46.

Let us turn to the uniform bound 4.43. In 4.39, we set $\varphi_T = T^\epsilon$ to have

$$\begin{aligned} &\int_\Omega \Delta_h \boldsymbol{\Psi}^\epsilon T^\epsilon\, dx + \int_\Omega k_T \nabla T^\epsilon \cdot \nabla T^\epsilon\, dx + \\ &+ \int_\Omega K(x)\, \mathsf{Z}_T(T^\epsilon)\, c_w \lambda_w^\epsilon(S^\epsilon, T^\epsilon) \nabla p_w^\epsilon \cdot \nabla T^\epsilon dx + \int_\Omega K(x)\, \mathsf{Z}_T(T^\epsilon)\, c_o \lambda_o^\epsilon(S^\epsilon, T^\epsilon) \nabla p_o^\epsilon \cdot \nabla T^\epsilon dx - \\ &- \int_\Omega K(x)\, \mathsf{Z}_T(T^\epsilon)\, c_w \varrho_w \lambda_w^\epsilon(S^\epsilon, T^\epsilon)\, \vec{g} \cdot \nabla T^\epsilon\, dx - \int_\Omega K(x)\, \mathsf{Z}_T(T^\epsilon)\, c_o \varrho_o \lambda_o^\epsilon(S^\epsilon, T^\epsilon)\, \vec{g} \cdot \nabla T^\epsilon\, dx + \\ &+ \int_\Omega c_w \eta \mathsf{Z}_T(T^\epsilon) \nabla (p_w^\epsilon - p_o^\epsilon) \cdot \nabla T^\epsilon\, dx + \int_\Omega c_o \eta \mathsf{Z}_T(T^\epsilon) \nabla (p_o^\epsilon - p_w^\epsilon) \cdot \nabla T^\epsilon\, dx = 0. \end{aligned} \tag{4.47}$$

Now using the uniform bounds 4.40 to 4.42, 4.46, the Cauchy-Schwartz, and Friedreich's inequalities, from 4.47, we obtain 4.43.

The uniform bounds 4.44 and 4.45 follow from 4.43, Remark 5, and equality 4.9. This completes the proof of Lemma 4.2.

4.5 | Step 3: passage to the limit as $\epsilon \to 0$

The uniform estimates established in the previous section imply the following convergence results.

Lemma 4.3. *Let $\{S^\epsilon\}_{\epsilon>0}, \{T^\epsilon\}_{\epsilon>0}, \{\mathsf{P}^\epsilon\}_{\epsilon>0}, \{p_w^\epsilon\}_{\epsilon>0}$, and $\{p_o^\epsilon\}_{\epsilon>0}$ be the sequences of saturation, temperature, global pressure, and the phase pressures, respectively. Then we have on a subsequence*

$$T^\epsilon \to T \quad \textit{weakly in } H^1(\Omega) \textit{ and a.e. in } \Omega\,; \tag{4.48}$$

$$\mathsf{P}^\epsilon \to \mathsf{P} \quad \textit{weakly in } H^1(\Omega) \textit{ and a.e. in } \Omega\,; \tag{4.49}$$

$$\beta^\epsilon(S^\epsilon) \to \beta(S) \quad \textit{weakly in } H^1(\Omega) \textit{ and a.e. in } \Omega\,; \tag{4.50}$$

$$\hat{P}_c(S^\epsilon) \to \hat{P}_c(S) \quad \textit{weakly in } H^1(\Omega)\,; \tag{4.51}$$

$$S^\epsilon \to S \textit{ a.e. in } \Omega\,; \tag{4.52}$$

$$p_w^\epsilon \to p_w \quad \textit{a.e. in } \Omega \qquad \textit{and} \qquad p_o^\epsilon \to p_o \quad \textit{a.e. in } \Omega\,, \tag{4.53}$$

and $p_o - p_w = \hat{P}_c(S)$ a.e. in Ω.

Proof of Lemma 4.3. The convergence 4.48 to 4.51 follow directly from Lemma 4.2. In 4.51, we have $\hat{P}_c(S^\epsilon) \to \xi^*$, for some $\xi^* \in H^1(\Omega)$, but the function $\hat{P}_c$ is invertible on $\mathbb{R}$, and we can define $S = (\hat{P}_c)^{-1}(\xi^*)$. This proves 4.51, and the continuity of the function $\hat{P}_c$ gives 4.52.

From $\nabla\beta^\epsilon(S^\epsilon) = -\alpha^\epsilon(S^\epsilon)\nabla\hat{P}_c(S^\epsilon)$ and uniform convergence on $\mathbb{R}\alpha^\epsilon \to \alpha$, we have

$$\nabla\beta^\epsilon(S^\epsilon) \to -\alpha(S)\nabla\hat{P}_c(S) = \nabla\beta(S),$$

and 4.50 is proved.

The phase pressures p_w^ϵ and p_o^ϵ are connected to the global pressure P^ϵ through the relations 2.12 and 2.14, namely,

$$p_o^\epsilon = \mathsf{P}^\epsilon + \int_1^{S^\epsilon} \frac{\lambda_w^\epsilon}{\lambda^\epsilon}(\xi, T^\epsilon)\, \widehat{P}_c'(\xi)\, d\xi, \quad p_w^\epsilon = \mathsf{P}^\epsilon - \int_1^{S^\epsilon} \frac{\lambda_o^\epsilon}{\lambda^\epsilon}(\xi, T^\epsilon)\, \widehat{P}_c'(\xi)\, d\xi. \tag{4.54}$$

By 4.48, 4.49, and 4.52, we obtain 4.53 where the limit functions p_o and p_w are given by

$$p_o = \mathsf{P} + \int_1^{S} \frac{\lambda_w}{\lambda}(\xi, T)\, \widehat{P}_c'(\xi)\, d\xi, \quad p_w = \mathsf{P} - \int_1^{S} \frac{\lambda_o}{\lambda}(\xi, T)\, \widehat{P}_c'(\xi)\, d\xi, \tag{4.55}$$

and $p_o - p_w = \widehat{P}_c(S)$ is obviously satisfied.

Now we are in position to complete the proof of Theorem 4.1. To this end, we have to pass to the limit in 4.37 to 4.39 as $\epsilon \to 0$ and then prove the maximum principle for the saturation and the temperature functions.

Let us consider the second term on the left-hand side of 4.37. It follows from 4.8 that

$$\int_\Omega K(x)\lambda_w^\epsilon(S^\epsilon, T^\epsilon)\nabla p_w^\epsilon \cdot \nabla\varphi_w\, dx$$
$$= \int_\Omega K(x)(\lambda_w^\epsilon(S^\epsilon, T^\epsilon)\nabla \mathsf{P}^\epsilon - \Lambda_1^\epsilon(S^\epsilon, T^\epsilon)\nabla\beta^\epsilon(S^\epsilon) + \lambda_w^\epsilon(S^\epsilon, T^\epsilon)\mathsf{B}_o^\epsilon(S^\epsilon, T^\epsilon)\nabla T^\epsilon) \cdot \nabla\varphi_w\, dx.$$

Using the uniform convergence of the functions λ_w^ϵ, Λ_1^ϵ, and B_o^ϵ (see Remark 5) and Lemma 4.3, we get

$$\int_\Omega K(x) \quad \lambda_w^\epsilon(S^\epsilon, T^\epsilon)\nabla p_w^\epsilon \cdot \nabla\varphi_w\, dx \to$$
$$\int_\Omega K(x)(\lambda_w(S, T)\nabla \mathsf{P} - \Lambda_1(S, T)\nabla\beta(S) + \lambda_w(S, T)\mathsf{B}_o(S, T)\nabla T) \cdot \nabla\varphi_w\, dx$$
$$= \int_\Omega K(x)\lambda_w(S, T)\nabla p_w \cdot \nabla\varphi_w\, dx,$$

where p_w is given by 4.55. In the same way, for the second term on the left-hand side of 4.38, we get

$$\int_\Omega K(x)\lambda_o^\epsilon(S^\epsilon, T^\epsilon)\nabla p_o^\epsilon \cdot \nabla\varphi_o\, dx \to \int_\Omega K(x)\lambda_o(S, T)\nabla p_o \cdot \nabla\varphi_o\, dx.$$

By Lemma 4.3, we can pass to the limit in all other terms in 4.37 to 4.39. Thus, we can conclude that there exists $\{p_w^h, p_o^h, T^h\}$ solution to

$$\int_\Omega \left\{\Phi(x)\Delta_h S^h\ \varphi_w + \left[K(x)\lambda_w(S^h, T^h)\left(\nabla p_w^h - \varrho_w\vec{g}\right) + \eta\nabla\left(p_w^h - p_o^h\right)\right] \cdot \nabla\varphi_w\right\}\quad dx = 0; \tag{4.56}$$

$$\int_\Omega \left\{-\Phi(x)\Delta_h S^h\ \varphi_o + \left[K(x)\lambda_o(S^h, T^h)\left(\nabla p_o^h - \varrho_o\,\vec{g}\right) - \eta\nabla\left(p_w^h - p_o^h\right)\right] \cdot \nabla\varphi_o\right\}\ dx = 0; \tag{4.57}$$

$$\int_\Omega \left\{\Delta_h \boldsymbol{\Psi}^h\varphi_T + \left(K(x)\ Z_T(T^h)\ \left[c_w\lambda_w(S^h, T^h)\left(\nabla p_w^h - \varrho_w\vec{g}\right) + c_o\varrho_o\lambda_o(S^h, T^h)\left(\nabla p_o^h - \varrho_o\vec{g}\right)\right]\right) \cdot \nabla\varphi_T\right.$$
$$\left. + \left[(c_o - c_w)\ \eta\ Z_T(T^h)\ \nabla\left(p_o^h - p_w^h\right) + k_T\nabla T^h\right] \cdot \nabla\varphi_T\right\}\ dx = 0 \tag{4.58}$$

for all $\varphi_w, \varphi_o, \varphi_T \in H_{\Gamma_1}^1(\Omega)$.

Let us prove the maximum principle for the saturation. As in Lemma 2.5 from Khalil and Saad,[43] one can obtain the following result.

Lemma 4.4. (Maximum principle for the saturation)
Let $0 \leqslant S^\star \leqslant 1$. Then under the conditions of Theorem 4.1, we have

$$0 \leqslant S^h \leqslant 1 \quad a.e.\ in\ \ \Omega. \tag{4.59}$$

Proof of Lemma 4.4. We start the proof by establishing the lower bound in 4.59. To this end, we introduce the function

$$\boldsymbol{\varphi}_w^0(x,t) \stackrel{\text{def}}{=} \min\left\{S^h, 0\right\}. \tag{4.60}$$

(Note that because $\widehat{P}_c(S^h) \in H^1(\Omega)$ then $\boldsymbol{\varphi}_w^0 \in H^1_{\Gamma_1}(\Omega)$.) We plug the function $\boldsymbol{\varphi}_w^0$ in 4.56. From the extension by zero of the wetting phase mobility for $S \leqslant 0$, we obtain that

$$\int_\Omega \left\{ \Phi(x) \frac{\mathsf{Z}(S^h) - S^\star}{h} \boldsymbol{\varphi}_w^0 - \eta \widehat{P}_c'(S^h) \nabla S^h \cdot \nabla \boldsymbol{\varphi}_w^0 \right\} \, dx = 0.$$

The integral is evaluated only for $S^h < 0$ so that the extension of the capillary pressure 4.6 gives

$$\int_\Omega \left\{ -\Phi(x) \frac{S^\star}{h} \boldsymbol{\varphi}_w^0 + \eta |\widehat{P}_c'(S^h)| \nabla \boldsymbol{\varphi}_w^0 \cdot \nabla \boldsymbol{\varphi}_w^0 \right\} \, dx = 0.$$

This gives $\boldsymbol{\varphi}_w^0 = 0$, and the lower bound in 4.59 is proved. The upper bound is proved by plugging $\boldsymbol{\varphi}_w^0(x,t) = \max\left\{S^h - 1, 0\right\}$ in 4.57. This completes the proof of Lemma 4.4.

Finally, we turn to the proof of the maximum principle for the temperature. Namely, we have the following:

Lemma 4.5. (Maximum principle for the temperature)
Let $\boldsymbol{\Psi}^\star$ be defined as follows:

$$\boldsymbol{\Psi}^\star \stackrel{def}{=} c_w \, \Phi(x) \, S^\star T^\star + c_o \, \Phi(x) \, (1 - S^\star) \, T^\star + c_s \, [1 - \Phi(x)] \, T^\star \quad \text{with} \quad 0 \leqslant S^\star \leqslant 1, \quad T_m \leqslant T^\star \leqslant T_M. \tag{4.61}$$

Then under the conditions of Theorem 4.1, we have

$$T_m \leqslant T^h \leqslant T_M \quad \text{a.e. in} \quad \Omega. \tag{4.62}$$

Proof of Lemma 4.5. We start the proof by establishing the upper bound in 4.62. To this end, we introduce the cutoff function defined by

$$\boldsymbol{\vartheta}_M^h(x,t) \stackrel{\text{def}}{=} \max\left\{T^h - T_M, 0\right\}. \tag{4.63}$$

First, we observe that$T^h \in H^1_{\Gamma_1}(\Omega)$ implies $\boldsymbol{\vartheta}_M^h \in H^1_{\Gamma_1}(\Omega)$. Then we plug the function $\boldsymbol{\vartheta}_M^h$ in 4.58 and taking into account 4.61, we get

$$\begin{aligned} &\int_\Omega \left\{ \frac{\Phi}{h} \, c_w \left[S^h \, \mathsf{Z}(T^h) - S^\star \, T^\star\right] + \frac{\Phi}{h} c_o \left[(1 - S^h) \, \mathsf{Z}(T^h) - (1 - S^\star) T^\star\right] + \frac{[1 - \Phi]}{h} c_s \left[\mathsf{Z}(T^h) - T^\star\right] \right\} \boldsymbol{\vartheta}_M^h dx \\ &\quad + \int_\Omega K(x) \, \mathsf{Z}(T^h) \, \left[c_w \lambda_w(S^h, T^h) \left(\nabla p_w^h - \varrho_w \vec{\mathrm{g}}\right) + c_o \lambda_o(S^h, T^h) \left(\nabla p_o^h - \varrho_o \vec{\mathrm{g}}\right) \right] \cdot \nabla \boldsymbol{\vartheta}_M^h \, dx + \end{aligned} \tag{4.64}$$

$$+ \int_\Omega \left[c_w \, \eta \, \mathsf{Z}(T^h) \, \nabla\left(p_w^h - p_o^h\right) + c_o \, \eta \, \mathsf{Z}(T^h) \, \nabla\left(p_o^h - p_w^h\right) + k_T \nabla T^h \right] \cdot \nabla \boldsymbol{\vartheta}_M^h \, dx = 0. \tag{4.65}$$

Let us denote the first term on the left-hand side of Equation 4.65 by J_1^h. Then we can rearrange it in the following way:

$$\begin{aligned} \mathsf{J}_1^h = &\int_\Omega \left\{ \frac{\Phi(x)}{h} \, c_w \, (S^h - S^\star) \mathsf{Z}(T^h) - \frac{\Phi(x)}{h} \, c_o \, (S^h - S^\star) \mathsf{Z}(T^h) \right\} \, \boldsymbol{\vartheta}_M^h \, dx \\ &+ \int_\Omega \left\{ \frac{\Phi(x)}{h} \, (c_w \, S^\star + c_o \, (1 - S^\star))(\mathsf{Z}(T^h) - T^\star) + \frac{[1 - \Phi(x)]}{h} \, c_s \, \left[\mathsf{Z}(T^h) - T^\star\right] \right\} \, \boldsymbol{\vartheta}_M^h \, dx. \end{aligned}$$

Because all the integrals in 4.65 are equal to zero for $T^h < T_M$, we can set $\mathsf{Z}(T^h) = T_M$ in 4.65, and the above transformation of the first integral gives

$$\begin{aligned}
&\int_\Omega \left\{ \frac{\Phi(x)}{h}\, c_w\, (S^h - S^\star)T_M - \frac{\Phi(x)}{h}\, c_o\, (S^h - S^\star)T_M \right\} \vartheta_M^h\, dx \\
&+ \int_\Omega \left\{ \frac{\Phi(x)}{h}\, (c_w\, S^\star + c_o\, (1 - S^\star))(T_M - T^\star) + \frac{[1 - \Phi(x)]}{h}\, c_s\, \left[T_M - T^\star\right] \right\} \vartheta_M^h\, dx \\
&+ \int_\Omega K(x)\, T_M\, \left[c_w \lambda_w(S^h, T^h)\big(\nabla p_w^h - \varrho_w \vec{g}\big) + c_o \lambda_o(S^h, T^h)\big(\nabla p_o^h - \varrho_o \vec{g}\big)\right] \cdot \nabla \vartheta_M^h\, dx \\
&+ \int_\Omega \left[c_w\, \eta\, T_M\, \nabla\big(p_w^h - p_o^h\big) + c_o\, \eta\, T_M\, \nabla\big(p_o^h - p_w^h\big) + k_T \nabla T^h\right] \cdot \nabla \vartheta_M^h\, dx = 0.
\end{aligned} \tag{4.66}$$

We now set the test function ϑ_M^h in 4.56 and 4.57, multiply them by $c_w T_M$ and $c_o T_M$, respectively, and subtract them from 4.66. We obtain

$$\int_\Omega \left\{ \frac{\Phi(x)}{h}\, (c_w\, S^\star + c_o\, (1 - S^\star))(T_M - T^\star) + \frac{[1 - \Phi(x)]}{h}\, c_s\, \left[T_M - T^\star\right] \right\} \vartheta_M^h\, dx + \int_\Omega k_T \nabla \vartheta_M^h \cdot \nabla \vartheta_M^h\, dx = 0.$$

This equality gives $\nabla \vartheta_M^h = 0$ and by the boundary condition $\vartheta_M^h = 0$. The upper bound in 4.62 is proved.

The lower bound in 4.62 is proved by similar arguments using the cutoff function $\vartheta_m(x,t) \overset{\text{def}}{=} \min\left\{T - T_m, 0\right\}$. This completes the proof of Lemma 4.5.

We can now eliminate the function Z_S and Z_T from 4.56 to 4.58, and Theorem 4.1 is proved.

5 | EXISTENCE RESULT FOR THE NONDEGENERATE SYSTEM

In this section, we prove the existence result for the nondegenerate system 3.15 with the boundary conditions 3.16 and the initial conditions 3.17. The main result of this section is the following theorem.

Theorem 5.1. *Let assumptions (A.1) to (A.8) be fulfilled. Then there exists* $\{p_o^\eta, p_w^\eta, S^\eta, T^\eta\}$ *such that*

(I) The functions $p_o^\eta, p_w^\eta, S^\eta, T^\eta$ *have the following regularity properties:*

$$p_o^\eta \in L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)), \quad p_w^\eta \in L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)), \quad \textit{and} \quad T^\eta \in L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.1}$$

$$1 - S^\eta \in L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.2}$$

$$\frac{\partial}{\partial t}(\Phi S^\eta) \in L^2(0, \mathcal{T}; H^{-1}(\Omega)), \quad \frac{\partial \boldsymbol{\Psi}^\eta}{\partial t} \in L^2(0, \mathcal{T}; H^{-1}(\Omega)). \tag{5.3}$$

(II) The maximum principle for the saturation holds:

$$0 \leqslant S^\eta \leqslant 1 \quad \textit{a. e. in} \quad Q. \tag{5.4}$$

(III) The maximum principle for the temperature holds:

$$T_m \leqslant T^\eta \leqslant T_M \quad \textit{a.e. in} \quad Q. \tag{5.5}$$

(IV) For any $\varphi_w, \varphi_o, \varphi_T \in C^1([0,\mathcal{T}]; H^1(\Omega))$ satisfying $\varphi_w = \varphi_o = \varphi_T = 0$ on $\Gamma_1 \times (0,\mathcal{T})$ and $\varphi_w(x,\mathcal{T}) = \varphi_o(x,\mathcal{T}) = 0$, we have

$$
\begin{aligned}
&-\int_Q \Phi(x) S^\eta \frac{\partial \varphi_w}{\partial t}\ dx\,dt - \int_\Omega \Phi(x) S^0(x) \varphi_w(x,0)\ dx \\
&+\int_Q K(x)\lambda_w(S^\eta,T^\eta)(\nabla p_w^\eta - \varrho_w \vec{g})\cdot\nabla\varphi_w\ dx\,dt + \eta\int_Q \nabla(p_w^\eta - p_o^\eta)\cdot\nabla\varphi_w\ dx\,dt = 0;
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
&\int_Q \Phi(x) S^\eta \frac{\partial \varphi_o}{\partial t}\ dx\,dt + \int_\Omega \Phi(x) S^0(x) \varphi_o(x,0)\ dx \\
&+\int_Q K(x)\lambda_o(S^\eta,T^\eta)(\nabla p_o^\eta - \varrho_o \vec{g})\cdot\nabla\varphi_o\ dx\,dt + \eta\int_Q \nabla(p_o^\eta - p_w^\eta)\cdot\nabla\varphi_o\ dx\,dt = 0;
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&-\int_Q \mathbf{\Psi}^\eta \frac{\partial \varphi_T}{\partial t}\ dx\,dt - \int_\Omega \mathbf{\Psi}^0 \varphi_T(x,0)\ dx \\
&+\int_Q \left\{ T^\eta K(x)\Big[c_w\lambda_w(S^\eta,T^\eta)\big(\nabla p_w^\eta - \varrho_w\vec{g}\big) + c_o\lambda_o(S^\eta,T^\eta)\big(\nabla p_o^\eta - \varrho_o\vec{g}\big)\Big] + k_T\nabla T^\eta \right\}\cdot\nabla\varphi_T\ dx\,dt \\
&+\eta\int_Q \left\{ c_w\ T^\eta\ \nabla\big(p_w^\eta - p_o^\eta\big) + c_o\ T^\eta\ \nabla\big(p_o^\eta - p_w^\eta\big)\right\}\cdot\nabla\varphi_T\ dx\,dt = 0.
\end{aligned}
\tag{5.8}
$$

The outline of the proof is as follows. First, in Section 5.1, we establish the uniform estimates for the solutions to the system 4.56 to 4.58, and then in Section 5.2, we obtain the corresponding compactness results with respect to the parameter h. Then we pass to the limit as $h \to 0$ to complete the proof of Theorem 5.1.

5.1 | Uniform estimates

The proof is based on a semidiscretization method in the time variable proposed in Alt and Luckhaus[44] and then applied in the study of 2-phase flows in some related literature.[35,36,41,45,46] Let $T > 0, N \in \mathbb{N}$ and $h = T/N$, and let us define

$$
p_{w,h}^0 = p_w^0,\quad p_{o,h}^0 = p_o^0 \quad \text{and} \quad T_h^0 = T^0 \quad \text{a.e. in } \Omega. \tag{5.9}
$$

For all $n \in [0, N-1]$, we consider the triple of functions $\{p_{w,h}^n, p_{o,h}^n, T_h^n\} \in \left[L^2(\Omega)\right]^3$ with

$$
0 \leqslant S_h^n \leqslant 1,\quad T_m \leqslant T_h^n \leqslant T_M,\quad \mathbf{\Psi}_h^n \stackrel{\text{def}}{=} \big(c_w S_h^n + c_o[1-S_h^n]\big)\ T_h^n\ \Phi + c_s\ [1-\Phi]\ T_h^n,
$$

and then define $\{p_{w,h}^{n+1}, p_{o,h}^{n+1}, T_h^{n+1}\}$ as the weak solution of the following system of equations:

$$
\Phi\Delta_h^n\ S_h^{n+1} - \operatorname{div}\left\{ K\lambda_w(S_h^{n+1},T_h^{n+1})\left(\nabla p_{w,h}^{n+1} - \varrho_w\vec{g}\right) + \eta\nabla\left(p_{w,h}^{n+1} - p_{o,h}^{n+1}\right)\right\} = 0; \tag{5.10}
$$

$$
-\Phi\Delta_h^n\ S_h^{n+1} - \operatorname{div}\left\{ K\lambda_o(S_h^{n+1},T_h^{n+1})\left(\nabla p_{o,h}^{n+1} - \varrho_o\ \vec{g}\right) + \eta\ \nabla\left(p_{o,h}^{n+1} - p_{w,h}^{n+1}\right)\right\} = 0; \tag{5.11}
$$

$$
\begin{aligned}
&\Delta_h^n\mathbf{\Psi}_h^{n+1} - \operatorname{div}\left\{K\ T_h^{n+1}\left[c_w\lambda_w(S_h^{n+1},T_h^{n+1})\big[\nabla p_w^{n+1} - \varrho_w\ \vec{g}\big] + c_o\lambda_o(S_h^{n+1},T_h^{n+1})\big[\nabla p_o^{n+1} - \varrho_o\ \vec{g}\big]\right]\right\} - \\
&-\operatorname{div}\left\{c_w\ \eta\ T_h^{n+1}\ \nabla\left(p_{w,h}^{n+1} - p_{o,h}^{n+1}\right) + c_o\ \eta\ T_h^{n+1}\ \nabla\left(p_{o,h}^{n+1} - p_{w,h}^{n+1}\right)\right\} - \operatorname{div}(k_T\nabla T_h^{n+1}) = 0,
\end{aligned}
\tag{5.12}
$$

where

$$\Delta_h^n\, S_h^{n+1} \overset{\text{def}}{=} \frac{S_h^{n+1} - S_h^n}{h}, \quad \Delta_h^n\, \mathbf{\Psi}_h^{n+1} \overset{\text{def}}{=} \frac{\mathbf{\Psi}_h^{n+1} - \mathbf{\Psi}_h^n}{h}.$$

The system 5.10 to 5.12 is completed with the following boundary conditions:

$$\begin{cases} p_{o,h}^{n+1} = p_{w,h}^{n+1} = T_h^{n+1} = 0 \quad \text{on} \quad \Gamma_1; \\ \vec{q}_{w,h}^{\,(n+1)} \cdot \vec{\nu} = \vec{q}_{o,h}^{\,(n+1)} \cdot \vec{\nu} = k_T \nabla T_h^{n+1} \cdot \vec{\nu} = 0 \quad \text{on} \quad \Gamma_2; \end{cases} \tag{5.13}$$

where

$$\vec{q}_{w,h}^{\,(n+1)} \overset{\text{def}}{=} -K(x)\ \lambda_w(S_h^{n+1}, T_h^{n+1})\big[\nabla p_{w,h}^{n+1} - \vec{g}\big] - \eta \nabla \left(p_{w,h}^{n+1} - p_{o,h}^{n+1}\right);$$

$$\vec{q}_{o,h}^{\,(n+1)} \overset{\text{def}}{=} -K(x)\ \lambda_o(S_h^{n+1}, T_h^{n+1})\big[\nabla p_{o,h}^{n+1} - \varrho_o\, \vec{g}\big] - \eta\ \nabla \left(p_{o,h}^{n+1} - p_{w,h}^{n+1}\right).$$

The sequence $\{p_{w,h}^{n+1}, p_{o,h}^{n+1}, T_h^{n+1}\}$ is well defined for all $n \in [0, N-1]$ because of Theorem 4.1. Thus, for given $T_m \leqslant T_h^n \leqslant T_M$ and $S_h^n \in [0,1]$, we construct $\{p_{w,h}^{n+1}, p_{o,h}^{n+1}, T_h^{n+1}\} \in \left[H_{\Gamma_1}^1(\Omega)\right]^3$ so that $S_h^{n+1} \in [0,1]$ and $T_m \leqslant T_h^{n+1} \leqslant T_M$.

In Lemma 5.1 below, we obtain uniform with respect to h estimates for $\{p_{w,h}^{n+1}, p_{o,h}^{n+1}, T_h^{n+1}\}$.

Lemma 5.1. *The solutions of 5.10 to 5.13 satisfy the bound:*

$$-\frac{1}{h}\int_\Omega \Phi(x)\left\{F(S_h^{n+1}) - F(S_h^n)\right\}\, dx + \eta \int_\Omega \left|\nabla \left(p_{o,h}^{n+1} - p_{w,h}^{n+1}\right)\right|^2 dx +$$
$$+ \int_\Omega \left\{\lambda_w(S_h^{n+1}, T_h^{n+1})\left|\nabla p_{w,h}^{n+1}\right|^2 + \lambda_o(S_h^{n+1}, T_h^{n+1})\left|\nabla p_{o,h}^{n+1}\right|^2\right\}\, dx \leqslant C; \tag{5.14}$$

$$\frac{1}{2h}\int_\Omega \left\{\mathbf{\Psi}_h^{n+1} T_h^{n+1} - \mathbf{\Psi}_h^n T_h^n\right\}\, dx + \int_\Omega k_T \nabla T_h^{n+1} \cdot \nabla T_h^{n+1}\, dx \leqslant C, \tag{5.15}$$

where

$$F(s) \overset{def}{=} \int_0^s P_c(\varsigma)\, d\varsigma \tag{5.16}$$

and C is a constant that does not depend on h and η.

Proof of Lemma 5.1. To derive 5.14, we use a standard energy estimate as in derivation 4.46 in the proof of Lemma 4.2. We only need to estimate the time difference term

$$-\int_\Omega \Phi(x)\Delta_h^n S_h^{n+1} P_c(S_h^{n+1})\, dx = -\int_\Omega \Phi(x)\frac{S_h^{n+1} - S_h^n}{h} P_c(S_h^{n+1})\, dx.$$

But using the monotonicity of the capillary pressure function, we get

$$\left(S_h^{n+1} - S_h^n\right) P_c\left(S_h^{n+1}\right) \leqslant \int_{S_h^n}^{S_h^{n+1}} P_c(s)\, ds = F\left(S_h^{n+1}\right) - F\left(S_h^n\right).$$

We obtain, therefore,

$$-\int_\Omega \Phi(x)\Delta_h^n S_h^{n+1} P_c\left(S_h^{n+1}\right)\, dx \geqslant -\frac{1}{h}\int_\Omega \Phi(x)\{F\left(S_h^{n+1}\right) - F\left(S_h^n\right)\}\, dx,$$

which completes the proof of 5.14.

To prove 5.15, we multiply 5.12 by T_h^{n+1} and subtract 5.10 multiplied by $(c_w/2)\left(T_h^{n+1}\right)^2$ and 5.11 multiplied by $(c_o/2)\left(T_h^{n+1}\right)^2$. After cancelation of terms, we get

$$\int_\Omega \left\{ \Delta_h^n \boldsymbol{\Psi}_h^{n+1} T_h^{n+1} - \frac{c_w}{2} \Phi \Delta_h^n \, S_h^{n+1} \left(T_h^{n+1}\right)^2 + \frac{c_o}{2} \Delta_h^n \, S_h^{n+1} \left(T_h^{n+1}\right)^2 \right\} \, dx + \int_\Omega k_T \nabla T_h^{n+1} \cdot \nabla T_h^{n+1} \, dx = 0.$$

Note that the terms multiplied by Φc_w in the time difference term can be estimated as follows:

$$\begin{aligned} S_h^{n+1}\left(T_h^{n+1}\right)^2 - S_h^n T_h^n T_h^{n+1} - \frac{1}{2}\left(S_h^{n+1} - S_h^n\right)\left(T_h^{n+1}\right)^2 &= \frac{1}{2} S_h^{n+1}\left(T_h^{n+1}\right)^2 - S_h^n \left(T_h^n T_h^{n+1} - \frac{1}{2}\left(T_h^{n+1}\right)^2 \right) \\ &\geqslant \frac{1}{2} S_h^{n+1}\left(T_h^{n+1}\right)^2 - \frac{1}{2} S_h^n \left(T_h^n\right)^2, \end{aligned}$$

where we have used the estimate $ab - b^2/2 \leqslant a^2/2$. In the same way, for the terms multiplied by Φc_o, we have

$$[1 - S_h^{n+1}]\left(T_h^{n+1}\right)^2 - \left[1 - S_h^n\right] T_h^n T_h^{n+1} + \frac{1}{2}\left(S_h^{n+1} - S_h^n\right)\left(T_h^{n+1}\right)^2 \geqslant \frac{1}{2}\left[1 - S_h^{n+1}\right]\left(T_h^{n+1}\right)^2 - \frac{1}{2}\left[1 - S_h^n\right]\left(T_h^n\right)^2.$$

Finally, for c_s terms, we have

$$\left([1-\Phi] T_h^{n+1} - [1-\Phi] T_h^n\right) T_h^{n+1} \geqslant [1-\Phi] \left(\frac{1}{2}\left(T_h^{n+1}\right)^2 - \frac{1}{2}\left(T_h^n\right)^2 \right).$$

From these estimates, we derive

$$\int_\Omega \left\{ \Delta_h^n \boldsymbol{\Psi}_h^{n+1} T_h^{n+1} - \frac{c_w}{2} \Phi \Delta_h^n \, S_h^{n+1} \left(T_h^{n+1}\right)^2 + \frac{c_o}{2} \Delta_h^n \, S_h^{n+1} \left(T_h^{n+1}\right)^2 \right\} dx \geqslant \frac{1}{2h} \int_\Omega \left(\boldsymbol{\Psi}_h^{n+1} T_h^{n+1} - \boldsymbol{\Psi}_h^n T_h^n \right) dx.$$

This proves 5.15, and Lemma 5.1 is proved.

Now, for a given sequence $\{u_h^n\}_n$, we define the following functions:

$$u^h(t) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} u_h^{n+1} \, \mathbf{1}_{(nh,(n+1)h]}(t) \quad \forall t \in (0, \mathcal{T}) \quad \text{with} \quad u^h(0) = u_0^h, \tag{5.17}$$

where $\mathbf{1}_{(nh,(n+1)h]}(t)$ denotes the characteristic function of the interval $(nh, (n+1)h]$. Next we define the function

$$\widetilde{u}^h(t) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} \left[\left(1 + n - \frac{t}{h}\right) u_h^n + \left(\frac{t}{h} - n\right) u_h^{n+1} \right] \mathbf{1}_{[nh,(n+1)h]}(t) \quad \forall t \in [0, \mathcal{T}], \tag{5.18}$$

which satisfies

$$\frac{\partial \widetilde{u}^h}{\partial t} = \frac{1}{h} \sum_{n=0}^{N-1} \left(u_h^{n+1} - u_h^n\right) \mathbf{1}_{(nh,(n+1)h)}(t) \quad \forall t \in [0, \mathcal{T}] \setminus \cup_{n=0}^{N} \{n\, h\}. \tag{5.19}$$

The following uniform estimates hold true.

Lemma 5.2. *Let $p_w^h, p_o^h, S^h, T^h, \boldsymbol{\Psi}^h$ be the functions defined in 5.17 by $p_{w,h}^n, p_{o,h}^n, S_h^n, T_h^n, \boldsymbol{\Psi}_h^n$ and $\widetilde{S}^h, \widetilde{\boldsymbol{\Psi}}^h$ be the functions defined in 5.18 by $S_h^n, \boldsymbol{\Psi}_h^n$. Then*

$$\{S^h\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^1(\Omega)); \tag{5.20}$$

$$\{p_w^h\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.21}$$

$$\{p_o^h\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.22}$$

$$\{T^h\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.23}$$

$$\{P_c(S^h)\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.24}$$

$$\{\partial_t(\Phi \widetilde{S}^h)\}_{h>0} \ \text{is uniformly bounded in} \ L^2(0, \mathcal{T}; H^{-1}(\Omega)); \tag{5.25}$$

$$\{\partial_t \widetilde{\Psi}^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,\mathcal{T};H^{-1}(\Omega)). \tag{5.26}$$

Proof of Lemma 5.2. By summing the inequality 5.14 multiplied by h, we get

$$\begin{aligned}\int_Q \left\{ \lambda_w(S^h,T^h)\left|\nabla p_w^h\right|^2 + \lambda_o(S^h,T^h)\left|\nabla p_o^h\right|^2 \right\}\, dx\, dt + \eta \int_Q |\nabla P_c(S^h)|^2\, dx\, dt \leqslant C \\ + \int_\Omega \Phi(x)\left|F(S^h(0)) - F(S^h(\mathcal{T}))\right|\, dx \leqslant C,\end{aligned} \tag{5.27}$$

where C is a constant that does not depend on h and η, which proves 5.24. From 5.24 and (A.3), we get the estimate 5.20.

From the inequality 5.15, in the same way, we get

$$\int_Q |\nabla T^h|^2\, dx \leqslant C + \frac{1}{2}\int_\Omega |\Psi_h(\mathcal{T})T_h(\mathcal{T}) - \Psi_h(0)T_h(0)|\, dx \leqslant C, \tag{5.28}$$

where C is a constant that does not depend on h and η, which proves 5.23.

Taking into account the definition of the nonisothermal global pressure 2.12 and 2.14, the bounds 5.27 and 5.28, the relation 2.24, Lemma 2.1, and the bound 2.21, we find that

$$\int_Q |\mathsf{P}^h|^2\, dx\, dt \leqslant C, \quad \int_Q |\beta(S^h)|^2\, dx\, dt \leqslant C, \tag{5.29}$$

where C is a constant that does not depend on h and η. Then the bounds 5.21 and 5.22 are the consequence of relations 2.14, 5.27-5.29, and Lemma 2.1.

Finally, the uniform bounds 5.25 and 5.26 follow directly from the weak formulation of the problem and the previous uniform estimates. Lemma 5.2 is proved.

Remark 6. If the initial conditions S^0 and T^0 belong to $H^1(\Omega)$, then the uniform bounds 5.20 and 5.23 imply the same bounds for the linearly interpolated functions $\tilde{S}^h$ and $\tilde{T}^h$.

Lemma 5.3. *The following convergence results hold as $h \to 0$:*

$$\|S^h - \widetilde{S}^h\|^2_{L^2(Q)} \to 0 \quad \text{and} \quad \|\Psi^h - \widetilde{\Psi}^h\|^2_{L^2(Q)} \to 0. \tag{5.30}$$

Proof of Lemma 5.3. The proof of the lemma is based on an abstract result stated in Lemma 3.2 from paper.[47]

5.2 | Compactness results

In this section, we obtain compactness and convergence results that will be used in the proof of the main existence theorem. Notice that the previous results obtained in Galusinski and Saad[46] and Khalil and Saad[41,43] are not sufficient for our purposes. In these papers, the method proposed earlier in Chavent and Jaffré[2] for the constant porosity function is generalized to the case of the porosity function belonging to the class $W^{1,\infty}$. The proof is essentially based on Simon's embedding theorem for the spaces of functions depending on the space and time variables (see Simon[37]). However, the assumption that the porosity function is from the space $W^{1,\infty}$ is not admissible for the homogenization of 2-phase flow in porous media made of different types of rock. Below we propose our own approach to this problem. Namely, we have the following compactness lemma.

Lemma 5.4. (Compactness lemma)
Let $\Phi_i = \Phi_i(x)$ $(i = 1, 2, \ldots, I)$ be functions such that $\Phi_i \in L^\infty(\Omega)$. Let $\{v_i^\epsilon\}_{\epsilon>0} \subset L^2(Q)$ $(i = 1, 2, \ldots I)$ be given families of functions satisfying the following properties:

1. *for each i, $\{v_i^\epsilon\}_{\epsilon>0}$ is uniformly bounded in the space $L^2(0,\mathcal{T};W^{\sigma_i,p_i}(\Omega))$, with $0 < \sigma_i \leqslant 1, p_i \geqslant 2$.*

2. *the family of functions* $\left\{\sum_{i=1}^{I}\Phi_i v_i^\epsilon\right\}_{\epsilon>0}$ *satisfy the following uniform estimate:*

$$\left\|\partial_t\left(\sum_{i=1}^{I}\Phi_i v_i^\epsilon\right)\right\|_{L^2(0,\mathcal{T};H^{-1}(\Omega))}\leqslant C. \tag{5.31}$$

Then $\left\{\sum_{i=1}^{I}\Phi_i v_i^\epsilon\right\}_{\epsilon>0}$ *is relatively compact in* $L^2(Q)$.

Proof of Lemma 5.4. It follows from Theorem 1 in Simon[37] that the family $\{\sum_{i=1}^{I}\Phi_i v_i^\epsilon\}_{\epsilon>0}$ is relatively compact in $L^2(0,\mathcal{T};H^{-1}(\Omega))$.

Let us prove the following statement (see Lemma 8 in Simon[37]): $\forall\eta>0, \exists N>0$ such that

$$\begin{aligned}&\forall(v_1,\dots,v_I)\in W^{\sigma_1,p_1}(\Omega)\times\cdots\times W^{\sigma_I,p_I}(\Omega),\\&\left\|\sum_{i=1}^{I}\Phi_i v_i\right\|_{L^2(\Omega)}\leqslant\eta\sqrt{\sum_{i=1}^{I}\|v_i\|^2_{W^{\sigma_i,p_i}(\Omega)}}+N\left\|\sum_{i=1}^{I}\Phi_i v_i\right\|_{H^{-1}(\Omega)}.\end{aligned} \tag{5.32}$$

To prove 5.32, let us introduce the set:

$$V_n\stackrel{\text{def}}{=}\left\{(v_1,\dots,v_I)\in[L^2(\Omega)]^I\;:\;\|\sum_{i=1}^{I}\Phi_i v_i\|_{L^2(\Omega)}<\eta+n\|\sum_{i=1}^{I}\Phi_i v_i\|_{H^{-1}(\Omega)}\right\}\quad(n\in\mathbb{N}).$$

The sets V_n are open in $[L^2(\Omega)]^I$; they increase with n ($V_n\subset V_{n+1}$) and their union covers whole $[L^2(\Omega)]^I$. The unit sphere S in $W^{\sigma_1,p_1}(\Omega)\times\cdots\times W^{\sigma_I,p_I}(\Omega)$ is relatively compact in $[L^2(\Omega)]^I$, and therefore, there exists N such that $S\subset V_N$, which gives 5.32 for all $(v_1,\dots,v_I)\in W^{\sigma_1,p_1}(\Omega)\times\cdots\times W^{\sigma_I,p_I}(\Omega)$ with unit norm. The inequality for every $(v_1,\dots,v_I)\in W^{\sigma_1,p_1}(\Omega)\times\cdots\times W^{\sigma_I,p_I}(\Omega)$ follows by multiplication by any positive number. This proves 5.32.

Because the family $\left\{\sum_{i=1}^{I}\Phi_i v_i^\epsilon\right\}_{\epsilon>0}$ is a relatively compact set in $L^2(0,\mathcal{T};H^{-1}(\Omega))$, then for any $\delta>0$, we can find a finite number of parameters $\{\epsilon_1,\epsilon_2,\dots\}$ such that for any ϵ there exists ϵ_k satisfying the following inequality:

$$\left\|\sum_{i=1}^{I}\Phi_i v_i^\epsilon-\sum_{i=1}^{I}\Phi_i v_i^{\epsilon_k}\right\|_{L^2(0,\mathcal{T};H^{-1}(\Omega))}\leqslant\delta.$$

Then 5.32 implies

$$\begin{aligned}\left\|\sum_{i=1}^{I}\Phi_i v_i^\epsilon-\sum_{i=1}^{I}\Phi_i v_i^{\epsilon_k}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}&\leqslant\sqrt{2}\eta\sqrt{\sum_{i=1}^{I}\|v_i^\epsilon-v_i^{\epsilon_k}\|^2_{L^2(0,\mathcal{T};W^{\sigma_i,p_i}(\Omega))}}\\&\quad+\sqrt{2}N\left\|\sum_{i=1}^{I}\Phi_i v_i^\epsilon-\sum_{i=1}^{I}\Phi_i v_i^{\epsilon_k}\right\|_{L^2(0,\mathcal{T};H^{-1}(\Omega))}\\&\leqslant\sqrt{2}(\eta C+N\delta),\end{aligned}$$

where C results from uniform boundedness of $\{v_i^\epsilon\}_{\epsilon>0}$ in the space $L^2(0,\mathcal{T};W^{\sigma_i,p_i}(\Omega))$. For given δ', we chose $\eta=\delta'/2\sqrt{2}C$ and $\delta=\delta'/2\sqrt{2}N$ and obtain

$$\left\|\sum_{i=1}^{I}\Phi_i v_i^\epsilon-\sum_{i=1}^{I}\Phi_i v_i^{\epsilon_k}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}\leqslant\delta',$$

which proves the relative compactness of $\{\sum_{i=1}^{I}\Phi_i v_i^\epsilon\}_{\epsilon>0}$ in $L^2(Q)$.

We apply Lemma 5.4 in this section to obtain the compactness results for the sequences $\{S^h\}$ and $\{\boldsymbol{\Psi}^h\}$ and to prove the strong convergence of the sequence $\{T^h\}$ in $L^2(Q)$. We have

Lemma 5.5. *The sequences* $\{S^h\}$ *and* $\{\boldsymbol{\Psi}^h\}$*, defined in Lemma 5.2, are relatively compact in* $L^2(Q)$.

Proof of Lemma 5.5. For the sequence $\{\tilde{S}^h\}$, the compactness follows directly form Lemma 5.4 by taking $I=1$ and the bounds 5.25 and 5.20 (see also Remark 6). Lemma 5.3 gives the relative compactness of the sequence $\{S^h\}$.

The function $\boldsymbol{\Psi}^h=\Phi(c_wS^hT^h+c_o(1-S^h)T^h)+(1-\Phi)c_sT^h$ can be considered as a sum of 2 functions with $\Phi_1=\Phi,\Phi_2=1-\Phi$ and $v_1^h=c_wS^hT^h+c_o(1-S^h)T^h, v_2^h=c_sT^h$. Then Lemma 5.4 can be applied directly to the sequence $\{\tilde{\boldsymbol{\Psi}}^h\}$; thanks to the estimates 5.26, 5.23, 5.20 and the maximum principle for the saturation and the temperature. Lemma 5.3 gives relative compactness of the sequence $\{\boldsymbol{\Psi}^h\}$.

Now we turn to the strong convergence result for the sequence $\{T^h\}_{h>0}$.

Proposition 5.1. (Strong convergence of the temperature)
On a subsequence

$$T^h \to T \quad \textit{strongly in } \ L^2(Q) \ \textit{ and } \ \textit{a.e. in } \ Q \quad \textit{with} \quad T \in L^2(0, \mathcal{T}; H^1(\Omega)). \tag{5.33}$$

Proof of Proposition 5.1. It follows from Lemma 5.5 that there exists a subsequence such that

$$S^h \to S \quad \text{strongly in } \ L^2(Q) \ \text{ and } \ \text{a.e. in } \ Q; \tag{5.34}$$

$$\boldsymbol{\Psi}^h \to \boldsymbol{\Psi} \quad \text{strongly in } \ L^2(Q) \ \text{ and } \ \text{a.e. in } \ Q, \tag{5.35}$$

where the limit saturation satisfies $0 \leqslant S \leqslant 1$. From $\Psi^h = \psi(S^h)T^h$ and $\psi(S^h) = (c_w S^h + c_o[1 - S^h])\Phi(x) + c_s[1 - \Phi(x)]$, we know that

$$T^h = \frac{\boldsymbol{\Psi}^h}{\psi(S^h)} \to \frac{\boldsymbol{\Psi}}{\psi(S)} \overset{\text{def}}{=} \hat{T} \quad \text{a.e. in } \ Q.$$

The pointwise convergence ensures that the limit temperature $\hat{T}$ satisfies $T_m \leqslant \hat{T} \leqslant T_M$. It follows from 5.23 that

$$T^h \to T \quad \text{weakly in } \ L^2(Q),$$

for some $T \in L^2(0, T; H^1(\Omega))$, and therefore, we have $\hat{T} = T$. We have proved that $\Psi = \psi(S)T$.

From the boundedness of the saturation, we have

$$0 < c_s \left[1 - \phi^+\right] \leqslant \psi(S^h), \psi(S) \leqslant c_w + c_o + c_s. \tag{5.36}$$

Then

$$\int_Q |T^h - T|^2 \, dx \, dt = \int_Q \frac{1}{\psi^2(S^h)} |\psi(S^h) \, T^h - \psi(S^h) \, T|^2 \, dx \, dt \leqslant$$

$$\leqslant \frac{2}{c_s^2 \left[1 - \phi^+\right]^2} \left[\int_Q |\boldsymbol{\Psi}^h(S^h, T^h) - \boldsymbol{\Psi}(S, T)|^2 \, dx \, dt + \int_Q |T|^2 \, |\psi(S) - \psi(S^h)|^2 \, dx \, dt \right].$$

Now the statement of the proposition immediately follows from 5.34, 5.35. Proposition 5.1 is proved. □

Our goal is to construct a solution to the evolution problem 3.15 by passing to the limit, as $h \to 0$, in the above elliptic problem.

Lemma 5.6. (Convergence results with respect to h)
Up to a subsequence, the following convergence results hold as $h \to 0$:

$$p_w^h \to p_w^\eta \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.37}$$

$$p_o^h \to p_o^\eta \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)); \tag{5.38}$$

$$S^h \to S^\eta \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^1(\Omega)) \ \ \textit{and a.e. in } \ Q; \tag{5.39}$$

$$T^h \to T^\eta \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^1_{\Gamma_1}(\Omega)) \ \ \textit{and a.e. in } \ Q; \tag{5.40}$$

$$\boldsymbol{\Psi}^h \to \boldsymbol{\Psi}^\eta \ \ \textit{strongly in } \ L^2(Q) \ \ \textit{and a.e. in } \ Q, \tag{5.41}$$

where $\boldsymbol{\Psi}^\eta = (c_w S^\eta + c_o[1 - S^\eta])T^\eta \Phi(x) + c_s[1 - \Phi(x)]T^\eta$;

$$\partial_t(\Phi \, \widetilde{S}^h) \to \partial_t(\Phi \, S^\eta) \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^{-1}(\Omega)); \tag{5.42}$$

$$\partial_t \widetilde{\boldsymbol{\Psi}}^h \to \partial_t \boldsymbol{\Psi}^\eta \ \ \textit{weakly in } \ L^2(0, \mathcal{T}; H^{-1}(\Omega)). \tag{5.43}$$

Moreover,

$$0 \leqslant S^\eta \leqslant 1, \quad T_m \leqslant T^\eta \leqslant T_M \quad a.e. \ in \ \ \Omega_T. \tag{5.44}$$

Proof of Lemma 5.6. The weak convergence in 5.37-5.40, 5.42, and 5.43 follows directly from Lemma 5.2. The strong and a.e. convergence in 5.39 and 5.41 follows directly from Lemma 5.5, while strong and a.e. convergence in 5.40 follows from Proposition 5.1. The boundedness 5.44 follows from the boundedness of the sequences $\{S^h\}$ and $\{T^h\}$ and the pointwise convergence. Lemma 5.6 is proved.

Now we are in position to complete the proof of Theorem 5.1. First, we observe that in view of the definitions of $S^h, p_w^h, p_o^h, T^h, \widetilde{S}^h, \widetilde{\Theta}^h$, and $\widetilde{\mathbf{\Psi}}^h$ from the system 5.10 to 5.12, we obtain the following system of equations:

$$\begin{cases} \Phi(x)\frac{\partial \widetilde{S}^h}{\partial t} - \text{div } \left\{K(x)\lambda_w(S^h,T^h)\left(\nabla p_w^h - \varrho_w \vec{g}\right) + \eta\ \nabla(p_w^h - p_o^h)\right\} = 0; \\ -\Phi(x)\frac{\partial \widetilde{S}^h}{\partial t} - \text{div } \left\{K(x)\lambda_o(S^h,T^h)\left(\nabla p_o^h - \varrho_o\ \vec{g}\right) + \eta\ \nabla(p_o^h - p_w^h)\right\} = 0; \\ \frac{\partial \widetilde{\mathbf{\Psi}}^h}{\partial t} - \ \text{div}\left\{K(x)T^h\left[c_w\lambda_w(S^h,T^h)\left(\nabla p_w^h - \varrho_w\vec{g}\right) + c_o\lambda_o(S^h,T^h)\left(\nabla p_o^h - \varrho_o\vec{g}\right)\right]\right\} - \\ -c_w\ \eta\ \ \text{div }\left\{T^h\nabla\left(p_w^h - p_o^h\right)\ \right\} - c_o\ \eta\ \ \text{div }\left\{T^h\nabla\left(p_o^h - p_w^h\right)\right\} - \ \text{div }(k_T\nabla T^h) = 0; \\ P_c\left(S^h\right) = p_o^h - p_w^h. \end{cases} \tag{5.45}$$

Now considering the weak formulation of the system 5.45 and taking into account Lemma 5.6, we pass to the limit as $h \to 0$ and obtain Equations 5.6 to 5.8, which represent the weak formulation of the system 3.15 to 3.17. This ends the proof of Theorem 5.1.

6 | PROOF OF THE MAIN RESULT: THE DEGENERATE SYSTEM

The goal of this section is to prove the main result of this work, ie, Theorem 3.1. The proof is based on Theorem 5.1 established in the previous section and the compactness results from Propositions 6.1 and 5.1. First, in Section 6.1, we establish the uniform estimates for the solutions to system 3.15 and obtain the corresponding compactness results with respect to the parameter η. Then in Section 6.2, we complete the proof of Theorem 3.1.

6.1 | Uniform estimates and compactness results

The a priori estimates for the solutions of problem 3.15 are given by the following lemma.

Lemma 6.1. *The sequences* $\{S^\eta\}_{\eta>0}, \{p_w^\eta\}_{\eta>0}, \{p_o^\eta\}_{\eta>0}, \{T^\eta\}_{\eta>0}, \{P^\eta\}_{\eta>0}$ *are such that*

$$0 \leqslant S^\eta \leqslant 1, \quad T_m \leqslant T^\eta \leqslant T_M \quad a.\ e.\ in \quad Q; \tag{6.1}$$

$$\{P^\eta\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(0,\mathcal{T};H^1(\Omega)); \tag{6.2}$$

$$\{T^\eta\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(0,\mathcal{T};H^1(\Omega)); \tag{6.3}$$

$$\{\sqrt{\eta}\ \ \nabla P_c(S^\eta)\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(Q); \tag{6.4}$$

$$\left\{\sqrt{\lambda_w(S^\eta,T^\eta)}\ \nabla p_w^\eta\right\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(Q); \tag{6.5}$$

$$\left\{\sqrt{\lambda_o(S^\eta,T^\eta)}\ \nabla p_o^\eta\right\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(Q); \tag{6.6}$$

$$\{\beta(S^\eta)\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(0,\mathcal{T};H^1(\Omega)); \tag{6.7}$$

$$\{\partial_t(\Phi S^\eta)\}_{\eta>0} \quad \textit{is uniformly bounded in} \ \ L^2(0,\mathcal{T};H^{-1}(\Omega)); \tag{6.8}$$

$$\{\partial_t \boldsymbol{\Psi}^\eta\}_{\eta>0} \quad \textit{is uniformly bounded in} \quad L^2(0,\mathcal{T};H^{-1}(\Omega)). \tag{6.9}$$

Proof of Lemma 6.1. The maximum principle 6.1 is preserved through passing to the limit. Bounds 6.2, 6.3, and 6.7 are consequences of 5.28, 5.29, and the weak lower semicontinuity of the norms.

To prove 6.4, 6.5, and 6.6, we set $\varphi_w = p_w$ in Equation 5.6 and $\varphi_o = p_o$ in Equation 5.7, and after localization in time, we get for a.e. $t \in (0, T)$

$$\begin{aligned}-\langle \partial_t(\Phi S^\eta), P_c(S^\eta)\rangle &+ \int_\Omega K(x)\lambda_w(S^\eta,T^\eta)\left[\nabla p_w^\eta - \varrho_w \vec{g}\right]\cdot \nabla p_w^\eta \, dx \\ +\int_\Omega K(x)\lambda_o(S^\eta,T^\eta)\left[\nabla p_o^\eta - \varrho_o\, \vec{g}\right]\cdot \nabla p_o^\eta \, dx &+ \int_\Omega \eta |\nabla P_c(S^\eta)|^2 \, dx = 0,\end{aligned} \tag{6.10}$$

where $\langle , \rangle$ stands for the duality product in $(H^1_{\Gamma_1}(\Omega))' \times H^1_{\Gamma_1}(\Omega)$. Using the integration lemma from Gagneux and Madaune-Tort,[4] pp 31, we get

$$\langle \partial_t(\Phi S^\eta), P_c(S^\eta)\rangle = \frac{d}{dt}\int_\Omega \Phi(x)\mathsf{F}(S^\eta)\, dx \quad \text{with} \quad \mathsf{F}(s) = \int_0^s P_c(\varsigma)\, d\varsigma.$$

By integrating 6.10 over the interval $(0, \mathcal{T})$, using condition (A.3), as in the proof of Lemma 4.2, we obtain inequality

$$\int_Q \left\{ K(x)\lambda_w(S^\eta,T^\eta)|\nabla p_w^\eta|^2 + K(x)\lambda_o(S^\eta,T^\eta)|\nabla p_o^\eta|^2 + \eta|\nabla P_c(S^\eta)|^2 \right\} dx\, dt \leqslant C, \tag{6.11}$$

which gives the estimates 6.4, 6.5, and 6.6.

The proof of the bounds 6.8 and 6.9 can be done in a standard way (see, eg, Amaziane et al[48]) by using the estimates 6.2 to 6.7. Lemma 6.1 is proved.

Proposition 6.1. (Compactness result)
The families $\{S^\eta\}_{\eta>0}, \{\boldsymbol{\Psi}^\eta\}_{\eta>0}, \{T^\eta\}_{\eta>0} \subset L^2(Q)$ are relatively compact in $L^2(Q)$.

Proof of Proposition 6.1. From 6.1, 6.7, and (A.7), it is easy to show that the family $\{S^\eta\}_{\eta>0}$ is uniformly bounded in $L^{2/\theta}(0,\mathcal{T};W^{s\theta,2/\theta}(\Omega))$ for any $0 < s < 1$. Then the compactness of the family $\{S^\eta\}_{\eta>0}$ follows from 6.8 and Lemma 5.4.

Because T^η is uniformly bounded in $L^2(0,\mathcal{T};H^1(\Omega))$, it is not difficult to show that $c_w ST + c_o(1-S)T$ is uniformly bounded in $L^{2/\theta}(0,\mathcal{T};W^{s\theta,2/\theta}(\Omega))$. An application of Lemma 5.4 gives the relative compactness of the family $\{\boldsymbol{\Psi}^\eta\}_{\eta>0}$, and further application of Proposition 5.1 gives the relative compactness of the temperature $\{T^\eta\}_{\eta>0}$. Proposition 6.1 is proved. □

Now from Lemma 6.1 and Proposition 6.1, we deduce all the convergence results required for the passage to the limit as $\eta \to 0$ in 3.15.

Lemma 6.2. *The sequences $\{S^\eta\}_{\eta>0}, \{T^\eta\}_{\eta>0}, \{\mathsf{P}^\eta\}_{\eta>0}$ are such that up to a subsequence,*

$$0 \leqslant S \leqslant 1, \quad T_m \leqslant T \leqslant T_M \quad \textit{a.e. in} \quad Q; \tag{6.12}$$

$$S^\eta \to S \quad \textit{strongly in} \quad L^2(Q) \quad \textit{and} \quad \textit{a.e. in} \quad Q; \tag{6.13}$$

$$T^\eta \to T \quad \textit{strongly in} \quad L^2(Q) \quad \textit{and} \quad \textit{a.e. in} \quad Q; \tag{6.14}$$

$$T^\eta \to T \quad \textit{weakly in} \quad L^2(0,\mathcal{T};H^1(\Omega)); \tag{6.15}$$

$$\mathsf{P}^\eta \to \mathsf{P} \quad \textit{weakly in} \quad L^2(0,\mathcal{T};H^1(\Omega)); \tag{6.16}$$

$$\beta(S^\eta) \to \beta(S) \quad \textit{weakly in} \quad L^2(0,\mathcal{T};H^1(\Omega)). \tag{6.17}$$

$$\Psi^\eta \to \Psi \quad strongly\ in \ \ L^2(Q) \ \ and \ \ a.e.\ in \ \ Q; \tag{6.18}$$

$$\partial_t(\Phi S^\eta) \to \partial_t(\Phi S) \quad weakly\ in \ \ L^2(0,\mathcal{T};H^{-1}(\Omega)); \tag{6.19}$$

$$\partial_t \Psi^\eta \to \partial_t \Psi \quad weakly\ in \ \ L^2(0,\mathcal{T};H^{-1}(\Omega)). \tag{6.20}$$

6.2 | Convergence of the approximating solutions

We have to pass to the limit, as $\eta \to 0$, in the weak formulation of problem 3.15 to 3.17, which is given by 5.6 to 5.8.

Let us analyze the limit as $\eta \to 0$ in Equation 5.6. The first term in 5.6 converges to the desired limit because of 6.13. The third term in 5.6 can be rewritten using 2.23 as

$$\int_Q K(x)\lambda_w(S^\eta,T^\eta)\nabla p_w^\eta \cdot \nabla\varphi_w \ \ dx\ dt = \int_Q K(x)\left\{\lambda_w(S^\eta,T^\eta)\nabla \mathsf{P}^\eta - \Lambda_1(S^\eta,T^\eta)\ \nabla\beta(S^\eta) + \lambda_w(S^\eta,T^\eta)\mathsf{B}_w(S^\eta,T^\eta)\ \nabla T^\eta\right\}\cdot\nabla\varphi_w\ dx\ dt.$$

Taking into account Lemma 6.2, we find

$$\begin{aligned}
&\lim_{\eta\to 0}\int_Q K(x)\lambda_w(S^\eta,T^\eta)\nabla p_w^\eta\cdot\nabla\varphi_w \ \ dx\ dt\\
&= \int_Q K(x)\left\{\lambda_w(S,T)\nabla\mathsf{P} - \Lambda_1(S,T)\ \nabla\beta(S) + \lambda_w(S,T)\mathsf{B}_w(S,T)\ \nabla T\right\}\cdot\nabla\varphi_w\ dx\ dt\\
&= \int_Q K(x)\lambda_w(S,T)\nabla p_w\cdot\nabla\varphi_w\ dx\ dt,
\end{aligned}$$

where the limit wetting phase pressure p_w is defined through the limit global pressure P, the limit temperature T, and the limit saturation S by 2.14, namely, $p_w = \mathsf{P} + \mathsf{G}_w(S,T)$. In the same way, defining the limit nonwetting phase pressure by 2.12, we get

$$\lim_{\eta\to 0}\int_Q K(x)\lambda_o(S^\eta,T^\eta)\nabla p_o^\eta\cdot\nabla\varphi_o \ \ dx\ dt = \int_Q K(x)\lambda_o(S,T)\nabla p_o\cdot\nabla\varphi_o \ \ dx\ dt.$$

We can now pass to the limit in the gravity term in 5.6, and the last term tends to zero because of 6.4. In that way, we obtain Equation 3.8. In the same way, we pass to the limit as $\eta \to 0$ in 5.7 and 5.8 and obtain 3.9 and 3.10. Note that the property 3.1 of the phase pressures follows form the regularity of the global pressure P, function $\beta(S)$, temperature T, and Equation 2.24. This completes the proof of Theorem 3.1.

7 | CONCLUDING REMARKS

We have presented a weak formulation and an existence result for a degenerate system modeling nonisothermal immiscible incompressible 2-phase flow through heterogeneous porous media. The extension to a porous medium made of several types of rocks, ie, the porosity, the absolute permeability, the capillary, and relative permeabilities curves are different in each type of porous media, is straightforward by using the approach developed in Amaziane et al.[35] The study still needs to be improved by developing a general approach to incorporating compressibility of both phases. This study was intended as a first step to the homogenization of nonisothermal immiscible incompressible 2-phase flow through heterogeneous reservoirs. These more complicated cases appear in the applications. Further work on these important issues is in progress.

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