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Asymptotic Behaviour of Ground States for Mixtures of Ferromagnetic and Antiferromagnetic Interactions in a Dilute Regime

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Abstract We consider randomly distributed mixtures of bonds of ferromagnetic and antiferromagnetic type in a two-dimensional square lattice with probability $1-p$ and p, respectively, according to an i.i.d. random variable. We study minimizers of the corresponding nearest-neighbour spin energy on large domains in $\mathbb{Z}^2$. We prove that there exists p_0 such that for $p \leq p_0$ such minimizers are characterized by a majority phase; i.e., they take identically the value 1 or -1 except for small disconnected sets. A deterministic analogue is also proved.

Keywords Ising models · Random spin systems · Ground states · Asymptotic analysis of periodic media

Mathematics Subject Classification 82B20 · 82D30 · 49K45 · 60D05

1 Introduction

We consider a prototypical system mixing attractive and repulsive interactions on a lattice through an energy defined on 'spin functions'; i.e., functions u defined on the nodes of the lattice only taking the values -1 and $+1$. In the language of Statistical Mechanics, our energy is given by a mixtures of nearest-neighbour bonds of ferromagnetic and antiferromagnetic type. The ferromagnetic bonds favour equal values of u, while antiferromagnetic bonds favour opposite values. In this sense, they mimic an attractive and repulsive behaviour, respectively. These bonds are randomly distributed in a two-dimensional square lattice, they form a family

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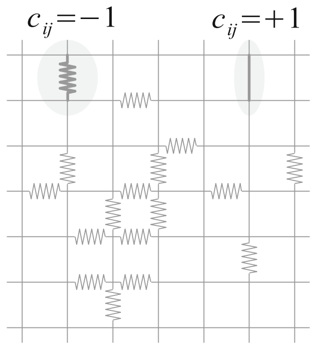

Fig. 1 Representation of a portion of spin system for some $c_{ij} = c^{\omega}_{ij}$

of i.i.d. random variables c_{ij}, $|i - j| = 1$, taking on values $+1$ and -1 with probability $1 - p$ and p, respectively. For each realization ω of that random variable, we consider, for each bounded region D, the energy

$$F^{\omega}(u, D) = -\sum_{i,j} c^{\omega}_{ij} u_i u_j,$$

where the sum runs over nearest-neighbours in the square lattice contained in D, $u_i \in \{-1, +1\}$ denote the values of a spin function, and $c^{\omega}_{ij} \in \{-1, +1\}$ are interaction coefficients corresponding to the realization. A portion of such a system is pictured in Fig. 1: ferromagnetic bonds; i.e, when $c^{\omega}_{ij} = 1$, are pictured as straight segments, while antiferromagnetic bonds are pictured as wiggly ones (as in the two examples highlighted by the gray regions, respectively).

In this paper we analyze ground states; i.e., absolute minimizers, for such energies. This is a fundamental first step in order to define an overall continuum energy: if ground states can be parameterized by a finite number of parameters, then it is sometimes possible to define the behaviour of the system beyond absolute minimization through an effective interfacial energy via a discrete-to-continuum process [10]. The description of ground states is a non trivial issue since in general they are *frustrated*; i.e., the energy cannot be separately minimized on all pairs of nearest neighbors. In other words, minimizing arrays $\{u_j\}$ may not satisfy simultaneously $u_i = u_j$ for all i, j such that $c^{\omega}_{ij} = +1$ and $u_i = -u_j$ for all i, j such that $c^{\omega}_{ij} = -1$. However, in [15] it is shown that if the antiferromagnetic links are contained in well-separated compact regions, then the ground states are characterized by a "majority phase"; i.e., they mostly take only the value 1 (or -1) except for nodes close to the "antiferromagnetic islands". In the case of random interactions we show that this is true in the *dilute case*; i.e., when the probability p of antiferromagnetic interactions is sufficiently small. More precisely, we show that there exists p_0 such that if p is not greater than p_0 then almost surely for all sufficiently large regular bounded domain $D \subset \mathbb{R}^2$ the minimizers of the energy $F(\cdot, D)$ are characterized by a majority phase (Theorem 10). An analogous result is proved to hold also for configurations whose energy is close enough to the minimal energy (see Theorem 12) and, at low temperature, for configurations whose probability does not deviate much from the maximal one (see Theorem 14).

The proof of our result relies on a scaling argument as follows: we remark that proving the existence of majority phases is equivalent to ruling out the possibility of large interfaces separating zones where a ground state u equals 1 and -1, respectively. Such interfaces may exist only if the percentage of antiferromagnetic bonds on the interface is larger than $1/2$. We then estimate the probability of such an interface with a fixed length and decompose a separating interface into portions of at least that length, to prove a contradiction if p is small enough. This is an argument close to the so-called Peierls argument in Statistical Physics (see, e.g., [4,25,27]). Though a terminology particular to Percolation Theory could have been used here, we have preferred to use a self-contained notation in view of possible applications to other attractive-repulsive systems not defined simply on spin functions, and to an interesting deterministic counterpart of the result above. Indeed, the probabilistic proof carries over also to a deterministic periodic setting; i.e., for energies

$$F(u, D) = -\sum_{i,j} c_{ij} u_i u_j$$

such that $c_{ij} \in \{-1, +1\}$ and there exists $N \in \mathbb{N}$ such that $c_{i+k\, j+k} = c_{ij}$ for all i and $j \in \mathbb{Z}^2$ and $k \in N\mathbb{Z}^2$. In this case ground states of F may sometimes be characterized more explicitly and exhibit various types of configurations, that are all possible independently of the percentage of antiferromagnetic bonds: up to boundary effects, there can be a finite number of periodic textures, or configurations characterized by layers of periodic patterns in one direction, or we might have arbitrary configurations of minimizers with no periodicity (see the examples in [10]). We show that "generically" these situations are exceptional in the dilute case: there exists p_0 such that if the percentage p of antiferromagnetic interactions is not greater than p_0 then the proportion of N-periodic systems $\{c_{ij}\}$ such that the minimizers of the energy $F(\cdot, D)$ are characterized by a majority phase for all $D \subset \mathbb{R}^2$ bounded domain large enough tends to 1 as N tends to $+\infty$. The probabilistic arguments are substituted by a combinatorial computation, which also allows a description of the size of the separating interfaces in terms of N.

This work is part of a general discrete-to-continuum analysis of variational problems in lattice systems (see [7] for an overview), most results dealing with spin systems focus on ferromagnetic Ising systems at zero temperature, both on a static framework (see [2,8, 9,19]) and a dynamic framework (see [12,16–18]). In that context, random distributions of bonds have been considered in [14,15] (see also [13]), and their analysis is linked to some recent advances in Percolation Theory (see [3,20,21,23,28]). A first paper dealing with antiferromagnetic interactions is [1], where non-trivial oscillating ground states are observed and the corresponding surface tensions are computed. A related variational motion of crystalline mean-curvature type has been recently described in [11], highlighting new effect due to surface microstructure. The classification of periodic systems mixing ferromagnetic and antiferromagnetic interactions that can be described by surface energies is the subject of [10]. In [15], as mentioned above, the case of well-separated antiferromagnetic island is studied. We note that in those papers the analysis is performed by a description of a macroscopic surface tension, which provides the energy density of a continuous surface energy obtained as a discrete-to-continuum Γ-limit [6]. This limit is performed by scaling the energy F on lattices with vanishing lattice space. That description is possible thanks to a precise knowledge of minimizers. That is not the case of the present paper, and is the reason why we do not address the formulation in terms of the Γ-limit but only study ground states. However, the fact that ground states may be parameterized with the majority phase, suggests that the discrete-to-continuum limit may still be represented as an interfacial energy.

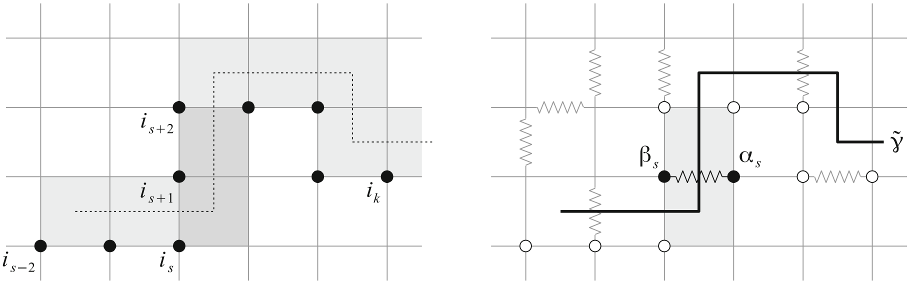

Fig. 2 A path γ and the corresponding curve $\tilde{\gamma}$

2 Random Media

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a *Bernoulli bond percolation* model in $\mathbb{Z}^2$. This means that to each bond (i, j), $i, j \in \mathbb{Z}^2$, $|i - j| = 1$, in $\mathbb{Z}^2$ we associate a random variable c_{ij} and assume that these random variables are i.i.d. and that they take on the value $+1$ with probability $1 - p$, and the value -1 with probability p, where $0 < p < 1$. The detailed description of the Bernoulli bond percolation model can be found for instance in [26].

We denote by $\mathcal{N}$ the set of *nearest neighbors*

$$\mathcal{N} = \{\{i, j\} : i, j \in \mathbb{Z}^2, |i - j| = 1\}$$

and, for each $\{i, j\}$ in $\mathcal{N}$, $[i, j]$ will be the closed segment with endpoints i and j.

Definition 1 (*Random stationary spin system*) A (ferromagnetic/antiferromagnetic) *spin system* is a realization of the random function $c(\{i, j\}) = c_{ij}(\omega) \in \{\pm 1\}$ defined on $\mathcal{N}$. In what follows we do not indicate explicitly the dependence on ω and simply write c_{ij}. The pairs $\{i, j\}$ with $c_{ij} = +1$ are called *ferromagnetic bonds*, the pairs $\{i, j\}$ with $c_{ij} = -1$ are called *antiferromagnetic bonds*.

2.1 Estimates on Separating Paths

We say that a finite sequence $(i_0, \ldots, i_k)$ is a *path* in $\mathbb{Z}^2$ if $\{i_s, i_{s+1}\} \in \mathcal{N}$ for any $s = 0, \ldots, k-1$ and $\{i_s, i_{s+1}\} \neq \{i_t, i_{t+1}\}$ for any $s \neq t$. The path is *closed* if $i_0 = i_k$. The number k is the *length* of γ, denoted by $l(\gamma)$, and we call $\mathcal{P}_k$ the set of the *paths with length* k. To each path $\gamma \in \mathcal{P}_k$ we associate the corresponding curve $\tilde{\gamma}$ of length k in $\mathbb{R}^2$ given by

$$\tilde{\gamma} = \bigcup_{s=0}^{k-1} [i_s, i_{s+1}] + \Big(\frac{1}{2}, \frac{1}{2}\Big). \tag{1}$$

Note that $\tilde{\gamma}$ is a closed curve if and only if γ is closed. In Fig. 2 we picture a path (the dotted sites of the left-hand side) and the corresponding curve (on the right-hand side picture).

Note that we may have self-intersecting paths as in Fig. 3.

Given two paths $\gamma = (i_0, \ldots, i_k)$ and $\delta = (j_0, \ldots, j_h)$, if $i_k = j_0$ and the sequence $(i_0, \ldots, i_k, j_1, \ldots, j_h)$ is a path, the latter is called the *concatenation* of γ and δ and it is noted by $\gamma * \delta$.

We note that for each s the intersection $(i_s + [0, 1]^2) \cap (i_{s+1} + [0, 1]^2)$ is a segment with endpoints $\{\alpha_s, \beta_s\} \in \mathcal{N}$; then, given a spin system $\{c_{ij}\}$, for each path $\gamma = (i_0, \ldots, i_k)$ we

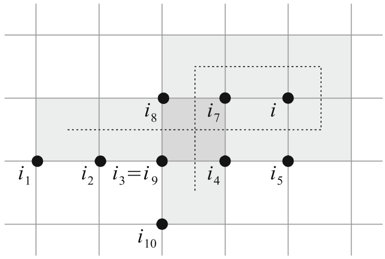

Fig. 3 Example of a self-intersecting path

can define the *number of antiferromagnetic bonds* of γ as

$$\mu(\gamma) = \mu(\gamma, \{c_{ij}\}) = \#\{s \in \{0, \ldots, k-1\} : c_{\alpha_s \beta_s} = -1\}. \tag{2}$$

If $\tilde{\gamma}$ is the curve corresponding to γ defined above, then the number $\mu(\gamma)$ counts the antiferromagnetic interactions "intersecting" $\tilde{\gamma}$ (see Fig. 2).

Definition 2 (*Separating paths*) A path γ of length k is a *separating path* for a spin system $\{c_{ij}\}$ if $\mu(\gamma) > k/2$.

Definition 3 (*α-separating paths*) Given $\alpha \in (0, \frac{1}{2}]$ we say that a path γ of length k is an *α-separating path* for a spin system $\{c_{ij}\}$ if $\mu(\gamma) > \alpha k$.

Remark 4 The terminology separating path evokes the fact that only closed separating paths may enclose (separate) regions where a minimal $\{u_i\}$ is constant. Indeed, if we have $u_i = 1$ on a finite set A of nodes in $\mathbb{Z}^2$ which is connected (i.e., for every pair i, j of points in A there is a path of points in A with i as initial point and j as final point) and $u_i = -1$ on all neighbouring nodes, then the boundary of A (i.e., the set of points $i \in A$ with a nearest neighbour not in A) determines a path. If such a path is not separating then the function $\tilde{u}$ defined as $\tilde{u}_i = -u_i$ for $i \in A$ and $\tilde{u}_i = u_i$ elsewhere has an energy strictly lower than u.

Remark 5 For a path γ of length $l(\gamma) = k$ the probability that γ be separating can be estimated as follows

$$\mathbf{P}\{\mu(\gamma) > k/2\} \le p^{k/2} 2^k. \tag{3}$$

Indeed, the probability that c_{ij} is equal to -1 at $k/2$ fixed places is equal to $p^{k/2}$. Since $\binom{k}{k/2}$ does not exceed 2^k, the desired estimates follows.

The following statement gives un upper bound for the length of separating paths in a large square $[0, n]^2$, $n \gg 1$. This is a Percolation Theory result, and its proof relies on the so-called path-counting argument that is widely used in Percolation Theory, see for instance [26, Section 1.4], [24].

Lemma 6 *There exists $p_0 > 0$ such that for any $\varkappa > 0$ and for all $p < p_0$ almost surely for sufficiently large n in a cube $Q_n = [0, n]^2$ there is no separating path γ with $l(\gamma) \ge (\log(n))^{1+\varkappa}$.*

Proof The number of paths of length k starting at the origin is not greater than 3^k. Therefore, in view of (3) the probability that there exists a separating path of length k that starts at the origin is not greater than $p^{k/2} 2^k 3^k$. Letting $p_0 = (1/12)^2$ we have

$$p^{k/2} 2^k 3^k \le 2^{-k} \qquad \text{for all } p \le p_0.$$

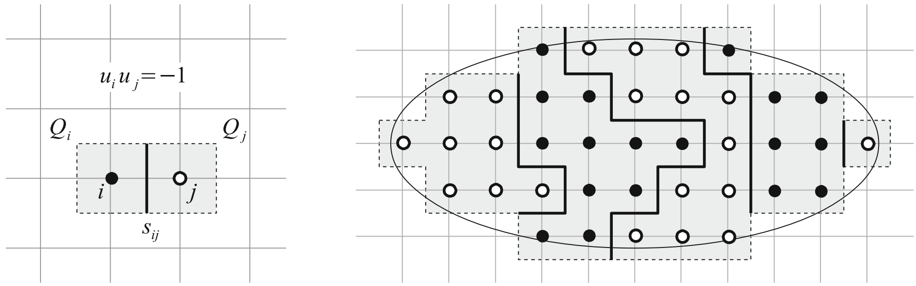

Fig. 4 Left: construction of s_{ij}; right: representation of the interface $\Sigma(u)$, with D the interior of the ellipse and $q(D)$ the grey region

Then, if $p \leq p_0$, the probability that there exists a separating path of length k in a cube Q_n does not exceed $n^2 2^{-k}$. For $k \geq \log(n)^{1+\varkappa}$ this yields

$$\mathbf{P}\{\text{there exists a separating path } \gamma \subset Q_n \text{ of length } k\} \\ \leq n^2 2^{-\log(n)^{1+\varkappa}} = n^{2-c_1 \log(n)^{\varkappa}}$$

with $c_1 = \log 2$. Finally, summing up in k over the interval $[\log(n)^{1+\varkappa}, n^2]$ we obtain

$$\mathbf{P}\{\text{there exists a separating path } \gamma \subset Q_n \text{ such that } l(\gamma) \geq \log(n)^{1+\varkappa}\} \\ \leq n^{4-c_1 \log(n)^{\varkappa}}.$$

Since for large n the right-hand side here decays faster than any negative power of n, the desired statement follows from the Borel-Cantelli lemma. □

In the same way one can prove the following statement.

Lemma 7 *For any $\alpha \in (0, \frac{1}{2}]$ there exists $p_0 = p_0(\alpha) > 0$ such that for any $\varkappa > 0$ and for all $p < p_0$ almost surely for sufficiently large n in a cube $Q_n = [0, n]^2$ there is no α-separating path γ with $l(\gamma) \geq (\log(n))^{1+\varkappa}$.*

2.2 Geometry of Minimizers in the Random Case

Let D be a bounded open subset of $\mathbb{R}^2$ and $u\colon D \cap \mathbb{Z}^2 \to \{\pm 1\}$. Then, denoting by $\mathcal{N}(D)$ the set of nearest neighbors in D, $F(u, D)$ is defined by

$$F(u, D) = -\sum_{\{i,j\} \in \mathcal{N}(D)} c_{ij} u_i u_j. \tag{4}$$

Note that the energy depends on ω through c_{ij}. We will characterize the almost-sure (quenched) behaviour of ground states for such energies.

We define the *interface* $S(u)$ as

$$S(u) = S(u; D) = \{\{i, j\} \in \mathcal{N}(D) : u_i u_j = -1\};$$

we associate to each pair $\{i, j\} \in S(u)$ the segment of the dual lattice $s_{ij} = \overline{Q}_i \cap \overline{Q}_j$, where Q_i is the coordinate unit open square centered at i, and consider the set

$$\Sigma(u) = \Sigma(u; D) = \bigcup_{\{i,j\} \in S(u)} s_{ij}. \tag{5}$$

If we extend the function u in $\bigcup_{i \in D \cap \mathbb{Z}^2} Q_i$ by setting $u = u_i$ in Q_i, and define

$$q(D) = \mathrm{int}\Big(\bigcup_{i \in D \cap \mathbb{Z}^2} \overline{Q}_i \Big), \tag{6}$$

where $\mathrm{int}(\cdot)$ denotes the interior part of a subset of $\mathbb{R}^2$, then the set $\Sigma(u) \cap q(D)$ turns out to be the jump set (i.e., the set of discontinuity points) of u and we can write

$$\Sigma(u) = \partial \overline{\{u = 1\}} \cap \partial \overline{\{u = -1\}}.$$

A pictorial explanation of the objects just introduced is contained in Fig. 4.

In the following remark we recall some definitions and classical results related to the notion of graph which will be useful to establish properties of the connected components of $\partial \overline{\{u = 1\}}$. For references on this topic, see for instance [5].

Remark 8 (*Graphs and two-coloring*) We say that a triple $G = (V, E, r)$ is a *multigraph* when V (*vertices*) and E (*edges*) are finite sets and r (*endpoints*) is a map from E to $V \otimes V$, where $\otimes$ denotes the symmetric product. The *order* of a vertex v is $\#\{e \in E : r(e) = x \otimes v \text{ for some } x \in V\} + \#\{e \in E : r(e) = v \otimes v\}$, so that the loops are counted twice. A *walk* in the graph G is a sequence of edges $(e_1, \dots, e_n)$ such that there exists a sequence of vertices $(v_0, \dots, v_n)$ with the property $r(e_i) = v_{i-1} \otimes v_i$ for each i; if moreover $v_n = v_0$, then the walk is called a *circuit*. The multigraph G is *connected* if given $v \neq v'$ in V there exists a walk connecting them, that is a walk such that $v_0 = v$ and $v_n = v'$ in the corresponding sequence of vertices.

We say that G is *Eulerian* if there is a circuit containing every element of E exactly once (Eulerian circuit). A classical theorem of Euler (see [5, Chap. 3] and [22] for the original formulation) states that G is Eulerian if and only if G is connected and the order of every vertex is even.

A multigraph G is *embedded in* $\mathbb{R}^2$ if $V \subset \mathbb{R}^2$ and the edges are simple curves in $\mathbb{R}^2$ such that the endpoints belong to V and two edges can only intersects at the endpoints. An embedded graph is Eulerian if and only if the union of the edges $\bigcup_{e \in E} e$ is connected, and its complementary in $\mathbb{R}^2$ can be *two-colored*, that is $\mathbb{R}^2 \setminus \bigcup_{e \in E} e$ is the union of two disjoint sets B and W such that $\partial B = \partial W = \bigcup_{e \in E} e$.

Remark 9 (*Eulerian circuits in* $\partial \overline{\{u = 1\}}$) Let C be a connected component of $\partial \overline{\{u = 1\}}$. We can see C as a connected embedded graph whose vertices are the points in $(\mathbb{Z}^2 + (1/2, 1/2)) \cap C$ and two vertices share an edge if there is a unit segment in C connecting them. By construction, $\mathbb{R}^2 \setminus C$ can be *two-colored*, hence C is an Eulerian circuit (see Remark 8). Recalling the definition of path and the definition (1), this corresponds to say that there exists a closed path η such that $\tilde{\eta} = C$.

The asymptotic description of minimizers is given by the following result.

Theorem 10 *Let G be a Lipschitz bounded domain in $\mathbb{R}^2$ and let p_0 be given by Lemma 6. Let $p < p_0$, and let u_ε be a minimizer for $F(\cdot, \frac{1}{\varepsilon} G)$. Then for any $\varkappa > 0$ almost surely for all sufficiently small $\varepsilon > 0$ either $\overline{\{u_\varepsilon = 1\}}$ or $\overline{\{u_\varepsilon = -1\}}$ is composed of connected components K_i such that the length of the boundary of each K_i is not greater than $|\log(\varepsilon)|^{1+\varkappa}$.*

The proof of the theorem essentially relies on Proposition 11 below.

We say that a path $\gamma \in \mathcal{P}_k$ *is in the interface* $S(u)$ if the corresponding $\tilde{\gamma}$ is contained in $\Sigma(u)$.

Proposition 11 *For any* $\Lambda > 0$ *a.s. for sufficiently small* $\varepsilon > 0$ *and for any open bounded subset* $D \subset (-\Lambda/\varepsilon, \Lambda/\varepsilon)^2$ *such that the distance between the connected components of* $\partial q(D)$ *is greater than* $|\log(\varepsilon)|^{1+\varkappa}$ *for a minimizer* u *of* $F(\cdot, D)$ *there is no path in the interface* $S(u)$ *of length greater than* $|\log(\varepsilon)|^{1+\varkappa}$.

Proof We use the so called Peierls argument. Let $\gamma \in \mathcal{P}_k$ be a path in the interface $S(u)$ with $k \geq |\log(\varepsilon)|^{1+\varkappa}$. We denote by C the connected component of $\partial\overline{\{u = 1\}}$ containing $\tilde{\gamma}$. Remark 9 ensures that $C = \tilde{\eta}$ where η is a closed path; hence, up to extending γ in η, we can assume without loss of generality that γ is a path of maximal length in the interface $S(u)$.

We start by considering the case when all the connected components of $q(D)$ are simply connected. In this case the Peierls argument applies in a straightforward way. First we show that there exists a closed path $\sigma = \sigma(\gamma)$ such that $\tilde{\gamma} \subset \tilde{\sigma} \subset C$ and

$$\tilde{\sigma} \cap \Sigma(u) = \tilde{\gamma}. \tag{7}$$

Indeed, if γ is closed, then we set $\sigma = \gamma$. Otherwise, $\tilde{\gamma}$ connects two points in $\partial q(D)$. Since all the connected components of $q(D)$ are simply connected, these endpoints belong to the same connected component of $\partial q(D)$. Then we can choose a path δ such that $\tilde{\delta}$ lies in $\partial q(D)$ and has the same endpoints as $\tilde{\gamma}$. Recalling the notion of concatenation of paths, we can define σ as $\gamma * \delta$. This yields (7).

Since σ is a closed path, then $\tilde{\sigma}$ is a closed properly self-intersecting curve so that all the vertices of the corresponding embedded graph have even order. Remark 8 ensures that the embedded graph corresponding to $\tilde{\sigma}$ is Eulerian, hence its complementary $\mathbb{R}^2 \setminus \tilde{\sigma}$ can be two-colored, that is it is the union of two disjoint sets B and W such that $\partial B = \partial W = \tilde{\sigma}$. Setting $\tilde{u}$ as the extension to $\cup_{i \in D \cap \mathbb{Z}^2} Q_i$ of the function

$$\tilde{u}_i = \begin{cases} u_i & \text{in } B \cap D \cap \mathbb{Z}^2 \\ -u_i & \text{in } W \cap D \cap \mathbb{Z}^2 \end{cases}$$

it follows that $F(u, D) - F(\tilde{u}, D) = 2(l(\gamma) - 2\mu(\gamma))$, where $\mu(\gamma)$ stands for the number of antiferromagnetic interactions in γ as defined in (2). Since u minimizes $F(\cdot, D)$, we can conclude that $\mu(\gamma) \geq l(\gamma)/2$; that is, γ is a separating path of length greater than $|\log(\varepsilon)|^{1+\varkappa}$, contradicting Lemma 6 and completing the proof in the special case.

In the general case we show that there exists a closed path $\sigma = \sigma(\gamma)$ such that $\tilde{\gamma} \subset \tilde{\sigma} \subset C$ and the following property holds:

$$\begin{aligned} &\text{there exist } r \text{ different paths } \eta_1, \ldots, \eta_r \text{ such that } \tilde{\sigma} \cap \Sigma(u) = \bigcup_{t=1}^r \tilde{\eta}_t \\ &\qquad\qquad\text{and } \; l(\eta_t) \geq |\log(\varepsilon)|^{1+\varkappa}/2 \text{ for all } t. \end{aligned} \tag{8}$$

If γ is closed or $\tilde{\gamma}$ connects two points in the same connected componnets of $\partial q(D)$, then we define σ in the same was as in the special case considered above. In this case $r = 1$ and $\gamma = \eta_1$.

It remains to construct σ when the endpoints of $\tilde{\gamma}$ belong to different connected components of $\partial q(D)$. We consider the set V of the connected components of $\partial q(D)$ and the set E of the connected components of $\tilde{\eta} \cap \Sigma(u)$ (note that $\tilde{\gamma} \in E$). By the existence of the path η, each element of E is a curve connecting two (possibly equal) elements of V, then (V, E) is a multigraph. Since $\tilde{\eta}$ is a closed curve containing $\tilde{\gamma}$, it realizes in the graph an Eulerian circuit containing $\tilde{\gamma}$. Therefore, there exists a minimal Eulerian circuit $(\tilde{\gamma} = \tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r)$ and, by minimality, the order of each vertex touched by this circuit is 2 (see Remark 8). Denoting by Δ_t the vertex shared by $\tilde{\eta}_t$ and $\tilde{\eta}_{t+1}$ for $t < r$, and by Δ_r the vertex shared by $\tilde{\eta}_r$ and $\tilde{\eta}_1$, for each t we can find a path δ_t such that $\tilde{\delta}_t \subset \Delta_t$ and such that the path $\sigma = \delta_1 * \eta_1 * \cdots * \delta_r * \eta_r$ is closed and satisfies property (8).

Then there exist disjoint sets B and W such that $\mathbb{R}^2 \setminus \tilde{\sigma} = B \cap W$ and $\partial B = \partial W = \tilde{\sigma}$. Setting

$$\tilde{u}_i = \begin{cases} u_i & \text{in } B \cap D \cap \mathbb{Z}^2 \\ -u_i & \text{in } W \cap D \cap \mathbb{Z}^2 \end{cases} \tag{9}$$

it follows that $F(u, D) - F(\tilde{u}, D) = 2\sum_{t=1}^{r}(l(\eta_t) - 2\mu(\eta_t))$. Since u minimizes $F(\cdot, D)$, then, for at least one index t, $\mu(\eta_t) \geq l(\eta_t)/2$; that is, η_t is a separating path of length greater than $|\log(\varepsilon)|^{1+\varkappa}$. This contradicts Lemma 6. □

We turn to the proof of Theorem 10.

Proof of Theorem 10 Letting $G_\varepsilon = q(\frac{1}{\varepsilon}G)$, we consider the connected components of the interface $\Sigma(u_\varepsilon)$. Since each of them corresponds to a path, they are either closed curves, denoted by C^i_ε for $i = 1, \ldots, n$, or curves with the endpoints in ∂G_ε, denoted by D^j_ε for $j = 1, \ldots, m$. Since for ε small enough the distance between two connected components of ∂G_ε is greater than $|\log(\varepsilon)|^{1+\varkappa}$, Proposition 11 ensures that in both cases the length of such curves is less than $|\log(\varepsilon)|^{1+\varkappa}$.

The distance between the endpoints of a component D^j_ε is less than $|\log(\varepsilon)|^{1+\varkappa}$, and, since G is Lipschitz, for ε small enough we can find a path in ∂G_ε with the same endpoints and length less than $\tilde{C}|\log(\varepsilon)|^{1+\varkappa}$. This gives a closed path S^j_ε with length less than $\tilde{C}|\log(\varepsilon)|^{1+\varkappa}$ containing D^j_ε.

The set $\mathbb{R}^2 \setminus \left(\bigcup_i C^i_\varepsilon \cup \bigcup_j D^j_\varepsilon\right)$ has exactly one unbounded connected component, which we call P_ε. The function u_ε is constant in $P_\varepsilon \cap G_\varepsilon$. Assuming that this constant value is 1, then $\partial\overline{\{u_\varepsilon = -1\}}$ is contained in $\left(\bigcup_i C^i_\varepsilon \cup \bigcup_j D^j_\varepsilon\right)$ and the boundary of every connected component K^i_ε of $\overline{\{u_\varepsilon = -1\}}$ has length less than $C|\log(\varepsilon)|^{1+\varkappa}$. □

Consider now an arbitrary configuration u_ε such that

$$F\left(u_\varepsilon, \frac{1}{\varepsilon}G\right) \leq \min_u F\left(u, \frac{1}{\varepsilon}G\right) + C|\log(\varepsilon)| \tag{10}$$

for some $C > 0$. The asymptotic structure of this configurations is given by

Theorem 12 *Let G be a Lipschitz bounded domain in $\mathbb{R}^2$ and let $p_0(\frac{1}{4})$ be given by Lemma 7. Let $p < p_0$, and let u_ε satisfy estimate (10) with some constant $C > 0$. Then for any $\varkappa > 0$ almost surely for all sufficiently small $\varepsilon > 0$ either $\overline{\{u_\varepsilon = 1\}}$ or $\overline{\{u_\varepsilon = -1\}}$ is composed of connected components K_i such that the length of the boundary of each K_i is not greater than $|\log(\varepsilon)|^{1+\varkappa}$.*

The proof of this theorem relies on the statement similar to that of Proposition 11

Proposition 13 *For any $\Lambda > 0$ and any $C > 0$ a.s. for sufficiently small $\varepsilon > 0$, for any open bounded subset $D \subset (-\Lambda/\varepsilon, \Lambda/\varepsilon)^2$ such that the distance between the connected components of $\partial q(D)$ is greater than $|\log(\varepsilon)|^{1+\varkappa}$ and for any configuration u_ε that satisfies the inequality*

$$F(u_\varepsilon, D) \leq \min_u F(u, D) + C|\log(\varepsilon)| \tag{11}$$

there is no path in the interface $S(u_\varepsilon)$ of length greater than $|\log(\varepsilon)|^{1+\varkappa}$.

Proof The proof of this statement is similar to that of Proposition 11.

If we assume that there exists a path $\gamma \in \mathcal{P}_k$ in the interface $S(u_\varepsilon)$ with $k \geq |\log(\varepsilon)|^{1+\varkappa}$, then there exists a closed path $\sigma = \sigma(\gamma)$ such that $\tilde{\gamma} \subset \tilde{\sigma} \subset C$ and the following property holds:

$$\text{there exist } r \text{ different paths } \eta_1, \dots, \eta_r \text{ such that } \tilde{\sigma} \cap \Sigma(u_\varepsilon) = \bigcup_{t=1}^r \tilde{\eta}_t \\ \text{and } l(\eta_t) \geq |\log(\varepsilon)|^{1+\varkappa}/2 \text{ for all } t. \tag{12}$$

If we define $\tilde{u}_\varepsilon$ by (9) then $F(u_\varepsilon, D) - F(\tilde{u}_\varepsilon, D) = 2\sum_{t=1}^r (l(\eta_t) - 2\mu(\eta_t))$. Since $\mu(\eta_t) \leq \frac{1}{4} l(\eta_t)$ for all sufficiently small ε, then we have

$$F(u_\varepsilon, D) - F(\tilde{u}_\varepsilon, D) \geq \sum_{t=1}^r l(\eta_t) \geq |\log(\varepsilon)|^{1+\varkappa}.$$

This contradicts inequality (11). □

The proof of Theorem 12 is similar to that of Theorem 10. We leave the details to the reader.

We now consider the Ising model at low temperature. Define the configuration probability by

$$P_\beta(u) = (Z_\beta)^{-1} e^{-\beta F(u, \frac{1}{\varepsilon} G)},$$

where $\beta > 0$ is inverse temperature, and $Z_\beta = \sum_v e^{-\beta F(v, \frac{1}{\varepsilon} G)}$. Denote $P_{\beta,\max}$ the maximum configuration probability $\max_u P_\beta(u)$.

As an immediate consequence of Theorem 12 we obtain the following statement.

Theorem 14 *Under the assumptions of Theorem 12 if $p < p_0$ and the configuration probability $P_\beta(u_\varepsilon)$ of u_ε satisfies the estimate*

$$P_\beta(u_\varepsilon) \geq e^{-C\beta|log(\varepsilon)|} P_{\beta,\max}$$

with some constant $C > 0$, then for any $\varkappa > 0$ almost surely for all sufficiently small $\varepsilon > 0$ either $\overline{\{u_\varepsilon = 1\}}$ or $\overline{\{u_\varepsilon = -1\}}$ is composed of connected components K_i such that the length of the boundary of each K_i is not greater than $|\log(\varepsilon)|^{1+\varkappa}$.

3 Periodic Media

We now turn our attention to a deterministic analog of the problem discussed above, where random coefficients are substituted by periodic coefficients and the probability of having antiferromagnetic interactions is replaced by their percentage.

3.1 Estimates on the Number of Antiferromagnetic Interactions Along a Path

In order to prove a deterministic analogue of Theorem 10, we need to give an estimate of the length of separating paths corresponding to the result stated in Lemma 6. We start with the definition of a periodic spin system in the deterministic case given on the lines of Definition 1.

Definition 15 (*Periodic spin system*) With fixed $N \in \mathbb{N}$, a deterministic (ferromagnetic/antiferromagnetic) *spin system* is a function $c(\{i, j\}) = c_{ij} \in \{\pm 1\}$ defined on $\mathcal{N}$.

The pairs $\{i, j\}$ with $c_{ij} = +1$ are called *ferromagnetic bonds*, the pairs $\{i, j\}$ with $c_{ij} = -1$ are called *antiferromagnetic bonds*. We say that a spin system is *N-periodic* if

$$c(\{i, j\}) = c(\{i + (N, 0), j + (N, 0)\}) = c(\{i + (0, N), j + (0, N)\}).$$

In the sequel of this section, when there is no ambiguity we use the same terminology and notation concerning the random case given in Sect. 2.

Definition 16 (*Spin systems with given antiferro proportion*) For $p \in (0, 1)$ we consider the set $\mathcal{C}_p(N)$ of N-periodic spin systems $\{c_{ij}\}$ such that the number of antiferromagnetic interactions in $[0, N]^2$ is $\lfloor 2pN^2 \rfloor$, and for any $\lambda \in (0, 1)$ we define

$$\mathcal{B}_p^\lambda(N) = \Big\{\{c_{ij}\} \in \mathcal{C}_p(N) : \exists\, \gamma \in \mathcal{P}_k(N) \text{ separating path for } \{c_{ij}\} \text{ with } k \geq \lambda N\Big\} \quad (13)$$

where $\mathcal{P}_k(N)$ is the set of paths $\gamma = (i_0, \dots, i_k) \in \mathcal{P}_k$ such that $i_s \in [0, N]^2$ for each s.

Proposition 17 *There exists $p_0 > 0$ such that for every $p < p_0$ and $\lambda > 0$:*

$$\lim_{N \to +\infty} \frac{\#\mathcal{B}_p^\lambda(N)}{\#\mathcal{C}_p(N)} = 0.$$

Proof We fix a path $\gamma \in \mathcal{P}_k(N)$ with $\lambda N \leq k \leq 4p_N N^2$, where $p_N = \frac{\lfloor 2pN^2 \rfloor}{2N^2}$. Then, the number of spin systems $\{c_{ij}\}$ in $\mathcal{C}_p(N)$ for which γ is a separating path depends only on k and it is given by

$$f_p(k, N) = \sum_{j=k/2}^{\min\{k, 2p_N N^2\}} \binom{k}{j} \binom{2N^2 - k}{2p_N N^2 - j}.$$

Since

$$\#\mathcal{B}_p^\lambda(N) \leq \sum_{k=\lfloor \lambda N \rfloor}^{4p_N N^2} \#\{\mathcal{P}_k(N)\} f_p(k, N) \;\leq \sum_{k=\lfloor \lambda N \rfloor}^{4p_N N^2} 3^k N^2 f_p(k, N)$$

$$\#\mathcal{C}_p(N) = \binom{2N^2}{2p_N N^2}$$

we get the estimate

$$\frac{\#\mathcal{B}_p^\lambda(N)}{\#\mathcal{C}_p(N)} \leq \sum_{m=m(\lambda)}^{m(N)} \left(\sum_{k=2^m N}^{2^{m+1} N} 3^k N^2 f_p(k, N) \binom{2N^2}{2p_N N^2}^{-1} \right),$$

where $m(\lambda) = \lfloor \log_2(\lambda) \rfloor - 1$ and $m(N) = \lfloor \log_2(4p_N N) \rfloor - 1$. Noting that

$$\binom{2p_N N^2}{j} \binom{2(1 - p_N)N^2}{k - j} \leq \binom{2p_N N^2}{k/2} \binom{2(1 - p_N)N^2}{k/2}$$

for each $j = k/2, \dots, \min\{k, 2p_N N^2\}$, we get

$$\begin{aligned} f_p(k, N) \binom{2N^2}{2p_N N^2}^{-1} &= \sum_{j=k/2}^{\min\{k, 2p_N N^2\}} \binom{k}{j} \binom{2N^2 - k}{2p_N N^2 - j} \binom{2N^2}{2p_N N^2}^{-1} \\ &= \sum_{j=k/2}^{\min\{k, 2p_N N^2\}} \binom{2p_N N^2}{j} \binom{2(1 - p_N)N^2}{k - j} \binom{2N^2}{k}^{-1}, \quad (14) \\ &\leq \Big(\min\{k, 2p_N N^2\} - \frac{k}{2}\Big) g_p(k, N) \end{aligned}$$

where

$$g_p(k,N)=\binom{2p_NN^2}{k/2}\binom{2(1-p_N)N^2}{k/2}\binom{2N^2}{k}^{-1}.$$

Now, we prove an estimate for $g_p(k,N)$, which we formulate as a separate lemma.

Lemma 18 *For any $k, N\in\mathbb{N}$ such that $k\le 4pN^2$ and for any $p\in(0,1/2)$ we have*

$$g_p(k,N)\le C(p)N^8(\theta(p))^k$$

with $\theta(p)=2e\sqrt{p(1-p)}$.

Proof of Lemma 18 Recalling that $n^ne^{1-n}\le n!\le n^{n+1}e^{1-n}$ for any n in $\mathbb{N}$, we get

$$\binom{2N^2}{k}^{-1}=\frac{k!(2N^2-k)!}{(2N^2)!}\le ek(2N^2-k)\left(\frac{k}{2N^2-k}\right)^k\left(\frac{2N^2-k}{2N^2}\right)^{2N^2}$$

and for $k\neq 4p_NN^2$

$$\begin{aligned}\binom{2p_NN^2}{k/2}&=\frac{(2p_NN^2)!}{(k/2)!(2p_NN^2-k/2)!}\\&\le\frac{2p_NN^2}{e}\left(2p_NN^2-\frac{k}{2}\right)^{k/2}\left(\frac{k}{2}\right)^{-k/2}\left(\frac{2p_NN^2}{2p_NN^2-k/2}\right)^{2p_NN^2}.\end{aligned}$$

Hence, the following estimate holds

$$\begin{aligned}g_p(k,N)\le{}&\frac{4p_N(1-p_N)}{e}N^4(2N^2-k)k\left(2\frac{(2p_NN^2-\frac{k}{2})^{1/2}(2(1-p_N)N^2-\frac{k}{2})^{1/2}}{2(1-p_N)N^2}\right)^k\\&\left(\frac{2N^2-k}{2N^2}\right)^{2N^2}\left(\frac{2p_NN^2}{2p_NN^2-k/2}\right)^{2p_NN^2}\left(\frac{2(1-p_N)N^2}{2(1-p_N)N^2-k/2}\right)^{2(1-p_N)N^2}.\end{aligned}$$

Recalling the inequalities

$$\left(\frac{x-a}{x}\right)^a e^{-a}\le\left(\frac{x-a}{x}\right)^x\le\left(\frac{x-a}{x}\right)^a$$

for a, x such that $0<a<x$, we get for any $k\neq 4p_NN^2$

$$\begin{aligned}\left(\frac{2N^2-k}{2N^2}\right)^{2N^2}&\le\left(\frac{2N^2-k}{2N^2}\right)^k\\\left(\frac{2p_NN^2}{2p_NN^2-k/2}\right)^{2p_NN^2}&\le\left(\frac{2p_NN^2}{2p_NN^2-k/2}\right)^{k/2}e^{k/2}\\\left(\frac{2(1-p_N)N^2}{2(1-p_N)N^2-k/2}\right)^{2(1-p_N)N^2}&\le\left(\frac{2(1-p_N)N^2}{2(1-p_N)N^2-k/2}\right)^{k/2}e^{k/2}.\end{aligned}$$

Since $p_N(1-p_N)\le p(1-p)$ for $p\in(0,1/2)$, the previous estimates give

$$\begin{aligned}g_p(k,N)&\le\frac{4p(1-p)}{e}N^4(2N^2-k)k\left(2e\sqrt{p_N(1-p_N)}\right)^k\\&\le\frac{32p^2(1-p)}{e}N^8\left(2e\sqrt{p_N(1-p_N)}\right)^k\end{aligned}$$

$$\leq \frac{32p^2(1-p)}{e} N^8 \left(2e\sqrt{p(1-p)}\right)^k$$

concluding the proof for $k \neq 4p_N N^2$. Note that $\theta(p) = 2e\sqrt{p(1-p)} \to 0$ for $p \to 0$.

It remains to check the case $k = 4p_N N^2$. Noting that for $p < 1/2$

$$\begin{aligned} g_p(4p_N N^2, N) &= \frac{(4p_N N^2)!(2(1-p_N)N^2)!}{(2p_N N^2)!(2N^2)!} \\ &\leq 8p_N(1-p_N)N^4(1-p_N)^{2N^2}\left(\frac{2p_N}{\sqrt{p_N(1-p_N)}}\right)^{4p_N N^2} \\ &\leq 8p(1-p)N^4\left(2e\sqrt{p(1-p)}\right)^{4p_N N^2} \end{aligned}$$

the thesis of Lemma 18 follows. □

Now, Lemma 18 allows to conclude the proof of the proposition. Indeed, applying the estimate on $g_p(k, N)$, we get from inequality (14)

$$\begin{aligned} \sum_{k=2^m N}^{2^{m+1}N} 3^k N^2 f_p(k, N) \binom{2N^2}{2p_N N^2}^{-1} &\leq pC(p)N^{12} \sum_{k=2^m N}^{2^{m+1}N} (3\theta(p))^k \\ &= pC(p)N^{12} \sum_{t=2^m}^{2^{m+1}} ((3\theta(p))^N)^t \\ &= pC(p)N^{12}((3\theta(p))^N)^{2^m} \frac{1-(3\theta(p))^{(2^m+1)N}}{1-(3\theta(p))^N} \\ &\leq C(p)N^{12}(3\theta(p))^{2^m N} \end{aligned}$$

for $p < 1/2$ and for N large enough (independent on m).

By summing over m, we get

$$\begin{aligned} \frac{\#\mathcal{B}_p^\lambda(N)}{\#\mathcal{C}_p(N)} &\leq C(p)N^{12} \sum_{m=m(\lambda)}^{m(N)} (3\theta(p))^{2^m N} \\ &\leq C(p)N^{12}(3\theta(p)^N)^{2^{m(\lambda)}-1} \sum_{m=m(\lambda)}^{m(N)} (3\theta(p)^N)^{2^m - 2^{m(\lambda)}+1} \\ &\leq C(p)N^{12}(3\theta(p))^{(2^{m(\lambda)}-1)N} \sum_{t=1}^{+\infty} ((3\theta(p))^N)^t \\ &\leq C(p)N^{12} \frac{(3\theta(p))^{2^{m(\lambda)}N}}{1-(3\theta(p))^N} \\ &\leq 2C(p)N^{12}((3\theta(p))^{2^{m(\lambda)}})^N \end{aligned} \tag{15}$$

which goes to 0 as $N \to +\infty$ if $3\theta(p) = 6e\sqrt{p(1-p)} < 1$. □

Remark 19 (*Translations*) Denoting by $\tilde{\mathcal{B}}_p^\lambda(N)$ the set of N-periodic spin systems $\{c_{ij}\}$ such that there exists a separating path for $\{c_{ij}\}$ in $z + [0, N]^2$ for some $z \in \mathbb{Z}^2$, then the estimate (15) implies

$$\lim_{N \to +\infty} \frac{\#\tilde{\mathcal{B}}_p^\lambda(N)}{\#\mathcal{C}_p(N)} = 0.$$

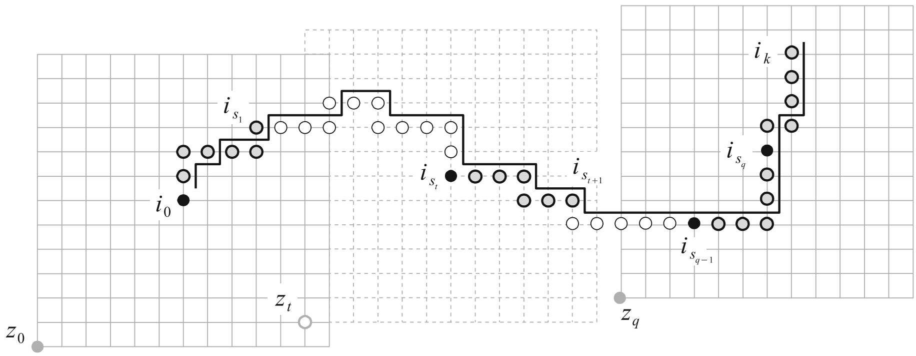

Fig. 5 Decomposition of γ

Now, we state the deterministic analogue of Lemma 6.

Proposition 20 *If the N-periodic spin system $\{c_{ij}\}$ belongs to $\mathcal{C}_p \setminus \tilde{\mathcal{B}}_p^{1/2}(N)$, then there is no separating path in $\mathbb{Z}^2$ of length greater than $N/2$.*

Proof Let $\gamma = (i_0, \ldots, i_k)$ be a path in $\mathcal{P}_k$ with $k \geq N/2$. We decompose γ as a concatenation of paths $\gamma_1 * \cdots * \gamma_{q-1} * \gamma_q$ with $l(\gamma_t) \geq N/2$ and each γ_t contained in a coordinate square $z + [0, N]^2$ for some $z \in \mathbb{Z}^2$ (see Fig. 5).

If N is even, setting $q = \lfloor \frac{2k}{N} \rfloor$ and $s_t = tN/2$ for $t = 0, \ldots, q$, we define

$$\gamma_t = (i_{s_{t-1}}, \ldots, i_{s_t}) \quad \text{for } t = 1, \ldots, q-1 \quad \text{and} \quad \gamma_q = (i_{s_{q-1}}, \ldots, i_k). \tag{16}$$

In this way, setting

$$z_t = i_{s_{t-1}} - \left(\frac{N}{2}, \frac{N}{2}\right) \quad \text{for } t = 1, \ldots, q-1$$
$$z_q = i_{s_q} - \left(\frac{N}{2}, \frac{N}{2}\right),$$

it follows that for any $t = 1, \ldots q$ γ_t is a path of length $l(\gamma_t)$ greater than $N/2$ contained in $z_t + [0, N]^2$. Since $\{c_{ij}\} \notin \tilde{\mathcal{B}}_p^{1/2}(N)$, the number of antiferromagnetic interactions $\mu(\gamma_t)$ is less than $l(\gamma_t)/2$ for any t. Hence $\mu(\gamma) \leq k/2$.

If N is odd, we pose $q = \lfloor \frac{2k}{N+1} \rfloor$ and $s_t = t(N+1)/2$ for $t = 0, \ldots, q$; defining the adjacent paths γ_t as in (16), by setting

$$z_t = i_{s_{t-1}} - \left(\frac{N-1}{2}, \frac{N-1}{2}\right) \quad \text{for } t = 1, \ldots, q-1$$
$$z_q = i_{s_q-1} - \left(\frac{N-1}{2}, \frac{N-1}{2}\right)$$

the result follows as in the previous case. □

3.2 Geometry of Minimizers

We conclude by stating the results concerning the geometry of the ground states, corresponding to Proposition 11 and Theorem 10 respectively. The main result states that for spin systems not in $\mathcal{B}_p^{1/2}$ the minimizers of F on large sets are characterized by a majority phase. Remark 19 then ensures that this is a generic situation for N large.

Theorem 21 *Let $N \in \mathbb{N}$, and let $\{c_{ij}\}$ be a N-periodic distribution of ferro/antiferromagnetic interactions such that $\{c_{ij}\} \notin \tilde{\mathcal{B}}_p^{1/2}$. Let G be a Lipschitz bounded open set and let u_ε be a minimizer for $F(\cdot, \frac{1}{\varepsilon}G)$. Then there exists a constant C depending only on G such that either $\overline{\{u_\varepsilon = 1\}}$ or $\overline{\{u_\varepsilon = -1\}}$ is composed of connected components K_ε^i such that the length of the boundary of each K_ε^i is not greater than CN.*

As for Theorem 10, the proof relies on the estimate of the length of paths in the interface, which in this case reads as follows.

Proposition 22 *Let $N \in \mathbb{N}$, and let $\{c_{ij}\}$ be a N-periodic distribution of ferro/antiferromagnetic interactions such that $\{c_{ij}\} \notin \tilde{\mathcal{B}}_p^{1/2}$. Let D be an open bounded subset of $\mathbb{R}^2$ such that the distance between the connected components of $\partial q(D)$ is greater than $N/2$. Let u be a minimizer for $F(\cdot, D)$. Then there is no path in the interface $S(u)$ of length greater than $N/2$.*

The steps of the proofs are exactly the same as in the random case, by substituting the applications of Lemma 6 with the corresponding applications of Proposition 20 (thus the logarithmic estimates become linear with N).

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